TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 325, Number 2, June 1991

ABSOLUTE CONTINUITY RESULTS FOR SUPERPROCESSES WITH SOME APPLICATIONS

STEVEN N. EVANS AND EDWIN PERKINS

ABSTRACT. Let X^1 and X^2 be instances of a measure-valued Dawson-Watanabe ξ-super process where the underlying spatial motions are given by a Borel right process, ξ, and where the branching mechanism has finite variance. A necessary and sufficient condition on X_0^1 and X_0^2 is found for the law of X_s^1 to be absolutely continuous with respect to the law of X_t^2. The conditions are the natural absolute continuity conditions on ξ, but some care must be taken with the set of times s, t being considered. The result is used to study the closed support of super-Brownian motion and give sufficient conditions for the existence of a nontrivial "collision measure" for a pair of independent super-Lévy processes or, more generally, for a super-Lévy process and a fixed measure. The collision measure gauges the extent of overlap of the two measures. As a final application, we give an elementary proof of the instantaneous propagation of a super-Lévy process to all points to which the underlying Lévy process can jump. This result is then extended to a much larger class of superprocesses using different techniques.

1. INTRODUCTION AND NOTATION

Before we can outline our results, we need to recall some salient details from Fitzsimmons [9] regarding the construction and regularity of the class of (ξ, φ)-superprocesses.

Suppose that E is a topological Lusin space (that is, a homeomorph of a Borel subset of a compact metric space), and $\mathscr{E}$ is the Borel σ-field of E. Denote by $M(E)$ the class of finite Borel measures on E. Define $\mathscr{M}(E)$ to be the σ-field of subsets of $M(E)$ generated by the maps $\mu \mapsto \langle \mu, f \rangle = \int \mu(dx) f(x)$ as f runs over $bp\mathscr{E}$, the class of bounded, nonnegative, $\mathscr{E}$-measurable functions. Let $\xi = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, \xi_t, P^x)$ be a Borel right Markov process with state space $(E, \mathscr{E})$ and semigroup $\{P_t\}$. Assume that $P_t 1 = 1$. Let $\varphi\colon E \times [0, \infty[\to \mathbb{R}$ be given by

$$\varphi(x, \lambda) = -b(x)\lambda - c(x)\lambda^2 + \int_0^\infty n(x, du)(1 - e^{-\lambda u} - \lambda u),$$

Received by the editors March 27, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60G30, 60G57, 60J25; Secondary 60G80, 60J60.
Key words and phrases. Superprocesses, absolute continuity, measure-valued diffusion, measure-valued branching process, random measure.
The work of the first author was supported in part by NSF Grant DMS 8701212. The work of the second author was supported in part by an NSERC operating grant.

where $c \geq 0$ and b are bounded and $\mathscr{E}$-measurable, and $n\colon E \times \mathscr{B}([0, \infty[) \to [0, \infty[$ is a kernel such that $\int_0^\infty n(\cdot, du)(u \vee u^2)$ is bounded.

For each $f \in bp\mathscr{E}$, the integral equation

$$v_t(x) = P_t f(x) + \int_0^t P_s(x, \varphi(\cdot, v_{t-s}))\, ds, \qquad t \geq 0, x \in E,$$

has a unique solution which we denote by $(t, x) \mapsto V_t f(x)$; and there exists a unique Markov kernel $\{Q_t\}$ on $(M(E), \mathscr{M}(E))$ with Laplace functionals

$$\int Q_t(\mu, d\nu) \exp(-\langle \nu, f \rangle) = \exp(-\langle \mu, V_t f \rangle)$$

for all $\mu \in M(E)$, $t \geq 0$, and $f \in bp\mathscr{E}$.

Write $M_0(E)$ for the set $M(E)$ topologized by the weak topology; that is, give $M(E)$ the weakest topology which makes all of the maps $\mu \mapsto \langle \mu, f \rangle$ continuous, where f runs through the bounded continuous functions on E. One can also consider a Ray-Knight compactification, $\overline{E}$ of E, and write $M_r(E)$ for $M(E)$ given the relative topology inherited from $M(\overline{E})$ with its weak topology. The Borel σ-fields of $M_0(E)$ and $M_r(E)$ coincide with $\mathscr{M}(E)$. There is a Markov process $X = (W, \mathscr{G}, \mathscr{G}_t, \Theta_t, X_t, \mathbb{P}^m)$ with the state space $(M(E), \mathscr{M}(E))$ and semigroup $\{Q_t\}$. Viewed as a process on $M_r(E)$, X is a Hunt process. Viewed as a process on $M_0(E)$, X is a right process and a Hunt process if ξ is. The process X is called the (ξ, φ)-superprocess.

Following the seminal paper of Watanabe [19], superprocesses have become the subject of an increasing amount of interest (see, for example, [3, 6, 7, 11, 17], and the references therein). While several of the results in [9] have precursors in the literature under more restrictive assumptions such as Feller hypotheses, we will use [9] as a general reference without mentioning earlier work.

The simplest and most studied superprocesses are those for which $\varphi(x, \lambda) = \text{constant} \cdot \lambda^2$, say $\varphi(x, \lambda) = -\lambda^2/2$. In this case, when ξ is a Feller process, X provides a description for the high-density limiting behavior of a population of individuals which have evolved in such a way that population size is a critical continuous time binary branching process—while in between births and deaths the individuals move around as independent copies of the process ξ, starting from the positions at which their parents died and gave them birth (see, for example, §9.4 of Ethier and Kurtz [8]). With this picture in mind, it is reasonable to expect that if ξ is such that when we look at a particle we cannot determine where it originated or when in its evolution we are observing it, then the same should in some sense be true for the ensemble described by X. The following result, which we prove in §2, formalizes this idea. (We use the notation $\varepsilon_m \in M(M(E))$ for the unit point mass at $m \in M(E)$.)

Theorem 1.1. *Assume that* $\varphi(x, \lambda) = -\lambda^2/2$. *Consider* $m_1, m_2 \in M(E)$ *and* $h \geq 0$. *The following are equivalent*:

(i) $m_1 P_t \ll m_2 P_{t+h}$, $\forall t > 0$;

(ii) $P^{m_1} \circ \theta_t^{-1} \ll P^{m_2} \circ \theta_{t+h}^{-1}$, $\forall t > 0$;
(iii) $\varepsilon_{m_1} Q_t \ll \varepsilon_{m_2} Q_{t+h}$, $\forall t > 0$;
(iv) $\mathbb{P}^{m_1} \circ \Theta_t^{-1} \ll \mathbb{P}^{m_2} \circ \Theta_{t+h}^{-1}$, $\forall t > 0$.

Absolute continuity results are more delicate for infinite-dimensional processes, so the interesting (and nontrivial) part of this result is the implication that (i) $\Rightarrow$ (iii) and (iv).

As well as being of intrinsic interest, Theorem 1.1 and the corollaries we draw from it in §2 are useful tools in studying the sample path behavior of X. For instance, in §3 we use these results to extend and simplify some of the results from Perkins [15, 16] on the closed support of super-Brownian motion. In §4, we define the concept of a "collision measure" that gauges the extent to which the masses of two measures are superimposed. We then use our absolute continuity results and some Fourier analysis to give sufficient conditions for the random measure X_t to be diffuse enough to possess a collision measure with respect to a fixed measure when ξ is a Lévy process in $\mathbb{R}^d$. In particular, this leads to sufficient conditions for the existence of a nontrivial collision measure for two independent super-Lévy processes at a fixed time.

We begin §5 still under the assumption that $\varphi(x, \lambda) = -\lambda^2/2$, and we use Theorem 1.1 to give a calculation-free proof of a result from [15]—to the effect that if ξ is a Lévy process, then the closed support of the random measure X_t contains any point to which ξ can jump. We then use totally different methods (that do not depend on our earlier results) to show that this behavior is an example of a phenomenon exhibited by all superprocesses.

Another application of Theorem 1.1 is the recent work of Tribe [18] on the degree of disconnectedness of the closed support of super-Brownian motion. There it was convenient to first work under the assumption that the starting point is a point mass, then use our results to show that these almost sure results hold independent of this assumption.

Because all of the results in §§2–5, except for those at the end of §5, include the hypothesis $\varphi(x, \lambda) = -\lambda^2/2$, we adopted it as a standing assumption for those sections and did not include it in the statements of our results. We alert the reader in the latter half of §5 when we return to considering general φ.

We end this section with some general lemmas. The first of these appears as Proposition 2.7 in Fitzsimmons [9], but since we use it so often we state it here for ease of reference.

Lemma 1.2. *For each $t \geq 0$, $\mu \in M(E)$, and $f \in bp\mathscr{E}$, we have*

$$\int Q_t(\mu, d\nu)\langle \nu, f\rangle = \mu P_t^b f$$

and

$$\int Q_t(\mu, d\nu)\langle \nu, f\rangle^2 = (\mu P_t^b f)^2 + \int_0^t \mu P_s^b(\hat{c} \cdot (P_{t-s}^b f)^2)\, ds,$$

where $\hat{c} = 2c + \int_0^\infty n(\cdot, du)u^2$ *and* $\{P_t^b\}$ *is the semigroup on* $bp\mathscr{E}$ *determined by*

$$P_t^b f(x) = P^x\left[\exp\left(-\int_0^t b(\xi_s)\,ds\right) f(\xi_t)\right].$$

Lemma 1.3. *Suppose that* $m \in M(E)$ *and* $D \in \mathscr{E}$. *Set* $\beta = \|b\|_\infty$ *and* $\gamma = \|\hat{c}\|_\infty$, *where* $\hat{c}$ *is as in Lemma* 1.2. *For* $t > 0$,

$$\mathbb{P}^m(\langle X_t, 1_D\rangle > 0) \geq e^{-4\beta t}(1 + \gamma t/mP_t1_D)^{-1},$$

where we interpret $0^{-1} = \infty$ *and* $\infty^{-1} = 0$.

Proof. Set $\beta = \|b\|_\infty$. From Lemma 1.2 we have

$$\mathbb{P}^m(\langle X_t, 1_D\rangle) \leq e^{\beta t} mP_t1_D \tag{1.3.1}$$

and

$$\mathbb{P}^m(\langle X_t, 1_D\rangle) \geq e^{-\beta t} mP_t1_D. \tag{1.3.2}$$

Also,

$$\begin{aligned}\mathbb{P}^m(\langle X_t, 1_D\rangle^2) &\leq (e^{\beta t} mP_t1_D)^2 + e^{\beta t}\gamma\int_0^t mP_s^b(P_{t-s}^b 1_D)\,ds \\ &\leq (e^{\beta t} mP_t1_D)^2 + e^{2\beta t}\gamma t mP_t1_D.\end{aligned} \tag{1.3.3}$$

If $mP_t1_D = 0$, then the result is clear from (1.3.1). Otherwise, note from the Cauchy-Schwarz inequality that

$$\begin{aligned}[\mathbb{P}^m(\langle X_t, 1_D\rangle)]^2 &= [\mathbb{P}^m(\langle X_t, 1_D\rangle, \langle X_t, 1_D\rangle > 0)]^2 \\ &\leq \mathbb{P}^m(\langle X_t, 1_D\rangle^2)\mathbb{P}^m(\langle X_t, 1_D\rangle > 0),\end{aligned}$$

and then apply (1.3.2) and (1.3.3). □

Lemma 1.4. *Suppose that* $\varphi(x, \lambda) = -\lambda^2/2$. *For each* $u > 0$, *there exists a unique kernel* $R_u(x, A)$, $x \in E$, $A \in \mathscr{M}(E)$, *such that*

$$R_u(x, \{0\}) = 0, \quad \forall x,$$
$$R_u(x, M(E)) = 2u^{-1}, \quad \forall x,$$

and

$$V_u f(x) = \int R_u(x, d\nu)[1 - \exp(-\langle \nu, f\rangle)].$$

Under $\mathbb{P}^m$, $m \in M(E)$, X_u *has the same law as* $\int \eta_u^m(d\nu)\nu$, *where* η_u^m *is a Poisson random measure on* $M(E)$ *with the finite intensity* $\int m(dx)R_u(x, \cdot)$.

Proof. This is a slight extension of Proposition III.1.1 of [7] and of the remarks preceding that result. The lemma follows from the Lévy-Hincin representation for infinitely divisible random measures in Chapter 6 of [12], just as in the above reference. □

2. Absolute continuity results

Recall that, until further notice, we are taking $\varphi(x, \lambda) = -\lambda^2/2$.

Proof of Theorem 1.1. Clearly, (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii). By the Markov property, (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv). Lemma 1.2 shows that (iii) $\Rightarrow$ (i). (If $A \in \mathscr{E}$, consider $B = \{m \in M(E): m(A) > 0\}$.) We are left with showing that (i) $\Rightarrow$ (iii).

Suppose that (i) holds. Fix $t_1 > 0$, set $t_2 = t_1 + h$, and choose $r \in]0, t_1[$. For $u > 0$, it follows from Lemma 1.4 that if we set

$$p_u^n(x_1, \ldots, x_n; B) = \int \cdots \int \prod_{i=1}^{n} R_u(x_i, d\nu_i) 1_B(\nu_1 + \cdots + \nu_n)$$

for $n \in \mathbb{N}$, $(x_1, \ldots, x_n) \in E^n$ and $B \in \mathscr{M}(E)$, then for any $m \in M(E)$

$$\text{(2.1.0)} \qquad Q_u(m, B) = 0 \Leftrightarrow$$
$$p_u^n(x_1, \ldots, x_n; B) = 0, \qquad m^{(n)}\text{-a.a. } (x_1, \ldots, x_n), \quad \forall n,$$

where $m^{(n)}$ is the n-fold product measure $m \times \cdots \times m$. Consequently, by the Markov property,

$$Q_{t_i}(m_i, B) = 0 \Leftrightarrow$$
$$p_r^n(x_1, \ldots, x_n; B) = 0, \qquad (X_{t_i - r})^{(n)}\text{-a.a. } (x_1, \ldots, x_n),$$
$$\mathbb{P}^{m_i}\text{-a.s.}, \quad \forall n;$$

and (iii) will follow if we can show that for all $n \in \mathbb{N}$ and $C \in \mathscr{E}^n$ such that

$$\text{(2.1.1)} \qquad \mathbb{P}^{m_1}((X_{t_2 - r})^{(n)}(C)) = 0,$$

we also have

$$\text{(2.1.2)} \qquad \mathbb{P}^{m_2}((X_{t_1 - r})^{(n)}(C)) = 0.$$

To prove that (2.1.1) implies (2.1.2), we will fix n, take C such that (2.1.1) holds, then use Theorem 1.1$'$ of Dynkin [6] (with some minor modifications in notation) to give explicit formulae for the left-hand sides of both equations and show that (2.1.2) also holds.

Following Dynkin, we consider a directed graph with a set A of arrows and a set V of vertices. We write $a: v \to v'$ to indicate that the arrow a begins at the vertex v and ends at the vertex v'. For $v \in V$, set $a_+(v)$ (respectively, $a_-(v)$) to be the number of $v' \in V$ for which there exists $a \in A$ such that $a: v' \to v$ (respectively $a: v \to v'$). We say that the graph is a *diagram* if we have a disjoint partition $V = V_- \cup V_0 \cup V_+$, where

$$v \in V_- \Leftrightarrow a_+(v) = 0, \quad a_-(v) = 1,$$
$$v \in V_0 \Leftrightarrow a_+(v) = 1, \quad a_-(v) = 2,$$
$$v \in V_+ \Leftrightarrow a_+(v) = 1, \quad a_-(v) = 0.$$

Fix an arbitrary diagram such that V_+ consists of n vertices. We label each of the k "entrance" vertices $v \in V_-$ with a pair $(0, x_v)$, $x_v \in E$. We order the n "exit" vertices V_+ and label the ith vertex with the pair $(t_1 - r, z_i)$, $z_i \in E$. We label the remaining "interior" vertices $v \in V_0$ with pairs (s_v, y_v), $s_v \in]0, \infty[$, $y_v \in E$. Let D_1 denote the resulting labeled diagram with ordered exits. If $a: v \to v'$, where v and v' have labels (s, w) and (s', w'), respectively, then we set

$$p_a^{D_1} = \begin{cases} P_{s'-s}(w, dw'), & \text{if } s' \geq s, \\ 0, & \text{if } s' < s. \end{cases}$$

Put

$$\pi_{D_1}(t_1 - r, (s_v)_{v \in V_0}, (x_v)_{v \in V_-}) = \int \prod_{a \in A} p_a^{D_1} 1_C(z_1, \dots, z_n),$$

where the integration is over all $(y_v)_{v \in V_0}$ and $(z_i)_{1 \leq i \leq n}$.

Let D_2 denote the labeled diagram with ordered exits formed from the same diagram with the same ordering of exits, only now label the exit vertices with the pairs $\{(t_2 - r, z_i): 1 \leq i \leq n\}$ and the interior vertices with the pairs $\{(s_v + h, y_v): v \in V_0\}$. The labeling of the entrance vertices is unchanged. Define $p_a^{D_2}$ and π_{D_2} in an analogous manner using this new labeling scheme. Theorem 1.1′ of [6] and (2.1.1) imply that

$$\begin{aligned} &\int m_2^{(k)}(dx) \pi_{D_2}(t_2 - r, (s_v + h)_{v \in V_0}, (x_v)_{v \in V_-}) = 0, \\ &\prod_{v \in V_0} ds_v\text{-a.a.}, \qquad (s_v) \in]0, \infty[^{V_0}. \end{aligned} \tag{2.1.3}$$

Fix $(s_v) \in]0, \infty[^{V_0}$ not belonging to the Lebesgue null set on which (2.1.3) fails to hold and choose $0 < s < \min\{s_v : v \in V_0\} \wedge (t_1 - r)$. Now let D_3 denote the labeled diagram with ordered exits formed from the same diagram as D_1 and D_2 with the same ordering of exits and labeling of entrances, only now label the exit vertices with the pairs $\{(t_1 - r - s, z_i): 1 \leq i \leq n\}$ and the interior vertices with the pairs $\{(s_v - s, y_v): v \in V_0\}$. Define π_{D_3} in the same manner as π_{D_1} and π_{D_2}. The Markov property applied to the left side of (2.1.3) gives that

$$\int (m_2 P_{s+h})^{(k)}(dx) \pi_{D_3}(t_1 - r - s, (s_v - s)_{v \in V_0}, (x_v)_{v \in V_-}) = 0,$$

and hence, by hypothesis,

$$\int (m_1 P_s)^{(k)}(dx) \pi_{D_3}(t_1 - r - s, (s_v - s)_{v \in V_0}, (x_v)_{v \in V_-}) = 0,$$

also. If we again apply the Markov property, we find that

$$\int m_1^{(k)}(dx) \pi_{D_1}(t_1 - r, (s_v)_{v \in V_0}, (x_v)_{v \in V_-}) = 0. \tag{2.1.4}$$

Since (2.1.4) holds for $\prod_{v\in V_0} ds_v$-a.a. $(s_v) \in]0, \infty[^{V_0}$, it follows from Theorem 1.1′ of [6] that (2.1.2) holds, as required. □

Remarks. By the Markov property, hypothesis (i) of Theorem 1.1 is equivalent to the existence of a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \downarrow 0$ as $n \to \infty$, and $m_1 P_{t_n} \ll m_2 P_{t_n+h}$ for all n.

It is obvious from the proof of Theorem 1.1 that if $\varepsilon_{m_1} Q_t \ll \varepsilon_{m_2} Q_{t+h}$ for a fixed $t > 0$, then $m_1 P_t \ll m_2 P_{t+h}$, also. A counterexample is given in the §6, Appendix, which shows that the converse implication is false, in general. The counterexample, however, leaves open the question of whether $m_1 P_s \ll m_2 P_{s+h}$ for some $0 < s < t$ implies that $\varepsilon_{m_1} Q_t \ll \varepsilon_{m_2} Q_{t+h}$. We conjecture that the answer is no, in general. Suppose, however, that $B \in \mathscr{M}(E)$ has the property

$$(*) \qquad \forall n \in \mathbb{N}, \quad \sum_{i=1}^{n} \nu_i \in B \Rightarrow \nu_i \in B \quad \text{for some } i \leq n;$$

(for example, $B = \{\nu : \nu(C) > 0\}$ for $C \in \mathscr{E}$), then if $m_1 P_s \ll m_2 P_{s+h}$ for some $s < t$, we have that $\varepsilon_{m_2} Q_{t+h}(B) = 0$ implies that $\varepsilon_{m_1} Q_t(B) = 0$—the short proof which only uses Lemma 1.4 and the Markov property is left to the reader. (Hint: Show that

$$(**) \qquad \varepsilon_m Q_u(B) > 0 \Leftrightarrow \int m(dx) R_u(x, B) > 0.)$$

The counterexample in the Appendix also shows that even for sets satisfying $(*)$, we cannot obtain the conclusion of the previous sentence under the assumption that $m_1 P_t \ll m_2 P_{t+h}$.

Theorem 1.1 has the following two straightforward consequences.

Notation. For $t \geq 0$, let $\mathscr{F}_{r,\infty} = \sigma\{\xi_s : s \geq t\}$ and $\mathscr{G}_{r,\infty} = \sigma\{X_s : s \geq t\}$.

Corollary 2.2. *Given* $m_1, m_2 \in M(E)$, *the following are equivalent*:

(i) $m_1 P_t \ll m_2 P_t$, $\forall t > 0$;
(ii) $P^{m_1}|_{\mathscr{F}_{t,\infty}} \ll P^{m_2}|_{\mathscr{F}_{t,\infty}}$, $\forall t > 0$;
(iii) $\varepsilon_{m_1} Q_t \ll \varepsilon_{m_2} Q_t$, $\forall t > 0$;
(iv) $\mathbb{P}^{m_1}|_{\mathscr{G}_{t,\infty}} \ll \mathbb{P}^{m_2}|_{\mathscr{G}_{t,\infty}}$, $\forall t > 0$.

Corollary 2.3. *Given* $m \in M(E)$ *we have that*

(i) $mP_s \ll mP_t$, $\forall 0 < s \leq t$, *if and only if*
(ii) $\varepsilon_m Q_s \ll \varepsilon_m Q_t$, $\forall 0 < s \leq t$.

Corollary 2.4. *We have that*

(i) $\varepsilon_x P_s \ll\gg \varepsilon_y P_t$ *for all* $x, y \in E$, $s, t > 0$, *if and only if*
(ii) $\varepsilon_{m_1} Q_s \ll\gg \varepsilon_{m_2} Q_t$ *for all* $m_1, m_2 \in M(E)$, $s, t > 0$.

Proof. In view of Lemma 1.2, we only need consider (i) $\Rightarrow$ (ii). Fix $m_1, m_2 \in M(E)$ and $A \in \mathscr{M}(E)$ such that $\varepsilon_{m_2} Q_t(A) = 0$. If $s \leq t$, then $\varepsilon_{m_1} Q_s(A) = 0$ by

Theorem 1.1. Assume now that $s > t$. Theorem 1.1 implies that $\varepsilon_{X_{s-t}} Q_t(A) = 0$, $\mathbb{P}^{m_1}$-a.s., and hence $\varepsilon_{m_1} Q_s(A) = 0$ by the Markov property. □

We end this section with a simple first application of these results.

Corollary 2.5. *Suppose that for all $x, y \in E$ and $s, t > 0$, we have $\varepsilon_x P_s \ll\gg \varepsilon_y P_t$. Then for all $m \in M(E)$ and $u > 0$, we have that in the topologies $M_0(E)$ and $M_r(E)$, the closed support of $\varepsilon_m Q_u$ is $M(E)$.*

Proof. Fix $m \in M(E)$ and $u > 0$. Suppose that $G \subset M(E)$ is an open neighborhood of $m' \in M(E)$ in either of the topologies. For $u' > 0$ sufficiently small, it is clear from the right continuity of the paths of X that $\varepsilon_{m'} Q_{u'}(G) > 0$. From Corollary 2.4, we see that $\varepsilon_m Q_u \ll\gg \varepsilon_{m'} Q_{u'}$, so $\varepsilon_m Q_u(G) > 0$ also and the result follows. □

3. The closed support of super-Brownian motion

In this section we still consider the case $\varphi(x, \lambda) = -\lambda^2/2$ and suppose, moreover, that ξ is a Brownian motion on $\mathbb{R}^d$ for some $d \geq 1$.

Let $S_t = \operatorname{supp} X_t$, the closed support of X_t, $t \geq 0$. Theorem 1.4 of Perkins [15] shows that $\{S_t : t \geq 0\}$ has right-continuous paths with left limits in the space of compact sets equipped with the Hausdorff metric. Moreover, $S_t \subset S_{t-}$ for all $t > 0$, and if we adjoin an isolated point Δ to $\mathbb{R}^d$, there is an $\mathbb{R}^d \cup \{\Delta\}$-valued, optional process, Z, such that $S_{t-} \backslash S_t = \{Z_t\} \cap \mathbb{R}^d$ for all $t > 0$. We can think of $\{Z_t : Z_t \neq \Delta\}$ as the set of sites at which a "colony" of X becomes extinct.

When $d \geq 3$, the precise asymptotics for $\mathbb{P}^m(X_t(\{y : |y - x| \leq \varepsilon\}) > 0)$ as $\varepsilon \downarrow 0$ given in [4] certainly imply that $\mathbb{P}^m(x \in S_t) = 0$ for all $x \in \mathbb{R}^d$ and $t > 0$. We now extend this latter result to $d = 2$.

Notation. For $\nu \in M(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, define $\tau_y(\nu) \in M(\mathbb{R}^d)$ by $\tau_y(\nu)(A) = \int \nu(dx) 1_A(x + y)$.

Theorem 3.1. *If $d \geq 2$, then $\mathbb{P}^m(x \in S_t) = 0$ for all $m \in M(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t > 0$.*

Proof. It is clear from the integral equation defining the Laplace functional of $\{Q_t\}$ that

$$\varepsilon_{\tau_y(m)} Q_t = \varepsilon_m Q_t \circ \tau_y^{-1}, \quad \forall y \in \mathbb{R}^d, \forall t \geq 0. \tag{3.1.1}$$

Assume $\mathbb{P}^m(x \in S_t) > 0$. Theorem 1.1 shows that $\mathbb{P}^{\tau_y(m)}(x \in S_t) > 0$ for every $y \in \mathbb{R}^d$. Combining this with (3.1.1) and the fact that $x \in \operatorname{supp} \tau_y(X_t)$ if and only if $x - y \in S_t$, we find that $\mathbb{P}^m(x - y \in S_t) > 0$ for all $y \in \mathbb{R}^d$. Thus, applying Fubini's theorem, S_t has positive Lebesgue measure with positive $\mathbb{P}^m$-probability, which contradicts Theorem 1.3 of [16]. □

Let $G = \{(t, x): x \in S_{t-}, t > 0\} \cup \{(0, S_0)\}$ be the closed graph of S, and let $H = \{(t, Z_t): t > 0, Z_t \in \mathbb{R}^d\}$ be the set of space-time "extinction points" of X.

Theorem 3.2. *If $d \geq 2$, then H is dense in G $\mathbb{P}^m$-a.s. for all $m \in M(\mathbb{R}^d)$.*

Proof. Given Theorem 3.1, the proof given for $d \geq 3$ in Theorem 4.8 of [15] extends to $d = 2$ (see the remark following that result). □

We can use Corollary 2.2 to give a shorter standard proof of the following result, which was proved in [15] by means of a more cumbersome nonstandard argument.

Theorem 3.3. *If $A \subset \mathbb{R}^d$ is Lebesgue null, then*

$$A \cap \left(\bigcup_{t>0} S_{t-} \backslash S_t \right) = \varnothing, \quad \mathbb{P}^m\text{-a.s.},$$

for all $m \in M(\mathbb{R}^d)$.

Proof. Let $D = \bigcup_{t>0} S_{t-} \backslash S_t$ and $B = \{t: t > 0, Z_t \in \mathbb{R}^d\}$. Then B is an optional set with countable sections (since $t \mapsto S_t$ has only countably many jumps) and so, from page 167 of [5], there exists a countable set of stopping times $\{T_k\}_{k=1}^{\infty}$ such that $B = \bigcup_{k=1}^{\infty} [T_k]$, where $[T_k]$ denotes the graph of T_k. Let $Z(\infty) = \Delta$ and $Y_k = Z(T_k)$. Then

$$D = \{Y_k : k \in \mathbb{N}\} \cap \mathbb{R}^d.$$

Assume that $A \subset \mathbb{R}^d$ is Lebesgue measurable and $\mathbb{P}^m(A \cap D \neq \varnothing) > 0$. Corollary 2.2 implies that for every $y \in \mathbb{R}^d$, $\mathbb{P}^{\tau_y(m)}(A \cap D \neq \varnothing) > 0$, and so, by the observation (3.1.1), $\mathbb{P}^m(A \cap (D - y) \neq \varnothing) > 0$. Hence, if λ denotes Lebesgue measure on $\mathbb{R}^d$ we have

$$(\lambda \times \mathbb{P}^m) \left(\bigcup_{k=1}^{\infty} \{(y, w): Y_k(w) - y \in A\} \right) > 0.$$

Choose $k \in \mathbb{N}$ such that

$$(\lambda \times \mathbb{P}^m)(\{(y, w): Y_k(w) - y \in A\}) > 0.$$

By Fubini's theorem, with positive $\mathbb{P}^m$-probability

$$0 < \lambda(\{y: Y_k(w) - y \in A\}) = \lambda(A),$$

as required. □

Remark. Let $R = \bigcup_{t>0} S_t$ and $\overline{R} = \bigcup_{\delta>0} \mathrm{cl}(\bigcup_{t \geq \delta} S_t) = \bigcup_{t>0} S_{t-}$. Given a Borel set $A \subset \mathbb{R}^d$, it follows easily from Theorem 3.3 that $\mathbb{P}^m(R \cap A \neq \varnothing) > 0$ if and only if $\mathbb{P}^m(\overline{R} \cap A \neq \varnothing) > 0$ (see Theorem 5.9 of [15]). This reduction and its analogue for k-multiple points was helpful in [15] because $\overline{R}$ is technically easier to handle than R.

4. Collision measures for super-Lévy processes

Suppose that we have two measures $\nu_1, \nu_2 \in M(\mathbb{R}^d)$, and we wish to construct another measure which captures the degree to which the masses of ν_1 and ν_2 overlap. There is a classical object known as the *Hellinger measure* which performs this role. This is the measure $\nu \in M(\mathbb{R}^d)$ defined by

$$\nu(A) = \int_A (\nu_1 + \nu_2)(dx) \left[\frac{d\nu_1(x)}{d(\nu_1 + \nu_2)}\right]^{\frac{1}{2}} \left[\frac{d\nu_1(x)}{d(\nu_1 + \nu_2)}\right]^{\frac{1}{2}}$$

for $A \subset \mathbb{R}^d$ Borel. Unfortunately, although the Hellinger measure is well defined for all pairs ν_1, ν_2, it will be null if $\nu_1 \perp \nu_2$. The following notion is an attempt to remedy this difficulty for certain singular pairs of measures by introducing a more discerning means of gauging overlap.

Definition. Let c_d denote the Lebesgue measure of the unit ball in $\mathbb{R}^d$. Given $\nu_1, \nu_2 \in M(\mathbb{R}^d)$ and $\varepsilon > 0$, we define $K^\varepsilon(\nu_1, \nu_2) \in M(\mathbb{R}^d)$ by

$$K^\varepsilon(\nu_1, \nu_2)(A) = (c_d \varepsilon^d)^{-1} \iint_{\{|x-y| \le \varepsilon\}} \nu_1(dx)\nu_2(dy) 1_A\left(\frac{x+y}{2}\right)$$

for $A \subset \mathbb{R}^d$ Borel. If there exists $K(\nu_1, \nu_2) \in M(\mathbb{R}^d)$ such that $K^\varepsilon(\nu_1, \nu_2) \to K(\nu_1, \nu_2)$ in $M_0(\mathbb{R}^d)$ as $\varepsilon \downarrow 0$, we say that $K(\nu_1, \nu_2)$ is the *collision measure* for ν_1 and ν_2.

Example. Suppose that $\nu_1(dx) = f(x)\,dx$ for some bounded continuous, nonnegative, integrable function f and $\nu_2 \in M(\mathbb{R}^d)$ is arbitrary. Then it is not hard to see that the collision measure exists and is given by $K(\nu_1, \nu_2)(dx) = f(x)\nu_2(dx)$.

Our aim in this section is to combine our absolute continuity results with some real analysis to find sufficient conditions under which collision measures exist for a fixed measure and the random measures produced by a super-Lévy process. We remind the reader once again that we are still considering the case $\varphi(x, \lambda) = -\lambda^2/2$.

Notation. Given $m \in M(\mathbb{R}^d)$, let $\hat{m}(z) = \int m(dx)\exp(iz \cdot x)$, $z \in \mathbb{R}^d$, denote the Fourier transform of m. If ξ is a Lévy process on $\mathbb{R}^d$ we define, as usual, the exponent of ξ to be the function ψ given by $\exp(-t\psi(z)) = \widehat{\varepsilon_0 P_t}(z)$.

Lemma 4.1. *Suppose that ξ is a Lévy process on $\mathbb{R}^d$ with exponent ψ. Then*

$$\mathbb{P}^m(|\widehat{X}_t(z)|^2) = |\hat{m}(z)|^2 \exp(-2t \operatorname{Re} \psi(z)) + m(\mathbb{R}^d) h(t, z),$$

where

$$h(t, z) = \begin{cases} (1 - \exp(-2t \operatorname{Re} \psi(z))(2 \operatorname{Re} \psi(z))^{-1}, & \operatorname{Re} \psi(z) > 0, \\ t, & \operatorname{Re} \psi(z) = 0. \end{cases}$$

Proof. Consider $f = f_1 + if_2$, with f_1, f_2 bounded and Borel. From Lemma 1.2 we have

$$\mathbb{P}^m(|\langle X_t, f\rangle|^2) = |mP_t f|^2 + \int_0^t mP_s(|P_{t-s}f|^2)\, ds.$$

Applying this result to $f(x) = \exp(iz \cdot x)$, we get

$$\begin{aligned}\mathbb{P}^m(|\widehat{X}_t(z)|^2) &= \left|\int m(dx)\exp(iz\cdot x - t\psi(z))\right|^2 \\ &\quad + \int_0^t \iint m(dx)P_s(x, dy)|\exp(iz\cdot y - (t-s)\psi(z))|^2\, ds \\ &= |\hat{m}(z)|^2 \exp(-2t\operatorname{Re}\psi(z)) \\ &\quad + m(\mathbb{R}^d)\int_0^t \exp(-2(t-s)\operatorname{Re}\psi(z))\, ds,\end{aligned}$$

and the lemma follows. □

Theorem 4.2. *Suppose that ξ is a Lévy process on $\mathbb{R}^d$ with exponent ψ and $m, \nu \in M(\mathbb{R}^d)\backslash\{0\}$. Assume that*

(i) *$mP_t \ll\gg \lambda$, $\forall t > 0$, where λ is Lebesgue measure, and*

(ii) *$\int(1 + \operatorname{Re}\psi(z))^{-1}|\hat{\nu}(z)|^2\, dz < \infty$.*

Then for each $t > 0$, $K(\nu, X_t)$ exists $\mathbb{P}^m$-a.s. and is nonnull with positive $\mathbb{P}^m$-probability for all $t > 0$.

Proof. Fix $t > 0$. Set $Y = \nu * \widetilde{X}_t$, where $\widetilde{X}_t(A) = X_t(-A)$. Note that $h(t, z) + e^{-2t\operatorname{Re}\psi(z)} \le c(t)(1 + \operatorname{Re}\psi(z))^{-1}$. From this, (ii), and Lemma 4.1, we see that $\mathbb{P}^m \int|\widehat{Y}(z)|^2\, dz < \infty$, and so, by Plancherel's theorem, $Y \ll \lambda$, $\mathbb{P}^m$-a.s.

Let f be a nonnegative, bounded, uniformly continuous function on $\mathbb{R}^d$. Define $\nu_f \in M(E)$ by $\nu_f(dx) = f(x)\nu(dx)$ and set $Y_f = \nu_f * \widetilde{X}_t$. Put

$$L_f(y) = \lim_{\varepsilon\downarrow 0}(c_d\varepsilon^d)^{-1} Y_f(\{x : |x - y| \le \varepsilon\})$$

for $y \in \mathbb{R}^d$ if the limit exists. We have $Y_f \ll \lambda$, $\mathbb{P}^m$-a.s. and hence, by a standard differentiation theorem, $L_f(y)$ exists for λ-a.a. y, $\mathbb{P}^m$-a.s. From the observation (3.1.1), it follows that $L_f(0)$ exists $\mathbb{P}^{\tau_y(m)}$-a.s. for λ-a.a. y. Note that (i) is equivalent to the condition

$$\tau_y(m)P_s \ll\gg mP_s, \quad \forall y \in \mathbb{R}^d, s > 0,$$

so from Corollary 2.2 we see that

$$L_f(0) \text{ exists } \mathbb{P}^m\text{-a.s.} \tag{4.2.1}$$

Observe that when $f \equiv 1$,

$$\mathbb{P}^m \int L_1(y)\, dy = \mathbb{P}^m\langle Y, 1\rangle = \langle m, 1\rangle\langle \nu, 1\rangle > 0,$$

and arguments similar to those of the previous paragraph establish that

$$\mathbb{P}^m(L_1(0) > 0) > 0. \tag{4.2.2}$$

As

$$Y_f(\{x\colon |x| \le \varepsilon\}) = \iint_{\{|x-y|\le\varepsilon\}} \nu(dx)X_t(dy)f(x),$$

it is clear from (4.2.1) the existence of $L_1(0)$, and the continuity of f that

$$L_f(0) = \lim_{\varepsilon\downarrow 0}\langle K^\varepsilon(\nu, X_t), f\rangle \text{ exists } \mathbb{P}^m\text{-a.s.} \tag{4.2.3}$$

Since (4.2.3) hold simultaneously for all f belonging to a given countable weak-convergence determining class, $K(\nu, X_t)$ exists $\mathbb{P}^m$-a.s. Moreover, from (4.2.2) we conclude that

$$\mathbb{P}^m(\langle K(\nu, X_t), 1\rangle > 0) > 0. \quad \square$$

It will sometimes be convenient to consider approximate identities other than $(c_d\varepsilon^d)^{-1}1(|x| \le \varepsilon)$.

Definition. A collection of symmetric functions $g_\varepsilon\colon \mathbb{R}^d \to [0, \infty[, \ \varepsilon \in]0, 1]$ is a differentiation system (d.s.) if, for any $\nu \in M(\mathbb{R}^d)$ such that $\nu \ll \lambda$,

$$\lim_{\varepsilon\to 0}\int \nu(dx)g_\varepsilon(x-y) = \frac{d\nu}{d\lambda}(y), \quad \lambda\text{-a.a. } y.$$

If $\nu_1, \nu_2 \in M(\mathbb{R}^d)$ and $\{g_\varepsilon\}$ is a d.s., define $K(g_\varepsilon, \nu_1, \nu_2) \in M(\mathbb{R}^d)$ by

$$K(g_\varepsilon, \nu_1, \nu_2)(A) = \iint \nu_1(dx)\nu_2(dy)g_\varepsilon(x-y)1_A((x+y)/2)$$

for $A \in \mathbb{R}^d$ Borel. If there exists $K^g(\nu_1, \nu_2) \in M(\mathbb{R}^d)$ such that $K(g_\varepsilon, \nu_1, \nu_2) \to K^g(\nu_1, \nu_2)$ in $M_0(\mathbb{R}^d)$ as $\varepsilon \downarrow 0$, we say $K^g(\nu_1, \nu_2)$ is the $\{g_\varepsilon\}$-collision measure for ν_1 and ν_2.

Remark. $g_\varepsilon(x) = (c_d\varepsilon^d)^{-1}1(|x| \le \varepsilon)$ is a d.s., so the above definition extends that of a collision measure. It is easy to check that

$$\tilde{g}_\varepsilon(x) = \exp\{-|x|^2/2\varepsilon\}(2\pi\varepsilon)^{-d/2}$$

is also a d.s.

Theorem 4.3. *Assume the hypotheses of Theorem 4.2. If $\{g_\varepsilon\}$ is a d.s. then for each $t > 0$, $K^g(\nu, X_t)$ exists and equals $K(\nu, X_t)$, $\mathbb{P}^m$-a.s.*

Proof. Proceed as in the previous proof but now define

$$L_f^g(y) = \lim_{\varepsilon\downarrow 0}\int Y_f(dx)g_\varepsilon(x-y)$$

for $y \in \mathbb{R}^d$, if the limit exists. Since $\{g_\varepsilon\}$ is a d.s., we see that $L_f^g(y)$ exists and equals $L_f(y)$ for λ-a.a. y, $\mathbb{P}^m$-a.s. Argue as in the derivation of (4.2.1) to see that

$$L_f^g(0) \text{ exists and equals } L_f(0), \quad \mathbb{P}^m\text{-a.s.}$$

Since $L_f(0) = \langle K(\nu, x_t), f\rangle$ by the proof of Theorem 4.2, it remains to show that

$$\lim_{\varepsilon\downarrow 0}\langle K(g_\varepsilon, \nu, X_t)f\rangle = L_f^g(0), \quad \mathbb{P}^m\text{-a.s.}$$

or equivalently

$$\lim_{\varepsilon\downarrow 0}\iint \nu(dx)X_t(dy)g_\varepsilon(x-y)\left(f\left(\frac{x+y}{2}\right) - f(x)\right) = 0, \quad \mathbb{P}^m\text{-a.s.}$$

By the uniform continuity of f and the existence of $L_1^g(0)$, it suffices to show that for every $\delta > 0$,

$$\lim_{\varepsilon\downarrow 0}\int_{\{|z|\geq\delta\}} \nu * \widetilde{X}_t(dz)g_\varepsilon(z) = 0, \quad \mathbb{P}^m\text{-a.s.} \tag{4.2.4}$$

Since $\nu * \widetilde{X}_t \ll \lambda$, $\mathbb{P}^m$-a.s., we do know that

$$\lim_{\varepsilon\downarrow 0}\int_{\{|z|\geq\delta\}} \nu * \widetilde{X}_t(dz)g_\varepsilon(z-y) = 0, \quad \text{for } \lambda\text{-a.a. } y, |y| < \delta, \ \mathbb{P}^m\text{-a.s.} \tag{4.2.5}$$

Now argue as in the derivation of (4.2.1) to obtain (4.2.4) from (4.2.5). □

Corollary 4.4. *Assume that ξ_1 and ξ_2 are Lévy processes on $\mathbb{R}^d$ with exponent functions ψ_1 and ψ_2, and semigroups P_t^1 and P_t^2, respectively. Let X^i be the ξ_i-superprocess starting at $m_i \in M(\mathbb{R}^d)\backslash\{0\}$. Assume that X^1, X^2 are independent and*

(i) $m_i P_t^i \ll\gg \lambda$, $\forall t > 0$ *for* $i = 1$ *or* 2,
(ii) $\int \prod_{i=1}^2 (1 + \operatorname{Re}\psi_i(z))^{-1} dz < \infty$.

Then for all $t > 0$ and any d.s. $\{g_\varepsilon\}$, $K(X_t^1, X_t^2) = K^g(X_t^1, X_t^2)$ exists $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$-a.s. and is nonnull with positive $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$-probability.

Proof. By the symmetry of $K^g(\cdot, \cdot)$ we may assume that (i) holds for $i = 1$. Apply Theorems 4.2 and 4.3 with $\nu = X_t^1$. Use Lemma 4.1 to see that $\mathbb{P}^{m_1}(|\widehat{X}_t^1(z)|^2) \leq c(t)(1 + \operatorname{Re}\psi_1(z))^{-1}$, and hence condition (ii) above implies (ii) of Theorem 4.2, $\mathbb{P}^{m_1}$-a.s. Since $\mathbb{P}^{m_1}(X_t^1 \neq 0) > 0$, the result follows from Theorem 4.2 and 4.3. □

If ξ_i is a symmetric stable process of index $\alpha_i \in]0, 2]$, then $\operatorname{Re}\psi_i(z) = c_i|z|^{\alpha_i}$, and so the above implies

Corollary 4.5. *Let X^i $(i = 1, 2)$ be independent ξ_i-superprocesses starting at $m_i \neq 0$, $i = 1, 2$, where ξ_i is a symmetric stable process in $\mathbb{R}^d$ of index α_i. If $\alpha_1 + \alpha_2 > d$, then for all $t > 0$ and any d.s. $\{g_\varepsilon\}$, $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$-a.s., the measures $K^g(X_t^1, X_t^2)$ and $K(X_t^1, X_t^2)$ exist and are equal. The measure $K(X_t^1, X_t^2)$ is nonnull with positive $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$-probability.*

The use of these collision measures to model point interactions between two populations will be the subject of a forthcoming paper.

5. Propagation of the support process

We begin by using our absolute continuity results to give an intuitively appealing proof of the following theorem, which appears as Theorem 1.5 in [15]. Our proof avoids the detailed estimates and Borel-Cantelli arguments of the original, and is one of the rare instances in which qualitative information about the sample path behavior of X can be gleaned more or less directly from the description of X as the solution to a martingale problem (see, for example, Theorem 4.1 of [9]). We remind the reader that we are taking $\varphi(x, \lambda) = -\lambda^2/2$. Later we will drop this assumption and, at the expense of a more complex proof, obtain more general results for superprocesses constructed from underlying processes that are not necessarily Lévy. (Here ν^{*k} denotes the k-fold convolution of ν with itself.)

Theorem 5.1. *Suppose that ξ is a Lévy process on $\mathbb{R}^d$ with Lévy measure ν. Then for all $m \in M(\mathbb{R}^d)$,*

$$\bigcup_{k=1}^{\infty}[\operatorname{supp}\nu^{*k} * X_t] \subset \operatorname{supp} X_t, \quad \mathbb{P}^m\text{-}a.s., \forall t > 0.$$

Proof. Let A denote the strong infinitesimal generator of ξ with domain $\mathscr{D}(A)$. Fix a nonnegative function $f \in C^\infty(\mathbb{R}^d)$ with compact support. From Theorem IV.4.1 of Gihman and Skorohod [10], we find that $f \in \mathscr{D}(A)$, and if $x \notin \operatorname{supp} f$, then

$$Af(x) = \int \nu(dy) f(x+y). \tag{5.1.1}$$

Set

$$C = \{\mu \in M(\mathbb{R}^d) \colon \langle \mu, f\rangle = 0, \langle \mu, Af\rangle \neq 0\}.$$

We claim first of all that

$$\int_0^\infty 1_{X_s \in C}\, ds = 0, \quad \mathbb{P}^m\text{-a.s.} \tag{5.1.2}$$

Note from Theorem 4.1 and Corollaries 4.3 and 4.7 of [9] that if we define

$$Z_t = \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t \langle X_s, Af\rangle\, ds,$$

then Z is a continuous, square integrable $\mathbb{P}^m$-martingale with quadratic variation

$$\langle Z, Z\rangle_t = \int_0^t \langle X_s, f^2\rangle\, ds.$$

Let L_t^x be the version of the local time of $\langle X_t, f\rangle$ constructed in Théorème 2 of Yor [20]. On the one hand we have from Corollaire 2 of [20] that

$$\begin{aligned} L_t^0 &= \lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \int_0^t 1_{(0\le\langle X_s, f\rangle\le\varepsilon)} \langle X_s, f^2\rangle\, ds \\ &= \limsup_{\varepsilon\downarrow 0} \varepsilon^{-1} \int_0^t 1_{(0<\langle X_s, f\rangle\le\varepsilon)} \langle X_s, f^2\rangle\, ds \\ &\le \|f\| \limsup_{\varepsilon\downarrow 0} \int_0^t 1_{(0<\langle X_s, f\rangle\le\varepsilon)}\, ds \\ &= 0. \end{aligned}$$

On the other hand, since $L_t^x = 0$ for $x < 0$, it is clear from Théorème 2(iv)(c) of [20] that

$$L_t^0 = 2\int_0^t 1_{(\langle X_s, f\rangle = 0)} \langle X_s, Af\rangle\, ds,$$

and (5.1.2) follows immediately.

Suppose for the moment that $m(dx) = (2\pi)^{-d/2} \exp(-|x|^2/2)\, dx$. An easy argument shows that $mP_t \ll\gg m$ for all $t > 0$. Applying Theorem 1.1, we find that $\varepsilon_m Q_s \gg \varepsilon_m Q_t$, $s \ge t > 0$, and so we can strengthen (5.1.2) in this case to conclude that

$$\mathbb{P}^m(X_t \in C) = 0, \quad \forall t > 0.$$

If $B \in \mathbb{R}^d$ is an open ball, we can choose f so that $\{x: f(x) > 0\} = B$, and thus, recalling (5.1.1),

$$\mathbb{P}^m(\langle X_t, 1_B\rangle = 0, \langle \nu * X_t, 1_B\rangle \ne 0) = 0, \quad \forall t > 0.$$

Since this will be true for all open balls belonging to a given countable base for the topology of $\mathbb{R}^d$, it follows that

$$\varepsilon_m Q_t(D) = 1, \quad \forall t > 0, \tag{5.1.3}$$

for this particular measure m, where

$$D = \{\mu \in M(\mathbb{R}^d): \operatorname{supp} \nu * \mu \subset \operatorname{supp} \mu\}.$$

Applying Lemma 1.4 gives, in the notation of that result,

$$R_t(x, M(\mathbb{R}^d)\backslash D) = 0, \quad \text{Lebesgue-a.a. } x, \forall t > 0$$

(use (2.1.0) with $n = 1$). However, an argument similar to the one leading to (3.1.1) implies that

$$R_t(x + y, \cdot) = R_t(x, \cdot) \circ \tau_y^{-1},$$

and since

$$\tau_y(M(\mathbb{R}^d)\backslash D) = M(\mathbb{R}^d)\backslash D, \quad \forall y \in \mathbb{R}^d,$$

we conclude that

$$R_t(x, M(\mathbb{R}^d)\backslash D) = 0, \quad \forall x, \forall t > 0.$$

Another application of Lemma 1.4, combined with the observation

$$\operatorname{supp}\left(\sum_{i=1}^{n}\mu_i\right)=\bigcup_{i=1}^{n}\operatorname{supp}\mu_i,\qquad \mu_i\in M(\mathbb{R}^d),\ n\in\mathbb{N},$$

establishes that (5.1.3) holds for a general measure $m\in M(\mathbb{R}^d)$. (See (**) in the Remarks following the proof of Theorem 1.1 with $B=D^c$.) A straightforward induction finishes the proof (cf., the completion of the proof of Theorem 1.5 in [15]). □

We now drop the assumption that $\varphi(x,\lambda)=-\lambda^2/2$. Our promised extension of Theorem 5.1 is Corollary 5.3 below, which will be a consequence of the following theorem.

Notation. Given a set $F\subset E$ which is closed in the relative Ray topology, we define $J(F)$ to be the set of points $x\in E$ such that for every Ray relatively open set $G\ni x$, there exists a Ray relatively open set H for which $H\cap F\neq\varnothing$ and $\limsup_{t\downarrow 0}\inf_{y\in H}P_t 1_G(y)/t>0$. We remark that $J(F)$ is closed in the relative Ray topology. Given $\mu\in M(E)$, we denote by $\operatorname{supp}_r\mu$ the closed support of μ in the relative Ray topology.

Theorem 5.2. *For all* $m\in M(E)$

$$\bigcup_{k=1}^{\infty}J^k(\operatorname{supp}_r X_t)\subset\operatorname{supp}_r X_t,\qquad \mathbb{P}^m\text{-}a.s.,\ \forall t>0.$$

Proof. Fix $m\in M(E)$ and $t>0$. Consider two sets G and H which are open in the relative Ray topology of E. As X is an $M_r(E)$-valued Hunt process, the paths of X are continuous at t, $\mathbb{P}^m$-a.s., and so the process $\{\langle X_s,1_H\rangle : s\geq 0\}$ has lower semicontinuous paths at t, $\mathbb{P}^m$-a.s.

Suppose that

$$\limsup_{s\downarrow 0}\inf_{y\in H}P_s 1_G(y)/s>0. \tag{5.2.1}$$

We have, from Lemma 1.3 and the almost sure continuity of the paths of X at t, that

$$\begin{aligned}1_{\langle X_t,1_G\rangle>0}&=\lim_{s\uparrow t}\mathbb{P}^m(\langle X_t,1_G\rangle>0\,|\,X_u:0\leq u\leq s)\\&=\lim_{s\uparrow t}\mathbb{P}^{X_s}(\langle X_{t-s},1_G\rangle>0)\\&\geq\limsup_{s\uparrow t}e^{-4\beta(t-s)}\left[1+\gamma(t-s)/(\langle X_s,1_H\rangle\inf_{y\in H}P_{t-s}1_G(y))\right]^{-1}\\&>0,\quad\mathbb{P}^m\text{-a.s.},\end{aligned}$$

when $\langle X_t,1_H\rangle>0$.

We have shown that if (5.2.1) holds, then

$$H\cap\operatorname{supp}_r X_t\neq\varnothing\Rightarrow G\cap\operatorname{supp}_r X_t\neq\varnothing,\qquad\mathbb{P}^m\text{-a.s.}$$

Letting G and H range over all pairs of sets chosen from a given countable base for the relative Ray topology of E gives

$$J(\operatorname{supp}_r X_t) \subset \operatorname{supp}_r X_t, \quad \mathbb{P}^m\text{-a.s.}$$

As $A \subset B$ clearly implies that $J(A) \subset J(B)$, an induction completes the proof. □

Before we can use Theorem 5.2 to extend Theorem 5.1, we need to recall the notion of a Lévy system (see Benveniste and Jacod [1]). Suppose that ξ is a Hunt process. There exists a pair (N, I) with the following properties:

(i) The map $N: E \times \mathscr{E} \to [0, \infty[$ is a kernel such that $N(x, \{x\}) = 0$, $x \in E$.

(ii) The process I is a continuous additive functional.

(iii) For each nonnegative $\mathscr{E} \times \mathscr{E}$-measurable function f, if we put

$$A_t^f = \sum_{s \leq t} f(\xi_{s-}, \xi_s) 1_{\xi_{s-} \neq \xi_s}$$

and

$$\widetilde{A}_t^f = \int_0^t I(ds) \int N(\xi_s, dy) f(\xi_s, y),$$

then $P^\mu(A_t^f) = P^\mu(\widetilde{A}_t^f)$ for all $t \geq 0$ and initial measures μ. Moreover, if $P^\mu(A_t^f) < \infty$ for all t, then $A^f - \widetilde{A}^f$ is a P^μ-martingale.

The pair (N, I) is not unique. In fact, $N(y, \cdot)$ is arbitrary for y off the support of I (see Blumenthal and Getoor [2, p. 215]). To avoid pathologies, we could therefore require that $N(y, \cdot) = 0$ for $y \notin \operatorname{supp}(I)$. In fact, we require a somewhat stronger condition (ii) in the next result.

Corollary 5.3. *Suppose ξ is a Feller process with Lévy system (N, I). Assume that if $U \subset E$ is open and $z \in E$ satisfy $N(z, U) > 0$, then there are open neighborhoods of z, $H \subset B$, such that*

(i) $\inf_{y \in B} N(y, U) > 0$;

(ii) $\limsup_{t \downarrow 0} \inf_{y \in H} P^y(1(\tau_{B^c} > t) I_t / t) > 0$, *where* $\tau_{B^c} = \inf\{t \geq 0: \xi_t \in B^c\}$.

Then for each $m \in M(E)$ and $t > 0$,

$$\bigcup_{r=1}^{\infty} \operatorname{supp}\left(\int \cdots \int X_t(dx_1) N(x_1, dx_2) \cdots N(x_r, \cdot)\right) \subset \operatorname{supp} X_t, \quad \mathbb{P}^m\text{-a.s.}$$

Proof. It suffices to show that for any measure μ we have $\operatorname{supp} \mu N \subset J(\operatorname{supp} \mu)$ (note in this case that the Ray topology and the original topology coincide). Choose $x \in \operatorname{supp} \mu N$, and let G be any open neighborhood of x. There exists $z \in \operatorname{supp} \mu$ such that $N(z, G) > 0$. Choose an open set U such that $\operatorname{cl}(U)$ is a compact subset of G and $N(z, U) > 0$. Let $H \subset B$ be open neighborhoods of z satisfying (i) and (ii), and let

$$\alpha = \inf_{y \in B} N(y, U) > 0,$$

$$\beta = \lim_{t_k \downarrow 0} \inf_{y \in H} P^y(1(\tau_{B^c} > t_k) I_{t_k} / t_k) > 0$$

for some appropriate sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \downarrow 0$ as $k \to \infty$. Then

(5.3.1)
$$\begin{aligned} P_t 1_G(y) &\geq P^y(\tau_U \leq t, \xi_t \in G) \\ &= P^y(P_{t-\tau_U} 1_G(\xi_{\tau_U}), \tau_U \leq t), \end{aligned}$$

and by the Feller property

(5.3.2)
$$\lim_{s \downarrow 0} \inf_{y \in \mathrm{cl}(U)} P_s 1_G(y) = 1.$$

Set

$$A_t = \sum_{s \leq t} 1_{(\xi_{s-} \in B, \xi_s \in U)}$$

and

$$\widetilde{A}_t = \int_0^t N(\xi_s, U) 1_{(\xi_s \in B)} \, dI_s.$$

Then, since A increases by unit jumps, we have

(5.3.3)
$$\begin{aligned} e^{-A_t} &= 1 + \sum_{s \leq t} (e^{-A_{s-}-1} - e^{-A_{s-}}) \Delta A_s \\ &= 1 + (e^{-1} - 1) \int_0^t e^{-A_{s-}} \, dA_s, \end{aligned}$$

and hence

(5.3.4)
$$\begin{aligned} \inf_{y \in H} P^y(\tau_U \leq t_k) &\geq \inf_{y \in H} P^y(A_{t_k} \geq 1) \geq \inf_{y \in H} P^y(1 - e^{-A_{t_k}}) \\ &= \inf_{y \in H} (1 - e^{-1}) P^y \left(\int_0^{t_k} e^{-A_{s-}} \, dA_s \right) \\ &= (1 - e^{-1}) \inf_{y \in H} P^y \left(\int_0^{t_k} e^{-A_{s-}} 1(\xi_s \in B) N(\xi_s, U) \, dI_s \right) \\ &\geq (\alpha/2) \inf_{y \in H} P^y \left(\int_0^{t_k} e^{-A_{s-}} 1(\xi_s \in B) \, dI_s \right) \\ &\geq (\alpha/2) \inf_{y \in H} P^y(1(\tau_{B^c} > t_k) I_{t_k}) \\ &\geq (\alpha\beta/3) t_k \end{aligned}$$

for t_k small enough. Now (5.3.1), (5.3.2), and (5.3.4) together imply that $\limsup_{t \downarrow 0} \inf_{y \in H} P_t 1_G(y)/t > 0$, and hence $x \in J(\operatorname{supp} \mu)$ as required. □

Corollary 5.4. *The conclusion of Corollary* 5.3 *continues to hold if hypothesis* (ii) *is replaced by*

(ii$'$) *There exists* $c > 0$ *such that* $I_t \geq ct$ *for all* $t > 0$ P^y*-a.s. for all* $y \in E$.

Proof. It is easy to use the Feller property to derive (ii) from (ii$'$). □

Examples. (i) Suppose that ξ is a Lévy process on a separable, locally compact, Abelian group. If ν is the Lévy measure of ξ, then the conditions of Corollary 5.4 hold, with $N(x, C) = \nu(C - x)$ and $I(t) = t$. In particular, when the group is the additive group of $\mathbb{R}^d$ and $\varphi(x, \lambda) = -\lambda^2/2$, we recover Theorem 5.1.

(ii) Suppose that ξ is a Feller process with strong infinitesimal generator A of the form

$$Af(x) = \rho \int \mu(x, dy)(f(y) - f(x)),$$

where $\rho > 0$, and μ is a probability kernel such that $x \mapsto \int \mu(x, dy)f(y) \in \widehat{C}(E)$ whenever $f \in \widehat{C}(E)$ ($\equiv$ the space of continuous functions vanishing at infinity). Then the conditions of Corollary 5.4 hold, with $N(x, C) = \rho\mu(x, C)$ and $I(t) = t$. In particular, if ξ is a Markov chain with all states stable, then for all $m \in M(E)$ and $t > 0$, we have

$x \in \operatorname{supp} X_t \Rightarrow y \in \operatorname{supp} X_t$ for all $x, y \in E$ such that $P^x(\exists t > 0 : \xi_t = y) > 0$, $\mathbb{P}^m$-a.s.

(iii) Suppose that $E = \mathbb{R}^d$ and the domain of the strong infinitesimal generator, A, of ξ contains all C^∞ functions with compact support. Following [14], we find that there exists a kernel N such that $Af(x) = \int N(x, dy)f(y)$ whenever f is a C^∞ function with a compact support that does not contain x. Moreover, if we set $I(t) = t$, then (N, I) is a Lévy system for ξ. Also, since $x \mapsto \int N(x, dy)f(y)$ is continuous on $\mathbb{R}^d \backslash (\operatorname{supp} f)$ when f is C^∞ with compact support, it is easy to show that condition (i) of Corollary 5.3 holds. In particular, from the remarks about generators at the beginning of the proof of Theorem 5.1, it is clear that we can recover Theorem 5.1 from this example.

6. Appendix

In the remarks following the proof of Theorem 1.1, we promised to give an example showing that $m_1 P_T \ll m_2 P_{T+h}$ for fixed $T > 0$ and $h \geq 0$ does not imply that $\varepsilon_{m_1} Q_T \ll \varepsilon_{m_2} Q_{T+h}$. For the sake of concreteness, we take $T = 1$ and $h = 0$, but our counterexample can easily be modified for other values. Recall that we are once again taking $\varphi(x, \lambda) = -\lambda^2/2$.

Let $E = [0, 1] \cup \{2\}$ with the relative Euclidean topology, and suppose that ξ is the Feller process with semigroup

$$P_t f(x) = f((x + t) \wedge 1) \exp(-(1 - x) \wedge t) + f(2)[1 - \exp(-(1 - x) \wedge t)], \quad x \in [0, 1],$$

$$P_t f(2) = f(2).$$

That is, at $x \in [0, 1[$ the process ξ drifts to the right with unit speed or jumps to the trap 2 at unit rate. The state 1 is also a trap.

Take $m_1 = \varepsilon_1 + \varepsilon_2$ and $m_2 = \varepsilon_0$. We have

$$m_1 P_1(\{1\}) = m_1 P_1(\{2\}) = 1$$

and

$$m_2 P_1(\{1\}) = e^{-1} = 1 - m_2 P_2(\{2\}),$$

so that $m_1 P_1 \ll\gg m_2 P_1$. Observe from Lemma 1.2 that for each $t \in [0, 1]$ we have $[0, 1] \cap \operatorname{supp} X_t \subset \{t\}$, $\mathbb{P}^{m_2}$-a.s. By an argument similar to the one used in the proof of Theorem 5.2, we see that

$$\begin{aligned} 1_{(X_1(\{2\})>0)} &\geq \liminf_{s\uparrow 1}[1+(1-s)/[X_s(\{s\})(1-e^{s-1})]]^{-1} \\ &= [1+1/X_1(\{1\})]^{-1} > 0, \quad \mathbb{P}^{m_2}\text{-a.s.} \end{aligned}$$

when $X_1(\{1\}) > 0$. Thus $\mathbb{P}^{m_2}(\operatorname{supp} X_1 = \{1\}) = 0$.

On the other hand, it is clear from inspection of the Laplace functionals of X that under $\mathbb{P}^{m_1}$ the processes $X_t(\{1\})$ and $X_t(\{2\}) = X_t(E) - X_t(\{1\})$ are just a pair of independent continuous state branching processes (see, for example, §4.3.5 of [13]). In particular, $\mathbb{P}^{m_1}(\operatorname{supp} X_1 = \{1\}) > 0$, and so $\varepsilon_{m_1} Q_1$ is not absolutely continuous with respect to $\varepsilon_{m_2} Q_1$.

References

1. A. Benveniste and J. Jacod, *Systèmes de Lévy des processus de Markov*, Invent. Math. **21** (1973), 183–198.
2. R. M. Blumenthal and R. K. Getoor, *Markov processes and potential theory*, Academic Press, New York and London, 1968.
3. D. A. Dawson, *The critical measure diffusion process*, Z. Wahrsch. Verw. Gebiete **40** (1977), 125–145.
4. D. A. Dawson, I. Iscoe and E. Perkins, *Super-Brownian motion: path properties and hitting probabilities*, Probab. Theory Related Fields **83** (1989), 135–206.
5. C. Dellacherie and P. A. Meyer, *Probability and potential*, North-Holland Math. Stud., no. 29, North-Holland, Amsterdam, 1978.
6. E. B. Dynkin, *Representation for functionals of superprocesses by multiple stochastic integrals, with applications to self-intersection local times*, In: Colloque Paul Lévy sur les processus stochastiques, Astérisque **157–158** (Société Mathématique de France) (1988).
7. N. El Karoui and S. Roelly-Coppoletta, *Study of a general class of measure-valued branching processes: A Lévy-Hincin representation*, preprint.
8. S. N. Ethier and T. G. Kurtz, *Markov processes: Characterization and convergence*, Wiley, New York, 1986.
9. P. J. Fitzsimmons, *Construction and regularity of measure-valued Markov branching processes*, Israël J. Math **64** (1988), 337–361.
10. I. I. Gihman and A. V. Skorohod, *The theory of stochastic processes* II, Springer-Verlag, New York, Heidelberg, and Berlin, 1975.
11. I. Iscoe, *A weighted occupation time for a class of mesaure-valued branching processes*, Probab. Theory Related Fields **71** (1986), 85–116.
12. O. Kallenberg, *Random measures*, 3rd ed., Akademie-Verlag, Berlin; Academic Press, New York, 1983.
13. F. B. Knight, *Essentials of Brownian motion and diffusion*, Math. Surveys Monogr., no. 18, Amer. Math. Soc., Providence, R.I., 1981.
14. P. A. Meyer, *Intégrales stochastiques*, Séminaire de Probabilités I, Lecture Notes in Math., Vol. 39, Springer-Verlag, New York, Heidelberg, and Berlin, 1967.
15. E. Perkins, *Polar sets and multiple point for super-Brownian motion*, Ann. Probab. **18** (1990), 453–491.
16. ——, *The Hausdorff measure of the closed support of super-Brownian motion*, Ann. Inst. H. Poincaré **25** (1989), 205–224.
17. ——, *A space-time property of a class of measure-valued diffusions*, Trans. Amer. Math. Soc. **305** (1988), 743–795.

18. R. Tribe, Ph.D. thesis, Univ. of British Columbia.

19. S. Watanabe, *A limit theorem of branching processes and continuous state branching processes*, J. Math. Kyoto Univ. **8** (1968), 141–167.

20. M. Yor, *Continuité des temps locaux*, Temps Locaux, Astérisque, Vols. 52–53, Soc. Math. France, Paris, 1978.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903

Current address: Department of Statistics, 367 Evans Hall, University of California, Berkeley, California 94720

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, V6T 1Z4, CANADA