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THE SYMBOLIC REPRESENTATION OF BILLIARDS WITHOUT BOUNDARY CONDITION

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ABSTRACT. We consider a dynamical system with elastic reflections in the whole plane and show that such a dynamical system can be represented as a symbolic flow over a mixing subshift of finite type. This fact enables us to prove an analogue of the prime number theorem for the closed orbits of such a dynamical system.

INTRODUCTION

Let $O_1, O_2, \dots, O_L$ $(L \geq 3)$ be a finite number of bounded domains in $\mathbb{R}^2$ with smooth boundaries $\partial O_1, \partial O_2, \dots, \partial O_L$. We assume that the closures $\overline{O_j} = O_j \cup \partial O_j$ of O_j are strictly convex and mutually disjoint. Consider the motion of a particle in the exterior domain $O = \mathbb{R}^2 \setminus \bigcup_{j=1}^L \overline{O_j}$, which obeys the law of reflection: "the particle moves along the straight line with unit speed in O and reflects at the boundary $\partial O = \bigcup_{j=1}^L \partial O_j$ so that the angle of the reflection coincides with the angle of the incidence." We can describe this motion of a particle by a dynamical system (a flow) S_t on the unit tangent bundle over O. We call the flow S_t a billiard without boundary condition in the light of the Sinai's billiard in [6], which is defined on the unit tangent bundle over 2-torus T^2, i.e., the billiard with periodic boundary condition.

The purpose of this paper is to prove the following theorems:

Theorem 1. *Under the hypotheses* (H.1) *and* (H.2) (*see* §2), *the flow* S_t *restricted to the nonwandering set can be represented as a symbolic flow* σ_t *over an appropriate subshift of finite type so that the corresponding closed orbits have the same period* (*see Proposition* 3.1).

Theorem 2. *Under the hypotheses* (H.1) *and* (H.2), *there is a positive constant* h *such that an analogue of the prime number theorem*

$$\#\{\gamma\,;\, \exp[hT_\gamma] \leq x\} \cdot \frac{\log x}{x} \to 1 \qquad (x \to \infty)$$

holds, where γ *and* T_γ *denote the prime closed orbit of* S_t *and its period respectively.*

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It is not hard to see that Theorem 2 is obtained by combining Theorem 1 and the result in Parry and Pollicott [5] (see §3).

It will be meaningful to note that the closed orbits of the dynamical system S_t make the essential contribution to the singular support of the distributional function $\sum_{\lambda_i \in \mathrm{Spec}\,\Delta} \cos \lambda_i^{1/2} t$ (see [1]) and they are closely related to the poles of the scattering matrix as mentioned in Ikawa [2] and [3] etc., where $\mathrm{Spec}\,\Delta$ denotes the set of eigenvalues of the selfadjoint realization of the Laplace operator with the appropriate boundary condition.

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1. Preliminaries

In this section we prepare the basic notions for the later convenience.

Let $O_1, O_2, \ldots, O_L$ $(L \geq 3)$ be a finite number of bounded domains in $\mathbb{R}^2$ as in the beginning of Introduction. We denote by $S\mathbb{R}^2 = \mathbb{R}^2 \times S^1 = \{(q, v) \in \mathbb{R}^2 \times \mathbb{R}^2;\ |v| = 1\}$ the unit tangent bundle over $\mathbb{R}^2$ and $\pi : S\mathbb{R}^2 \to \mathbb{R}^2$ the natural projection, where $|\cdot|$ denotes the usual Euclidean norm. Choose a point $q_j \in \partial O_j$ for each j and define the following quantities for $x = (q, v) \in \partial O = \bigcup_{j=1}^L \partial O_j$:

$$(1.1)\qquad \begin{cases} \xi_0(x) = j, \quad \text{if } q \in \partial O_j; \\ r(x) = \text{the arclength between } q_{\xi_0(x)} \text{ and } q, \\ \qquad\qquad \text{measured clockwise along the curve } \partial O_{\xi_0(x)}; \\ \phi(x) = \text{the angle between the vector } v \text{ and the} \\ \qquad\qquad \text{unit innernormal } n(q) \text{ of } \partial O_{\xi_0(x)} \text{ at } q, \\ \qquad\qquad \text{measured unitclockwise}. \end{cases}$$

Therefore $\pi^{-1}(\partial O)$ is parametrized as

$$(1.2)\qquad \begin{aligned} \pi^{-1}(\partial O) = \{(j, r, \phi);\ & 1 \leq j \leq L, \\ & 0 \leq r < \text{ the perimeter of } \partial O_j, \text{ and } 0 \leq \phi < 2\pi\}. \end{aligned}$$

Put

$$(1.3)\qquad \begin{cases} M_- = \{x \in \pi^{-1}(\partial O),\ \frac{\pi}{2} \leq \phi(x) \leq \frac{3}{2}\pi\}, \text{ and} \\ M_+ = \{x \in \pi^{-1}(\partial O),\ \frac{\pi}{2} < \phi(x) + \pi < \frac{3}{2}\pi \bmod 2\pi\}. \end{cases}$$

We introduce the following equivalence relation '$\sim$' to $\pi^{-1}(\partial O)$:

$$(1.4)\qquad x \sim y \quad \text{if and only if } \mathrm{Inv}(x) = y \text{ or } x = y,$$

where $\mathrm{Inv} : \pi^{-1}(\partial O) \to \pi^{-1}(\partial O)$ is defined by

$$(1.5)\qquad \mathrm{Inv}(j, r, \phi) = (j, r, \pi - \phi) \bmod 2\pi.$$

It is natural to identify $(\pi^{-1}(\partial O))/\sim$ with M_- and we often use this identification without specification. Put

$$M = \pi^{-1}(O) \cup (\pi^{-1}(\partial O)/\sim) = \pi^{-1}(O) \cup M_- . \tag{1.6}$$

Now we recall the notion of the billiard without boundary condition. Consider the motion of a particle which moves along the straight line with unit speed in $O = \mathbb{R}^2 \setminus \bigcup_{j=1}^{L} \overline{O_j}$ and reflects at the boundary $\partial O = \bigcup_{j=1}^{L} \partial O_j$ according to the law of reflection: the angle of reflection coincides with that of incidence. Then the motion determines a dynamical system (a flow) on M in a canonical way (see Remark 1.2 below). We call it a billiard without boundary condition.

Remark 1.1. It is easy to see that M_- and M_+ denote the set of the incidental vectors and the set of the reflection vectors respectively.

We define the first collision time τ_+ and the last collision time τ_- by

$$\begin{cases} \tau_+(x) = \inf\{t > 0\,,\ \pi(S_t x) \in \partial O\} \\ \tau_-(x) = \sup\{t < 0\,,\ \pi(S_t x) \in \partial O\}\,. \end{cases} \tag{1.7}$$

Here we regard $\tau_+(x)$ (resp. $\tau_-(x)$) as $+\infty$ (resp. $-\infty$) if the set in the definition is empty.

Remark 1.2. Let $x = (q\,, v)\,,\ M = \pi^{-1}(O) \cup M_-$. We note that the flow S_t is defined so that

$$S_t x = \begin{cases} (q + tv\,, v)\,, & (x \in \pi^{-1}(O))\,, \\ (q + t\tilde{v}\,, \tilde{v})\,, & (x \in M_-) \end{cases}$$

if $0 < t < \tau_+(x)$, where $\tilde{v}$ is determined by the formula $\mathrm{Invx} = (q\,, \tilde{v})$.

Put

$$\Omega = \{x \in M\,,\ \pi(S_t x) \in \partial O \text{ for both infinitely many } t > 0 \text{ and infinitely many } t < 0\}\,. \tag{1.8}$$

Clearly, Ω coincides with the nonwandering set of the flow S_t. Put

$$\Omega_0 = \pi^{-1}(O) \cap \Omega \quad \text{and} \quad \Omega_- = M_- \cap \Omega\,. \tag{1.9}$$

We define the local maps T and T^{-1} by

$$\begin{cases} T(x) = S_{\tau_+(x)}(x) & \text{if } \tau_+(x) < +\infty\,, \\ T^{-1}(x) = S_{\tau_-(x)}(x) & \text{if } \tau_-(x) > -\infty\,, \end{cases} \tag{1.10}$$

respectively.

It is not hard to see that the above notation T^{-1} is compatible with the definition of the inverse map of T and T is locally diffeomorphic.

Remark 1.3. Consider the flow S_t restricted to Ω. The set Ω_- and the first collision time τ_+ play the role of the Poincaré section and the Poincaré map respectively.

For $x \in M_-$ $(= \pi^{-1}(\partial O)/\sim)$, we put

$$\xi_j(x) = \xi_0(T^j x) \quad \text{if } T^j \text{ is defined}. \tag{1.11}$$

We call the sequence $\xi = (\xi_j)_{j=-\infty}^{\infty}$ the itinarary of $x \in \Omega_-$ if $\xi_j = \xi_j(x)$ and write ξ as $\xi(x)$.

We conclude this section by stating the following lemma. The proof is due to elementary calculation of the Jacobi matrix of T, and it can be found in [4]. Therefore we omit the proof.

Lemma 1.1. *Let C be a curve of class C^1 in M_- which is represented as $\{(j, r, \phi);\ \phi = \psi(r),\ a \le r \le b\}$, where ψ is a C^1-function in r. Assume that T (resp. T^{-1}) is defined and continuous on C. If the image $C_1 = TC$ (resp. $C_{-1} = T^{-1}C$) is represented as $\{(j_1, r_1, \phi_1);\ \phi_1 = \psi_1(r_1),\ a_1 \le r_1 \le b_1\}$ (resp. $\{(j_{-1}, r_{-1}, \phi_{-1});\ \phi_- = \psi_-(r_{-1}),\ a_{-1} \le r_{-1} \le b_{-1}\}$), where ψ_1 (resp. ψ_{-1}) is C^1-function in r_1 (resp. r_{-1}), then we have:*

(1.12)

$$\frac{d\psi_1}{dr_1} = k(r_1) - \frac{\cos\psi_1}{\cos\psi}\left(\frac{\tau_+(j, r, \phi)}{\cos\psi} - \left(\frac{d\psi}{dr} + k(r)\right)^{-1}\right)^{-1}$$

$$\left(\text{resp. } \frac{d\psi_{-1}}{dr_{-1}} = -k(r_{-1}) - \frac{\cos\psi_{-1}}{\cos\psi}\left(\frac{\tau_-(j, r, \phi)}{\cos\psi} - \left(\frac{d\psi}{dr} + k(r)\right)^{-1}\right)^{-1}\right),$$

$$\frac{dr_1}{dr} = -\frac{\cos\psi}{\cos\psi_1}\left(1 - \frac{\tau_+(j, r, \phi)}{\cos\psi}\left(\frac{d\psi}{dr} + k(r)\right)\right) \tag{1.13}$$

$$\left(\text{resp. } \frac{dr_{-1}}{dr} = -\frac{\cos\psi}{\cos\psi_{-1}}\left(1 - \frac{\tau_-(j, r, \phi)}{\cos\psi}\left(\frac{d\psi}{dr} - k(r)\right)\right)\right),$$

where $k(r)$ denotes the curvature of ∂O_j at (j, r, ϕ), etc.

2. Well-posedness of itinarary problem

From now on we assume:

(H.1) (convexity). *For each $j = 1, 2, \ldots, L$ boundary ∂O_j of O_j is a simple closed curve with nonvanishing curvature.*

(H.2) (no eclipse). *For any triple (j_1, j_2, j_3) of distinct indices,*

$$\operatorname{conv}[O_{j_1} \cup O_{j_2}] \cap O_{j_3} = \varnothing,$$

where $\operatorname{conv}[B]$ *denotes the convex hull of the set B.*

We introduce the following shift dynamical system. Let A be an $L \times L$-matrix with entries $A(i, j) = (1 - \delta_{ij})$, $1 \le i, j \le L$, where δ_{ij} denotes the

Kronecker's delta. Put

$$\Sigma = \Sigma_A = \left\{ \xi = (\xi_j)_{j=-\infty}^{\infty} \in \prod_{j=-\infty}^{\infty} \{1, 2, \dots, L\}; \right. \tag{2.1}$$
$$\left. A(\xi_j, \xi_{j+1}) = 1 \text{ for any } j \right\}.$$

We define $d_\rho : \Sigma \times \Sigma \to R$ by

$$d_\rho(\xi, \eta) = \rho^n \quad \text{if } \xi_j = \eta_j \text{ for } |j| < n \text{ and } \xi_n \neq \eta_n \text{ or } \xi_{-n} \neq \eta_{-n}. \tag{2.2}$$

It is easy to see that d_ρ becomes a metric on Σ which defines the same topology as the induced topology of Σ as a subset of the product space

$$\prod_{j=-\infty}^{\infty} \{1, 2, \dots, L\}.$$

The shift transformation $\sigma : \Sigma \to \Sigma$, $(\sigma\xi)_j = \xi_{j+1}$, $j \in Z$ is well defined and the shift dynamical system (Σ, σ) is a typical example of a mixing subshift of finite type (see [5]). Consider the billiard without boundary condition S_t. We say that a point $x \in \Omega_-$ solves the itinarary problem

$$\xi(y) = \xi \in \Sigma \tag{2.3}$$

or x is a solution of the itinarary problem (2.3) if the itinarary $\xi(x)$ of x coincides with the sequence ξ.

Under the hypotheses (H.1) and (H.2) we prove that the itinarary problem (2.3) is well-posed in the following sense.

Theorem 0 (the Lipschitz well-posedness of the itinarary problem). *If the hypotheses* (H.1) *and* (H.2) *are satisfied, there exists a unique* $x \in \Omega_-$ *which solves the itinarary problem* (2.3) *for any* $\xi \in \Sigma$. *In addition, if we denote by* $x(\xi)$ *the solution of the itinarary problem* (2.3), *there exist constants* $C > 0$ *and* $0 < \rho < 1$ *such that*

$$|\tau_+(x(\xi)) - \tau_+(x(\eta))| < C d_\rho(\xi, \eta) \quad \textit{for any } \xi, \eta \in \Sigma, \tag{2.4}$$

where d_ρ *denotes the metric on* Σ *defined by* (2.2).

We prepare an a priori estimate for the proof of Theorem 0.

Lemma 2.1 (a priori estimate). *Let* x *and* y *be elements in* M_-. *Assume* T^j *is well defined and* $\xi_j(x) = \xi_j(y)$ *for each* j *with* $-n \le j \le n$ $(n \ge 1)$. *Then the arclength* $r(x, y)$ *between* $\pi(x)$ *and* $\pi(y)$ *satisfies*

$$r(x, y) \le c_0 l (1 + \eta)^{-n}, \tag{2.5}$$

where c_0 *is a positive constant independent of* x *and* y,

$$l = \max\{\textit{the perimeter of } \partial O_j,\ j = 1, 2, \dots, L\} \quad \textit{and}$$
$$\eta = \min\{\textit{the distance between } \overline{O_i} \textit{ and } \overline{O_j},\ 1 \le i < j \le L\}$$
$$\times \min\{k(q),\ q \le \partial O_j,\ j = 1, 2, \dots, L\}.$$

Proof. Let C be a C^1-curve in M_- as in Lemma 1.1. We call it an increasing (resp. decreasing) curve if $\frac{d\psi}{dr} \geq 0$ (resp. $\frac{d\psi}{dr} \leq 0$). Assume that $x, y \in M_-$ satisfy the assumption of Lemma 2.1. Then we may write $T^j x = (r_j(x), \phi_j(x))$, $j = -n, -(n-1), \ldots, n-1, n$ without confusion. First we connect x and y by a line segment C_0 in M_-. We may assume that $r_0(x) < r_0(y)$. If $\phi_0(x) \leq \phi_0(y)$, C_0 becomes an increasing curve in M_-. Therefore it is not hard to show that T^j is continuous on C_0 and the image $C_j = T^j C_0$ turns out to be an increasing curve for each $j = 1, 2, \ldots, n$ in the same way as in the proof of Lemma 4.1 in [4]. Thus C_j can be expressed as $\{(r_j, \phi_j),\ \phi_j = \psi_j(r_j),\ a_j \leq r_j \leq b_j\}$ with $d\psi_j/dr_j \geq 0$ for each j. In virtue of the formula (1.13), we obtain

$$\frac{dr_n}{dr_0} = \frac{dr_n}{dr_{n-1}} \frac{dr_{n-1}}{dr_{n-2}} \cdots \frac{dr_1}{dr_0} = (-1)^n \frac{\cos\psi_n}{\cos\psi_0} \prod_{j=1}^{n} b_j, \tag{2.6}$$

where $b_j = (1 - \tau_+(T^j(r_0, \phi_0))(d\psi_j/dr_j + k(r_j))$. Since $d\psi_j/dr_j \geq 0$ for all $j = 0, 1, \ldots, n-1$, we have

$$|b_j| \geq 1 + \eta \tag{2.7}$$

from the formula (1.12). Thus we have

$$\left|\frac{dr_n}{dr_0}\right| \geq |\cos\phi_0|(1+\eta)^n. \tag{2.8}$$

Therefore we obtain

$$r(x, y) \leq |\cos\phi_0|^{-1} l(1+\eta)^{-n}. \tag{2.9}$$

On the other hand it is easy to show that $|\cos\phi_0|$ is bounded below by a positive constant which is independent of x and y in virtue of the hypotheses (H.1) and (H.2). Hence we have proved the inequality (2.5) when $\phi_0(x) \leq \phi_0(y)$. If $\phi_0(x) > \phi_0(y)$, we can prove the estimate (2.5) in the same manner, by using T^j, $-n \leq j \leq -1$, instead of T^j, $1 \leq j \leq n$. □

Now we can prove Theorem 0. The Lipschitz continuity (2.4) of the first collision time is an immediate consequence of Lemma 2.1 if we take $(1+\eta)^{-1}$ as ρ. The uniqueness of the itinarary problem follows from the estimate (2.4). Therefore it suffices to show the existence. First we assume that ξ is periodic, i.e., $\xi_{n+m} = \xi_n$ for some $m > 0$, for all $n \in Z$. Consider the following minimal value problem

$$l(q^0, q^1, \ldots, q^{m-1}) = \sum_{j=0}^{m-1} |q^j - q^{j-1}|, \tag{2.10}$$

$$q^j \in \partial O_{\xi_j},\quad j = 0, 1, \ldots, m-1,$$

where $q^{-1} = q^{m-1}$.

Hypotheses (H.1) and (H.2) imply that there exists $(p^0, p^1, \dots, p^{m-1})$ which minimizes $l(q^0, q^1, \dots, q^{m-1})$ in virtue of the Borzano Weierstrass theorem. The points $p^0, p^1, \dots, p^{m-1}$ have to satisfy the equations

$$\frac{\partial l}{\partial q_k^j} = \lambda_j \frac{\partial f_j}{\partial q_k^j}(p^j), \qquad j = 0, 1, \dots, m-1 \quad \text{and} \quad k = 1, 2, \dots, \tag{2.11}$$

where the curves ∂O_{ξ_j} are assumed to be represented as $f_j(q^j) = 0$ in the neighborhood of p^j, and λ_j denote the Lagrange multipliers.

The equations (2.11) are nothing but the law of reflections. Therefore the existence of the solution of the itinarary problem has been proved when ξ is periodic.

Let ξ be an element in Σ which is not periodic. Choose $\xi^m \in \Sigma$ which is periodic and $d_p(\xi^m, \xi) \to 0$, $(m \to \infty)$. Let x^m be the unique solution of the itinarary problem $\xi(x) = \xi^m$. The estimate (2.5) in Lemma 2.1 implies that

$$r(T^j x^m, T^j x^{m+1}) \le C\rho^{(\text{the period of } \xi^m) - |j|}$$

for $|j| \le$ the period of ξ^m. Therefore x^m converges to some $x \in M_-$ and x satisfies $\xi(x) = \xi$. Now the proof of Theorem 0 is complete.

Remark 2.1. It is not necessary to use the inequality (2.5) to show the existence of the solution of the itinarary problem. One can show it by use of the diagonal argument.

3. Proofs of results

The purpose of this section is to complete the proofs of Theorem 1 and Theorem 2 in Introduction. Note that we always assume the hypotheses (H.1) and (H.2).

Define a function f on Σ by

$$f(\xi) = \tau_+(x(\xi)), \quad \text{for } \xi \in \Sigma, \tag{3.1}$$

where $x(\xi)$ denotes the unique solution of the itinarary problem (2.3) as before. We denote by (Σ^f, σ_t) (simply σ_t) the symbolic flow over $\sum$ with ceiling function f. Precisely, Σ^f is the set $\{(\xi, s);\ \xi \in \Sigma,\ 0 \le s < f(\xi)\}$ with the identification $(\xi, f(\xi)) = (\sigma\xi, 0)$ for any $\xi \in \Sigma$, and the flow σ_t on Σ^f is defined so that

$$\sigma_t(\xi, s) = (\sigma^k \xi, u), \quad \text{if } \sum_{j=0}^{k-1} f(\sigma^j \xi) \le t + s < \sum_{j=0}^{k} f(\sigma^j \xi), \tag{3.2}$$

where $u = t + s - \sum_{j=0}^{k-1} f(\sigma^j \xi)$ and so on.

Theorem 1 and Theorem 2 follow from Proposition 3.1 and Proposition 3.2 below, respectively.

Proposition 3.1. *The map* $h : \Sigma^f \to \Omega$ *defined by* $h(\xi, s) = S_s(x(\xi))$ *gives the conjugacy between the flows* σ_t *and* S_t *restricted to* Ω *so that the corresponding closed orbits have the same period.*

Proof. Theorem 0 in §2 implies that the map $h_0: \Sigma \to \Omega_-$ defined by $h_0(\xi) = x(\xi)$ is a homeomorphism with $h_0(\sigma\xi) = x(\sigma\xi) = T(x(\xi)) = T(h_0(\xi))$. Therefore h_0 gives a conjugacy between the Poincaré maps σ of σ_t and T of S_t restricted to Ω. On the other hand, the corresponding points ξ and $h_0(\xi)$ have the same return time to Σ and Ω_- respectively from the definition of f. Hence h gives a conjugacy between σ_t and S_t restricted to Ω. Obviously the corresponding closed orbits have the same period. □

Proposition 3.2. *The ceiling function* f *cannot be represented as*

$$f = g \circ \sigma - g + aK, \tag{3.3}$$

where g *denotes a real valued function,* K *an integer valued function, and* a *a positive constant.*

Remark 3.1. Proposition 3.2 implies that the symbolic flow σ_t is topologically weak mixing. On the other hand the estimate (2.4) shows that the celling function f is Lipschitz continuous with respect to the metric d_ρ. Therefore we can obtain an analogue of the prime number theorem for the distribution of the prime closed orbits of the flow S_t by use of the zeta function

$$\begin{aligned}\zeta(s) &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\sigma^n\xi=\xi}\exp\left[-s\sum_{j=0}^{n-1}f(\sigma^j\xi)\right]\right)\\ &= \prod_{\gamma}(1-\exp[-sT_\gamma])^{-1},\end{aligned} \tag{3.4}$$

where γ denotes a prime closed orbit of S_t and T_γ is its period (see Parry and Pollicott [5]).

Proof of Proposition 3.2. Suppose that f can be represented as in (3.3). By using the similarity transformation we may assume that $a = 1$, i.e.,

$$f = g \circ \sigma - g + K, \tag{3.5}$$

where g is a real valued function and K is an integer valued function.

Since we already established the conjugacy in Proposition 3.1 we can identify the symbolic flow σ_t and the flow S_t restricted to Ω without confusion. The assumption (3.5) yields that every closed orbit of S_t has an integer period. Now we restrict ourselves to three domains O_1, O_2, and O_3.

For each $n \geq 1$, let $\xi^n = (\xi_j^n)_{j=-\infty}^{\infty}$ be the sequence in $\sum$ so that $\xi_0^n = 1$; $\xi_j^n = 2$ for odd $j \leq 4n-1$; $\xi_j^n = 3$ for even $j \leq 4n-2$; and $\xi_{m+4n}^n = \xi_m^n$ for any $m \in Z$. Let ξ^0 be the sequence in with $\xi_j^0 = 3$ for even j and $\xi_j^n = 2$ for odd j. We denote by $x^{n,j}$ the unique element in Ω_- which solves $\xi(x) = \sigma^j\xi^n$, and $q^{n,j} = \pi(x^{n,j})$. Namely, $q^{n,j}$ denotes the position where the jth collision occurs along the closed orbit $\gamma_n = (S_t x^{n,0})_t$ starting from $x^{n,0}$, for $n \geq 0$. We note that the period T_n of γ_n has the minimal property which appeared in the proof of Theorem 0 (see the minimal value problem (2.11) in §2). The minimal property of T_n and the uniqueness of the solution of the itinarary problem imply that γ_n must by symmetric, i.e., $q^{n,2n+j} = q^{n,2n-j}$ for $j \geq 1$. We claim that

$$T_{n+1} \geq T_n + 2T_0 + 1, \quad \text{for } n \geq 1. \tag{3.6}$$

Consider a fictitious motion of a particle such that the particle moves along the orbit γ_{n+1} until it collides at $q^{n+1,2n}$ and after that it returns to $q^{n+1,0}$ taking the same way as it has taken to reach $q^{n+1,2n}$. It will be more convenient to introduce the following notation:

$$q^{n+1,0} \to q^{n+1,1} \to \cdots \to q^{n+1,2n} \to q^{n+1,2n-1} \to \cdots \to q^{n+1,1} \to q^{n+1,0},$$

where $p \to q$ denotes that the fictitious particle moves from p to q.

Now we obtain a fictitious closed orbit γ_n' whose period T_n' is

$$l(q^{n+1,0}, q^{n+1,1}, \dots, q^{n+1,2n}, q^{n+1,2n-1}, \dots, q^{n+1,1}).$$

Therefore, $T_n < T_n'$ in virtue of the minimal value problem (2.11). Thus we have

$$T_{n+1} \geq T_n' + 2T_0 > T_n + 2T_0.$$

We used the fact that γ_{n+1} is symmetric to see the first inequality in the above. But T_n's are all integers by our assumption. Hence we obtain (3.6).

On the other hand we can show

$$T_{n+1} \leq T_n + 2T_0 + C'\rho^{2n} \tag{3.7}$$

where C' is a positive constant which is independent of n and $\rho = (1+\eta)^{-1}$ as before. Clearly the inequality (3.7) contradicts the inequality (3.6). Therefore the celling function f cannot be represented as in (3.5). This completes the proof of Proposition 3.2.

It remains to prove the inequality (3.7). We consider the following fictitious motion of a particle:

$$q^{n,0} \to q^{n,1} \to \cdots \to q^{n,2n} \to q^{0,0} \to q^{0,1} \to q^{0,0} \to q^{n,2n}$$
$$\to q^{n,2n-1} \to \cdots \to q^{n,1} \to q^{n,0}.$$

Then we obtain the fictitious closed orbit γ_{n+1}' whose fictitious period T_{n+1}' is

$$l(q^{n,0}, q^{n,1}, \dots, q^{n,2n}, q^{0,0}, q^{0,1}, q^{0,1}, q^{n,2n}, q^{n,2n-1}, \dots, q^{n,1}).$$

On the other hand we have

$$T_{n+1}' \leq T_n + T_0 + 2|q^{n,2n} - q^{0,0}|. \tag{3.8}$$

Here we used the fact that γ is symmetric. From the definition of γ_n and γ_0, $T^j x^{n,2n}$ and $T^j x^{0,1}$ belong to $\partial O_{\xi^0_{j+1}}$ for $|j| \leq 2n-1$. Therefore the arclength $r(x^{0,1}, x^{n,2n})$ between $q^{0,1} = \pi(x^{0,1})$ and $q^{n,2n} = \pi(x^{n,2n})$ is less than or equal to $C\rho^{2n-1}$ by the a priori estimate (2.5). Thus we obtain

$$2|q^{n,2n} - q^{0,0}| \le T_0 + 2C\rho^{2n-1} \tag{3.9}$$

in virtue of the triangle inequality.

The inequalities (3.8) and (3.9) imply the inequality (3.7) in virtue of the minimal property of T_{n+1}.

Now the proof is complete. □

Remark 3.2. In the proof of inequality (3.7) we used the fact that $\pi\gamma_0$ and $\pi\gamma_n$ cannot intersect. We restrict ourselves to note that it is an easy consequence of our hypotheses (H.1) and (H.2).

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