

C^∞ LOOP ALGEBRAS AND NONCOMMUTATIVE BOTT PERIODICITY

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ABSTRACT. We construct the noncommutative analogs $\Omega_\infty A$ and $\Omega_{\text{lip}} A$ of the C^∞ and Lipschitz loop spaces for a pro- C^* -algebra A equipped with a suitable dense subalgebra. With U_{nc} and P being the classifying algebras for K -theory earlier introduced by the author, we then prove that there are homotopy equivalences $\Omega_\infty U_{\text{nc}} \simeq P$ and $\Omega_\infty P \simeq U_{\text{nc}}$. This result is a noncommutative analog of Bott periodicity in the form $\Omega U \simeq \mathbf{Z} \times BU$ and $\Omega(\mathbf{Z} \times BU) \simeq U$.

INTRODUCTION

Bott periodicity has become quite familiar to operator algebraists in the form

$$(*) \quad K_0(SA) \cong K_1(A) \quad \text{and} \quad K_1(SA) \cong K_0(A).$$

Here A is a C^* -algebra, SA is its suspension $C_0(\mathbf{R}) \otimes A$, and K -theory is as in [3]. The statement $(*)$ is the noncommutative generalization of the topological Bott periodicity theorem

$$(**) \quad K^0(SX) \cong K^1(X) \quad \text{and} \quad K^1(SX) \cong K^0(X),$$

where X is a locally compact space and $SX = \mathbf{R} \times X$.

There is, however, another commutative form of Bott periodicity. To state it, we need the infinite unitary group U , its classifying space BU , and the loop space functor $X \mapsto \Omega X$. The group U is the direct limit of the unitary groups $U(n)$ of unitary $n \times n$ matrices, via the maps $u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$. The classifying space BU is $\varinjlim BU(n)$, as in [6, Proposition II.1.32]. Finally, if X is a pointed topological space, then ΩX is the set of continuous basepoint preserving functions from the circle S^1 to X , with the compact-open topology; see p. 37 of

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[16]. Then the other form of Bott periodicity is the assertion that there are homotopy equivalences

$$(***) \quad \Omega(\mathbf{Z} \times BU) \simeq U \quad \text{and} \quad \Omega U \simeq \mathbf{Z} \times BU,$$

relative to suitable basepoints. (See the last two lines of Theorem III.5.22 of [6], and the footnote to the theorem.) The purpose of this paper is to prove a noncommutative analog of (***) , with one slight modification: we use C^∞ loop spaces in place of continuous loop spaces.

To explain the relation between (**) and (***) , we introduce some notation. If X and Y are pointed spaces, then $[X, Y]_+$ denotes the set of homotopy classes of basepoint preserving continuous maps from X to Y . Furthermore, if X is a locally compact space, then by convention the basepoint in its one point compactification X^+ is taken to be the point at infinity. With these conventions, topological K -theory can be defined for a locally compact space X by

$$(***) \quad K^0(X) = [X^+, \mathbf{Z} \times BU]_+ \quad \text{and} \quad K^1(X) = [X^+, U]_+.$$

(Compare with Theorem II.1.33 and Corollary II.3.19 of [6], which give the unpointed version of this result.) Now the loop space is defined so as to satisfy the relation $[(SX)^+, Y]_+ \cong [X^+, \Omega Y]_+$ for locally compact X . Therefore (****) and (***) imply (**).

In §2.5 of [11], we constructed noncommutative analogs, called $W_\infty(q\mathbf{C})^+$ and U_{nc} , of $\mathbf{Z} \times BU$ and U respectively. These are pro- C^* -algebras, that is, inverse limits of C^* -algebras [9], but not C^* -algebras, as one should expect given that U and $\mathbf{Z} \times BU$ are direct limits of compact spaces but are not compact. As shown in [11], these algebras satisfy the noncommutative analog of (****), namely

$$[W_\infty(q\mathbf{C})^+, A^+]_+ \cong K_0(A) \quad \text{and} \quad [U_{\text{nc}}, A^+]_+ \cong K_1(A),$$

for any C^* -algebra A . Here A^+ is the unitization of A and $[A, B]_+$ is the set of homotopy classes of pointed homomorphisms from A to B , as defined in [11, §2.5]. In §2.6 of [11], we constructed a noncommutative analog Ω of the loop space functor, and we conjectured that there are homotopy equivalences of pro- C^* -algebras

$$\Omega(W_\infty(q\mathbf{C})^+) \simeq U_{\text{nc}} \quad \text{and} \quad \Omega U_{\text{nc}} \simeq W_\infty(q\mathbf{C})^+.$$

We have not quite been able to prove this, but in this paper, we define a noncommutative analog Ω_∞ of the C^∞ loop space of a manifold, and, with respect to appropriate subalgebras of smooth elements, we prove:

Theorem. *There are homotopy equivalences of pro- C^* -algebras*

$$\Omega_\infty(W_\infty(q\mathbf{C})^+) \simeq U_{\text{nc}} \quad \text{and} \quad \Omega_\infty U_{\text{nc}} \simeq W_\infty(q\mathbf{C})^+.$$

(See Theorem 6.1, which is a slightly more precise statement.) It can be easily shown that this theorem implies (*), in spite of the use of the C^∞ loop space.

This theorem should be regarded as an example in the as yet largely undeveloped subject of noncommutative homotopy theory. This subject was first introduced in [13 and 14], but those papers dealt only with C^* -algebras. They were thus doing something akin to homotopy theory restricted to the category of compact spaces. However, no better category was available at the time. Our theorem, we believe, is an example of what might be accomplished once the theory is freed from the restriction to the noncommutative analogs of the compact spaces.

Our proof uses the Yoneda lemma and the results of [12] to the effect that $W_\infty(q\mathbf{C})^+$ and U_{nc} are classifying algebras for the representable K -theory of σ - C^* -algebras (countable inverse limits of C^* -algebras). It thus actually uses (*), and so does not provide a new proof of (*). (We think that there should be a proof not using (*)—see Problem 6.6.) We work initially with Lipschitz loop algebras, because in the case at hand they are actually σ - C^* -algebras. We then prove that the Lipschitz loop algebras are homotopy equivalent to the C^∞ loop algebras; this step apparently does not work for continuous loop algebras.

This paper is organized as follows. In the first section, we define Lipschitz and C^∞ loop algebras relative to appropriate dense subalgebras, and prove their basic properties. In §2 we relate the C^∞ loop algebra of the algebra of continuous functions on a C^∞ manifold to the C^∞ loop space of the manifold, thus showing that we really have defined a noncommutative analog of the C^∞ loop space. (Unfortunately, this result does not work quite so well for the Lipschitz loop algebra.) The third section is devoted to some facts about homotopy dual group structures on loop algebras; these structures are what give group structures on sets of homotopy classes of homomorphisms.

The fourth section contains the key smoothing lemma, needed to replace continuous paths in σ - C^* -algebras by smooth ones. In §5 we go from smoothing paths of elements to smoothing paths of homomorphisms, and in §6 we prove the main theorem and discuss several open problems.

Our main reference for facts about pro- C^* -algebras and σ - C^* -algebras will be [9]. We will understand the word “homomorphism” to mean “continuous *-homomorphism.” We will be dealing with the classifying algebras $W_\infty(q\mathbf{C})^+$ (also called P) and U_{nc} ; these are defined in [11, §2.5] and also [12, §§2 and 3]. The loop algebra ΩA of a pointed pro- C^* -algebra A is constructed in [11, §2.6] and, in a different but equivalent way, in [12, Theorem 5.6]. Pointed (pro-) C^* -algebras are as in [11, §2.5] or [12, §5]. Finally, we give one warning on the terminology we use: in this paper, Lipschitz and smooth homotopies are homotopies in which each stage is a Lipschitz or smooth element or homomorphism, and in which the stages vary continuously, *not* homotopies which are Lipschitz or smooth in the direction of the homotopy. (See for example Definition 1.6.)

1. LIPSCHITZ AND C^∞ LOOP ALGEBRAS

In this section, we define and prove the basic properties of certain pro- C^* -algebras which are noncommutative analogs of Lipschitz and C^∞ loop spaces.

These Lipschitz and C^∞ loop algebras will depend on a choice of a dense subalgebra, just as the C^∞ loop space of a manifold depends on the C^∞ structure on the manifold. After establishing our notation and conventions concerning pro- C^* -algebras, we therefore begin by considering dense subalgebras of pro- C^* -algebras.

If A is an arbitrary pro- C^* -algebra, we let $S(A)$ denote the set of continuous C^* -seminorms on A , and for $p \in S(A)$ we write $\text{Ker}(p)$ for the set $\{a \in A : p(a) = 0\}$ and A_p for the C^* -algebra $A/\text{Ker}(p)$, with the norm induced by p . (Note that A_p is complete, by [9, Corollary 1.12].) We furthermore let κ_p denote the quotient map $A \rightarrow A_p$, and we let κ_{pq} denote the map $A_p \rightarrow A_q$ for $p \geq q$. Then we have a canonical isomorphism $A \cong \varprojlim_{p \in S(A)} A_p$. (See [9, §1].)

If A is a C^* -algebra, and S is a subset of A , then we denote by $\text{hol}(S)$ the smallest $*$ -subalgebra of A containing S which is closed under holomorphic functional calculus. (Functional calculus is done on the spectrum with respect to A .) Observe that $\text{hol}(S)$ can be obtained as $\bigcup_{n=0}^\infty B_n$, where B_0 is the $*$ -subalgebra of A generated by S , and where B_{n+1} is the $*$ -subalgebra of A generated by all elements $f(a)$, for $a \in B_n$ and f holomorphic on a neighborhood of $\text{sp}(a)$. Note that $\text{hol}(S)$ is dense in A if and only if B_0 is, and that $\text{hol}(S)$ does not change if A is replaced by some other C^* -algebra containing S .

1.1. Definition. Let A be a pro- C^* -algebra. Then an *admissible subalgebra* of A is a dense $*$ -subalgebra A_0 of A such that $A_0 = \varprojlim_{p \in S(A)} \text{hol}(\kappa_p(A_0))$. (This means that if $a \in A$ satisfies $\kappa_p(a) \in \text{hol}(\kappa_p(A_0))$ for all $p \in S(A)$, then $a \in A_0$.) If A and B are pro- C^* -algebras and A_0 and B_0 are admissible subalgebras of A and B respectively, then a *morphism* from (A, A_0) to (B, B_0) is a continuous $*$ -homomorphism $\varphi : A \rightarrow B$ such that $\varphi(A_0) \subset B_0$. If A_0 and B_0 are understood, we refer to φ as an *admissible morphism* from A to B .

For our purposes, the essential part of this definition is that A_0 be the inverse limit of its images in the algebra A_p . The condition on holomorphic functional calculus is included to make our development compatible with the ideas of Connes' noncommutative differential geometry [4].

1.2. Examples. (1) Let M be a C^∞ manifold, not necessarily compact, let $C(M)$ denote the pro- C^* -algebra of all continuous complex-valued functions on M , with the topology of uniform convergence on compact sets, and let $C^\infty(M)$ be the subalgebra consisting of all C^∞ functions on M . Then $C^\infty(M)$ is admissible in $C(M)$.

(2) Let M be as in (1), let $C_c(M)$ be the algebra of all continuous functions with compact support, and let $C_c^\infty(M) = C_c(M) \cap C^\infty(M)$. Then $C_c(M)$ and $C_c^\infty(M)$, and their unitizations $C_c(M)^+$ and $C_c^\infty(M)^+$, are dense $*$ -subalgebras of $C(M)$ which are closed under holomorphic functional calculus but not admissible.

We now give the basic properties of admissible subalgebras.

1.3. Lemma. (1) *If A is a C^* -algebra, then the admissible subalgebras of A are exactly the dense $*$ -subalgebras which are closed under holomorphic functional calculus.*

(2) *In the definition of admissibility, $S(A)$ may be replaced by any cofinal subset.*

(3) *An admissible subalgebra is closed under holomorphic functional calculus, as given in Proposition 1.9 of [9].*

(4) *An admissible subalgebra of a unital pro- C^* -algebra contains the identity.*

(5) *If S is a subset of a pro- C^* -algebra A which generates a dense $*$ -subalgebra of A , then there is a smallest admissible subalgebra A_0 of A containing S . It is given by $A_0 = \varprojlim \text{hol}(\kappa_p(S))$.*

The proofs are easy, and are omitted. Part (5) shows that it makes sense to talk about the admissible subalgebra generated by a subset S of a pro- C^* -algebra A , provided that S generates a dense $*$ -subalgebra of A . Note that this admissible subalgebra may be larger than the smallest $*$ -subalgebra containing S and closed under holomorphic functional calculus—see Example 1.2(2).

1.4. Definition. If A is a C^* -algebra, then an admissible subalgebra of A is called *finitely generated* if it has the form $\text{hol}(F)$ for a finite subset $F \subset A$. If A is a pro- C^* -algebra, then an admissible subalgebra A_0 is called *locally finitely generated* if $\text{hol}(\kappa_p(A_0))$ is finitely generated for all $p \in S(A)$.

The importance of this definition will become clear in Proposition 1.11 below.

1.5. Lemma. (1) *In the definition of locally finitely generated, $S(A)$ can be replaced by a cofinal subset.*

(2) *If A is a pro- C^* -algebra and $S \subset A$ is a subset which generates a dense $*$ -subalgebra and such that $\kappa_p(S)$ is finite for all $p \in S(A)$, then the smallest admissible subalgebra containing S is locally finitely generated.*

The proof is easy, and is omitted.

In analogy with the topological literature, and so as to obtain algebras with homotopy dual group structures, we will actually work in the category of pointed pro- C^* -algebras. (See §5 of [12].) Recall that the unitization functor $A \mapsto A^+$ is a category equivalence from pro- C^* -algebras and morphisms to pointed pro- C^* -algebras and pointed morphisms. We therefore define a *pointed admissible subalgebra* of a pointed pro- C^* -algebra A^+ to be a subalgebra of the form A_0^+ for an admissible subalgebra A_0 of A .

1.6. Definition. Let B be a pro- C^* -algebra, and let M be a C^∞ manifold, possibly with boundary. Then an element $a \in C(M) \otimes B$, equivalently a continuous function $a: M \rightarrow B$, will be called *smooth* if, for every $p \in S(B)$, the function from M to B_p defined by a is smooth. A homotopy $t \rightarrow a_t$ of smooth elements of $C(M) \otimes B$ is called *smooth* if it defines a smooth function from M to $C([0, 1]) \otimes B$. (This is the same as a continuous function from

$[0, 1]$ to $C^\infty(M, B)$; see Lemma 2.2. Note that we do *not* require smoothness in the $[0, 1]$ direction.)

If A is another pro- C^* -algebra and A_0 is an admissible subalgebra, then a homomorphism φ from A to $C(M) \otimes B$, or to a subalgebra of $C(M) \otimes B$, is called *smooth* if $\varphi(a)$ is smooth for all $a \in A_0$. A homotopy $t \mapsto \varphi_t$ is called *smooth* if it defines a smooth homomorphism

$$A \rightarrow C(M) \otimes (C([0, 1]) \otimes B).$$

(Again, we do *not* demand smoothness in the $[0, 1]$ direction.)

We will be particularly interested in the case in which B is a pointed pro- C^* -algebra and the homomorphisms take their values in the subalgebra ΣB of $C(S^1) \otimes B$. (Recall [12, p. 1080] that ΣB is the pointed suspension of B .) In this case, generalizing the notation Hom_+ for pointed morphisms and $[\ , \]_+$ for their homotopy classes, we write $\text{Hom}_+^\infty((A, A_0), \Sigma B)$ for the set of smooth pointed morphisms, and $[(A, A_0), \Sigma B]_+^\infty$ for the set of homotopy classes in $\text{Hom}_+^\infty((A, A_0), \Sigma B)$. We omit A_0 if it is understood.

We define *Lipschitz elements, Lipschitz homomorphisms, and Lipschitz homotopies* similarly, and write $\text{Hom}_+^{\text{lip}}((A, A_0), \Sigma B)$ and $[(A, A_0), \Sigma B]_+^{\text{lip}}$. Of course, in this case M can be any metric space. Note that the Lipschitz constants of the functions from M to B_p are allowed to depend on p . Also, it is *not* true that Lipschitz functions from M to $C([0, 1]) \otimes B$ are the same as continuous functions from $[0, 1]$ to $\text{Lip}(M, B)$; see Remark 2.3.

Note that a smooth morphism from A to $C(M, B)$ is, in the terminology of Definition 1.1, a morphism from (A, A_0) to $(C(M, B), C^\infty(M, B))$.

1.7. Proposition. *Let A be a pointed pro- C^* -algebra and let A_0 be a pointed admissible subalgebra. Then there exist functorial pointed pro- C^* -algebras $\Omega_\infty(A, A_0)$ and $\Omega_{\text{lip}}(A, A_0)$ such that there are natural isomorphisms, for any pointed pro- C^* -algebra B ,*

$$\text{Hom}_+(\Omega_\infty(A, A_0), B) \cong \text{Hom}_+^\infty((A, A_0), \Sigma B)$$

and

$$\text{Hom}_+(\Omega_{\text{lip}}(A, A_0), B) \cong \text{Hom}_+^{\text{lip}}((A, A_0), \Sigma B).$$

These algebras also satisfy

$$[\Omega_\infty(A, A_0), B]_+ \cong [(A, A_0), \Sigma B]_+^\infty$$

and

$$[\Omega_{\text{lip}}(A, A_0), B]_+ \cong [(A, A_0), \Sigma B]_+^{\text{lip}}.$$

The isomorphisms are of course natural with respect to pointed admissible morphisms in the variable A . We will explain the relationship between Ω_∞ and the C^∞ loop space in the next section.

Proof of Proposition 1.7. The proof is essentially the same as the proof of [12, Theorem 5.6]. In the C^∞ case, let A have the basepoint $\alpha: A \rightarrow \mathbb{C}$, and let

D be the set of isomorphism classes of pairs $(Z, (\pi_\zeta)_{\zeta \in S^1})$ in which Z is a pointed C^* -algebra, each π_ζ is a pointed morphism from A to Z , and the following conditions are satisfied:

- (1) $(a, \zeta) \mapsto \pi_\zeta(a)$ is (jointly) continuous.
- (2) $\pi_1 = \varepsilon_Z \circ \alpha$, where $\varepsilon_Z: C \mapsto Z$ is $\varepsilon_Z(1) = 1$.
- (3) Z is the C^* -algebra generated by $\bigcup_{\zeta \in S^1} \pi_\zeta(A)$.
- (4) $\zeta \mapsto \pi_\zeta(a)$ is C^∞ for all $a \in A_0$.

One then takes an inverse limit over D , exactly as in the proof of [12, Theorem 5.6]. The Lipschitz case is the same, using Lipschitz in place of C^∞ in condition (4). The homotopy statements are obtained by putting $C([0, 1]) \otimes B$ in place of B . Q.E.D.

1.8. *Remark.* It is clear from this proof that $\Omega_\infty A$ is the universal pro- C^* -algebra on generators $z(a, \zeta)$ (whose image in Z is $\pi_\zeta(a)$), subject to the relations given for ΩA in [11, §2.6] and the additional requirement that $\zeta \mapsto z(a, \zeta)$ be C^∞ for every $a \in A_0$. Similarly, for $\Omega_{\text{lip}} A$, the additional requirement is that $\zeta \mapsto z(a, \zeta)$ be Lipschitz. (For more on generators and relations, see [11, §1.3].) Thus, if $\varphi: A \rightarrow \Sigma B$ is a smooth or Lipschitz pointed morphism, then the corresponding homomorphism from $\Omega_\infty A$ or $\Omega_{\text{lip}} A$ to B is determined by $z(a, \zeta) \mapsto \varphi(a)(\zeta)$. We will use the notation $z(a, \zeta)$ for these elements throughout this paper. It will always be clear from the context whether they are supposed to be in ΩA , $\Omega_{\text{lip}} A$, or $\Omega_\infty A$.

It follows from these observations that these are natural homomorphisms

$$\Omega A \rightarrow \Omega_{\text{lip}}(A, A_0) \rightarrow \Omega_\infty(A, A_0),$$

given by $z(a, \zeta) \mapsto z(a, \zeta)$ in each case, and each having dense range.

1.9. *Remark.* There are analogs Ω_0 , $\Omega_{0,\infty}$, and $\Omega_{0,\text{lip}}$ of the loop algebra construction for algebras without basepoints. Letting SB denote the usual unpointed suspension of a pro- C^* -algebra B , and identifying it with $\{f: S^1 \rightarrow B: f(1) = 0\}$ for the purpose of defining smooth and Lipschitz homomorphisms, $\Omega_{0,\infty}$ satisfies

$$\text{Hom}(\Omega_{0,\infty} A, B) \cong \text{Hom}^\infty(A, SB) \quad \text{and} \quad [\Omega_{0,\infty} A, B] \cong [A, SB]^\infty,$$

with the obvious meanings of Hom^∞ and $[,]^\infty$. The relation to Ω_∞ is that there is a canonical isomorphism of pointed pro- C^* -algebras $(\Omega_{0,\infty} A)^+ \cong \Omega_\infty(A^+)$. The analogous statements hold for Ω_0 and $\Omega_{0,\text{lip}}$.

We should also point out that there is nothing special about Lipschitz or C^∞ here. For example, we could easily substitute C^r for C^∞ and Lip^α ($0 < \alpha < 1$) for Lipschitz throughout this entire paper. There are, however, crucial differences between Lipschitz and Lip^α on the one hand, and C^∞ , C^r , and continuous on the other hand, as will be seen in Proposition 1.11 below, and in §2.

1.10. **Proposition.** *Let A be a pointed pro- C^* -algebra, let A_0 be a pointed admissible subalgebra, and let D be a cofinal subset of $S(A)$. Then*

$$\Omega_{\text{lip}}(A, A_0) \cong \varprojlim_{p \in D} \Omega_{\text{lip}}(A_p, \text{hol}(\kappa_p(A_0))),$$

and similarly for C^∞ in place of Lipschitz.

Proof. It is sufficient (for the Lipschitz case) to prove that there is a natural isomorphism, for any pointed pro- C^* -algebra B ,

$$(*) \quad \text{Hom}_+(\varprojlim \Omega_{\text{lip}}(A_p, \text{hol}(\kappa_p(A_0))), B) \cong \text{Hom}_+^{\text{lip}}((A, A_0), \Sigma B).$$

(Compare with the Yoneda lemma, on page 61 of [8].) Now by the definition of an inverse limit, we have $\text{Hom}(C, \varprojlim_\alpha B_\alpha) \cong \varprojlim_\alpha \text{Hom}(C, B_\alpha)$, so we may take B to be a C^* -algebra in $(*)$. The left-hand side of $(*)$ is then

$$\varprojlim_{p \in D} \text{Hom}_+(\Omega_{\text{lip}}(A_p, \text{hol}(\kappa_p(A_0))), B) \cong \varprojlim_{p \in D} \text{Hom}_+^{\text{lip}}((A_p, \text{hol}(\kappa_p(A_0))), \Sigma B).$$

Let S denote the limit on the right-hand side of this isomorphism. Then we have to show that S is canonically isomorphic to $\text{Hom}_+^{\text{lip}}((A, A_0), \Sigma B)$. There is an obvious map $S \rightarrow \text{Hom}_+^{\text{lip}}((A, A_0), \Sigma B)$ which sends

$$\varphi \in \text{Hom}_+^{\text{lip}}((A_p, \text{hol}(\kappa_p(A_0))), \Sigma B)$$

to $\varphi \circ \kappa_p$. To construct an inverse for this map, let $\psi \in \text{Hom}_+^{\text{lip}}((A, A_0), \Sigma B)$ and choose p such that ψ factors through A_p , that is, $\psi = \varphi \circ \kappa_p$ for some $\varphi: A_p \rightarrow \Sigma B$. Then $\zeta \mapsto \varphi(a)(\zeta)$ is Lipschitz for every $a \in \kappa_p(A_0)$. Since holomorphic functional calculus and algebraic operations preserve Lipschitz functions, and since holomorphic functional calculus commutes with the homomorphisms $a \mapsto \varphi(a)(\zeta)$, we conclude from the construction of $\text{hol}(\kappa_p(A_0))$ preceding Definition 1.1 that $\zeta \mapsto \varphi(a)(\zeta)$ is Lipschitz for every $a \in \text{hol}(\kappa_p(A_0))$, so that $\varphi \in S$. It is obvious that on two maps are inverses of each other, so the Lipschitz case is proved.

The proof for the C^∞ case is exactly the same. Q.E.D.

1.11. **Proposition.** *Let A be a pointed σ - C^* -algebra, and let A_0 be a pointed admissible subalgebra which is locally finitely generated (Definition 1.4). Then $\Omega_{\text{lip}}(A, A_0)$ is a separable σ - C^* -algebra.*

The important point here is that $\Omega_{\text{lip}}(A, A_0)$ is a countable inverse limit of C^* -algebras. This would also hold for Lip^α in place of Lipschitz, but is surely false in general for $\Omega_\infty(A, A_0)$.

Proof of Proposition 1.11. By the previous proposition and the definition of locally finitely generated, and because a countable inverse limit of σ - C^* -algebras is a σ - C^* -algebra, it suffices to prove this result when A is a C^* -algebra and $A_0 = \text{hol}(F)$ for some finite set F . According to Section 1 of [2], we may now

form the universal C^* -algebra Z_n on the generators $z(a, \zeta)$ for $a \in A$ and $\zeta \in S^1$, subject to the relations

- (1) $\|z(a, \zeta_1) - z(a, \zeta_2)\| \leq n|\zeta_1 - \zeta_2|$ for $\zeta_1, \zeta_2 \in S^1$ and $a \in F$.
- (2) For each fixed $\zeta \in S^1$, the map $a \mapsto z(a, \zeta)$ is a $*$ -homomorphism. (That is, the elements $z(a, \zeta)$ satisfy all the algebraic relations which hold in A .)
- (3) $z(1, \zeta) = z(1, 1)$ for all $\zeta \in S^1$.
- (4) $z(a, 1) = z(\alpha(a) \cdot 1, 1)$ for all $a \in A$, where $\alpha: A \rightarrow \mathbf{C}$ is the basepoint of A .

(Compare with Remark 1.8, and with [11, Definition 2.6.2].) Here n is any positive integer. Note that Z_n has an identity $z(1, 1)$ and a basepoint sending $z(a, \zeta)$ to $\alpha(a)$ for all a, ζ .

We now define homomorphisms $\pi_\zeta: A \rightarrow Z_n$ by $\pi_\zeta(a) = z(a, \zeta)$ for $\zeta \in S^1$ and $a \in A$. Since F generates a dense $*$ -subalgebra of A , and since $\zeta \mapsto \pi_\zeta(a)$ is continuous for $a \in F$, it follows that $\zeta \mapsto \pi_\zeta(a)$ is continuous for $a \in A$. Also, $\zeta \mapsto \pi_\zeta(a)$ is Lipschitz for $a \in A_0 = \text{hol}(F)$ by an argument used in the proof of the previous proposition. We therefore have a canonical map $\varphi_n: \Omega_{\text{lip}}(A, A_0) \rightarrow Z_n$ for each n . We clearly have homomorphisms $\pi_n: Z_{n+1} \rightarrow Z_n$ (sending $z(a, \zeta)$ to $z(a, \zeta)$), and they satisfy $\pi_n \circ \varphi_{n+1} = \varphi_n$. We therefore obtain a homomorphism $\varphi: \Omega_{\text{lip}}(A, A_0) \rightarrow \varprojlim Z_n$.

Since $\varprojlim Z_n$ is a σ - C^* -algebra, it now suffices to prove that φ is an isomorphism. It is clearly enough to show that any homomorphism $\psi: \Omega_{\text{lip}}(A, A_0) \rightarrow B$, for any C^* -algebra B , factors through some Z_n . But this is obvious: we simply choose n larger than the Lipschitz constants of the finitely many functions $\zeta \mapsto \mu(a)(\zeta)$, for $a \in F$, where $\mu: A \rightarrow \Sigma B$ is the homomorphism corresponding to ψ . Q.E.D.

2. LOOP SPACES AND LOOP ALGEBRAS

The purpose of this section is to show that the C^∞ loop algebra constructed in the previous section is a reasonable noncommutative analog of the ordinary C^∞ loop space, by computing the abelianization of $\Omega_\infty(C(M), C^\infty(M))$. The results of this section are not used in the rest of this paper.

Throughout this section, we let M be a paracompact C^∞ manifold, with basepoint m_0 . Since M is paracompact, it is metrizable and can be embedded as a smooth submanifold of \mathbf{R}^n for some n . We will assume for convenience that an embedding has been chosen, but what we do will not depend on this choice. The C^∞ loop space $\Omega_\infty M$ (or, more correctly, $\Omega_\infty(M, m_0)$) is then the set of pointed C^∞ maps from $(S^1, 1)$ to (M, m_0) , with the topology of uniform convergence of all derivatives. This topology is in practice most easily obtained as the subspace topology that $\Omega_\infty(M, m_0)$ inherits as a subspace of $C^\infty(S^1, \mathbf{R}^n)$, the space of all C^∞ functions from S^1 to \mathbf{R}^n with the C^∞ topology.

2.1. Proposition. *The abelianization of $\Omega_\infty(C(M), C^\infty(M))$ is canonically isomorphic to $C(\Omega_\infty M)$.*

We do not claim that the map $\Omega_\infty(C(M), C^\infty(M)) \rightarrow C(\Omega_\infty M)$ is surjective, only that it has dense range. See [11, Definition 1.5.7] and the remark following it.

Proof of Proposition 2.1. To shorten the notation, we set

$$A = \Omega_\infty(C(M), C^\infty(M)).$$

By [9, Remark 2.10], the abelianization of A has the form $C(X)$ for some compactly generated completely Hausdorff space X , with the topology of uniform convergence on the members of a distinguished family F of compact subsets of X . Clearly X can be identified with the set of all unital homomorphisms $\varphi: A \rightarrow \mathbf{C}$, that is, with the set of all smooth pointed homomorphisms $\psi: C(M) \rightarrow C(S^1)$. One readily checks that $\zeta \mapsto \psi(f)(\zeta)$ is C^∞ for all C^∞ functions f if and only if the corresponding map from S^1 to M is C^∞ . This gives us a bijection $h: X \rightarrow \Omega_\infty M$. The proof now consists of showing that h is a homeomorphism and that $F = \{h^{-1}(K): K \subset \Omega_\infty M \text{ is compact}\}$.

We first prove that h^{-1} is continuous. Let $\alpha: A \rightarrow C(X)$ be the abelianization map. Also observe that there is a continuous homomorphism $\beta: A \rightarrow C(\Omega_\infty M)$, sending the standard generators $z(f, \zeta)$ to the functions $\omega \mapsto f(\omega(\zeta))$ for $\omega \in \Omega_\infty M$. (Notice that $C(\Omega_\infty M)$ is a pro- C^* -algebra, because $\Omega_\infty M$ is a subspace of the metrizable space $C^\infty(S^1, \mathbf{R}^n)$, hence metrizable, hence compactly generated by [17, 2.2].) The universal property of the abelianization now gives a factorization $\beta = \varphi \circ \alpha$ for some $\varphi: C(X) \rightarrow C(\Omega_\infty M)$, and clearly φ is composition with h^{-1} . By [9, Theorem 2.7 and Proposition 2.6], it follows that h^{-1} is continuous and that $h^{-1}(K) \in F$ for every compact subset $K \subset \Omega_\infty M$.

We now want to prove that h is continuous. By the definition of a distinguished family [9, Definition 2.5], it suffices to prove that $h|_L$ is continuous for all $L \in F$. Clearly we need only consider subsets L which also contain the basepoint of X . Now $L \in F$ means that $A \rightarrow C(L)$ is continuous, and hence a pointed morphism. Therefore the corresponding map $\varphi: C(M) \rightarrow \Sigma C(L)$ is smooth. This means that for all $f \in C^\infty(M)$, the map $\zeta \mapsto \varphi(f)(\zeta, \cdot)$ is C^∞ from S^1 to $C(L)$. It is equivalent (see Lemma 2.2 below) that $x \mapsto \varphi(f)(\cdot, x)$ is continuous from L to $C^\infty(S^1)$. Now $\varphi(f)(\zeta, x) = f(h(x)(\zeta))$, so we have shown that $x \mapsto f(h(x)(\cdot))$ is continuous from L to $C^\infty(S^1)$ for all $f \in C^\infty(M)$. Letting f run through the restrictions to M of the coordinate projections of \mathbf{R}^n , we see that $x \mapsto h(x)(\cdot)$ is continuous from L to $C^\infty(S^1)^n = C^\infty(S^1, \mathbf{R}^n)$. This means that $h|_L: L \rightarrow \Omega_\infty M$ is continuous, as desired.

It now also follows that $h(L)$ is compact, so that we have also shown that $L \in F$ if and only if $h(L)$ is compact. Thus h is a homeomorphism, and F is the set of all compact subsets of X . Q.E.D.

We now prove the lemma used in the proof just given.

2.2. Lemma. *Let K be a compact Hausdorff space, and let M be a C^∞ manifold. Then $C^\infty(M, C(K)) = C(K, C^\infty(M))$ when both are regarded as subsets of $C(M \times K)$ in the obvious way. Here $C^\infty(M)$ has the topology of uniform convergence of all derivatives on every compact subset of M .*

Proof. This is surely known, so we only sketch the proof. Implicit in the statement of the lemma are three ways of writing a function from $M \times K$ to \mathbb{C} . We will use subscripts to identify them, via the convention $f(m, x) = f_1(m)(x) = f_2(x)(m)$ for $x \in K$ and $m \in M$. Also note that, since M is locally compact, membership in $C^\infty(M, C(K))$ and $C(K, C^\infty(M))$ is local on M . Therefore we may assume that $M = \mathbb{R}^n$.

We will prove the statement for C^k in place of C^∞ , by induction on k . For $k = 0$, standard arguments using the compactness of K show that both sides are $C(M \times K)$. For $k = 1$, it is easy to show that if $f_1 \in C^1(M, C(K))$, then $f_2 \in C(K, C^1(M))$, by checking that $f_2(x)' \rightarrow f_2(x_0)'$ as $x \rightarrow x_0$, uniformly on compact subsets of \mathbb{R}^n . The other direction is a bit more tricky, and involves showing that $(f_1'(m))(x) = f_2(x)'(m)$. The hard part is showing that f_1' exists for the topology of $C(K)$, but the details are similar to the standard argument in the $k = 0$ case. The induction step is now done by observing that a function g is C^{k+1} if and only if g is C^1 and g' is C^k . Finally, the $k = \infty$ case is obtained by intersecting over all finite k . Q.E.D.

2.3. Remark. The analog of Lemma 2.2 for Lipschitz functions fails. For example, take $M = K = [0, 1]$ and set $f(s, t) = |s - t|$. Then f is a Lipschitz function from M to $C(K)$ and also from K to $C(M)$, but f is continuous neither from K to $\text{Lip}(M)$ nor from M to $\text{Lip}(K)$. (With $g(s)(t) = f(s, t)$, the function $g(s_1) - g(s_2)$ has Lipschitz constant 2 for any $s_1 \neq s_2$, and the same thing happens in the other order.)

One can check that the abelianization of $\Omega_{\text{lip}}(C(M), \text{Lip}(M))$, or equivalently of $\Omega_{\text{lip}}(C(M), C^\infty(M))$, is isomorphic to $C(X)$ for a space X in one-to-one correspondence with the set of Lipschitz loops on M . However, as a consequence of the failure of the analog of Lemma 2.2, one finds that the topology on X is not the topology of Lipschitz convergence of loops. This is one reason for preferring to use C^∞ loop algebras instead of Lipschitz loop algebras in the statement of our main theorem. The Lipschitz loop algebra, however, plays an important role in the proof because of Proposition 1.11.

3. HOMOTOPY DUAL GROUP STRUCTURES ON LOOP ALGEBRAS

In this paper, homotopy dual groups will always be taken in the category of pointed pro- C^* -algebras. That is, the conditions of [12, Definition 1.2] must be satisfied, with the additional provision that all maps and homotopies appearing in that definition be pointed. (See also the remark before Proposition 5.3 in [12].)

Let A be a homotopy dual group. Then there are two apparently rather different ways one might try to make ΩA or $\Omega_\infty A$ into a homotopy dual group. The purpose of this section is to give the relevant constructions, and to show, just as in the commutative case, that they are essentially the same.

3.1. Definition. If $(A, \mu_A, \iota_A, \chi_A)$ and $(B, \mu_B, \iota_B, \chi_B)$ are homotopy dual groups, then a *morphism* between them is a pointed morphism $\varphi: A \rightarrow B$ such that there are pointed homotopy equivalences $(\varphi * \varphi) \circ \mu_A \simeq \mu_B \circ \varphi$ and $\varphi \circ \iota_A \simeq \iota_B \circ \varphi$. (Note that $\chi_A = \chi_B \circ \varphi$ because φ is pointed.) The morphism φ is a *homotopy equivalence* if it has a homotopy inverse ψ in the category of pointed pro- C^* -algebras. It is automatic that ψ is also a morphism of homotopy dual groups.

We will frequently be dealing with free products, so for this section we establish the following notation. In a free product $A *_C A$, we will write $a^{(i)}$ for the image of $a \in A$ in the i th free factor, and similarly for more than two factors. If A_0 and B_0 are pointed admissible subalgebras of A and B , then we write $A_0 *_C B_0$ for the smallest pointed admissible subalgebra of $A *_C B$ containing A_0 and B_0 . (It exists by Lemma 1.3(5).) This “free product” is easily checked to be associative, so that expressions such as $A_0 *_C A_0 *_C A_0$ make sense.

3.2. Proposition. *Let A be a pointed pro- C^* -algebra, and let A_0 be a pointed admissible subalgebra. Then the algebras ΩA , $\Omega_{\text{lip}}(A, A_0)$, and $\Omega_\infty(A, A_0)$ all have natural structures of homotopy dual groups, and the homomorphisms $\Omega A \rightarrow \Omega_{\text{lip}}(A, A_0) \rightarrow \Omega_\infty(A, A_0)$ of Remark 1.8 are morphisms of homotopy dual groups.*

Note that we do not need A_0 to make ΩA into a homotopy dual group. It will be understood throughout this section that the statements not referring to the given admissible subalgebra do not depend on its presence.

Proof of Proposition 3.2. Let $\chi: \Omega A \rightarrow \mathbf{C}$ be the usual basepoint, defined by $\chi(z(a, \zeta)) = \alpha(a)$, where $\alpha: A \rightarrow \mathbf{C}$ is the basepoint of A . Define $\mu: \Omega A \rightarrow \Omega A *_C \Omega A$ by

$$\mu(z(a, \zeta)) = \begin{cases} z^{(1)}(a, \zeta^2), & \text{Im}(\zeta) \geq 0, \\ z^{(2)}(a, \zeta^2), & \text{Im}(\zeta) \leq 0. \end{cases}$$

(This definition makes sense because for $\text{Im}(\zeta) = 0$, we have $\zeta^2 = 1$, and in $\Omega A *_C \Omega A$, we have $z^{(1)}(a, 1) = \alpha(a)z^{(1)}(1, 1) = \alpha(a) \cdot 1 = z^{(2)}(a, 1)$.) Finally define $\iota: \Omega A \rightarrow \Omega A$ by $\iota(z(a, \zeta)) = z(a, \zeta^{-1})$. It is then easy to see that these definitions do in fact make ΩA into a homotopy dual group; for example, one can imitate the proof of Theorem 1.5.7 of [16]. Naturality is obvious.

Exactly the same definitions also make $\Omega_{\text{lip}}(A, A_0)$ into a homotopy dual group in such a way that $\Omega A \rightarrow \Omega_{\text{lip}}(A, A_0)$ is a morphism. For $\Omega_\infty(A, A_0)$, one must replace the function $\zeta \mapsto \zeta^2$ used in the definition of μ by a smooth

function f such that $|f(\zeta) - \zeta^2|$ is small, and such that $f(\zeta)$ is the constant 1 on neighborhoods of 1 and -1 . Using f in place of $\zeta \mapsto \zeta^2$ in the definition of $\mu: \Omega_{\text{lip}}(A, A_0) \rightarrow \Omega_{\text{lip}}(A, A_0) *_C \Omega_{\text{lip}}(A, A_0)$ clearly gives a homotopic map, so $\Omega_{\text{lip}}(A, A_0) \rightarrow \Omega_\infty(A, A_0)$ is also a morphism. Q.E.D.

We now want to examine what happens when A is already a homotopy dual group. We need a definition and a lemma.

3.3. Definition. Let $(A, \mu_0, \iota_0, \chi_0)$ be a homotopy dual group, and let A_0 be a pointed admissible subalgebra. We say that A_0 and the homotopy dual group structure are *compatible* if $\mu_0(A_0) \subset A_0 *_C A_0$, $\iota_0(A_0) \subset A_0$, and the (pointed) homotopies $\varphi = (\varphi_t): A \rightarrow C([0, 1]) \otimes (A *_C \cdots *_C A)$ whose existence is required by the definition of a homotopy dual group (Definition 1.2 of [12]) can all be chosen such that the following condition is satisfied. Let $s \mapsto \psi_s$ be a homotopy of homomorphisms, defined on an interval, from $A *_C \cdots *_C A$ to a C^* -algebra C , and suppose that $s \mapsto \psi_s(b)$ is smooth (respectively, Lipschitz) for all $b \in A_0 *_C \cdots *_C A_0$. Then it is required that $s \mapsto (\text{id}_{C([0, 1])} \otimes \psi_s)(\varphi(a))$ be smooth (respectively Lipschitz) for all $a \in A_0$.

3.4. Lemma. *Let A be a pointed pro- C^* -algebra, and let A_0 be a pointed admissible subalgebra. Then there are canonical isomorphisms*

$$\begin{aligned} \Omega(A *_C \cdots *_C A) &\cong \Omega A *_C \cdots *_C \Omega A, \\ \Omega_{\text{lip}}(A *_C \cdots *_C A, A_0 *_C \cdots *_C A_0) &\cong \Omega_{\text{lip}}(A, A_0) *_C \cdots *_C \Omega_{\text{lip}}(A, A_0), \end{aligned}$$

and

$$\Omega_\infty(A *_C \cdots *_C A, A_0 *_C \cdots *_C A_0) \cong \Omega_\infty(A, A_0) *_C \cdots *_C \Omega_\infty(A, A_0).$$

(Every free product has the same number of factors.)

Proof. For convenience of notation, we only do the case of two factors. In the first case, the assignments $z(a^{(i)}, \zeta) \mapsto z(a, \zeta)^{(i)}$ and $z(a, \zeta)^{(i)} \mapsto z(a^{(i)}, \zeta)$ are easily seen to extend to pointed morphisms which are inverse to each other. We want to use the same formulas in the second and third cases as well. If $a \in A_0$, then certainly $a^{(i)} \in A_0 *_C A_0$ for each i , so that $\zeta \mapsto z(a^{(i)}, \zeta)$ is Lipschitz from S^1 to $\Omega(A *_C A, A_0 *_C A_0)$. Therefore $z(a, \zeta)^{(i)} \mapsto z(a^{(i)}, \zeta)$ extends to a homomorphism

$$\Omega_{\text{lip}}(A, A_0) *_C \Omega_{\text{lip}}(A, A_0) \rightarrow \Omega_{\text{lip}}(A *_C A, A_0 *_C A_0).$$

In the other direction, consider the obvious homomorphism $\varphi: A *_C A \rightarrow \Sigma(\Omega_{\text{lip}}(A, A_0) *_C \Omega_{\text{lip}}(A, A_0))$. If $a \in A_0$, then $\zeta \mapsto \varphi(a^{(i)})(\zeta) = z(a, \zeta)^{(i)}$ is Lipschitz. It follows from Lemma 1.3(5) that $\zeta \mapsto \varphi(x)(\zeta)$ is also Lipschitz for $x \in A_0 *_C A_0$. So $z(a^{(i)}, \zeta) \mapsto z(a, \zeta)^{(i)}$ also extends to a homomorphism, obviously the inverse of the earlier one. The C^∞ case is done in exactly the same way. Q.E.D.

3.5. Proposition. *Let $(A, \mu_0, \iota_0, \chi_0)$ be a homotopy dual group in the category of pointed pro- C^* -algebras, and let A_0 be a compatible pointed admissible subalgebra of A . Let χ be the standard basepoint of ΩA , and also of $\Omega_{\text{lip}}(A, A_0)$ and $\Omega_{\infty}(A, A_0)$. Then $(\Omega A, \Omega\mu_0, \Omega\iota_0, \chi)$ is an abelian homotopy dual group, equivalent to the homotopy dual group structure on ΩA of Proposition 3.2, and the analogous statements hold for $(\Omega_{\text{lip}}(A, A_0), \Omega_{\text{lip}}\mu_0, \Omega_{\text{lip}}\iota_0, \chi)$ and $(\Omega_{\infty}(A, A_0), \Omega_{\infty}\mu_0, \Omega_{\infty}\iota_0, \chi)$.*

Proof. Throughout this proof, we will make the identifications implied by the previous lemma without comment. We first consider the statements involving ΩA (and not involving A_0). Note that $\chi = \Omega\chi_0$. ($\Omega\mathbb{C} = \mathbb{C}$, in analogy with the fact that the loop space of a one point space is again a one point space.) Therefore $(\Omega A, \Omega\mu_0, \Omega\iota_0, \chi)$ is a pointed homotopy dual group by functoriality.

We now compare it to the other structure on ΩA . Set $\mu = \Omega\mu_0$ and $\iota = \Omega\iota_0$, and let μ', ι' , and χ' be the structural maps from Proposition 3.2. Note that $\chi' = \chi$. Let $\varphi: \Omega A *_C \Omega A \rightarrow \Omega A *_C \Omega A$ be the flip map, given by $\varphi(x^{(1)}) = x^{(2)}$ and $\varphi(x^{(2)}) = x^{(1)}$ for $x \in \Omega A$. Then we claim that

$$(*) \quad (\mu' * \mu') \circ \mu = (\text{id} * \varphi * \text{id}) \circ (\mu * \mu) \circ \mu',$$

as homomorphisms from ΩA to $\Omega A *_C \Omega A *_C \Omega A *_C \Omega A$. In fact, one checks that if $a \in A$, and

$$\mu_0(a) = \lim_n p_n(b_{n1}^{(1)}, b_{n2}^{(1)}, \dots, c_{n1}^{(2)}, c_{n2}^{(2)}, \dots),$$

for $b_{n1}, \dots \in A$ and polynomials p_n in finitely many noncommuting variables, then the result of applying either side of $(*)$ to $z(a, \zeta)$ is

$$\lim_n p_n(z(b_{n1}, \zeta^2)^{(1)}, z(b_{n2}, \zeta^2)^{(1)}, \dots, z(c_{n1}, \zeta^2)^{(3)}, z(c_{n2}, \zeta^2)^{(3)}, \dots)$$

if $\text{Im}(\zeta) \geq 0$, and

$$\lim_n p_n(z(b_{n1}, \zeta^2)^{(2)}, z(b_{n2}, \zeta^2)^{(2)}, \dots, z(c_{n1}, \zeta^2)^{(4)}, z(c_{n2}, \zeta^2)^{(4)}, \dots)$$

if $\text{Im}(\zeta) \leq 0$.

Next, note that by definition the maps $(\text{id} * \chi) \circ \mu$, $(\text{id} * \chi) \circ \mu'$, $(\chi * \text{id}) \circ \mu$, and $(\chi * \text{id}) \circ \mu'$ are all homotopic to id , the identity map of A . Therefore, following the argument used in the proof of Theorem 1.6.8 of [16], we obtain

$$\begin{aligned} \mu &\simeq [((\text{id} * \chi) \circ \mu') * ((\chi * \text{id}) \circ \mu')] \circ \mu = (\text{id} * \chi * \chi * \text{id}) \circ (\mu' * \mu') \circ \mu \\ &= (\text{id} * \chi * \chi * \text{id}) \circ (\text{id} * \varphi * \text{id}) \circ (\mu * \mu) \circ \mu' \\ &= (\text{id} * \chi * \chi * \text{id}) \circ (\mu * \mu) \circ \mu' \simeq \mu'. \end{aligned}$$

Here, the third step is $(*)$, the fourth step follows from $(\chi * \chi) \circ \varphi = \chi * \chi$, and the last step is the reverse of the first two steps. Thus $\mu \simeq \mu'$. Also, we have

$$\begin{aligned} \mu &\simeq [((\chi * \text{id}) \circ \mu') * ((\text{id} * \chi) \circ \mu')] \circ \mu = (\chi * \text{id} * \text{id} * \chi) \circ (\mu' * \mu') \circ \mu \\ &= (\chi * \text{id} * \text{id} * \chi) \circ (\text{id} * \varphi * \text{id}) \circ (\mu * \mu) \circ \mu' \\ &= \varphi \circ (\chi * \text{id} * \text{id} * \chi) \circ (\mu * \mu) \circ \mu' = \varphi \circ \mu' \simeq \varphi \circ \mu, \end{aligned}$$

where the last step is $\mu' \simeq \mu$ and the second last step is the reverse of the first two steps. Therefore $(\Omega A, \mu, \iota, \chi)$ is homotopy abelian.

Finally, we check that $\iota \simeq \iota'$. Let $\varepsilon: C \rightarrow \Omega A$ be the homomorphism determined by $\varepsilon(1) = 1$, and let $\delta: \Omega A *_C \Omega A \rightarrow \Omega A$ be the homomorphism given by $\delta(x^{(\iota)}) = x$ for all x . Then one checks, using the definition of a homotopy dual group, that for any $\psi: \Omega A \rightarrow \Omega A$, one has $\delta \circ (\psi * (\varepsilon \circ \chi)) \circ \mu \simeq \delta \circ ((\varepsilon \circ \chi) * \psi) \circ \mu \simeq \psi$, and similarly for μ' in place of μ . Thus:

$$\begin{aligned} \iota' &\simeq \delta \circ ((\varepsilon \circ \chi) * \iota') \circ \mu \simeq \delta \circ [(\delta \circ (\iota * \text{id}) \circ \mu) * \iota'] \circ \mu \\ &\simeq \delta \circ [(\delta \circ (\iota * \text{id}) \circ \mu') * \iota'] \circ \mu' \\ &= \delta \circ (\delta * \text{id}) \circ (\iota * \text{id} * \iota') \circ (\mu' * \text{id}) \circ \mu' \\ &= \delta \circ [\iota * (\delta \circ (\text{id} * \iota') \circ \mu)] \circ \mu' \simeq \delta \circ (\iota * (\varepsilon \circ \chi)) \circ \mu' \simeq \iota, \end{aligned}$$

as desired. This completes the proof of the statements about ΩA .

For the other two cases, we note that the same arguments will work provided that we can show that $(\Omega_{\text{lip}}(A, A_0), \Omega_{\text{lip}}\mu_0, \Omega_{\text{lip}}\iota_0, \chi)$ and $(\Omega_\infty(A, A_0), \Omega_\infty\mu_0, \Omega_\infty\iota_0, \chi)$ are in fact homotopy dual groups. The requirements $\mu_0(A_0) \subseteq A_0 *_C A_0$ and $\iota_0(A_0) \subset A_0$ ensure that $\Omega_{\text{lip}}\mu_0$ is in fact a homomorphism from $\Omega_{\text{lip}}A$ to $\Omega_{\text{lip}}A *_C \Omega_{\text{lip}}A$, and similarly for ι_0 and in the C^∞ case. We must now verify the existence of the appropriate homotopies; we do only one case, namely homotopy associativity of $\Omega_\infty\mu_0$.

To keep the notation short, write $B = A *_C A *_C A$ and $B_0 = A_0 *_C A_0 *_C A_0$. Let $\varphi = (\varphi_t): A \rightarrow C([0, 1], B)$ be a homotopy from $(\mu_0 * \text{id}) \circ \mu_0$ to $(\text{id} * \mu_0) \circ \mu_0$, satisfying the appropriate condition in Definition 3.3. We want to define a homotopy $\bar{\varphi} = (\bar{\varphi}_t): \Omega_\infty A \rightarrow C([0, 1], \Omega_\infty B)$ by $\bar{\varphi}_t(z(a, \zeta)) = z(\varphi_t(a), \zeta)$. For $b \in C([0, 1], B)$, write $\tilde{z}(b, \zeta)$ for the function $t \rightarrow z(b(t), \zeta)$ in $C([0, 1], \Omega_\infty B)$. Then to prove that $\bar{\varphi}$ is defined, we must verify that $\zeta \mapsto \tilde{z}(\varphi(a), \zeta)$ is smooth for every $a \in A_0$. Writing

$$C([0, 1], \Omega_\infty B) = \varprojlim_{p \in S(\Omega_\infty B)} C([0, 1], (\Omega_\infty B)_p),$$

and using the construction of $\Omega_\infty B$ in the proof of Proposition 1.7, we see that it suffices to prove that for every C^* -algebra C and $\psi: B \rightarrow C(S^1, C)$, the function

$$\zeta \mapsto (\text{id}_{C([0, 1])} \otimes \psi)(\varphi(a))(\zeta)$$

is smooth from S^1 to $C([0, 1], C)$. This follows from Definition 3.3 because, by the same reasoning, the functions $\zeta \mapsto \psi(b)(\zeta)$ are smooth for all ψ and for all $b \in B_0$. Thus, $\bar{\varphi}$ is in fact well defined, and it is clearly a homotopy from $(\Omega_\infty\mu_0 * \text{id}) \circ \Omega_\infty\mu_0$ to $(\text{id} * \Omega_\infty\mu_0) \circ \Omega_\infty\mu_0$. Q.E.D.

4. SMOOTHING PATHS IN σ - C^* -ALGEBRAS

The purpose of this section is to prove certain approximation lemmas which we will use in the next sections to identify, for example, $[\Omega U_{\text{nc}}, A]_+$ with

$[\Omega_\infty U_{nc}, A]_+$ for a σ - C^* -algebra A and a suitable dense subalgebra of U_{nc} . (Here U_{nc} is as in [12].) The basic version of our main lemma says that any continuous path in a σ - C^* -algebra differs from a C^∞ path by a bounded path of arbitrarily small norm.

4.1. Lemma. *Let $\pi: B \rightarrow A$ be a surjective homomorphism of C^* -algebras, $b: [0, 1] \rightarrow B$ a continuous function, $\varepsilon > 0$, and $f: [0, 1] \rightarrow A$ a C^1 function such that $\|f - \pi(b)\| < \varepsilon$. Then there is a C^1 function $g: [0, 1] \rightarrow B$ such that $\|g - b\| < \varepsilon$ and $\pi(g) = f$.*

Proof. We first prove the following claim:

(*) If $a: [0, 1] \rightarrow A$ is C^1 , then there is a C^1 function $z: [0, 1] \rightarrow B$ such that $\pi(z) = a$ and $\|z\| < \|a\| + \varepsilon$.

To prove the claim, begin by choosing a continuous function $w: [0, 1] \rightarrow B$ such that $\pi(w) = a$ and $\|w\| < \|a\| + \varepsilon/5$. By convolving w (extended over \mathbf{R} using its values at 0 and 1) with an appropriate nonnegative C^∞ function on \mathbf{R} supported in a small neighborhood of 0 and with integral equal to 1, we obtain a C^∞ function $x: [0, 1] \rightarrow B$ such that $\|x - w\| < \varepsilon/5$ and $\|\pi(x'(t)) - a'(t)\| < \varepsilon/5$ for all $t \in [0, 1]$. Choose a continuous function $y: [0, 1] \rightarrow B$ such that $\|y\| < 2\varepsilon/5$ and $\pi(y(t)) = a'(t) - \pi(x'(t))$ for all t , and define

$$z(t) = w(0) + \int_0^t y(s) ds + x(t) - x(0).$$

Then

$$\pi(z(t)) = a(0) + \int_0^t a'(s) ds + \pi\left(-\int_0^t x'(s) ds + x(t) - x(0)\right) = a(t),$$

and

$$\begin{aligned} \|z(t)\| &\leq t\|y\| + \|x(t)\| + \|w(0) - x(0)\| < \frac{3\varepsilon}{5} + \|x\| \\ &\leq \frac{3\varepsilon}{5} + \|x - w\| + \|w\| < \varepsilon + \|a\|, \end{aligned}$$

as desired. Obviously z is C^1 , so the claim is proved.

We now prove the lemma. Choose a continuous function $h: [0, 1] \rightarrow B$ such that $\pi(h) = f - \pi(b)$ and $\|h\| < \varepsilon$. Set $\varepsilon_0 = \varepsilon - \|h\|$. Using a convolution as in the proof of the claim, choose a C^∞ function $k: [0, 1] \rightarrow B$ such that $\|k - (h + b)\| < \varepsilon_0/3$. Then $\|\pi(k) - f\| < \varepsilon_0/3$, so, by the claim (*), there is a C^1 function $l: [0, 1] \rightarrow B$ such that $\pi(l) = \pi(k) - f$ and $\|l\| < 2\varepsilon_0/3$. Then $g = k - l$ is a C^1 function such that $\pi(g) = f$, and

$$\|g - b\| = \|k - l - b\| \leq \|k - b - h\| + \|h\| + \|l\| < \frac{\varepsilon_0}{3} + \|h\| + \frac{2\varepsilon_0}{3} = \varepsilon. \quad \text{Q.E.D.}$$

In the next lemma, we replace C^1 by C^∞ .

4.2. Lemma. *Let $\pi: B \rightarrow A$ be a surjective homomorphism of C^* -algebras, $b: [0, 1] \rightarrow B$ a continuous function, $\varepsilon > 0$, and $f: [0, 1] \rightarrow A$ a C^∞ function such that $\|f - \pi(b)\| < \varepsilon$. Then there is a C^∞ function $g: [0, 1] \rightarrow B$ such that $\|g - b\| < \varepsilon$ and $\pi(g) = f$.*

Proof. We begin with the analog of the claim (*) in the proof of the previous lemma. Thus, let $a: [0, 1] \rightarrow A$ be C^∞ ; then we want to find a C^∞ function $z: [0, 1] \rightarrow B$ such that $\pi(z) = a$ and $\|z\| < \|a\| + \varepsilon$.

Start by using the previous lemma to choose a C^1 function $z_1: [0, 1] \rightarrow B$ such that $\pi(z_1) = a$ and $\|z_1\| < \|a\| + \varepsilon/2$. Inductively assume we are given functions $z_k: [0, 1] \rightarrow B$ for $1 \leq k \leq n$, with $\|z_k\| < \|a\| + \varepsilon/2$ as before, with $\pi(z_k) = a$ for all k , and with z_k a C^k function whose derivatives satisfy $\|z_k^{(l)} - z_{k+1}^{(l)}\| < \varepsilon/2^{k+1}$ for $0 \leq l \leq k$. We construct z_{n+1} extending the sequence such that the hypotheses are satisfied for $n + 1$. Using the previous lemma, choose a C^1 function $y_0: [0, 1] \rightarrow B$ such that $\pi(y) = a^{(n)}$ and $\|z_n^{(n)} - y_0\| < \varepsilon/2^{n+1}$. Define inductively

$$y_k(t) = z_n^{(n-k)}(0) + \int_0^t y_{k-1}(s) ds,$$

for $1 \leq k \leq n$, and set $z_{n+1} = y_n$. Then z_{n+1} is a C^{n+1} function from $[0, 1]$ to B , and $z_{n+1}^{(l)} = y_n^{(l)} = y_{n-l}$. Therefore

$$\|z_n^{(n)}(t) - z_{n+1}^{(n)}(t)\| = \|z_n^{(n)}(t) - y_0(t)\| < \frac{\varepsilon}{2^{n+1}},$$

so

$$\begin{aligned} \|z_n^{(n-1)}(t) - z_{n+1}^{(n-1)}(t)\| &= \left\| z_n^{(n-1)}(t) - z_n^{(n-1)}(0) - \int_0^t y_0(s) ds \right\| \\ &\leq \int_0^t \|z_n^{(n)}(s) - y_0(s)\| ds < \frac{\varepsilon}{2^{n+1}} \end{aligned}$$

and, inductively, $\|z_n^{(l)}(t) - z_{n+1}^{(l)}(t)\| < \varepsilon/2^{n+1}$ for all $l \leq n$ and all t . Also, it is easily checked inductively that $\pi(y_l) = a^{(n-l)}$, so that $\pi(z_{n+1}) = a$. Thus, z_{n+1} extends the sequence, and the induction is complete.

We can therefore assume that we have an infinite sequence z_1, z_2, \dots satisfying the conditions in the induction hypothesis. Then for each l , the sequence of derivatives $z_l^{(l)}, z_{l+1}^{(l)}, \dots$ satisfies $\|z_k^{(l)} - z_{k+1}^{(l)}\| < \varepsilon/2^{k+1}$. This sequence therefore has a uniform limit x_l , and each x_l is differentiable with $x'_l = x_{l+1}$. Setting $z = x_0 = \lim_k z_k$, we see that z is a C^∞ function from $[0, 1]$ to B such that $\pi(z) = a$, and

$$\|z\| \leq \|z_1\| + \sum_{k=1}^\infty \|z_{k+1} - z_k\| < \|a\| + \varepsilon,$$

as desired. This proves the C^∞ analog of (*) of the previous lemma.

The proof that the C^∞ analog of (*) implies the statement of the lemma is exactly the same as the analogous step in the proof of the previous lemma, and is omitted. Q.E.D.

4.3. Corollary. *Let A be a σ - C^* -algebra, let $a: [0, 1] \rightarrow A$ be a continuous function, and let $\varepsilon > 0$. Then there exists a C^∞ function $f: [0, 1] \rightarrow A$ such that $a - f$ is bounded and in fact $\|a - f\|_\infty < \varepsilon$.*

In this corollary, continuity and differentiability are with respect to the usual seminorm topology on A . The statement that $a - f$ is bounded and $\|a - f\|_\infty < \varepsilon$ means that

$$\sup_{p \in S(A)} \left(\sup_{t \in [0, 1]} p(a(t) - f(t)) \right) < \varepsilon;$$

this is, of course, a very strong condition. (See [9, §1] for the general definitions.)

Proof of Corollary 4.3. Write $A = \varprojlim A_n$, with the maps $\pi_n: A_{n+1} \rightarrow A_n$ and $\kappa_n: A \rightarrow A_n$ all surjective. Using the usual convolution argument, choose a C^∞ function $f_1: [0, 1] \rightarrow A_1$ such that $\|\kappa_1(a) - f_1\| < \varepsilon/2$. Then inductively choose, using the previous lemma, C^∞ functions $f_n: [0, 1] \rightarrow A_n$ such that $\pi_{n-1}(f_n) = f_{n-1}$ and $\|\kappa_n(a) - f_n\| < \varepsilon/2$. The sequence (f_n) defines a C^∞ function $f: [0, 1] \rightarrow A$ such that $\|\kappa_n(a) - \kappa_n(f)\| \leq \varepsilon/2$ for all n , so that $\|a - f\|_\infty < \varepsilon$. Q.E.D.

4.4. Remark. One might think that one should try to prove Lemmas 4.1 and 4.2 by constructing a C^1 or C^∞ right inverse to the quotient map π , along the lines of an argument used in [1]. If, however, h is a C^1 right inverse to π , then the derivative of h at 0 is a continuous linear right inverse to π . As far as we know, quotient maps of nonnuclear C^* -algebras need not in general even have continuous linear right inverses. In fact, our proofs work for quotient maps of general Banach spaces, which certainly need not have continuous linear right inverses. (Note that [1, §6] contains arguments for the C^1 case which are similar to ours.)

We now turn to the situations that we will actually use.

4.5. Lemma. *Let $\pi: B \rightarrow A$ be a surjective homomorphism of C^* -algebras, $b: S^1 \rightarrow B$ a continuous function, $\varepsilon > 0$, and $f: S^1 \rightarrow A$ a C^∞ function such that $\|f - \pi(b)\| < \varepsilon$. Then there is a C^∞ function $g: S^1 \rightarrow B$ such that $\|g - b\| < \varepsilon$ and $\pi(g) = f$.*

Proof. Define $f_0: [0, 2\pi + 1] \rightarrow A$ and $b_0: [0, 2\pi + 1] \rightarrow B$ by $f_0(t) = f(e^{it})$ and $b_0(t) = b(e^{it})$. By Lemma 4.3 there is a C^∞ function $h_0: [0, 2\pi + 1] \rightarrow B$ such that $\|h_0 - b_0\| < \varepsilon$ and $\pi(h_0) = f_0$. Choose a C^∞ function $\alpha: [0, 1] \rightarrow [0, 1]$ such that $\alpha(t) = 0$ for $t < 1/4$ and $\alpha(t) = 1$ for $t > 3/4$. Now set

$$g_0(t) = \begin{cases} \alpha(t)h_0(t) + (1 - \alpha(t))h_0(t + 2\pi), & 0 \leq t \leq 1, \\ h_0(t), & 1 \leq t \leq 2\pi, \\ (1 - \alpha(t - 2\pi))h_0(t) + \alpha(t - 2\pi)h_0(t - 2\pi), & 2\pi \leq t \leq 2\pi + 1. \end{cases}$$

One readily verifies that g_0 is a C^∞ function from $[0, 2\pi + 1]$ to B such that $\|g_0 - b_0\| < \varepsilon$, $\pi(g_0) = f_0$, and $g_0(t + 2\pi) = g_0(t)$ for $t \in [0, 1]$. Therefore there is a C^∞ function $g: S^1 \rightarrow B$ such that $g(e^{it}) = g_0(t)$ for $t \in [0, 2\pi + 1]$, and this is the desired function. Q.E.D.

For the next lemma, we recall some definitions from [12]. Let K be the algebra of compact operators on a separable infinite-dimensional Hilbert space. Regard K as the closure of the increasing union $K_0 = \bigcup_1^\infty M_k$, for appropriate standard embeddings of the algebras M_k of $k \times k$ matrices in K . If B is a pro- C^* -algebra, then [12, Definition 1.8] $K_0 \tilde{\otimes} B$ is the algebraic inverse limit $\varprojlim_{p \in S(B)} \bigcup_{k=1}^\infty M_k \otimes B_p$. It is a dense subalgebra of the tensor product $K \otimes B$ as defined in [9, §3].

For a pro- C^* -algebra B , we identify SB with $C_0(S^1 - \{1\}) \otimes B$. Then $(K_0 \tilde{\otimes} SB)^+$ becomes a subalgebra of $C(S^1) \otimes (K \otimes B)^+$, namely the algebra of functions $f: S^1 \rightarrow (K \otimes B)^+$ such that:

- (1) $f(1) = \lambda \cdot 1$ for some $\lambda \in \mathbb{C}$.
- (2) $f(\zeta) - \lambda \cdot 1 \in K \otimes B$ for all $\zeta \in S^1$, with λ as in (1).
- (3) For every $p \in S(B)$, the image of f in $C(S^1) \otimes (K \otimes B_p)^+$ actually lies in some $C(S^1) \otimes (M_{k(p)} \otimes B_p)^+$.

4.6. Lemma. *Let B be a σ - C^* -algebra, $b \in (K_0 \tilde{\otimes} SB)^+$, and $\varepsilon > 0$. Then there exists a C^∞ element $f \in (K_0 \tilde{\otimes} SB)^+$ such that $f - b$ is bounded, $\|f - b\|_\infty < \varepsilon$, and $f(1) = b(1)$.*

Proof. We work inside $C(S^1) \otimes (K \otimes B)$. First choose λ such that $b - \lambda \cdot 1 \in C(S^1) \otimes (K \otimes B)$; it clearly suffices to prove the lemma for $b - \lambda \cdot 1$ in place of b , so we assume that $b \in C(S^1) \otimes (K \otimes B)$. Thus in fact $b \in K_0 \tilde{\otimes} SB$. Now write $B = \varprojlim B_n$, with all maps $\kappa_n: B \rightarrow B_n$ and $\pi_n: B_{n+1} \rightarrow B_n$ surjective. Choose an increasing sequence $n \mapsto k(n)$ of integers such that $\kappa_n(b) \in M_{k(n)} \otimes SB$, which we identify in the obvious way with a subalgebra of $C(S^1) \otimes M_{k(n)} \otimes B \subset C(S^1) \otimes K \otimes B$.

We will construct by induction C^∞ functions $f_n: S^1 \rightarrow M_{k(n)} \otimes B$ such that $\|f_n - \kappa_n(b)\| < \varepsilon/2$ and $\pi_{n-1}(f_n) = f_{n-1}$. We obtain $f_1: S^1 \rightarrow M_{k(1)} \otimes B_1$ by convolving the function $\kappa_1(b)$ with a C^∞ nonnegative function on S^1 whose integral is 1 and whose support is contained in a sufficiently small neighborhood of 1. Given $f_n: S^1 \rightarrow M_{k(n)} \otimes B_n$, we regard it as a function from S^1 to $M_{k(n+1)} \otimes B_n$, and apply the previous lemma to the surjective map $M_{k(n+1)} \otimes B_{n+1} \rightarrow M_{k(n+1)} \otimes B_n$ and the function $\kappa_{n+1}(b)$ to obtain the desired f_{n+1} .

The functions (f_n) now form a coherent sequence which defines a C^∞ element f of $K_0 \tilde{\otimes} (C(S^1) \otimes B)$ satisfying $\|f - b\|_\infty < \varepsilon/2$. Replacing f by $f - f(1)$, we have a C^∞ element of $K_0 \tilde{\otimes} SB$ which still satisfies $\|f - b\|_\infty < \varepsilon$, as desired. Q.E.D.

5. SMOOTHING PATHS OF HOMOMORPHISMS

In this section, we use the last lemma of the previous section to obtain results about homomorphisms which are more directly related to the problem of identifying, for example, $[\Omega U_{nc}, A]_+$ with $[\Omega_\infty U_{nc}, A]_+$. First, we recall the constructions of the algebras P and U_{nc} , and specify the required dense subalgebras. We let qC be as in [5, §1], that is, the C^* -subalgebra of $C \star C$ generated by $e_0 - f_0$ and $e_0(e_0 - f_0)$, where e_0 and f_0 are the identities of the two copies of C in the free product. We take $(qC)_0$ to be the admissible $*$ -subalgebra of qC generated by $e_0 - f_0$ and $e_0(e_0 - f_0)$. Further let $S = \{f \in C(S^1) : f(1) = 0\}$, with S_0 the $*$ -subalgebra generated by the function $s(z) = z - 1$.

Also recall that in [12, §1], we constructed an adjoint functor W_∞ to $K_0 \tilde{\otimes} -$, and that we have $P = W_\infty(qC)^+$ [12, Definition 2.1] and $U_{nc} \cong W_\infty(S)^+$ [12, Proposition 4.2]. We further recall that the algebra $W_\infty(A)$ is generated by elements $x_\infty(a, i, j)$ for $a \in A$ and $i, j \geq 1$, such that the homomorphism from $W_\infty(A)$ to B corresponding to $\varphi : A \rightarrow K_0 \tilde{\otimes} B$ sends $x_\infty(a, i, j)$ to $\varphi(a)_{ij}$.

5.1. Definition. We let P_0 be the smallest admissible subalgebra of P containing 1 and all elements $x_\infty(e_0 - f_0, i, j)$ and $x_\infty(e_0(e_0 - f_0), i, j)$. We let $(U_{nc})_0$ be the smallest admissible subalgebra of U_{nc} containing 1 and all elements $x_\infty(s, i, j)$.

5.2. Proposition. P_0 and $(U_{nc})_0$ are pointed admissible subalgebras which are locally finitely generated and compatible with the homotopy dual group structures on P and U_{nc} .

Proof. Everything except compatibility is immediate. To verify compatibility, we observe that it holds in much greater generality. Indeed, let A be a C^* -algebra, equipped as in [12, Definition 1.10] with a homomorphism $\iota_0 : A \rightarrow A$ such that $\iota_0 \circ \iota_0 \simeq \text{id}_A$ and such that

$$a \mapsto \begin{pmatrix} \iota_0(a) & 0 \\ 0 & a \end{pmatrix}$$

is homotopic to the zero map from A to $M_2(A)$. Suppose further that A_0 is a dense $*$ -subalgebra generated as a $*$ -algebra by a set G , and that, if $\varphi = (\varphi_t)$ is one of the homotopies above, then each $\varphi(a)$ for $a \in G$ can be written as a polynomial in the noncommuting variables $G \cup G^*$, with coefficients in $C([0, 1])$ or $C([0, 1], M_2)$ as appropriate. Let $W_\infty(A)_0^+$ be the pointed admissible subalgebra of $W_\infty(A)^+$ generated by 1 and all $x_\infty(a, i, j)$ for $a \in A_0$. Then one readily verifies that the maps determining the homotopy dual group structure on $W_\infty(A)^+$ send $W_\infty(A)_0^+$ into $W_\infty(A)_0^+ \star_C W_\infty(A)_0^+$ or $W_\infty(A)_0^+$ as appropriate. Also, the homotopies used in the proof that $W_\infty(A)^+$ is a homotopy dual group are obtained by sending $x_\infty(a, i, j)$ to some linear combination of

the $x_\infty(a, k, l)$, $t \mapsto x_\infty(\varphi_t(a), k, l)$, or $t \mapsto x_\infty(\psi_t(a)_{mn}, k, l)$ (for φ, ψ the homotopies above), with coefficients in $C([0, 1])$. (See the proof of [12, Theorem 1.11].) Therefore the smoothness condition in Definition 3.3 follows from the conditions on the homotopies given above.

It remains only to verify the conditions above for the maps ι_0 under consideration. In the case $A = q\mathbf{C}$, the relevant map from A to A (called τ in the proof of [12, Proposition 1.15]) clearly satisfies $\tau(A_0) \subset A_0$ and $\tau \circ \tau = \text{id}_A$. A calculation further shows that the homotopy given in that proof from

$$a \mapsto \begin{pmatrix} \tau(a) & 0 \\ 0 & a \end{pmatrix}$$

to the zero map sends $e_0 - f_0$ to $(h_{ij}(t)(e_0 - f_0))_{i,j=1}^2$ and $e_0(e_0 - f_0)$ to $(k_{ij}(t)e_0(e_0 - f_0) + l_{ij}(t)(e_0 - f_0)^2)_{i,j=1}^2$ for certain continuous functions h_{ij} , k_{ij} , and l_{ij} . (They turn out to be combinations of trigonometric functions.) In particular, the required condition is satisfied. Using $\iota_0: S \rightarrow S$ as in [12, Proposition 4.2], and the homotopy given in its proof, we obtain the same result for S_0 . Q.E.D.

To simplify the notation, we will write $\Omega_\infty P$ and $\Omega_{\text{lip}} P$ for $\Omega_\infty(P, P_0)$ and $\Omega_{\text{lip}}(P, P_0)$, and similarly for U_{nc} .

In preparation for the next lemma, we generalize Definition 1.6 to algebras of the form $K_0 \tilde{\otimes} B$.

5.3. Definition. A homotopy of elements in an algebra of the form $K_0 \tilde{\otimes} B$ is a continuous path $t \mapsto b_t$ which defines an element of $K_0 \tilde{\otimes} (C([0, 1]) \otimes B)$. (This means that $b \in \varinjlim_k (\bigcup_k [C([0, 1]) \otimes M_k \otimes B_p])$, that is, for every $p \in S(B)$ there is $k(p)$ such that $\kappa_p(b_t) \in M_{k(p)} \otimes B_p$ for all t . See [12, Definition 1.8].) Similarly, homotopies of elements in $(K_0 \tilde{\otimes} B)^+$, $M_2(K_0 \tilde{\otimes} B)$, etc. are to lie in

$$\varinjlim_p \left(\bigcup_k [C([0, 1]) \otimes (M_k \otimes B_p)^+] \right), \varinjlim_p \bigcup_k (C([0, 1]) \otimes M_2(M_k \otimes B_p)),$$

etc. Also, a homotopy of homomorphisms from A to, say, $(K_0 \tilde{\otimes} B)^+$ is an assignment $t \mapsto \varphi_t$ such that, for every $a \in A$, $t \mapsto \varphi_t(a)$ is a homotopy of elements of $(K_0 \tilde{\otimes} B)^+$. We now define smooth and Lipschitz elements of $K_0 \tilde{\otimes} (C(M) \otimes B)$, homomorphisms to $K_0 \tilde{\otimes} (C(M) \otimes B)$, and their homotopies by analogy with Definition 1.6, regarding $K_0 \tilde{\otimes} (C(M) \otimes B)$ as a subalgebra of $C(M) \otimes (K \otimes B)$, and also imposing the conditions on homotopies given above. Regarding $(K_0 \tilde{\otimes} SB)^+$ as a subalgebra of $(K_0 \tilde{\otimes} [C(S^1) \otimes B])^+$ and extending our definitions in the obvious way, we obtain the notions of smooth and Lipschitz elements, homomorphisms, and homotopies in algebras such as $(K_0 \tilde{\otimes} SB)^+$ and $M_2((K_0 \tilde{\otimes} SB)^+)$.

5.4. Lemma. *Let A be either S or $q\mathbf{C}$, and let B be a σ - C^* -algebra. Then:*

- (1) *Every homomorphism from A to $K_0 \tilde{\otimes} SB$ is homotopic to a smooth homomorphism.*

(2) If $\varphi_0, \varphi_1: A \rightarrow K_0 \tilde{\otimes} SB$ are homotopic smooth (respectively, Lipschitz) homomorphisms, then the homotopy can be chosen to be smooth (respectively, Lipschitz).

(3) Let $\varphi_0, \varphi_1: A \rightarrow K_0 \tilde{\otimes} SB$ be homotopic Lipschitz homomorphisms, and let $t \mapsto \varphi_t$ be the homotopy of part (2). Further let $\psi: B \rightarrow C$ be a homomorphism to a pro- C^* -algebra C , and also write ψ for the corresponding homomorphism from $K_0 \tilde{\otimes} SB$ to $K_0 \tilde{\otimes} SC$. If $\psi \circ \varphi_0$ and $\psi \circ \varphi_1$ are smooth, then $t \mapsto \psi \circ \varphi_t$ is a smooth homotopy.

The dense subalgebras of S and qC needed for the definitions of smooth and Lipschitz homomorphisms are of course taken to be S_0 and $(qC)_0$ as defined at the beginning of this section.

Proof of Lemma 5.4. We will do the case $A = S$ first, since it is easier. First, note that (smooth, Lipschitz) homomorphisms from S to $K_0 \tilde{\otimes} SB$ are in bijective correspondence with (smooth, Lipschitz) pointed homomorphisms from $C(S^1) = S^+$ to $(K_0 \tilde{\otimes} SB)^+$, and that these in turn are in bijective correspondence with (smooth, Lipschitz) unitaries $u \in (K_0 \tilde{\otimes} SB)^+$ whose images in C are 1. (We call such unitaries *special*.) Furthermore, these correspondences all preserve homotopy classes. Therefore it suffices to prove parts (1) through (3) for special unitaries in $(K_0 \tilde{\otimes} SB)^+$.

For part (1) let $u \in (K_0 \tilde{\otimes} SB)^+$ be a special unitary, and use Lemma 4.6 to choose $f \in (K_0 \tilde{\otimes} SB)^+$ such that $\|f - u\|_\infty < 1/2$ and $f(1) = 1$. Set $f_t = tf + (1 - t)u$. Then the required homotopy is $t \mapsto u_t = f_t(f_t^* f_t)^{-1/2}$.

For part (2), let u_0 and u_1 be homotopic smooth (respectively, Lipschitz) special unitaries in $(K_0 \tilde{\otimes} SB)^+$, and let v be a homotopy. Using Lemma 4.6, choose a smooth element $b \in [K_0 \tilde{\otimes} S(C([0, 1]) \otimes B)]^+$ such that $\|v - b\|_\infty < 1/2$ and $b(1) = 1$.

We now regard elements of $K_0 \tilde{\otimes} S(C([0, 1]) \otimes B)$ and its unitization as functions from $[0, 1] \times S^1$ to $K_0 \tilde{\otimes} B$ and its unitization. (Of course, these functions must vanish or take values in $C \cdot 1$ at $(t, 1)$ for all $t \in [0, 1]$.) Then define $v_0 \in [K_0 \tilde{\otimes} S(C([0, 1]) \otimes B)]^+$ by

$$v_0(t, \zeta) = \begin{cases} v(0, \zeta), & t \in [0, \frac{1}{3}], \\ v(3t - 1, \zeta), & t \in [\frac{1}{3}, \frac{2}{3}], \\ v(1, \zeta), & t \in [\frac{2}{3}, 1], \end{cases}$$

and set

$$b_0(t, \zeta) = \begin{cases} (1 - 3t)u_0(\zeta) + 3tb(0, \zeta), & t \in [0, \frac{1}{3}], \\ b(3t - 1, \zeta), & t \in [\frac{1}{3}, \frac{2}{3}], \\ (3 - 3t)b(1, \zeta) + (3t - 2)u_1(\zeta), & t \in [\frac{2}{3}, 1]. \end{cases}$$

If u_0 and u_1 are C^∞ , then so is b_0 , as can be seen by differentiating the formula for it with respect to ζ . Similarly, if u_0 and u_1 are Lipschitz, then b_0 is Lipschitz (with respect to ζ , uniformly in t), with Lipschitz constant in

the seminorm p being smaller than or equal to the maximum of the Lipschitz constants for u_0, u_1 , and b in the seminorm p .

Since $v(i, \zeta) = u_i(\zeta)$ for $i = 0, 1$, we have $\|v_0 - b_0\|_\infty = \|v - b\|_\infty < \frac{1}{2}$. We can therefore set $u = b_0(b_0^*b_0)^{-1/2}$, and observe that u is a homotopy from u_0 to u_1 which has the same smoothness (respectively, Lipschitz) property as u_0 and u_1 . This is the desired homotopy.

For part (3), we continue with the notation used in the proof of part (2). If u_0 and u_1 are Lipschitz, and $\psi(u_0)$ and $\psi(u_1)$ are smooth, then we note that $\psi(u) = \psi(b_0)(\psi(b_0)^*\psi(b_0))^{-1/2}$ is smooth for the same reason that u is smooth when u_0 and u_1 are. This completes the proof of the case $A = S$.

We now do the case $A = qC$. First, fix an isomorphism $K_0 \cong M_2(K_0)$ [12, Lemma 1.13] and use it to identify $K_0 \tilde{\otimes} SB$ with $M_2(K_0 \tilde{\otimes} SB)$ and $K_0 \tilde{\otimes} SC$ with $M_2(K_0 \tilde{\otimes} SC)$. Next, call a projection $p \in M_2((K_0 \tilde{\otimes} SB)^+)$ special if $p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(K_0 \tilde{\otimes} SB)$. Call a homomorphism $\varphi: qC \rightarrow M_2(K_0 \tilde{\otimes} SB)$ special if it is the restriction to qC of a homomorphism $\varphi: C * C \rightarrow M_2((K_0 \tilde{\otimes} SB)^+)$ such that $\varphi(e_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\varphi(f_0)$ is a special projection. Thus, there is a bijective correspondence between (smooth, Lipschitz) special projections and (smooth, Lipschitz) special homomorphisms, and this correspondence preserves (smooth, Lipschitz) homotopies.

Parts (1) through (3) of the lemma hold for special projections, and hence for special homomorphisms, by essentially the same argument as for special unitaries. One needs only to replace the function $b \mapsto b(b^*b)^{-1/2}$ by the function $b \mapsto h((b + b^*)/2)$, using functional calculus with $h: \mathbf{R} - \{1/2\} \rightarrow \mathbf{R}$ given by $h(\alpha) = 0$ for $\alpha < 1/2$ and $h(\alpha) = 1$ for $\alpha > 1/2$.

We will next show how to construct, for any homomorphism $\varphi_0: qC \rightarrow M_2(K_0 \tilde{\otimes} SB)$, a homotopy $s \mapsto \varphi_s$ from φ_0 to a special homomorphism φ_1 . If $\psi: B \rightarrow C$ is any homomorphism, and if ψ also denotes the corresponding map from $M_2(K_0 \tilde{\otimes} SB)$ to $M_2(K_0 \tilde{\otimes} SC)$, then $s \mapsto \psi \circ \varphi_s$ will be a smooth (respectively, Lipschitz) homotopy whenever $\psi \circ \varphi_0$ is smooth (respectively, Lipschitz). Together with our result on special homomorphisms, this will immediately give us part (1). For part (2), we replace B by $C([0, 1]) \otimes B$, and apply our construction to the homotopy from φ_0 to φ_1 . Taking for ψ the maps of evaluation of 0 and 1 from $C([0, 1]) \otimes B$ to B , we see that this produces smooth (respectively, Lipschitz) homotopies from φ_0 and φ_1 to homotopic special homomorphisms, so that the result of the previous paragraph can be applied. Part (3) is immediate.

We now carry out the construction. It is based on the part of the proof of [12, Lemma 2.10] which is concerned with surjectivity of the map defined there. However, we do the two steps in the opposite order. We first claim that it suffices to deal only with homomorphisms and homotopies whose ranges are contained in $\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}: a \in K_0 \tilde{\otimes} SB\}$.

Let $\eta_1: M_2(K_0) \rightarrow M_2(K_0)$ be the composite of an isomorphism $M_2(K_0) \cong K_0$ with the inclusion map $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ of K_0 in $M_2(K_0)$. By [12, Lemma 1.12],

there is a homotopy $s \mapsto \eta_s$ of endomorphisms of $M_2(K_0)$ such that η_0 is the identity and η_1 is as already defined. Then $s \mapsto \varphi_s = (\eta_s \otimes \text{id}_{SB}) \circ \varphi$ defines a homotopy of homomorphisms from φ_0 to a homomorphism $\varphi_1: q\mathbf{C} \rightarrow M_2(K_0 \tilde{\otimes} SB)$ whose range is contained in $\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in K_0 \tilde{\otimes} SB \}$. This homotopy does not depend on B being a σ - C^* -algebra, is functorial in B for a fixed choice of $s \mapsto \eta_s$, and is smooth (respectively, Lipschitz) if φ_0 is. (For the last part, use $\|\eta_s\| \leq 1$ for all s .) Thus, the required smoothing property for homomorphisms $\psi: B \rightarrow C$ holds, and the claim of the previous paragraph is proved.

We can thus write

$$\varphi_0(x) = \begin{pmatrix} \bar{\varphi}_0(x) & 0 \\ 0 & 0 \end{pmatrix}$$

for some homomorphism $\bar{\varphi}_0: q\mathbf{C} \rightarrow K_0 \tilde{\otimes} SB$. As in [12, Proposition 1.16], we now identify $\mathbf{C} * \mathbf{C}$ with

$\left\{ a: [0, 1] \rightarrow M_2: a \text{ is continuous, } a(0) \text{ is diagonal, and } a(1) \in \mathbf{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$, $q\mathbf{C}$ with $\{a \in \mathbf{C} * \mathbf{C}: a(1) = 0\}$, and the generators e_0 and f_0 of $\mathbf{C} * \mathbf{C}$ with the functions

$$e_0(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f_0(t) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

Next, let $u_s \in UM_2(M(q\mathbf{C}))$, $e, f \in M_2(q\mathbf{C}^+)$, $e^{(s)}, f^{(s)} \in M_2(M(q\mathbf{C}))$, and $\gamma_s: q\mathbf{C} \rightarrow M_2(q\mathbf{C})$ all be as in the proof of [12, Lemma 2.10]. Thus, $\gamma_s(e_0 - f_0) = e^{(s)} - f^{(s)}$, $\gamma_s(e_0(e_0 - f_0)) = e^{(s)}(e^{(s)} - f^{(s)})$, and $s \mapsto \varphi_s = M_2(\bar{\varphi}_0) \circ \gamma_s$ is a homotopy from $M_2(\bar{\varphi}_0) \circ \gamma_0 = \varphi_0$ to a special homomorphism $\varphi_{\pi/2}: q\mathbf{C} \rightarrow M_2(K_0 \tilde{\otimes} SB)$.

This construction is functorial, so, just as above, we need only show that if φ_0 is smooth or Lipschitz, then the homotopy $\varphi = (\varphi_s)$ is smooth or Lipschitz. This means that, for $x = e_0 - f_0$ and for $x = e_0(e_0 - f_0)$, we must show that $\varphi(x)$ is a smooth or Lipschitz element of $M_2(K_0 \tilde{\otimes} S[C([0, \pi/2]) \otimes B])$.

Define $x_1, x_2, x_3, x_4 \in q\mathbf{C}$ by

$$\begin{aligned} x_1(t) &= \begin{pmatrix} 1-t & 0 \\ 0 & 0 \end{pmatrix}, & x_2(t) &= \begin{pmatrix} 0 & \sqrt{t(1-t)} \\ 0 & 0 \end{pmatrix}, \\ x_3(t) &= \begin{pmatrix} 0 & 0 \\ \sqrt{t(1-t)} & 0 \end{pmatrix}, & \text{and } x_4(t) &= \begin{pmatrix} 0 & 0 \\ 0 & t-1 \end{pmatrix}. \end{aligned}$$

Calculations show that

$$\begin{aligned} &\gamma_s(e_0 - f_0)(t) \\ &= \begin{pmatrix} 1-t & -(\cos s)\sqrt{t(1-t)} & 0 & -(\sin s)\sqrt{t(1-t)} \\ -(\cos s)\sqrt{t(1-t)} & (\cos^2 s)(t-1) & 0 & (\sin s)(\cos s)(t-1) \\ 0 & 0 & 0 & 0 \\ -(\sin s)\sqrt{t(1-t)} & (\sin s)(\cos s)(t-1) & 0 & (\sin^2 s)(t-1) \end{pmatrix}, \end{aligned}$$

while $\gamma_s(e_0(e_0 - f_0))$ has the same top row and all other entries zero. Therefore $\varphi(e_0 - f_0)$ and $\varphi(e_0(e_0 - f_0))$ are 2×2 matrices over $K_0 \tilde{\otimes} (C([0, \pi/2]) \otimes SB)$ whose entries have the form $\sum_1^4 h_i \otimes \bar{\varphi}_0(x_i)$ for appropriate continuous functions h_i on $[0, \pi/2]$. Now the easily verified relations

$$\begin{aligned} x_1 &= [e_0(e_0 - f_0)] \cdot [e_0 - f_0], & x_2 &= x_1 - e_0(e_0 - f_0), \\ x_4 &= x_1 - (e_0 - f_0)^2, & x_3 &= x_1 - x_2 + x_4 - (e_0 - f_0) \end{aligned}$$

show that $x_i \in (qC)_0$ for all i , so that φ is smooth (respectively Lipschitz) whenever φ_0 is. Q.E.D.

To relate these results to homomorphisms from P and U_{nc} , we need:

5.5. Lemma. *Let B be a pro- C^* -algebra. Then a homomorphism from $W_\infty(qC)$ (respectively $W_\infty(S)$) to SB is smooth or Lipschitz if and only if the corresponding homomorphism from qC (respectively, S) to $K_0 \tilde{\otimes} SB$ is smooth or Lipschitz.*

Proof. This follows immediately from the form of the correspondence between homomorphisms from $W_\infty(A)$ to SB and those from A to $K_0 \tilde{\otimes} SB$ (see the beginning of this section), and the relationships between the dense subalgebras $(qC)_0$ and P_0 and between S_0 and $(U_{nc})_0$. Q.E.D.

5.6. Corollary. *For $A = P$ or $A = U_{nc}$, and for any pointed σ - C^* -algebra B , the natural maps*

$$[A, \Sigma B]_+^\infty \rightarrow [A, \Sigma B]_+^{lip} \rightarrow [A, \Sigma B]_+$$

are bijective.

6. THE MAIN THEOREM

In this section, we prove the theorem stated in the introduction. We then discuss some consequences and open problems.

6.1. Theorem. *There are homotopy equivalences of homotopy dual groups $\Omega_\infty U_{nc} \simeq P$ and $\Omega_\infty P \simeq U_{nc}$.*

Proof. Both parts are essentially the same, so we do the second and make only brief remarks afterwards on the first.

The proof that $\Omega_\infty P \simeq U_{nc}$ consists of two steps, namely $\Omega_\infty P \simeq \Omega_{lip} P$ and $\Omega_{lip} P \simeq U_{nc}$. For the second of these, we observe that for every separable pointed σ - C^* -algebra A , there are natural isomorphisms of groups

$$(*) \quad [\Omega_{lip} P, A]_+ \cong [P, \Sigma A]_+^{lip} \cong [P, \Sigma A]_+ \cong \widetilde{RK}_0(\Sigma A) \cong \widetilde{RK}_1(A) \cong [U_{nc}, A]_+.$$

Here, the first step is an isomorphism of sets by Proposition 1.7, and is a group homomorphism when $\Omega_{lip} P$ is given the homotopy dual group structure induced from P . But by Proposition 3.5, this structure is equivalent to its usual homotopy dual group structure from Proposition 3.2. The remaining steps are, in order, Corollary 5.6, [12, Theorem 5.4(2)] (note that ΣA is separable), [10,

Theorem 3.4], and [12, Theorem 5.4(1)]. Now U_{nc} is a separable σ - C^* -algebra by construction, and $\Omega_{lip}P$ is a separable σ - C^* -algebra by Propositions 1.11 and 5.2. By the Yoneda Lemma (see [8, p. 61]), (*) therefore implies that $\Omega_{lip}P$ and U_{nc} are homotopy equivalent as homotopy dual groups.

We now prove that $\Omega_{\infty}P \simeq \Omega_{lip}P$. We have a homomorphism $\varphi: \Omega_{lip}P \rightarrow \Omega_{\infty}P$ by Remark 1.8, and it is a morphism of homotopy dual groups by Proposition 3.2. It therefore suffices to construct a pointed homotopy inverse for φ (compare Definition 3.1). We will in fact construct an ordinary homotopy inverse ψ for $\varphi: \Omega_{0, lip}W_{\infty}(qC) \rightarrow \Omega_{0, \infty}W_{\infty}(qC)$, with $\Omega_{0, lip}$ and $\Omega_{0, \infty}$ as in Remark 1.9.

To save space, we write $W = W_{\infty}(qC)$, $Q_{lip} = K_0 \tilde{\otimes} S\Omega_{0, lip}W$, and $Q_{\infty} = K_0 \tilde{\otimes} S\Omega_{0, \infty}W$. Remark 1.9 and Lemma 5.5 give us natural bijections

$$(**) \quad \text{Hom}(\Omega_{0, \infty}W, \Omega_{0, lip}W) \cong \text{Hom}^{\infty}(W, S\Omega_{0, lip}W) \cong \text{Hom}^{\infty}(qC, Q_{lip}).$$

Similar arguments give isomorphisms for all the horizontal arrows in the following two diagrams:

$$\begin{array}{ccc}
 \text{Hom}(\Omega_{0, lip}W, \Omega_{0, lip}W) & \rightarrow & \text{Hom}^{lip}(qC, Q_{lip}) \\
 \downarrow \varphi \circ - & & \downarrow \bar{\varphi} \circ - \\
 (***) \quad \text{Hom}(\Omega_{0, lip}W, \Omega_{0, \infty}W) & \rightarrow & \text{Hom}^{lip}(qC, Q_{\infty}) \\
 \uparrow - \circ \varphi & & \uparrow \\
 \text{Hom}(\Omega_{0, \infty}W, \Omega_{0, \infty}W) & \rightarrow & \text{Hom}^{\infty}(qC, Q_{\infty})
 \end{array}$$

and

$$\begin{array}{ccc}
 \text{Hom}(\Omega_{0, lip}W, \Omega_{0, lip}W) & \rightarrow & \text{Hom}^{lip}(qC, Q_{lip}) \\
 \uparrow - \circ \varphi & & \uparrow \\
 (***) \quad \text{Hom}(\Omega_{0, \infty}W, \Omega_{0, lip}W) & \rightarrow & \text{Hom}^{\infty}(qC, Q_{lip}) \\
 \downarrow \varphi \circ - & & \downarrow \bar{\varphi} \circ - \\
 \text{Hom}(\Omega_{0, \infty}W, \Omega_{0, \infty}W) & \rightarrow & \text{Hom}^{\infty}(qC, Q_{\infty})
 \end{array}$$

In (***) , the left vertical arrows are $\eta \mapsto \varphi \circ \eta$ at the top and $\eta \mapsto \eta \circ \varphi$ at the bottom. The upper right vertical arrow is composition with the homomorphism $\bar{\varphi}: Q_{lip} \rightarrow Q_{\infty}$ determined by φ , and the lower right vertical arrow is the inclusion of the smooth homomorphisms in the Lipschitz homomorphisms. The maps in (****) are defined analogously. One verifies that both the diagrams commute by following the reasoning (**) simultaneously for all horizontal arrows.

Let $\lambda_0 \in \text{Hom}^{lip}(qC, Q_{lip})$ correspond, under the appropriate analog of (**), to the identity map of $\Omega_{0, lip}W$. By Propositions 1.11 and 5.2, $S\Omega_{0, lip}W$ is a σ - C^* -algebra, so by Lemma 5.4, there is a Lipschitz homotopy $t \mapsto \lambda_t$ from λ_0 to a homomorphism $\lambda_1 \in \text{Hom}^{\infty}(qC, Q_{lip})$. Let $\psi: \Omega_{0, \infty}W \rightarrow \Omega_{0, lip}W$ be the homomorphism corresponding to λ_1 under (**).

The same reasoning as used to obtain (**) also shows that $\lambda = (\lambda_t)$ corresponds to a homotopy $\eta = (\eta_t)$ of homomorphisms from $\Omega_{0, lip}W$ to $\Omega_{0, lip}W$. By definition, $\eta_0 = \text{id}$, while the definition of ψ and the commutativity of the top square in (****) show that $\eta_1 = \psi \circ \varphi$. Thus $\psi \circ \varphi \simeq \text{id}_{\Omega_{0, lip}W}$.

Observe that $\bar{\varphi} \circ \lambda_0$ corresponds to $\text{id}_{\Omega_{0,\infty}W}$, by the commutativity of (***) . In particular, $\bar{\varphi} \circ \lambda_0$ is smooth. Also, λ_1 is smooth. Therefore, by part (3) of Lemma 5.4, $t \mapsto \bar{\varphi} \circ \lambda_t$ is a smooth homotopy, and therefore corresponds to a homotopy $t \mapsto \mu_t$ of endomorphisms of $\Omega_{0,\infty}W$. (One again imitates the proof of (**).) The definition of ψ , the commutativity of the bottom square in (***) , and the fact that its horizontal arrows are isomorphisms, show that $\mu_1 = \varphi \circ \psi$. Thus, μ is a homotopy from $\text{id}_{\Omega_{0,\infty}W}$ to $\varphi \circ \psi$. The proof that ψ is a homotopy inverse for φ is now complete, and therefore so is the proof that $\Omega_\infty P \simeq U_{\text{nc}}$.

The proof that $\Omega_\infty U_{\text{nc}} \simeq P$ is essentially the same: one must exchange P and U_{nc} , qC and S , RK_0 and RK_1 , and the two parts of [12, Theorem 5.4] wherever they appear. Q.E.D.

As an immediate consequence, we can remove the separability hypothesis in [12, Theorem 5.4].

6.2. Corollary. *For an arbitrary σ - C^* -algebra A , there are natural isomorphisms $\widetilde{RK}_0(A^+) \simeq [P, A^+]_+$ and $\widetilde{RK}_1(A^+) \simeq [U_{\text{nc}}, A^+]_+$.*

Proof. We only need to do the first of these, since the second was proved in [12] without separability. Now

$$\begin{aligned} \widetilde{RK}_0(A^+) &\cong \widetilde{RK}_1(\Sigma(A^+)) \cong [U_{\text{nc}}, \Sigma(A^+)]_+ \cong [U_{\text{nc}}, \Sigma(A^+)]_+^\infty \\ &\cong [\Omega_\infty U_{\text{nc}}, A]_+ \cong [P, A^+]_+, \end{aligned}$$

where the isomorphisms are, in order, Bott periodicity ([10, Theorem 3.4 (2)]), [12, Theorem 5.4 (1)], Corollary 5.6, Proposition 1.7, and $\Omega_\infty U_{\text{nc}} \simeq P$ (Theorem 6.1). Q.E.D.

6.3. Remark. It should be possible to derive the commutative case, namely $\Omega U \simeq \mathbf{Z} \times BU$ and $\Omega(\mathbf{Z} \times BU) \simeq U$, from Theorem 6.1. It is known, for example, that if M is a C^∞ manifold, then $\Omega_\infty M \simeq \Omega M$. One would then have to extend this result to cover U , which is a direct limit of C^∞ manifolds, and also the space X such that the abelianization of P is isomorphic to $C(X)$.

Our results also leave the following problems open.

6.4. Problem. Define appropriate C^∞ structures on the C^∞ loop algebras, and prove that the homotopy equivalences $\Omega_\infty U_{\text{nc}} \simeq P$ and $\Omega_\infty P \simeq U_{\text{nc}}$ respect these C^∞ structures. As a consequence, one should obtain, for example, a homotopy equivalence $\Omega_\infty(\Omega_\infty U_{\text{nc}}) \simeq U_{\text{nc}}$. Constructions with a second C^∞ loop algebra will be messier than the constructions in this paper, since we have exploited the one-dimensionality of S^1 in §4. Nevertheless, it should be possible to do this. For example, one should have $\text{Hom}_+(\Omega_\infty(\Omega_\infty, A), B) \cong \text{Hom}_+^\infty(A, C(S^2) \wedge B)$.

6.5. Problem. Prove or disprove the existence of homotopy equivalences $\Omega U_{\text{nc}} \simeq P$ and $\Omega P \simeq U_{\text{nc}}$, as conjectured in [11, §2.6]. Alternatively, with $S = C_0(S^1 - \{1\})$ and the functor W as in [12, Proposition 5.8], prove or disprove

the existence of homotopy equivalences $\Omega(W(S)^+) \simeq W(qC)^+$ and $\Omega(W(qC)^+) \simeq W(S)^+$. It is far from clear how to do this. We do not see how to generalize the results of §4 to uncountable inverse limits of C^* -algebras. Also, we have been unable to decide whether or not two homotopic projections in an uncountable inverse limit of C^* -algebras are necessarily unitarily equivalent. (Compare with [12, Lemma 2.4].)

Note, however, that ΩU_{nc} is at least weakly homotopy equivalent (compare [16, page 404]) to P in a rather strong sense: there is a map from ΩU_{nc} to P which induces an isomorphism $[\Omega U_{nc}, A]_+ \rightarrow [P, A]_+$ for all σ - C^* -algebras A . (A weak homotopy equivalence $Y \rightarrow Z$ determines a bijection $[X, Z] \rightarrow [X, Y]$ for all CW -complexes X . See [16, Corollary 23 on page 405].) A similar statement holds for ΩP and U_{nc} .

6.6. Problem. Our proof of Theorem 6.1 uses Bott periodicity for the representable K theory of σ - C^* -algebras. We believe that there should be a more direct proof of Theorem 6.1, not relying on this theorem. Such a proof would provide a new proof of Bott periodicity for C^* -algebras, which would be quite different in spirit from the existing proofs. We hope to investigate this possibility in the future.

In closing, we should also mention the possibility of generalizations to real K -theory (compare [7]), K -theory with $\mathbf{Z}/n\mathbf{Z}$ [15] or other coefficients, and functors of the form $KK^*(B, -)$ [7]. In the real case, the period is 8, so that many more classifying algebras need to be considered. Presumably one should instead try to prove directly that $\Omega_\infty^8 E \simeq E$ for some appropriate real σ - C^* -algebra E . To use the methods of this paper, one would of course need real representable K -theory.

For the theory with $\mathbf{Z}/n\mathbf{Z}$ coefficients, one should presumably use the commutative C^* -algebra Cn introduced in [15, §1], and define $\widetilde{RK}_*(A^+; \mathbf{Z}/n\mathbf{Z}) = \widetilde{RK}_*(A^+ \wedge Cn^+)$, which is the same theory as $RK_*(A \otimes Cn)$. (For the definition of $A \wedge B$, see [12, §5].) The same method used to construct ΩA gives a functor F on pro- C^* -algebras such that $\text{Hom}_+(A, B \wedge Cn^+) \cong \text{Hom}_+(FA, B)$. Then one has

$$(*) \quad \widetilde{RK}_0(A; \mathbf{Z}/n\mathbf{Z}) \cong [FP, A]_+ \quad \text{and} \quad \widetilde{RK}_1(A; \mathbf{Z}/n\mathbf{Z}) \cong [FU_{nc}, A]_+$$

for any pointed σ - C^* -algebra A . Using the methods of this paper, it should be possible to replace F in $(*)$ by an appropriate C^∞ analog F_∞ , and then to prove that

$$(**) \quad \Omega_\infty F_\infty U_{nc} \simeq F_\infty P \quad \text{and} \quad \Omega_\infty F_\infty R \simeq F_\infty U_{nc}.$$

Indeed, F_∞ ought to commute with Ω_∞ , so that $(**)$ would follow from Theorem 6.1.

The situation for the functors $KK^*(B, -)$ is much less clear. As shown in [12, §5], they have classifying algebras which are pro- C^* -algebras, but in general we do not know if these classifying algebras can be chosen to be σ - C^* -algebras.

Also, the smoothing lemmas in §5 of this paper only apply to homomorphisms from very special algebras. Progress for these functors would therefore seem to depend on a positive solution to Problem 6.5.

Added in proof. Since this paper was submitted, we have found a much more direct proof of the main theorem, as suggested in Problem 6.6.

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