TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 326, Number 1, July 1991

# CHARACTERIZATIONS OF TURBULENT ONE-DIMENSIONAL MAPPINGS VIA $\omega$-LIMIT SETS

MICHAEL J. EVANS, PAUL D. HUMKE, CHENG-MING LEE, AND RICHARD J. O'MALLEY

ABSTRACT. The structure of $\omega$-limit sets for nonturbulent functions is studied, and various characterizations for turbulent and chaotic functions are obtained. In particular, it is proved that a continuous function mapping a compact interval into itself is turbulent if and only if there exists an $\omega$-limit set which is a unilaterally convergent sequence

## 1. INTRODUCTION

Using a concept of turbulent function as introduced in [BC] (and elaborated in [C]), we show that a study of the structure and existence of certain types of $\omega$-limit sets can be used to obtain characterizations of these functions. In particular, it is seen that for nonturbulent functions the location and number of fixed points in any $\omega$-limit set is restricted. Further, it is shown that for nonturbulent functions there must be a balance, in a certain strict sense, between the up and down points in every $\omega$-limit set. This balance is based on the structure of the $\alpha$th derived sets of the $\omega$-limit set. Our results are immediately extendable to chaotic functions, and as one corollary (see Theorem 17) we obtain a result of Sarkovskii [S1] or [S2]. In addition it should be noted that Bruckner et al. [ABCP] have taken an alternate approach to the study of $\omega$-limit sets. Namely, given a set $A \subset [a, b]$, when is there a continuous map $f: [a, b] \to [a, b]$ and an $x \in [a, b]$ such that $\omega(x, f) = A$?

In the next section we include the definitions and statements of known results. The purpose is to make this paper readily accessible to the reader who has studied an introduction to the field such as [D] or [C].

In §3 we establish conditions which are shown to be sufficient for a function to be turbulent. Necessary conditions for a function to be nonturbulent are also discussed. In particular, the balance between the up and down points of such functions in every $\omega$-limit set is established in this section. Then various sufficient conditions for a function to be chaotic are considered in §4. Of interest

Received by the editors June 12, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26A18, 54H20; Secondary 58F20.

*Key words and phrases*. Turbulence, chaos, Sarkovskii stratification, $\omega$-limit set, unilaterally convergent sequence.

is the fact that the existence of a countably infinite $\omega$-limit set is such a sufficient condition.

Finally, in §5, we examine a natural extension of the technique of symbolic dynamics, as presented in [C, D] for strictly turbulent functions. Using the results obtained from this extension, it is shown that even the most restrictive sufficient conditions obtained in the previous sections are also necessary conditions. In particular, we obtain characterizations of turbulence such as the following:

A continuous function from a compact interval into itself is turbulent if and only if there exists an $\omega$-limit set which is a unilaterally convergent sequence.

At the end of the paper we augment the results discussed above by showing that any possible $\omega$-limit set behavior is realized by some continuous function in the case that the $\omega$-limit set is a convergent sequence.

## 2. Definitions, notations, and preliminary known results

Most of the terminology and notation used herein can also be found in [C]. However, since this reference may not be widely available, we have included alternate consistent references when possible. Further, although we restrict our attention to functions on a compact interval, many of the ideas are applicable to mappings on more general spaces and, as such, equivalent definitions can be found in standard texts such as [S].

For each natural number $n$ we let $f^n$ denote the $n$th iterate of $f$. It is standard to think of $f^0$ as the identity function. For a fixed $x \in I$ we denote the *trajectory of* $f$ at $x$ by $\gamma(x, f) = \{f^n(x): n = 0, 1, 2, \ldots\}$. Sometimes we will view $\gamma(x, f)$ as a set of points and sometimes as a sequence; however, the context will always indicate which is meant. In the case where $\gamma(x, f)$ has a finite range, $x$ is said to be an *eventually periodic* point. In particular, if there is a natural number $k$ such that $f^k(x) = x$, then $f$ is said to be *periodic* at $x$, and if $k$ is the least natural number such that $f^k(x) = x$, then $x$ is called a *periodic point with period* $k$. The set of all periodic points for $f$ is denoted by $P(f)$. In the special case where $k = 1$, i.e., when $f(x) = x$, $x$ is called a *fixed point of* $f$, and we let $F(f)$ denote the set of all fixed points of $f$. The *omega limit set of* $f$ at $x$ is denoted by $\omega(x, f)$ and is the set of all numbers $y$ for which there exists a sequence $\{f^{n_k}(x)\}$ such that as $k \to \infty$ we have $n_k \to \infty$ and $f^{n_k}(x) \to y$. A nonempty set $A \subset I$ is called *invariant* if $f(A) \subset A$ and *strongly invariant* if $f(A) = A$. A nonempty invariant closed set is called *minimal* if it contains no proper invariant closed subset.

Following [C], a point $x \in I$ is called an *up point* for the function $f$ provided $f(x) > x$. The collection of all up points for $f$ is denoted by $U(f)$. Similarly, $D(f)$ will denote the set of *down points* for $f$, i.e., the set of points $x$ for which $f(x) < x$.

A continuous function $f: I \to I$ is said to be *turbulent* if there exist closed subintervals $J$ and $K$ of $I$ such that $J$ and $K$ have at most one point in common and

$$J \cup K \subset f(J) \cap f(K).$$

If $J$ and $K$ can be disjoint, then $f$ is called *strictly turbulent*. The function $f$ is called *chaotic* if there exists a natural number $n$ for which $f^n$ is turbulent. This definition of turbulence seems to have first appeared in [BC]. Many of the results of [BC] appear in [C], also.

If $S$ and $T$ are subsets of the real line $R$, we let $S\backslash T$ denote the intersection of $S$ with the complement of $T$, $\overset{\circ}{S}$ denote the *interior* of $S$, $\overline{S}$ denote the *closure* of $S$, and $D(S)$ denote the *derived set* of $S$. For each natural number $n$ we let $D^n(S) = D(D^{n-1}(S))$, where we identify $D^0(S)$ with $S$. The set $D^n(S)$ is called the $n$th *derived set* of $S$. More generally, if $\alpha$ is any countable ordinal, the $\alpha$th *derived set* of $S$ is defined as $D^\alpha(S) = D(D^{\alpha-1}(S))$ if $\alpha$ is an isolated ordinal and as $D^\alpha(S) = \bigcap_{\beta<\alpha} D^\beta(S)$ if $\alpha$ is a limit ordinal.

Now we are in position to list some known results which will be used later either explicitly or implicitly. Throughout, $f$ will be a continuous function mapping a compact interval $I$ to itself.

**K1** (cf. [BC or C]). The function $f$ is turbulent if and only if there exist points $a, b, c \in I$ such that

$$f(a) = f(b) = a \quad \text{and} \quad f(c) = b,$$

and, in addition, either

$$\begin{aligned} &a < c < b, \text{ and} \\ &f(x) > a \quad \text{for } x \in (a, b), \text{ and} \\ &x < f(x) < b \quad \text{for } x \in (a, c), \end{aligned} \tag{1}$$

or the same with all the inequalities in (1) reversed.

**K2** (cf. [BC or C]). If $f$ is turbulent, then $f$ has periodic points of all periods.

**K3** (cf. [C]). If $x \in I$ and $\omega(x, f)$ is finite, then $\omega(x, f)$ is the orbit of a periodic point.

**K4** (cf. [C]). For each $x \in I$, the set $\omega(x, f)$ is strongly invariant.

**K5** (cf. [C]). Every nonempty compact invariant set contains a minimal set.

**K6** (cf. [C]). For each $x \in I$, if there exist two disjoint nonempty closed sets $A$, $B$ such that $\omega(x, f) = A \cup B$, then neither $A$ nor $B$ is invariant.

**K7** (cf. [C]). For each $x \in I$, if $\omega(x, f)$ is infinite, then no isolated point of $\omega(x, f)$ is periodic.

**K8** (cf. [S]). Every infinite minimal set is a perfect set.

**K9** (cf. [C]). If $f$ is nonturbulent, then for $x \in I$ one has

$$\sup(U(f) \cap \gamma(x, f)) \le \inf(D(f) \cap \gamma(x, f)),$$

and if $\gamma(x, f)$ contains a fixed point $z$, then

$$\sup(U(f) \cap \gamma(x, f)) \leq z \leq \inf(D(f) \cap \gamma(x, f)).$$

**K10** (cf. [C]). Suppose that $f$ is nonturbulent and let $c \in I$ and $n > 1$. If $f^n(c) \leq c < f(c)$, then $f(x) > x$ for every $x \in [f^n(c), c]$.

If $f(c) < c \leq f^n(c)$, then $f(x) < x$ for every $x \in [c, f^n(c)]$.

**K11** (cf. [C]). For every natural number $n$ and for any $x \in I$,

$$\omega(x, f) = \bigcup_{i=1}^{n} \omega(f^i(x), f^n) = \bigcup_{i=1}^{n} f^i(\omega(x, f^n)).$$

**K12** (cf. [BC or C]). For each natural number $n$ let $P_n$ denote the set of continuous functions $f$ which have a periodic point of period $n$, and let $T_n$ denote the set of all continuous $f$ such that $f^n$ is turbulent. Then

$$\begin{aligned} T_1 \subset P_3 \subset P_5 \subset \cdots \subset T_2 \subset P_6 \subset P_{10} \subset \cdots \subset T_4 \\ \subset P_{12} \subset P_{20} \subset \cdots \subset \cdots \subset P_8 \subset P_4 \subset P_2 \subset P_1, \end{aligned}$$

and $T_n = P_n$ provided $n$ is not a power of 2. Further, each of the inclusions above is strict.

## 3. Sufficient conditions for a function to be turbulent

First we prove a simple result which holds for any continuous function.

**Lemma 1.** *Let $f: I \to I$ be continuous and $x_0 \in I$.*

(1) *If $x_0 < f(x_0)$, then*

$$\sup\{f(u): u \in U(f) \cap \gamma(x_0, f)\} \geq \sup(D(f) \cap \gamma(x_0, f)).$$

(2) *If $f(x_0) < x_0$, then*

$$\inf\{f(d): d \in D(f) \cap \gamma(x_0, f)\} \leq \inf(U(f) \cap \gamma(x_0, f)).$$

*Proof.* We need only prove (1). Let $d \in D(f) \cap \gamma(x_0, f)$. Then there is a $k \geq 1$ such that $d = f^k(x_0)$, and in the sequence $\{x_0, f(x_0), f^2(x_0), \ldots, f^k(x_0)\}$, there is a last up point $f^j(x_0)$, $j < k$. We then have $f(f^j(x_0)) = f^{j+1}(x_0) \geq f^{j+2}(x_0) \geq f^k(x_0) = d$, and the result follows.

This lemma is now used to prove the following result, which extends the second part of K9 (see §2). Our focus will be on the condition that $\omega(x_0, f)$ contains a fixed point of $f$. For an extension of the first part of K9, see Proposition 5.

**Theorem 2.** *Let $f: I \to I$ be continuous and nonturbulent. If $x_0 \in I$ and $\omega(x_0, f)$ contains a fixed point $z$, then*

$$\sup(\gamma(x_0, f) \cap U(f)) \leq z \leq \inf(\gamma(x_0, f) \cap D(f)).$$

*Proof*. Let $a = \sup(\gamma(x_0, f) \cap U(f))$ and $b = \inf(\gamma(x_0, f) \cap D(f))$. Then by K9 $a \leq b$. We wish to show that $z \in [a, b]$. Suppose to the contrary that $z \notin [a, b]$. Then either $z < a$ or $z > b$, and we suppose that $z < a$. There exists a $j \geq 0$ such that $z < f^j(x_0) < f^{j+1}(x_0)$. If $k \geq j$ and $f^k(x_0) \leq z$, then by K10, we have $[f^k(x_0), f^j(x_0)] \subset U(f)$, which is impossible because $z \in [f^k(x_0), f^j(x_0)]$ with $z = f(z)$. Hence the set $\{k: k \geq 0 \text{ and } f^k(x_0) \leq z\}$ is finite. Let $x_1 = f^j(x_0)$. Then, $z \in \omega(x_1, f)$ and $z < f^n(x_1)$ for all $n \geq 0$. As $z \in \omega(x_1, f)$, there exists a sequence $\{n_k\}$ such that $f^{n_k}(x_1) \downarrow z$ and $f^{n_k}(x_1) < x_1$ for all $k \geq 1$. Since $f^{n_k}(x_1) \downarrow z$ and $z < x_1 = f^j(x_0) < f^{j+1}(x_0)$, we conclude from K10 that $(z, f^j(x_0)] \subset U(f)$. In particular, $f^{n_k}(x_1) \in U(f)$ for all $k$. As $f^{n_{k+1}}(x_1) \leq f^{n_k}(x_1)$, we cannot have $f^{n_k+n}(x_1) > f^{n_k}(x_1)$ for all $n \geq 1$. Thus there exists a smallest $l_k > 1$ such that $f^{n_k+l_k}(x_1) \leq f^{n_k}(x_1)$, $f^{n_k+l_k}(x_1) \to z$. Furthermore,

$$f^{n_k+l_k-1}(x_1) > f^{n_k}(x_1) \geq f^{n_k+l_k}(x_1) = f(f^{n_k+l_k-1}(x_1)),$$

so that $f^{n_k+l_k-1}(x_1) \in D(f)$ for each $k$. By passing to a subsequence if necessary, we can assume that $\{f^{n_k+l_k-1}(x_1)\}$ converges to some number $b_1$. Then $b_1 \geq b \geq a > z$, and $f(b_1) = \lim_{k\to\infty} f^{n_k+l_k}(x_1) = z$. As $f^{n_k+l_k-1}(x_1) \in D(f)$, it follows that $b_1 \leq \sup(\gamma(x_1, f) \cap D(f))$, so that by Lemma 1, there exists $y \in \gamma(x_1, f) \cap U(f) \subset [z, b_1]$ with $f(y) \geq b_1$. We now have $f(z) = z$, $f(b_1) = z$, and $f(y) \geq b_1$ for some $y \in [z, b_1]$, from which it follows that $f$ is turbulent. This contradiction completes the proof.

In the following theorem we consider the situation where $\omega(x_0, f)$ contains a fixed point $z$ and $x_0$ is not eventually fixed by $f$. If $x_0$ is eventually fixed by $f$, then both $\gamma(x_0, f) \cap U(f)$ and $\gamma(x_0, f) \cap D(f)$ are finite, $\omega(x_0, f) = \{z\}$ and both of the inequalities in Theorem 2 are strict or vacuous. Note also that conclusion (3) below is of particular importance when the $\omega$-limit set is infinite.

**Theorem 3.** *Suppose $f: I \to I$ is continuous and nonturbulent, $x_0 \in I$, $x_0$ is not eventually fixed, and $\omega(x_0, f)$ contains a fixed point $z$. Then*

(1) *if $\gamma(x_0, f) \cap U(f)$ is finite, then $\omega(x_0, f) = \{z\}$ and*

$$\sup(\gamma(x_0, f) \cap U(f)) < z = \inf(\gamma(x_0, f) \cap D(f));$$

(2) *if $\gamma(x_0, f) \cap D(f)$ is finite, then $\omega(x_0, f) = \{z\}$ and*

$$\sup(\gamma(x_0, f) \cap U(f)) = z < \inf(\gamma(x_0, f) \cap D(f));$$

(3) *otherwise*

$$\sup(\gamma(x_0, f) \cap U(f)) = z = \inf(\gamma(x_0, f) \cap D(f)).$$

*Proof*. Because $x_0$ is not eventually fixed by $f$ and $\omega(x_0, f)$ has a fixed point $z$, it follows that $\gamma(x_0, f) \cap U(f)$ is infinite or $\gamma(x_0, f) \cap D(f)$ is infinite. From this it is easy to establish cases (1) and (2).

In case (3) both $\gamma(x_0, f) \cap U(f)$ and $\gamma(x_0, f) \cap D(f)$ are infinite and since $z \in \omega(x_0, f)$, $z$ is a limit point of one of these sets. We assume there is a sequence $\{n_k\}$ such that $f^{n_k}(x_0) \in U(f)$ for each $n_k$ and $\{f^{n_k}(x_0)\}$ converges to $z$. From Theorem 2 it follows that $\sup(\gamma(x_0, f) \cap U(f)) \leq z$ and as $\gamma(x_0, f) \cap D(f)$ is infinite, we may further assume that for each $n_k$, $f(f^{n_k}(x_0)) \in D(f)$ But then by the continuity of $f$, $\{f^{n_k+1}(x_0)\}$ also converges to $z$. A second application of Theorem 2 now completes the proof.

**Corollary 4.** *If $f: I \to I$ is continous and $x_0 \in I$, with $\omega(x_0, f)$ containing at least two fixed points of $f$, then $f$ is turbulent.*

Not surprisingly the relationship between $U(f)$ and $D(f)$ in the set $\gamma(x_0, f)$ in K9 can be extended to the $\omega$-limit set, as well. That is the purpose of the next two propositions.

**Proposition 5.** *Let $f: I \to I$ be continuous and nonturbulent. Then*

$$\sup(U(f) \cap \omega(x_0, f)) \leq \inf(D(f) \cap \omega(x_0, f)).$$

*Proof.* If $x \in D(f) \cap \omega(x_0, f)$, then there is a sequence $\{n_k\}$ such that $\{f^{n_k}(x_0)\}$ converges to $x$. Since $f(x) < x$, it follows that for sufficiently large $n_k$, $f^{n_k+1}(x_0) < f^{f_n}(x_0)$. Hence, $x$ is the limit of down points of the trajectory of $x_0$. The analogous statement is true for up points and so the conclusion now follows from K9.

**Proposition 6.** *Let $f: I \to I$ be continuous and nonturbulent, $x_0 \in I$, and let*

$$J = (\inf[U(f) \cap \omega(x_0, f)], \sup[U(f) \cap \omega(x_0, f)]),$$
$$K = (\inf[D(f) \cap \omega(x_0, f)], \sup[D(f) \cap \omega(x_0, f)]).$$

*Then $f(x) > x$ for all $x \in J$, and $f(x) < x$ for all $x \in K$.*

*Proof.* We prove this result by contradiction and assume there is a fixed point $x$ of $f$ and two points $u_1$ and $u_2$ in $U(f) \cap \omega(x_0, f)$ with $u_1 < x < u_2$. As in the proof of Proposition 5, there is an integer $j$ such that $f^j(x_0) \in U(f)$ and $f^j(x_0) > x$. Since $u_1 \in \omega(x_0, f)$, there exists a $k > j$ such that $f^k(x_0) < x$. The result is now an immediate consequence of K10.

The previous results lead us to our first main result.

**Theorem 7.** *Let $f: I \to I$ be continuous, and suppose that there exists a point $x_0 \in I$ such that*

(i) *$\omega(x_0, f)$ is infinite,*
(ii) *$\omega(x_0, f)$ contains a fixed point $z$,*
(iii) *there is an $\varepsilon > 0$ such that either $(z, z+\varepsilon) \cap \omega(x_0, f)$ or $(z-\varepsilon, z) \cap \omega(x_0, f)$ is empty.*

*Then $f$ is turbulent.*

*Proof.* First note that since $\omega(x_0, f)$ is infinite, $z$ cannot be an isolated point of $\omega(x_0, f)$ by K7. By (iii) we may assume without loss of generality that $z$ is a right isolated point of $\omega(x_0, f)$. Then there exists a sequence $\{y_n\} \subset \omega(x_0, f)$ such that $y_n \to z$ and $y_n < z$, for every $n$. Hence there exists a subsequence $f^{n_k}(x_0)$ such that for every $k$, $f^{n_k}(x_0) < z$ and $f^{n_k}(x_0) \to z$.

Assume that $f$ is nonturbulent. Then by Theorem 2 we know that every term of $\gamma(x_0, f)$ less than $z$ is an up point of $f$ and every term of $\gamma(x_0, f)$ greater than $z$ is a down point of $f$.

Using the continuity of $f$ and the fact that $f(z) = z$, there is a $\delta_1 > 0$, such that $\delta_1 < z - \min\{t: t \in \omega(x_0, f)\}$ and if $z - \delta_1 < y < z$, then $f(y) < z + \varepsilon/2$. Further, there is a $\delta_2$ with $0 < \delta_2 < \delta_1$ such that if $z < y < z + \delta_2$, then $f(y) > z - \delta_1$.

Let $n$ be any integer such that $z - \delta_1 < f^n(x_0) < z$. Then $f^n(x_0)$ is an up point and hence $z - \delta_1 < f^n(x_0) < f^{n+1}(x_0)$. Since $\omega(x_0) \neq \{z\}$, there is a first $j > 0$ such that $f^{n+j}(x_0) > z > f^{n+j-1}(x_0) > \cdots > z - \delta_1$. Further, since $z - \delta_1 < f^{n+j-1}(x_0) < z$, we have $z < f^{n+j}(x_0) < z + \varepsilon/2$.

This establishes that the set $A = \{m_k: f^{m_k}(x_0) \in (z, z + \varepsilon/2)\}$ is an infinite set and since $\omega(x_0, f) \cap (z, z + \varepsilon)$ is empty, we must have $f^{m_k}(x_0) \to z$.

Let $M$ be chosen such that, if $m > M$ and $m \in A$, then $z < f^m(x_0) < z + \delta_2$.

Now let $B = \{n: z - \delta_1 < f^n(x_0) < z\}$. We know from the above that $B$ is also an infinite set. Let $n \in B$ with $n > M$. We claim that for every $j \geq 1$,

$$(*) \qquad z - \delta_1 < f^{n+j}(x_0) < z + \delta_2,$$

from which it will follow that $\omega(x_0, f) \subset [z - \delta_1, z + \delta_2]$, contrary to the definition of $\delta_1$. Hence the proof of this theorem will be complete if we establish inequality $(*)$.

We prove $(*)$ via mathematical induction. It is certainly true that $z - \delta_1 < f^n(x_0) < f^{n+1}(x_0) < z + \varepsilon/2$ and either $f^{n+1}(x_0) < z$ or $z < f^{n+1}(x_0) < z+\varepsilon/2$. In the first case $(*)$ is obviously true, and in the second case $n+1 > M$, so $n + 1 \in A$ and $z < f^{n+1}(x_0) < z + \delta_2$. (Note we cannot have $f^n(x_0) = z$.) This completes the proof when $j = 1$.

Assume that $(*)$ holds for $j = k$. We then have

$$z - \delta_1 < f^{n+k}(x_0) \neq z < z + \delta_2,$$

so that either $z - \delta_1 < f^{n+k}(x_0) < z$, or $z < f^{n+k}(x_0) < z + \delta_2$. In the first case, the argument above for $j = 1$ applies. In the second case, $f^{n+k}(x_0)$ is a down point of $f$ and hence

$$z - \delta_1 < f^{n+k+1}(x_0) < f^{n+k}(x_0) < z + \delta_2,$$

completing the induction. This completes the proof of Theorem 7.

Before proceeding we should remark that in light of K3 in §2 statements (i) and (ii) of Theorem 7 can be replaced by:

$\omega(x_0, f)$ contains a fixed point and at least one other point.

**Corollary 8.** *Let $f: I \to I$ be continuous and nonturbulent and let $x_0 \in I$ be such that $\omega(x_0, f)$ contains a fixed point $z$ amd $\omega(x_0, f) \neq \{z\}$. Then*

$$\sup[\omega(x_0, f) \cap U(f)] = z = \inf[\omega(x_0, f) \cap D(f)].$$

In order to fully utilize Theorem 7, we need to establish sufficient conditions for an $\omega$-limit set to contain a fixed point. This is done in Lemma 9 below. For convenience, a closed set which has only one accumulation point is called a *convergent sequence*, and if that sole accumulation point is one-sided, we call the sequence *unilaterally* convergent.

**Lemma 9.** *Let $f: I \to I$ be continuous and $x_0 \in I$ be such that $\omega(x_0, f)$ is infinite. Suppose that either*

(1) *$\omega(x_0, f)$ is a convergent sequence, or*
(2) *one of $\omega(x_0, f) \cap D(f)$ and $\omega(x_0, f) \cap U(f)$ is finite.*

*Then $\omega(x_0, f)$ contains a fixed point of $f$.*

*Proof. Case* (1): Let $z$ be the accumulation point of $\omega(x_0, f)$. Then any $y \in \omega(x_0, f)$, $y \neq z$, is an isolated point of $\omega(x_0, f)$, and hence, by K7, any such $y$ cannot be a periodic point. It follows that $\gamma(y, f)$ converges to $z$, and hence $z$ is a fixed point.

*Case* (2). The conclusion is obvious if both $\omega(x_0, f) \cap D(f)$ and $\omega(x_0, f) \cap U(f)$ are finite. Therefore we need only consider the case where $\omega(x_0, f) \cap D(f)$ is finite and $\omega(x_0, f) \cap U(f)$ is infinite. We will give a proof based on the inverse trajectory of a point. Let $y_1 \in \omega(x_0, f)$. Since $\omega(x_0, f)$ is a strongly invariant set, for each natural number $n$ we can choose a point $y_{n+1} \in \omega(x_0, f)$ such that $f(y_{n+1}) = y_n$. Note that if $y_n$ is a fixed point for any $n$, the result follows immediately. It is clear that any down point in $\omega(x_0, f)$ must be an isolated point of $\omega(x_0, f)$ and hence by K7 no such point can be a periodic point of $f$. Hence we may assume that there is an integer $N$ such that we have $y_n \in U(f)$ for all $n > N$. Consequently, $y_{n+1} < y_n$ for all $n > N$. Thus, there is a point $z \in \omega(x_0, f)$ such that $z = \lim y_n$ and it is easy to see that $z$ must be a fixed point.

As a direct consequence of Theorem 7 and Lemma 9 we have the following results.

**Corollary 10.** *Let $f: I \to I$ be continuous. Suppose that there exists a point $x_0 \in I$ such that $\omega(x_0, f)$ is a unilaterally convergent sequence. Then $f$ is turbulent.*

The case where $\omega(x_0, f)$ is merely a convergent sequence will be considered in Corollary 15.

**Corollary 11.** *Let $f: I \to I$ be continuous. Suppose that there exists a point $x_0 \in I$ such that $\omega(x_0, f)$ is infinite and one of $\omega(x_0, f) \cap D(f)$ and $\omega(x_0, f) \cap U(f)$ is finite. Then $f$ is turbulent.*

The above results point out that for a nonturbulent function $f$ there must be a certain balance between the up and down points of $f$ in every $\omega$-limit set. The sense of this balance is made precise in the following result.

**Theorem 12.** *Let $f: I \to I$ be continuous and nonturbulent. Then for each $x \in I$ and for each countable ordinal $\alpha \geq 0$,*

$$D^{\alpha}(\omega(x_0, f) \cap D(f)) = \varnothing \quad \textit{if and only if} \quad D^{\alpha}(\omega(x_0, f) \cap U(f)) = \varnothing.$$

The proof of the theorem depends on the following lemma, which we prove first.

**Lemma 13.** *Let $f: I \to I$ be continuous and let $x_0 \in I$. Then $D^{\alpha}(\omega(x_0, f)) \subset f(D^{\alpha}(\omega(x_0, f)))$ for every countable ordinal $\alpha$.*

*Proof.* The case $\alpha = 0$ is a consequence of K4. Let $\alpha > 0$ and suppose that the lemma is true for all ordinals $\beta < \alpha$, and let $x \in D^{\alpha}(\omega(x_0, f))$. Let $\{\beta_n\}$ be an increasing sequence of ordinals converging to $\alpha$ if $\alpha$ is a limit ordinal, and $\beta_n = \alpha - 1$ for every $n$ if $\alpha$ is a nonlimit ordinal. As $x \in D^{\alpha}(\omega(x_0, f))$, for each $n$ there is an $x_n \in D^{\beta_n}(\omega(x_0, f))$ within $1/n$ of $x$. By the induction assumption, there is a $y_n \in D^{\beta_n}(\omega(x_0, f))$ such that $f(y_n) = x_n$, and extracting a subsequence if necessary, we may assume that the sequence $\{y_n\}$ converges to $y$. As $\{\beta_n\}$ is increasing, $y \in \bigcap_{n=1}^{\infty} D^{\beta_n+1}(\omega(x_0, f)) = D^{\alpha}(\omega(x_0, f))$. By continuity it follows that $f(y) = x$, and this completes the proof.

*Proof of Theorem* 12. The case when $\alpha = 0$ is a consequence of Corollary 11. Proceeding inductively, we assume that the theorem is true for all $\beta < \alpha$, and initially consider the case where $\alpha$ is a limit ordinal. As $f$ is nonturbulent, $\omega(x_0, f)$ contains at most one fixed point of $f$, and this fixed point, if extant, is the $\sup(\omega(x_0, f) \cap U(f))$. Consequently, if

$$u = \inf(D^{\alpha}(\omega(x_0, f) \cap U(f))),$$

then $u$ is a minimum and is an element of $D^{\alpha}(\omega(x_0, f) \cap U(f)) \cap U(f)$. But, $u \in f(D^{\alpha}(\omega(x_0, f)))$ and as $D^{\alpha}(\omega(x_0, f))$ coincides with $D^{\alpha}(\omega(x_0, f) \cap U(f))$ with the possible exception of at most one fixed point, this is impossible.

If $\alpha$ is an isolated ordinal, then without loss of generality we may assume that $D^{\alpha-1}(\omega(x_0, f) \cap D(f))$ is finite while $D^{\alpha-1}(\omega(x_0, f) \cap U(f))$ is infinite. If $D^{\alpha}(\omega(x_0, f) \cap U(f))$ contains more than one point, then there is a point $x_1 \in D^{\alpha}(\omega(x_0, f))$ which is mapped by $f$ to the least element of $D^{\alpha}(\omega(x_0, f))$. However this implies that $x_1$ is either in $D^{\alpha}(\omega(x_0, f) \cap U(f))$ and is a down point of $f$ or a fixed point of $f$. As $D^{\alpha}(\omega(x_0, f))$ contains no down point, both options are impossible. Hence, we may assume that $D^{\alpha}(\omega(x_0, f) \cap U(f))$ consists of exactly one point, $z$. As

$$f(D^{\alpha}(\omega(x_0, f))) \supset D^{\alpha}(\omega(x_0, f))$$

and $\omega(x_0, f)$ has at most one fixed point, it follows that $z$ is fixed by $f$. Therefore, we can conclude that

$$D^{\alpha-1}(\omega(x_0, f)) = \{a_n : n = 1, 2, \ldots\} \cup \{z\} \cup \{b_n : n = 1, 2, \ldots, k\},$$

where $\{a_n\}$ is an increasing sequence of up points converging to $z$, $z$ is a fixed point, and each $b_n$ is a down point of $f$. We consider the $b_n$ to be ordered so that $f(b_1) = a_1$, $f(b_i) \in \{a_n\}$ for $i = 1, 2, \ldots, k^* \leq k$, and $f(b_i) \notin \{a_n\}$ for $i > k^*$. There exists a first index $n_1$ such that $f^{n_1}(a_1) \neq a_{n_1+1}$, because if $f^n(a_1) \neq a_{n+1}$ for every $n$, then $f^{-1}(b^*) = \varnothing$, where $b^* = \max\{b_1, b_2, \ldots, b_k\}$. Now,

$$a_{n_n+1} \notin f(\{a_n : n = 1, 2, \ldots\}) = \{f(b_1), f(a_1), f^2(a_1), \ldots, f^{n_1-1}(a_1)\}.$$

As $\{a_n\} \subset U(f)$, $a_{n_1+1} \notin f(\{a_n\})$, and as a consequence, there is an $i_2 \in \{1, 2, \ldots, k^*\}$ such that $f(b_{i_2}) = a_{n_1+1}$. An argument analogous to that above shows that there is a first index $n_2$ such that $f^{n_2}(a_{n_1+1}) \neq a_{n_1+n_2+1}$. Again,

$$a_{n_1+n_2+1} \notin f(\{a_n\}),$$

and as $\{a_n\} \subset U(f)$, there is an $i_3 \in \{1, 2, \ldots, k^*\}$ such that $f(b_{i_3}) = a_{n_1+n_2+1}$. Continuing inductively, we obtain a point $a^* = a_{n_1+\cdots+n_{k^*}+1}$ such that $a^* \notin f(\{a_n\})$, and also that $a^* \neq f(b_i)$ for $i = 1, 2, \ldots, k^*$. However,

$$D^{\alpha-1}(\omega(x_0, f)) \subset f(D^{\alpha-1}(\omega(x_0, f))),$$

which is a contradiction. This completes the proof of Theorem 12.

## 4. Sufficient conditions for a function to be chaotic

In §3 we found several conditions which are sufficient to prove turbulence, and in §5 we verify that these are also necessary. In this section we use the results of §3 to develop conditions sufficient to prove a given function is chaotic.

**Theorem 14.** *Let $f: I \to I$ be continuous, and suppose that there exists a point $x_0 \in I$ such that*

(i) *$\omega(x_0, f)$ is infinite, and*
(ii) *$\omega(x_0, f)$ contains a fixed point $z$.*

*Then $f^2$ is turbulent.*

*Proof.* For convenience, we let $I = [0, 1]$. Let $z$ be a fixed point in $\omega(x_0, f)$, and suppose that the function $f^2$ is nonturbulent. Then $f$ is also nonturbulent. As $\omega(x_0, f)$ contains points other than $z$, the terms of $\gamma(x_0, f)$ are all distinct, and it follows from Theorem 2 that both $[0, z] \cap \gamma(x_0, f) \subset U(f)$ and $[z, 1] \cap \gamma(x_0, f) \subset D(f)$. As $\omega(x_0, f)$ is infinite, one of the sets $[0, z] \cap \omega(x_0, f)$ or $[z, 1] \cap \omega(x_0, f)$ must contain at least two points. Suppose $y_1, y_2 \in \omega(x_0, f)$ with $y_1 < y_2 < z$. Then there exists an integer $n \geq 0$ such that $y_1 < f^n(x_0) < f^{n+1}(x_0)$. Then there exists $k > n + 1$ such that $f^k(x_0) < f^n(x_0)$ (and $f^k(x_0) < f^{k+1}(x_0)$). Therefore, by Theorem 2 in Chapter II of [C], the trajectory $\gamma(f^n(x_0), f)$ satisfies the condition that $f^i(f^n(x_0)) < f^j(f^n(x_0))$ for all even $i$ and all odd $j$. Since $f^n(x_0) \in [0, z]$, we conclude that $f^{2m}(f^n(x_0)) \in U(f)$ and $f^{2m}(f^{n+1}(x_0)) \in D(f)$ for all $m \geq 0$, and hence both $\gamma(f^n(x_0), f^2) \subset [0, z)$ and $\gamma(f^{n+1}(x_0), f^2) \subset (z, 1]$. Applying K11, we obtain

$$\begin{aligned} z \in \omega(x_0, f) &= \omega(f^n(x_0), f^2) \cup \omega(f^{n+1}(x_0), f^2) \\ &= \omega(f^n(x_0), f^2) \cup f(\omega(f^n(x_0), f^2)). \end{aligned}$$

Hence, both $\omega(f^n(x_0), f^2)$ and $\omega(f^{n+1}(x_0), f^2)$ are infinite, and either $z \in \omega(f^n(x_0), f^2)$ or $z \in \omega(f^{n+1}(x_0), f^2)$. Suppose $z \in \omega(f^n(x_0), f^2)$. As $z$ is a fixed point of $f^2$, it follows from Theorem 2 that $\gamma(f^n(x_0), f^2) \cap D(f^2) \subset [z, 1]$. However, $\gamma(f^n(x_0), f^2) \subset [0, z)$, so that $\omega(f^n(x_0), f^2) \cap D(f^2)$ is finite. Hence by Corollary 11, $f^2$ is turbulent. This contradiction completes the proof.

As a consequence of Theorem 14 and Lemma 9, we have the following result.

**Corollary 15.** *Let $f: I \to I$ be continuous. If there exists a point $x_0 \in I$ such that $\omega(x_0, f)$ is a convergent sequence, then $f^2$ is turbulent.*

In §5, we prove the converse of Corollary 15 (see Theorem 24). Hence by the stratification result K12, there is a function $f$ such that $f^{2n-1}$ is nonturbulent for all positive integers $n$, but for which there is an $x_0$ where $\omega(x_0, f)$ is a convergent sequence. Indeed, such examples are not hard to construct. Thus the conclusions of the above two results are the best possible in the sense of the stratification result K12.

It is useful to combine Theorems 7 and 14 in a concise fashion which describes the role of periodic points in $\omega$-limit sets. In fact, taking K3 and K12 into account, we have the following result.

**Summary Theorem 16.** *Let $f: I \to I$ be continuous, and $x_0 \in I$ be such that $\omega(x_0, f)$ contains a periodic point $z$ of $f$ of period $n$. Then one of the following must hold:*

(a) $\omega(x_0, f) = \{z, f(z), \dots, f^{n-1}(z)\}$;

(b) *$\omega(x_0, f)$ is infinite and $f^{2n}$ is turbulent. Moreover, $f^n$ is turbulent if some element of the orbit $\{z, f(z), \dots, f^{n-1}(z)\}$ is a one-sided accumulation point of the set $\omega(f^i(x_0), f^n)$ for some $i$.*

**Theorem 17.** *Let $f: I \to I$ be continuous, and let $x_0 \in I$ be such that one of the following conditions is satisfied:*

(1) *$\omega(x_0, f)$ is infinite and contains a periodic point;*

(2) *$\omega(x_0, f)$ contains two periodic orbits;*

(3) *$\omega(x_0, f)$ contains a periodic point of period not equal to a power of 2;*

(4) *$\omega(x_0, f)$ is countably infinite.*

*Then $f$ is chaotic.*

*Proof.* Cases (1) and (2) are consequences of Summary Theorem 16, and case (3) follows from K12. Suppose that (4) holds. By K4 and K5, $\omega(x_0, f)$

contains a minimal subset $M$. Either $M$ is finite or $M$ is perfect by K8. Yet every perfect set is uncountable. It follows that $M$ must be finite, and thus is a periodic orbit. The result follows from case (1).

The following theorem is again presented to make the previous results concise. Type $2^\infty$ means $f$ possesses periodic orbits of period $2^n$ for $n = 0, 1, 2, 3, \ldots,$ and no other periods.

**Summary Theorem 18.** *Let $f: I \to I$ be continuous, and let $x_0 \in I$ be such that $\omega(x_0, f)$ is infinite. Then exactly one of the following holds true:*

(a) *$f$ is chaotic.*

(b) *$f$ is of type $2^\infty$ and $\omega(x_0, f)$ is uncountably infinite without any periodic points.*

*Proof*. Suppose that $f$ is neither chaotic nor of type $2^\infty$. Then there is an $M > 0$ such that for every $x \in P(f)$ the period of $x$ does not exceed $2^M$. Hence, $P(f)$ is closed. By hypothesis, $\omega(x_0, f) \cap P(f)$ is empty. By [C, Chapter IV, Lemma 7], if $J$ is any interval disjoint from $P(f)$, then $\omega(x_0, f)$ has at most one point in $J$. This implies that $\omega(x_0, f)$ is countably infinite, and hence $f$ is chaotic, which is a contradiction.

We remark that part (1) of Theorem 17 and part (b) of Summary Theorem 18 were considered by Sarkovskii in [S1].

The identity $\omega(x_0, f) = \omega(x_0, f^2) \cup f(\omega(x_0, f^2))$ was used in the proof of Theorem 14 to establish that $f^2$ is turbulent. Also, if $\omega(x_0, f)$ contains a fixed point, then that point is in the intersection of $\omega(x_0, f^2)$ and $f(\omega(x_0, f^2))$. Surprisingly, the intersection of these two sets can be useful in determining when an odd iterate of $f$ is turbulent.

**Theorem 19.** *Let $f: I \to I$ be continuous, and let $x_0 \in I$ be such that $\omega(x_0, f^2) \cap f(\omega(x_0, f^2))$ contains at least two points. Then $f$ has a periodic point of odd period $q > 1$ and hence $f^q$ is turbulent.*

*Proof*. For simplicity we assume that $I = [0, 1]$. Suppose that $A = \omega(x_0, f^2) \cap f(\omega(x_0, f^2)) \supset \{a, b\}$. If $\omega(x_0, f) = \{a, b\}$, then $f(a) = b$ and $f(b) = a$, implying that $\omega(x_0, f^2) \cap f(\omega(x_0, f^2))$ is empty. Hence $\omega(x_0, f)$ contains at least three points.

If $f$ is turbulent, then $f$ has cycles of every odd length. Thus we may assume $f$ is nonturbulent. As $A \subset \omega(x_0, f)$ and $f$ is nonturbulent, $A$ contains at most one fixed point. As $A$ has at least two points, is strongly invariant under $f$, and is closed, it is easy to see that $A$ contains points from both $U(f)$ and $D(f)$. Suppose $a \in A \cap U(f)$, $b \in A \cap D(f)$, and $c \in \omega(x_0, f) \setminus \{a, b\}$. We consider three cases, depending on the relative positions of these points.

*Case* 1. Suppose $a < b < c$. Let $X$ and $Y$ be disjoint intervals containing $b$ and $c$ respectively such that $X \subset D(f)$. There exist an even integer $n$ and an odd integer $m$ such that both $f^n(x_0)$ and $f^m(x_0)$ belong to $X$. As $c \in \omega(x_0, f)$, there is an integer $j > \max\{n, m\}$ such that $f^j(x_0) \in Y$. Then both

$$f^{n+1}(x_0) < f^n(x_0) < f^j(x_0) \tag{1}$$

and

$$f^{m+1}(x_0) < f^m(x_0) < f^j(x_0). \tag{2}$$

Either $j-n$ or $j-m$ is odd and we assume the latter. Setting $x_1 = f^m(x_0)$, (2) becomes

$$f(x_1) < x_1 < f^{j-m}(x_1).$$

It now follows from [C, Chapter II, Theorem 1] (cf. also Theorem A in [BC]) that $f$ contains a $j-m$ cycle.

*Case* 2. Suppose $c < a < b$. This case reduces to Case 1, by considering $g(x) = f(1-x)$ as we have assumed that $I = [0, 1]$.

*Case* 3. Suppose $a < c < b$. Let $X$, $Y$, and $Z$ be pairwise disjoint intervals containing $a$, $c$, and $b$ respectively such that $f(X) \subset U(f)$ and $f(Z) \subset D(f)$. Let $x_1 = f^j(x_0) \in Y$. Then either $x_1 \in U(f)$ or $x_1 \in D(f)$ and we suppose the former. Let $n > j$ be odd and $m > j$ be even such that both $f^n(x_0)$ and $f^m(x_0)$ are in $X$. One of $n-j$ or $m-j$ is odd and we suppose the latter. Then $f^{m-j}(x_1) < x_1 < f(x_1)$ and, as in Case 1, it follows that $f$ has an $m-j$ cycle. Thus, the proof is complete.

## 5. Necessary conditions and symbolic dynamics

To obtain some necessary conditions for a function to be turbulent, we wish to present a scheme of symbolic dynamics that can be applied to turbulent functions. This scheme can be viewed as a modification of the one presented in [C, Chapter II] for strictly turbulent functions. Our scheme will require the following characterization of turbulent functions which is slightly different from K1. For convenience, we will let $I$ denote the unit interval $[0, 1]$.

Let $g: I \to I$ be turbulent. Then, upon setting $f(x) = g(x)$ or $f(x) = 1 - g(1-x)$, we will have the existence (cf. K1) of four points $a, b, c, c^*$ such that $a < c \le c^* < b$ and

(i) $f(a) = a = f(b)$,

(ii) $f(c) = f(c^*) = b$,

(iii) $F(f) \cap (a, c)$ is empty,

(iv) $b \notin f((a, c)) \cup f((c^*, b))$ and $a \notin f((c^*, b))$.

Now we begin to specify the notation needed. Let $\Sigma_n$ denote the set of all $n$-long sequences of 0's and 1's and let $\Sigma$ denote the set of all sequences of 0's and 1's. If $\alpha \in \Sigma$, we denote the restriction of $\alpha$ to the first $n$ coordinates by $\alpha|_n$. If $\alpha \in \Sigma$ (or $\Sigma_k$), we define the shift mapping $\sigma$ by $\sigma(\alpha(n)) = \alpha(n+1)$ for $n = 1, 2, \ldots$ (or $n = 1, 2, \ldots, k-1$). Note that $\sigma$ maps $\Sigma$ to $\Sigma$ and $\Sigma_k$ to $\Sigma_{k-1}$. Inductively we shall associate a closed subinterval of $[a, b]$ with each finite sequence $\alpha \in \bigcap_{n=0}^{\infty} \Sigma_n$ in such a way that if $m < n$ and $\beta \in \Sigma$, the interval associated with $\beta|_n$ is contained in the interval associated with $\beta|_m$.

To begin the induction we first define $J_\varnothing = [a, b]$, $J_0 = [a, c]$, and $J_1 = [c^*, b]$. Then $J_0$ and $J_1$ are nonoverlapping subintervals of $J_\varnothing$ and each shares exactly one endpoint with $J_\varnothing$. Further, if $x \in (a, c) \cup (c^*, b)$, then $f(x) \in (a, b)$. These properties of descendant intervals will be maintained throughout the construction.

Now suppose that for $k = 0, 1, \ldots, n-1$, $J_{\alpha|_k 0}$ and $J_{\alpha|_k 1}$ have been defined in such a way that:

(1) For each $\alpha \in \Sigma$ and each $k = 0, 1, \ldots, n-1$, $J_{\alpha|_k 0}$ and $J_{\alpha|_k 1}$ are closed, nonoverlapping, and share exactly one endpoint with $J_{\alpha|_k}$;

(2) $J_{\alpha|_k 0} \cup J_{\alpha|_k 1} \subset J_{\alpha|_k}$ for $\alpha \in \Sigma$ and $k = 0, 1, \ldots, n-1$;

(3) $f(J_{\alpha|_k}) = J_{\sigma(\alpha|_k)}$ and $f(\mathring{J}_{\alpha|_k}) = \mathring{J}_{\sigma(\alpha|_k)}$ for $\alpha \in \Sigma$ and $k = 1, 2, \ldots, n$.

For every $\alpha \in \Sigma$ we must define $J_{\alpha|_{(n+1)}}$ in such a way as to satisfy (1), (2) for $k = n$ and (3) for $k = n+1$. Let $\alpha \in \Sigma$. Then $f(J_{\alpha|_n}) = J_{\sigma(\alpha|_n)}$ and both $J_{\sigma(\alpha|_n)0}$ and $J_{\sigma(\alpha|_n)1}$ are contained in $J_{\sigma(\alpha|_n)}$. As a first case, suppose that $\alpha(n+1) = 0$, and denote $J_{\alpha|_n}$ by $[A, B]$ and $J_{\sigma(\alpha|_n)}$ by $[A^*, B^*]$. From (1) above we know that exactly one of $A^*$ or $B^*$ is an endpoint of $J_{\sigma(\alpha|_n)0}$. We suppose the former and denote $J_{\sigma(\alpha|_n)0}$ by $[A^*, C^*]$. It follows from (2) and (3) that exactly one point from $[A, B]$ maps by $f$ to $A^*$ and that point is either $A$ or $B$. If $f(B) = A^*$, define $J_{\alpha|_{(n+1)}} = [C, B]$, where $C = \max\{x \in [A, B]: f(x) = C^*\}$. If $f(A) = A^*$, define $J_{\alpha|_{(n+1)}} = [A, C]$, where $C = \min\{x \in [A, B]: f(x) = C^*\}$. It is easy to see that (1), (2), and (3) are satisfied for $\alpha \in \Sigma$ and all appropriate $k = 0, 1, 2, \ldots, n+1$.

For each $\alpha \in \Sigma$, define $I_\alpha = \bigcap_{n=0}^{\infty} J_{\alpha|_n}$. Then, for every $\alpha \in \Sigma$ we have

(4) $I_\alpha$ is a closed interval (or a point);

(5) $f(I_\alpha) = I_{\sigma(\alpha)}$.

**Proposition 20.** *If $\alpha \in \Sigma$ is such that $\alpha(n) = 0$ for all $n \geq N$, then $I_\alpha$ is a point.*

*Proof.* This is true for $N = 1$ because if $\alpha = \overline{0}$ (the sequence consisting of all 0's), then the right endpoints of $J_{\alpha|_n}$ form a decreasing sequence of up points of $f$. In addition, $f(I_\alpha) \subset I_\alpha$ and so the limit point of this sequence converges to a fixed point of $f$. As there is but one fixed point of $f$ in $J_0$, the result follows. If $I_{1\overline{0}}$ were a nondegenerate interval, then $J_1$ would contain a nondegenerate interval which maps to $a$. However, $b$ is the sole point in $J_1$ mapping to $a$ by (3). Hence, $I_{1\overline{0}}$ is a single point, namely $\{b\}$. The result now follows by using (3) inductively.

Now we define a relation $\approx$ on $\Sigma$ by defining $\alpha \approx \beta$ if $I_\alpha \cap I_\beta$ is not empty.

**Proposition 21.** *The relation $\approx$ is an equivalence relation, and if $\alpha \approx \beta$, then either $\alpha = \beta$ or there is an $n$ such that $\alpha(m) = \beta(m)$ for all $m < n$, $\alpha(n) \neq \beta(n)$, $\alpha(n+1) = \beta(n+1) = 1$, and $\alpha(m) = \beta(m) = 0$ for all $m \geq n+2$.*

*Proof*. For convenience, we write $\alpha_k = \alpha(k)$ and $\beta_k = \beta(k)$. Suppose $\alpha \approx \beta$ and $\alpha \neq \beta$. Let $n$ be the first coordinate where $\alpha_n \neq \beta_n$, and suppose first that $n = 1$. Without loss of generality we may assume $\alpha_1 = 0$. Then $\beta_1 = 1$ and $J_{\alpha_1\alpha_2} \cap J_{\beta_1\beta_2} \neq \varnothing$. Since $J_{\alpha_1\alpha_2} \subset J_{\alpha_1}$ and $J_{\beta_1\beta_2} \subset J_{\beta_1}$, the only possible common point of $J_{\alpha_1\alpha_2}$ and $J_{\beta_1\beta_2}$ is $c$. Since $f(J_{\alpha_1\alpha_2}) = J_{\alpha_2}$ and $f(J_{\beta_1\beta_2}) = J_{\beta_2}$ it follows that $J_{\alpha_2}$ and $J_{\beta_2}$ both contain $b = f(c)$. Hence $\alpha_2 = \beta_2 = 1$. Thus $c = c^*$ and $J_{01} \cap J_{11} = \{c\}$. Since also $J_{01\alpha_3} \cap J_{11\beta_2} \neq \varnothing$, it follows by similar reasoning that $\alpha_3 = \beta_3 = 0$ and $J_{010} \cap J_{110} = \{c\}$. In fact, $\alpha_k = \beta_k$ for all $k > 2$.

If $n > 1$ then $I_{\sigma^{n-1}(\alpha)} \cap I_{\sigma^{n-1}(\beta)} \neq \varnothing$, since $f^{n-1}(I_\alpha) \cap f^{n-1}(I_\beta) \neq \varnothing$, and so by what we have already proved

$$\alpha = \alpha_1\alpha_2 \cdots \alpha_{n-1}\alpha_n 1\overline{0}, \qquad \beta = \alpha_1\alpha_2 \cdots \alpha_{n-1}\beta_n 1\overline{0},$$

where $\alpha_n \neq \beta_n$. It now follows from Proposition 20 that $I_\alpha = \{p\} = I_\beta$. Thus $\alpha \approx \beta$ if and only if $I_\alpha = I_\beta$. Hence $\approx$ is an equivalence relation.

Let $\Sigma^{\approx}$ denote the quotient space of $\Sigma \bmod \approx$. The equivalence class containing $\alpha$ is denoted by $\overset{\approx}{\alpha}$. If $\alpha \approx \beta$ and $\alpha \neq \beta$, then $I_\alpha = I_\beta = \{p\}$ so that both $I_{\sigma(\alpha)}$ and $I_{\sigma(\beta)}$ contain $f(p)$. Hence, $I_{\sigma(\alpha)} \cap I_{\sigma(\beta)}$ is nonempty and so $\sigma(\alpha) \approx \sigma(\beta)$. From this it follows that the shift operator, $\sigma$, factors through the quotient map; that is, $\overset{\approx}{\sigma}: \Sigma^{\approx} \to \Sigma^{\approx}$ defined by $\overset{\approx}{\sigma}(\overset{\approx}{\alpha}) = \overset{\approx}{\sigma(\alpha)}$ is well defined and continuous. The topology on $\Sigma^{\approx}$ is generated by the total order $<$ on $\Sigma^{\approx}$ defined by

$$\overset{\approx}{\alpha} < \overset{\approx}{\beta} \quad \text{if } \max(I_\sigma) < \min(I_\beta).$$

Let $E = \bigcup_{\alpha\in\Sigma} I_\alpha = \bigcap_{n=1}^{\infty} \bigcup_{\alpha\in\Sigma_n} J_\alpha$ with the induced topology from the real line. We define $x \cong y$ if there is an $\alpha \in \Sigma$ such that $x, y \in I_\alpha$. It follows from Proposition 21 that $\cong$ is transitive, and hence, $\cong$ is an equivalence relation. Further, if $x \cong y$, then $(x, y) \subset E$ and $z \cong x$ for every $z \in (x, y)$. Consequently, the topology on the quotient space $E \bmod \cong$, denoted $E^{\cong}$, is generated by the total order

$$\overset{\cong}{x} < \overset{\cong}{y} \quad \text{if whenever } x_1 \in \overset{\cong}{x} \text{ and } y_1 \in \overset{\cong}{y} \text{ we have } x_1 < y_1.$$

If $x \cong y$ and $x \neq y$, then there is a unique $\overset{\approx}{\alpha} \in \Sigma^{\approx}$ such that $x, y \in I_\alpha$. It follows from (5) that

$$\overset{\cong}{f(x)} = \overset{\cong}{f(y)} = I_{\sigma(\alpha)}.$$

Hence for each $x \in E$ defining $\overset{\cong}{f}(\overset{\cong}{x}) = \overset{\cong}{f(x)}$, we see that $\overset{\cong}{f}$ is well defined from $E^{\cong}$ to $E^{\cong}$.

Now define $\Gamma: E^{\cong} \to \Sigma^{\approx}$ by $\Gamma(\overset{\cong}{x}) = \overset{\approx}{\alpha}$ whenever $x \in I_\alpha$. If $x \neq y$ and $y \in \overset{\cong}{x}$, then there is a unique $\alpha$ such that both $x$ and $y$ are in $I_\alpha$. Consequently, $\Gamma$ is well defined. Further, if $\alpha \neq \beta$ but $\overset{\approx}{\alpha} = \overset{\approx}{\beta}$, then $I_\alpha = I_\beta = \{p\}$.

Hence, $\Gamma^{-1}(\overset{\approx}{\alpha}) = \overset{\cong}{p} = \{p\}$ which shows that $\Gamma$ is one-to-one. Also, $\Gamma$ is order preserving, for if $\overset{\cong}{x} < \overset{\cong}{y}$, $x \in I_\alpha$, and $y \in I_\beta$, then $\max(I_\alpha) < \min(I_\beta)$. As $\Gamma$ clearly maps onto $\Sigma^{\approx}$ we conclude that $\Gamma$ is a homeomorphism. As a final remark we note that $\overset{\cong}{f}$ is conjugate to $\overset{\approx}{\sigma}$ via the homeomorphism $\Gamma$. This is because

$$\Gamma^{-1} \circ \overset{\approx}{\sigma} \circ \Gamma(\overset{\cong}{x}) = \Gamma^{-1} \circ \overset{\approx}{\sigma}(\overset{\approx}{\alpha}) = I_{\sigma(\alpha)},$$

where $x \in I_\alpha$. Also, for $x \in I_\alpha$

$$\overset{\cong}{f}(\overset{\cong}{x}) = \overset{\cong}{\widetilde{f(x)}} = I_{\sigma(\alpha)}.$$

Theorem 22 below is a summary of this discussion.

**Theorem 22.** *The function* $\Gamma: E^{\cong} \to \Sigma^{\approx}$ *defined by* $\Gamma(\overset{\cong}{x}) = \overset{\approx}{\alpha}$, *where* $x \in I_\alpha$, *is a homeomorphism. Further, the following diagram commutes*:

$$\begin{tikzcd}
E^{\cong} \arrow[r, "\Gamma"] \arrow[d, "\overset{\cong}{f}"] & \Sigma^{\approx} \arrow[d, "\overset{\approx}{\sigma}"] \\
E^{\cong} \arrow[r, "\Gamma"] & \Sigma^{\approx}
\end{tikzcd}$$

It should be noted that if a semiconjugacy was all that was desired then the above approach could be modified: instead of the quotient space $E^{\cong}$, only the set $X$ of all endpoints of all $I_\alpha$ enters into the picture (cf. [C]).

Finally, Theorem 22 enables us to use the structure of $\Sigma$ to prove that the sufficient conditions considered earlier are necessary.

**Corollary 23.** *If* $f: I \to I$ *is turbulent, there is a point* $x_0 \in [0, 1]$ *such that* $\omega(x_0, f)$ *is a unilaterally convergent sequence.*

*Proof.* Let $\alpha_0 = (01001000100001\cdots) \in \Sigma$. Then

$$\omega(\alpha_0, \sigma) = \{\overline{0}, 1\overline{0}, 01\overline{0}, 001\overline{0}, \dots\},$$

which is a convergent sequence in $\Sigma$. It is easy to see that $\omega(\overset{\approx}{\alpha}_0, \overset{\approx}{\sigma})$ consists of the equivalence classes of the elements of $\omega(\alpha_0, \sigma)$ and is a unilaterally convergent sequence in $E^{\cong}$, and hence, $\omega(x_0, f)$ has the same property in $[0, 1]$, whatever $x_0$ is selected from $I_{\alpha_0}$.

Further, combining Corollaries 10 and 15, we have

**Theorem 24.** *Let* $f: I \to I$ *be continuous. Then*

(i) $f$ *is turbulent if and only if there is a point* $x_0 \in I$ *such that* $\omega(x_0, f)$ *is a unilaterally convergent sequence.*

(ii) $f^2$ *is turbulent if and only if there is a point* $x_0 \in I$ *such that* $\omega(x_0, f)$ *is a convergent sequence.*

The symbolic dynamics of Theorem 22 can be used to obtain other necessary conditions for a function to be turbulent. For example, we have the following result, which weakens the hypothesis of Proposition 2 in Chapter II of [C] from strictly turbulent to just turbulent.

**Corollary 25.** *If $f$ is turbulent, then there is a point $x_0 \in I$ such that $\omega(x_0, f)$ is uncountably infinite, and for each positive integer $m$, contains a periodic point of period $m$ or $2m$.*

Using Theorem 22, the proof of the corollary is identical to that of the aforementioned proposition in [C] and, as such, is omitted here. Instead, we give another result to further reveal the richness of possibilities for itineraries of the shift operator in the sequence space $\Sigma$. It follows, then, via Theorem 22 that the itineraries for any turbulent function enjoy an analogous richness. First, we establish some notation. A sequence which begins with a certain initial segment $s$ and terminates with all 0's is denoted by $(s\overline{0})$, and a sequence which begins with $n$ repeated copies of $s$ and terminates in all 0's is denoted by $(s_n\overline{0})$. If $n \in \mathbf{N}$, $p_n$ denotes the segment $0_n 1$, where $\mathbf{N}$ denotes the set of all the natural numbers.

**Theorem 26.** *For every countable ordinal $\beta > 0$ and every infinite subset $A \subset \mathbf{N}$, there exists a point $s^*(\beta, A) = s^* \in \Sigma$ such that:*

(1) $D^\beta(\omega(s^*, \sigma)) = \{(\overline{0})\}$.

(2) *If $s \in \omega(s^*, \sigma)$, then $s$ terminates in $\overline{0}$ and $(0_n s) \in \omega(s^*, \sigma)$ for each positive integer $n$.*

(3) *If $n \notin A$, then $s^*$ does not contain the segment $1p_n$.*

(4) *For every positive integer $n$, $(0_n 1\overline{0}) \in \omega(s^*, \sigma)$.*

(5) *The point $s^*$ can be written as an aglutination of $p_n$, where $n \in A$.*

*Proof*. The proof uses transfinite induction on $\beta$. Suppose that $\beta = 1$ and enumerate $A = \{n_1, n_2, \ldots\}$. Let $s^* = (p_{n_1} p_{n_2} \cdots)$. Then

$$\omega(s^*, \sigma) = \{(0_n 1\overline{0}) : n = 0, 1, 2, \ldots\} \cup \{(\overline{0})\},$$

and as such, conditions (1) through (5) above are satisfied.

Now, suppose $\beta > 1$, $A = \{n_1, n_2, \ldots\}$, and the theorem holds for all $\alpha < \beta$. As a first case we assume $\beta = \gamma + 1$ is an isolated ordinal. By the induction assumptions, there is an $s' = s'(\gamma, A)$ satisfying conditions (1) through (5) for the ordinal $\gamma$ and the set $A$. By condition (5) we can write $s' = (p_{m_1} p_{m_2} \cdots)$, where each $m_i \in A$. We define $b_k$ to be that segment of $s'$ beginning with $p_{m_k}$ and ending with $p_{m_{k^*}}$, where $k^*$ is the first index larger than $k$ with $m_{k^*} > k^2$. We define

$$s^*(\beta, A) = s^* = (p_{n_1} b_1 p_{n_1} b_2 p_{n_2} b_3 p_{n_1} b_4 p_{n_2} b_5 p_{n_3} b_6 p_{n_1} \cdots).$$

Then,

$$\omega(s^*, \sigma) = \{(p_n s) : s \in \omega(s', \sigma) \text{ and } n \in \mathbf{N}\}.$$

As such, $D((\omega^*, \sigma)) = \omega(s', \sigma)$ which implies that

$$D^\beta(\omega(s^*, \sigma)) = D^\gamma(\omega(s', \sigma)) = \{(\overline{0})\}.$$

The remaining conditions are easily verified.

The case when $\beta$ is a limit ordinal is slightly more complicated. Suppose that for each $m$, $\alpha_m < \beta$ and that $\beta = \lim_{m\to\infty} \alpha_m = \beta$. Partition $A$ into countably many disjoint countable sets $A_m$. Applying the induction assumptions, there are points $s'_m(\alpha_m, A_m) = s'_m$ satisfying conditions (1) through (5). From condition (5) we can write $s'_m = (p^m_{m_1} p^m_{m_2} \cdots)$, where each $p^m_{m_i} \in A_m$. As in the isolated ordinal case we utilize certain segments of the $s'_m$. Denote by $b^m_k$ that segment of $s'_m$ which begins with $p^m_{m_k}$ and ends with $p^m_{m_{k^*}}$, where $k^*$ is the first index larger than $k$ with $m_{k^*} > k^2$. Now set

$$s^*(\beta, A) = s^* = (b^1_1 b^2_1 b^1_2 b^3_1 b^2_2 b^1_3 \cdots).$$

From this definition it is easy to see that $\bigcup_{n=1}^{\infty} \omega(s', \sigma) \subset \omega(s^*, \sigma)$. We claim that these two sets are equal. To demonstrate this, let $s_0 \in \omega(s^*, \sigma)$, and suppose $\{\sigma^{n_k}(s^*)\} \to s_0$. If $s_0$ contains fewer than two 1's, then $s_0$ is in each set $\omega(s^*, \sigma)$. Hence, we may assume that $s_0$ contains two 1's. We suppose that the first 1 occurs in the $j$th coordinate and that the second occurs in the $(j+i)$th coordinate. It follows that $s^*$ contains the fixed segment $1p_i$ infinitely often. As each segment $b^m_k$ terminates in a block of at least $k^2$ zeros followed by a lone one, we can conclude that there are infinitely many segments $b^m_k$ which contain the subsegment $1p_i$. However, $1p_i \in A_m$ for at most one index $m$, and it follows that there is a unique index, $m_0$, such that $1p_i \in A_{m_0}$. From this it is easy to see that $s_0$ has the form $(0_f s)$ for some nonnegative integer $f$ and some $s \in \omega(s'_{m_0}, \sigma)$. Condition (2) implies that $s_0$ itself is in $\omega(s'_{m_0}, \sigma)$, and this completes the proof of the theorem.

**Corollary 27.** *Let $f: I \to I$ be turbulent and let $\alpha$ be a countable ordinal number. Then there is a point $x_\alpha \in I$ such that $D^\alpha(\omega(x_\alpha, f))$ is a point.*

Knowing that the existence of a unilaterally convergent $\omega$-limit set characterizes turbulent functions, it is interesting to see how a continuous function can be constructed, given information about the desired behavior of that function on a set which is a convergent sequence. We end this paper by giving such a construction.

**Example 28.** *Let $p: \mathbf{N} \cup \{0\} \to \mathbf{N} \cup \{0\}$ be such that $p$ is onto, has no finite cycles other than $p(0) = 0$, and $p^{-1}(n)$ is finite for every $n \in \mathbf{N}$. Let $\{a_n : n = 1, 2, \ldots\}$ be an arbitrary sequence of distinct points in the compact interval $I$ converging to the point $a_0 \in I \setminus \{a_n : n = 1, 2, \ldots\}$. Then there is a continuous function $f: I \to I$ and a point $x_0 \in I$ such that $\omega(x_0, f) = \{a_n : n = 0, 1, \ldots\}$ and $f(a_n) = a_{p(n)}$ for every $n \in \mathbf{N} \cup \{0\}$.*

*Proof.* First, we describe a rather simple encoding idea. For an arbitrary countable discrete set $\{a_n : n \in \mathbf{N}\} \subset I$, we find sequences $\{a^k_n : n, k \in \mathbf{N}\}$ such that

(1) $\lim_{k\to\infty} a^k_n = a_n$,

(2) $a_n^k = a_m^i$ only if $n = m$ and $k - i$, and

(3) if $\{n_i\}$ is unbounded and $\lim_{i\to\infty} a_{n_i}^{k_i} = a$ for some sequence $\{k_i\}$, then

$$a \in \overline{\{a_n : n \in \mathbf{N}\}} \setminus \{a_n : n \in \mathbf{N}\}.$$

Now, suppose $s : \mathbf{N} \to \mathbf{N}$ satisfies the condition that $s^{-1}(n)$ is infinite for each $n \in \mathbf{N}$. We use $s$ to define a surjection $f_s$ from $\{a_n^k : n, k \in \mathbf{N}\}$ to itself as follows. For each $n$, writing $s^{-1}(n)$ as an increasing sequence, $\{m_n^1, m_n^2, m_n^3, \dots\}$, and for each $k$, we define

$$f_s(a_n^k) = a_{s(m_n^k+1)}^{i+1},$$

where $i$ is the number of terms in the set $\{1, 2, 3, \dots, m_n^k\}$ which map to $s(m_n^k + 1)$ by $s$. For example, if $s$ is the sequence $\{1, 2, 1, 2, 3, 1, 2, 3, 4, \dots\}$, then $f_s(a_1^3) = a_2^3$ because "the sixth position 1 is the 3rd occurrence of 1 and the seventh position is 2, which occurs 2 times in the first six terms of the sequence $s$". Similarly, $f_s(a_3^2) = a_4^1$, $f(a_2^1) = a_1^2$, and so on.

Now, under the hypothesis of Example 28, we illustrate how to find a correct encoding sequence $s : \mathbf{N} \to \mathbf{N}$ such that the corresponding function $f_s$ can be extended to a function we are looking for. As $p$ has no finite cycles other than 0, for each pair of natural numbers $m$, $M$ there is a finite sequence of natural numbers $B_m^M = \{b_1, b_2, \dots, b_k\}$ such that:

(4) $b_1 > M + m$ and either $b_k > M + m$ or $p(b_k) = 0$.

(5) $b_j = m$ for some $1 < j \le k$.

(6) $p(b_i) = b_{i+1}$ for $1 \le i < k$.

Juxtaposing these "blocks" $B_m^M$ appropriately, we define $s$ as

$$s = \{B_1^2 B_2^1 B_1^3 B_2^2 B_3^1 B_1^4 B_2^3 B_3^2 B_4^1 \cdots\}.$$

Then using this $s$, one can show that the function $f_s$ defined on the set $\{a_n^k : n, k \in \mathbf{N}\}$ can be extended to a function $f$ which is defined and continuous on the set $A = \{a_n^k : n, k \in \mathbf{N}\} \cup \{a_n : n \in \mathbf{N}\} \cup \{a_0\}$ in such a manner that $f(a_n) = a_{p(n)}$ for all nonnegative integers $n$. As the set $A$ is a closed subset of the compact interval $I$, the function $f$ can be extended continuously to all of $I$ by letting $f$ be linear on those intervals contiguous to $A$ (and constant on the end intervals if necessary). Finally, letting $x_0 = a_1^1$, we see that the conclusion of Example 28 follows.

## References


[ABCP] S. J. Agronsky, A. M. Bruckner, J. G. Ceder and T. L. Pearson, *The structure of ω-limit sets for continuous functions*, Real Analysis Exchange **15** (1989–90), 483–510.

[BC] L. S. Block and W. A. Coppel, *Stratification of continuous maps of an interval*, Trans. Amer. Math. Soc. **297** (1986), 587–604.

[C] W. A. Coppel, *Continuous maps of an interval*, Xeroxed notes, 1984. [It has been reported that an extended version of the notes, by L. S. Block and W. A. Coppel, is being prepared for publication.]

[D] R. L. Devaney, *An introduction to chaotic dynamical systems*, Benjamin/Cummings, 1986.

[S] K. S. Sibirsky, *Introduction to topological dynamics*, Noordhoff, 1975.

[S1] A. N. Sarkovskii, *The behavior of a map in a neighborhood of an attracting set*, Ukrain. Mat. Z. **18** (1966), 60–83; Amer. Math. Soc. Transl. **97** (1970), 227–258.

[S2] ——, *The partially ordered system of attracting sets*, Soviet Math. Dokl. **7** (1966), 1384–1386.

(M. J. Evans) Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695

(P. D. Humke) Department of Mathematics, St. Olaf College, Northfield, Minnesota 55057

(C.-M. Lee and R. J. O'Malley) Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201