

A MEASURE OF SMOOTHNESS RELATED TO THE LAPLACIAN

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ABSTRACT. A K -functional on $f \in C(\mathbb{R}^d)$ given by

$$\tilde{K}(f, t^2) = \inf(\|f - g\| + t^2 \|\Delta g\|; g \in C^2(\mathbb{R}^d))$$

will be shown to be equivalent to the modulus of smoothness

$$\tilde{w}(f, t) = \sup_{0 < h \leq t} \left\| 2df(x) - \sum_{i=1}^d [f(x + he_i) + f(x - he_i)] \right\|.$$

The situation for other Banach spaces of functions on \mathbb{R}^d will also be resolved.

1. INTRODUCTION

The Laplacian of $f(x)$, $x \in \mathbb{R}^d$ given by

$$(1.1) \quad \Delta f(x) = \frac{\partial^2}{\partial x_1^2} f(x) + \cdots + \frac{\partial^2}{\partial x_d^2} f(x)$$

was related recently [3] to the discrete Laplacian given by

$$(1.2) \quad \tilde{\Delta}_h f(x) = 2df(x) - \sum_{i=1}^d (f(x + he_i) + f(x - he_i))$$

where e_i is a fixed given orthonormal base of \mathbb{R}^d . It was shown in [3] that for $C(\mathbb{R}^d)$, $L_p(\mathbb{R}^d)$ where $1 < p < \infty$ or $L_1(\mathbb{R}^d)$ the inequality

$$(1.3) \quad \|h^{-2} \tilde{\Delta}_h f\| \leq M$$

(where M is independent of h) is equivalent to the statement: Δf exists in the weak (Sobolev) sense as an element of $L_\infty(\mathbb{R}^d)$, $L_p(\mathbb{R}^d)$, $1 < p < \infty$, or $\mathcal{M}(\mathbb{R}^d)$ (the space of finite measures) respectively and $\|\Delta f\| \leq M_1$.

For a space B of functions on \mathbb{R}^d on which translation is a continuous isometry, we may define the natural K -functional

$$(1.4) \quad \tilde{K}(f, t^2)_B = \inf_{g \in S} \{\|f - g\|_B + t^2 \|\Delta g\|_B\}$$

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where $g \in S$ means that $\frac{\partial}{\partial x_i} g$ and $(\frac{\partial}{\partial x_i})^2 g$ exist as strong derivatives in B . If the space B is continuously imbedded in \mathcal{D}' , the space of Schwartz distributions, we define $\tilde{K}_*(f, t^2)_B$ by replacing S in (1.4) by S^* , where S^* is the collection of all g , such that $\Delta g \in B$ and Δg is taken in the weak (distributional) sense. It turns out that $\tilde{K}(f, t^2)_B$ is equivalent to $\tilde{K}_*(f, t^2)_B$. The main result of the paper is the equivalence of the K -functional \tilde{K} given by (1.4) with the modulus of smoothness given by

$$(1.5) \quad \tilde{\omega}(f, t)_B = \sup_{0 < h \leq t} \left\| 2df(x) - \sum_{i=1}^d [f(x + he_i) + f(x - he_i)] \right\|_B$$

for some given orthonormal base $\{e_i\}_{i=1}^d$ of R^d . That is, it will be shown that

$$(1.6) \quad \tilde{\omega}(f, t)_B \sim \tilde{K}(f, t^2)_B,$$

which means

$$C^{-1} \tilde{\omega}(f, t)_B \leq \tilde{K}(f, t^2)_B \leq C \tilde{\omega}(f, t)_B.$$

Proving (1.6), we see that (1.5) is independent of the orthonormal base $\{e_i\}_{i=1}^d$.

For $L_p(R^d)$ the K -functional

$$(1.7) \quad K_2(f, t^2)_{L_p(R^d)} = \inf_{g \in S} \left(\|f - g\|_{L_p(R^d)} + t^2 \sum_{i,j} \left\| \frac{\partial^2}{\partial x_i \partial x_j} g \right\|_{L_p(R^d)} \right)$$

where g has distributional derivatives and $(\partial^2 / \partial x_i \partial x_j)g \in L_p(R^d)$ was shown to be equivalent to

$$(1.8) \quad \omega^2(f, t)_{L_p(R^d)} = \sup_{0 < h \leq t} \sup_{\substack{e \in R^d \\ |e|=1}} \|f(x + he) - 2f(x) + f(x - he)\|_{L_p(R^d)}$$

(see [1, pp. 337–339]). In fact the equivalence between (1.7) and (1.8) is valid with the same proof if $L_p(R^d)$ is replaced by a Banach space of functions on R^d for which translation is a continuous isometry and S is replaced by the class of functions for which $(\partial^2 / \partial x_i \partial x_j)g$ is a strong derivative in B or when $B \subset \mathcal{D}'$ by the class of functions g for which the distributional derivative $(\partial^2 / \partial x_i \partial x_j)g$ is in B .

We should note that $\tilde{K}(f, t^2)_{C(R^d)}$ and $K_2(f, t^2)_{C(R^d)}$ are not equivalent, and also that $\tilde{K}(f, t^2)_{L_1(R^d)}$ and $K_2(f, t^2)_{L_1(R^d)}$ are not equivalent. It is virtually obvious that $\tilde{K}(f, t^2)_{L_p(R^d)}$ and $K_2(f, t^2)_{L_p(R^d)}$ are equivalent for $1 < p < \infty$. A further relation between $\tilde{K}(f, t^2)_B$ and $K_2(f, t^2)_B$, and therefore between $\tilde{\omega}(f, t)_B$ and $\omega^2(f, t)_B$, is that the Besov spaces generated by them are equivalent. This means that the K -functionals $\tilde{K}(f, t^2)_B$ and $K_2(f, t^2)_B$ differ only at optimal or near optimal rate.

As characterization of a given K -functional by the behaviour of the function is considered quite a basic problem in interpolation of spaces, I hope the present somewhat combinatorial result will be of interest.

2. SOME INEQUALITIES

Throughout this section we will assume that $\{e_i\}_{i=1}^d$ is a given fixed orthonormal base of R^d . We define the operator Δ_v^2 for $v \in R^d$

$$(2.1) \quad \Delta_v^2 f(x) \equiv f(x+v) - 2f(x) + f(x-v), \quad \Delta_v^{2l+2} f(x) \equiv \Delta_v^2(\Delta_v^{2l} f(x)).$$

A Banach space B of functions or distributions on R^d is called translation invariant if

$$(2.2) \quad \|f(\cdot + v)\|_B = \|f(\cdot)\|_B \quad \text{for all } v \in R^d.$$

We can now state and prove the following result which will be helpful later.

Theorem 2.1. *For a Banach space of functions or distributions on R^d which satisfies (2.2), $\Delta_{he_j}^{2s_j}$ given by (2.1) and $\tilde{\Delta}_h$ given by (1.2) we have*

$$(2.3) \quad \left\| \sum \Delta_{he_{i_1}}^{2s_1} \cdots \Delta_{he_{i_m}}^{2s_m} f(x) \right\|_B \leq C(m, s) \sum_{k=1}^{s_1 + \cdots + s_m = s} \|\tilde{\Delta}_{kh} f\|_B$$

where the sum on the left of (2.3) is taken on all possibilities $e_{i_1} \cdots e_{i_m}$ such that $e_{i_s} \neq e_{i_n}$ for $r \neq s$ and where $e_{i_j} \in \{e_i\}_{i=1}^d$.

Remark. Note that for $\tilde{\Delta}_h f$ to be defined by (1.2) it is sufficient that B is a Banach space of distribution on R^d . Observe that in the sum (2.3) two different orders of $e_{i_1} \cdots e_{i_r}$ are two entries in the sum.

Proof. We first prove our theorem for $m = 1$. We can easily obtain the identity

$$(2.4) \quad \sum_{i=1}^d \Delta_{he_i}^{2s} f = \sum_{l=0}^{s-1} \binom{2s}{l} (-1)^{l+1} \tilde{\Delta}_{(s-l)h} f.$$

This will imply (2.3) for $m = 1$. We can now write the identity

$$(2.5) \quad \left(\sum_{i=1}^d \Delta_{he_i}^{2s_i} \right) \left(\sum_{\substack{e_{i_r} \neq e_{i_l} \\ r \neq l}} \Delta_{he_{i_1}}^{2s_1} \cdots \Delta_{he_{i_{l-1}}}^{2s_{l-1}} \right) f \\ = \sum_{\substack{i_r \neq i_l \\ r \neq l}} \Delta_{he_{i_1}}^{2s_1} \cdots \Delta_{he_{i_l}}^{2s_l} f + \sum_{j=1}^{l-1} \sum_{\substack{i_r \neq i_l \\ r \neq l}} \Delta_{he_{i_1}}^{2s_1} \cdots \Delta_{he_{i_j}}^{2s_j+2s_l} \cdots \Delta_{he_{i_{l-1}}}^{2s_{l-1}} f.$$

Assuming (2.3) for $m < l$ and utilising (2.2), we have

$$\begin{aligned} & \left\| \left(\sum_{i=1}^d \Delta_{he_i}^{2s_i} \right) \sum_{\substack{i_r \neq i_t \\ r \neq t}} \Delta_{he_1}^{2s_1} \cdots \Delta_{he_{l-1}}^{2s_{l-1}} f \right\| \\ & \leq C(l-1, s_1 + \cdots + s_{l-1}) 4^{s_l} d \sum_{k=1}^{s_1 + \cdots + s_{l-1}} \|\tilde{\Delta}_{kh} f\| \end{aligned}$$

and

$$\left\| \sum_{\substack{i_r \neq i_t \\ r \neq t}} \Delta_{he_{i_1}}^{2s_1} \cdots \Delta_{he_{i_j}}^{2s_j + 2s_l} \cdots \Delta_{he_{i_{l-1}}}^{2s_{l-1}} f \right\| \leq C(l-1, s_1 + \cdots + s_l) \sum_{k=1}^{s_1 + \cdots + s_l} \|\tilde{\Delta}_{kh} f\|,$$

which, together with (2.5), implies (2.3) for $m = l$. \square

We can now prove

Theorem 2.2. *For a Banach space B of functions or distributions on R^d satisfying (2.2) and $f \in B$ we have*

$$(2.6) \quad \|\Delta_{he_i}^{2d} f\|_B \leq C(d) \sum_{k=1}^d \|\tilde{\Delta}_{kh} f\|_B$$

for $i = 1, \dots, n$.

Proof. We may assume with no loss of generality that $i = 1$. We now write

$$\begin{aligned} -\Delta_{he_1}^2(\tilde{\Delta}_h) &= \Delta_{he_1}^2 \left(\sum_{i=1}^d \Delta_{he_i}^2 \right) = \Delta_{he_1}^4 + \Delta_{he_1}^2 \sum_{i=2}^d \Delta_{he_i}^2 \\ &= \Delta_{he_1}^4 + \sum_{i \neq j} \Delta_{he_i}^2 \Delta_{he_j}^2 - \sum_{\substack{i \neq j \\ i, j \neq 1}} \Delta_{he_i}^2 \Delta_{he_j}^2. \end{aligned}$$

Using

$$\|\Delta_{he_1}^2 \tilde{\Delta}_h f\| \leq 4 \|\tilde{\Delta}_h f\|$$

and

$$\left\| \sum_{i \neq j} \Delta_{he_i}^2 \Delta_{he_j}^2 f \right\| \leq C(2) (\|\tilde{\Delta}_{2h} f\| + \|\tilde{\Delta}_h f\|),$$

we have

$$\left\| \Delta_{he_1}^4 f - \sum_{\substack{i \neq j \\ i, j \neq 1}} \Delta_{he_i}^2 \Delta_{he_j}^2 f \right\| \leq C_1(2) \sum_{l=1}^2 \|\tilde{\Delta}_{lh} f\|.$$

As for $d = 2$, the set $i \neq j$ and $i, j \neq 1$ is empty, (2.6) is proved for $d = 2$.

We continue and show by induction that

$$(2.7) \quad \left\| \Delta_{he_1}^{2k} f + (-1)^{k-1} \sum_{\substack{i_r \neq i_t \\ r \neq t \\ i_r \neq 1}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_k}}^2 f \right\| \leq C_1(k) \sum_{l=1}^k \|\tilde{\Delta}_{lh} f\|.$$

One can operate on the expression inside the norm on the left of (2.7) to obtain

$$\begin{aligned} & \Delta_{he_1}^2 (\Delta_{he_1}^{2k} f + (-1)^{k-1} \sum_{\substack{i_r \neq i_t \\ r \neq t \\ i_r \neq 1}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_k}}^2 f) \\ &= \Delta_{he_1}^{2(k+1)} f + (-1)^k \sum_{\substack{i_r \neq i_t \\ r \neq t \\ i_r \neq 1}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_{k+1}}}^2 f + (-1)^{k-1} \sum_{\substack{i_r \neq i_t \\ r \neq t}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_{k+1}}}^2 f. \end{aligned}$$

As (2.3) implies

$$\left\| \sum_{\substack{i_r \neq i_t \\ r \neq t}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_{k+1}}}^2 f \right\| \leq C \left(\sum_{l=1}^{k+1} \|\tilde{\Delta}_{lh} f\| \right),$$

we have (2.7) for $k + 1$. For $k = d$ the sum

$$\sum_{\substack{i_r \neq i_t \\ r \neq t \\ i_r \neq 1}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_k}}^2$$

is a sum on an empty set, and our theorem is proved. \square

We will now obtain an estimate which, while as stated will be a multivariate result, is in fact a result about functions of one variable.

A Banach space of functions on R^d is called strongly continuous if

$$(2.8) \quad \|f(\cdot + he) - f(\cdot)\|_B = o(1), \quad h \rightarrow 0 \text{ for all unit vectors } e \text{ in } R^d.$$

Theorem 2.3. *Suppose B is a Banach space of functions on R^d for which translation is a strongly continuous isometry, that is (2.2) and (2.8) are satisfied. Then for $0 < u < t$ we can find $\alpha_i(y)$ ($y \in R$) fixed even polynomials of degree $2i$ such that*

$$(2.9) \quad \left\| \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t} \right) \Delta_{te}^{2i} f - \Delta_{ue}^2 f \right\|_B \leq M \sup_{0 < h \leq t} \|\Delta_{he}^{2k} f\|_B.$$

Proof. We define

$$F(s) = \langle f(\cdot + se), g(\cdot) \rangle \quad \text{for } g \in B^*, s \in R,$$

where B^* is the dual space to B . Obviously, $F(s) \in C(R)$. We first show that (2.9) for F , that is,

$$(2.10) \quad \left\| \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t} \right) \Delta_t^{2i} F - \Delta_u^2 F \right\|_{C(R)} \leq M \sup_{0 < h \leq t} \|\Delta_h^{2k} F\|_{C(R)}$$

implies (2.9) in general. Of course, when we deal with $F(s)$ which is a function on $C(R)$, we may write Δ_t^2 rather than Δ_{te}^2 with e the unit vector in $R = R^1$. We choose g such that $\|g\|_{B^*} = 1$ to obtain

$$\begin{aligned} \left| \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t} \right) \Delta_t^{2i} F(0) - \Delta_u^2 F(0) \right| &\leq \left\| \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t} \right) \Delta_t^{2i} F - \Delta_u^2 F \right\|_{C(R)} \\ &\leq M \sup_{0 < h \leq t} \|\Delta_h^{2k} F\|_{C(R)} \leq M \sup_{0 < h \leq t} \|\Delta_{he}^{2k} f\|_B. \end{aligned}$$

We can now choose a g (of those that satisfy $\|g\|_{B^*} = 1$) that depends on u, t, f and $\varepsilon > 0$ such that

$$\begin{aligned} \left| \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t} \right) \Delta_{te}^{2i} f - \Delta_{ue}^2 f \right|_B - \varepsilon &\leq \left| \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t} \right) \Delta_t^{2i} F(0) - \Delta_u^2 F(0) \right| \\ &\leq M \sup_{0 < h \leq t} \|\Delta_{he}^{2k} f\|_B, \end{aligned}$$

and as ε is arbitrary, we complete the proof of (2.9) given (2.10). In fact, we established that the present result is essentially about functions on R .

We now construct α_i such that the operator

$$(2.11) \quad O_{t,u,k} \equiv \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t} \right) \Delta_t^{2i} - \Delta_u^2$$

annihilates polynomials of degree $2k - 1$. It is easy to see, and probably helpful in understanding the direction of the proof, that $\alpha_1(u/t) = u^2/t^2$ and that $(u^2/t^2)\Delta_t^2 - \Delta_u^2$ annihilates polynomials of degree 3. The construction follows by induction. We assume that we have constructed for $i < l$ even polynomials of degree $2i$, $\alpha_i(y)$ so that $O_{t,u,l}$ annihilates polynomials of degree $2l - 1$. As Δ_t^{2l} also annihilates polynomials of degree $2l - 1$, we construct $\alpha_l(y)$ so that at $x = 0$ we have

$$\alpha_l \left(\frac{u}{t} \right) \Delta_t^{2l} x^{2l} = \alpha_l \left(\frac{u}{t} \right) (2l)! t^{2l} = -O_{t,u,l} x^{2l}.$$

As $O_{t,u,l}$ annihilates polynomials of degrees $2l - 1$, it is sufficient to calculate $O_{t,u,l}(x - a)^{2l}$ at a or $O_{t,u,l} x^{2l}$ at 0 (which is equal), and therefore,

$$-O_{t,u,l} x^{2l} = 2u^{2l} - 2 \sum_{i=1}^{l-1} \alpha_i \left(\frac{u}{t} \right) \left\{ \sum_{r=0}^i \binom{2i}{r} (-1)^r (i-r)^{2l} \right\} t^{2l}.$$

As $\alpha_i(\frac{u}{t})$ is $\frac{-1}{(2l)!} O_{t,u,l} x^{2l}$, it is even polynomial in $(\frac{u}{t})$ of degree $2l$, and in fact we can effectively estimate its coefficients. We have now shown that $O_{t,u,l+1}$ annihilates polynomials of degree $2l$. As $O_{t,u,l+1}(x-a)^{2l+1}$ at a is equal to zero for all a , and as polynomials can be expanded in $(x-a)^j$, we have constructed α_i as desired.

We now recall (see [1, p. 339, Theorem 4.12] for instance) that for any function $F \in C(R)$ we can find a function G_t such that $G_t^{(2k-1)} \in A.C._{loc}$, $G_t^{(2k)} \in L_\infty$,

$$\|F - G_t\|_{C(R)} \leq A \sup_{0 < h \leq t} \|\Delta_h^{2k} F\|_{C(R)}$$

and

$$t^{2k} \|G_t^{(k)}\| \leq \sup_{0 < h \leq t} \|\Delta_h^{2k} F\|_{C(R)}.$$

As the operator $O_{t,u,k}$ is linear, we can estimate it separately on $F - G_t$ and G_t . The estimate of $O_{t,u,k}(F - G_t)$ is trivial since $O_{t,u,k}$ is bounded. The estimate of $O_{t,u,k}G_t$ follows from Taylor's formula, the annihilation of $(x-a)^j$ for $j < 2k$ and easy computation on the remainder. \square

In fact, we can prove an analogue of Theorem 2.3 for a Banach space of distribution (generalized functions) on R^d , that is a space $B \subset \mathcal{D}'$ (where \mathcal{D}' is the Schwartz space of distributions) such that B is continuously imbedded in \mathcal{D}' .

Theorem 2.4. *Suppose B is a Banach space of distributions which is strongly, weakly or weakly* continuous and on which translations are isometries. Then for the $\alpha_i(y)$ ($y \in R$) of Theorem 2.3, (2.9) is valid.*

Proof. We recall that B is weakly continuous if

$$(2.12) \quad \langle f(\cdot + v) - f(\cdot), g(\cdot) \rangle = o(1), \quad |v| \rightarrow 0 \text{ for all } f \in B \text{ and } g \in B^*$$

and weakly* continuous if $B = X^*$ and

$$(2.12)' \quad \langle f(\cdot + v) - f(\cdot), g(\cdot) \rangle = o(1), \quad |v| \rightarrow 0 \text{ for all } f \in B \text{ and } g \in X.$$

We now define $F(s)$ as in Theorem 2.3 for B which is strongly or weakly continuous and by

$$F(s) = \langle f(\cdot + se), g(\cdot) \rangle \quad \text{for } g \in X, f \in B \text{ where } B = X^*$$

for B which is weakly* continuous. We now follow the first part of the proof of Theorem 2.3 to show that (2.10) is sufficient for our result. The assertion (2.10) was proved while proving Theorem 2.3. \square

3. THE UPPER ESTIMATE

We recall the definition $\tilde{K}(f, t^2)_B$ and $\tilde{K}_*(f, t^2)_B$. The K -functional $\tilde{K}(f, t^2)_B$ is given by

$$\tilde{K}(f, t^2)_B = \inf_{g \in S} (\|f - g\|_B + t^2 \|\Delta g\|_B)$$

where S is the class of elements of B for which the strong first two derivatives exist and are in B . The K -functional $\tilde{K}_*(f, t^2)_B$ is defined for $B \subset \mathcal{D}'$ (B is continuously imbedded in the Schwartz space of distributions \mathcal{D}') with S^* replacing S , and S^* is the class of elements of \mathcal{D}' for which the weak Laplacian belongs to B . We note that it is possible that $\tilde{K}(f, t^2) = O(t^2)$ and at the same time $(\partial/\partial x_i)^2 f$ is not in B or does not exist even locally.

Theorem 3.1. *Suppose $f \in B$, translations on B are isometries and the expressions $\tilde{K}(f, t^2)_B$ and $\tilde{\omega}(f, t)_B$ are given by (1.4) and (1.5) respectively. Then we have for a Banach space B over R^d for which translations are strongly continuous*

$$(3.1) \quad \tilde{K}(f, t^2)_B \leq C\tilde{\omega}(f, t)_B.$$

If $B \subset \mathcal{D}'$ and translations on B are strongly, weakly or weakly* continuous, then

$$(3.1)' \quad \tilde{K}_*(f, t^2)_B \leq C\tilde{\omega}(f, t)_B.$$

Proof. Using the definitions of $\tilde{K}(f, t^2)$ and $\tilde{K}_*(f, t^2)$,

$$(3.2) \quad \tilde{K}(f, (at)^2) \leq a^2\tilde{K}(f, t^2), \text{ and } \tilde{K}_*(f, (at)^2) \leq a^2\tilde{K}_*(f, t^2) \quad \text{for } a > 1.$$

As d is a fixed finite integer, it is sufficient to prove

$$(3.3) \quad \tilde{K}(f, t^2) \leq C\tilde{\omega}(f, td^2) \quad \text{and} \quad \tilde{K}_*(f, t^2) \leq C\tilde{\omega}(f, td^2).$$

To prove (3.3) we construct the function

$$(3.4) \quad g_t(x) = t^{-2d} \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} f(x + (u_1 + u_2)e_1 + \cdots + (u_{2d-1} + u_{2d})e_d) du_1 \cdots du_{2d}$$

and show that

$$(3.5) \quad \|g_t - f\| \leq C\tilde{\omega}(f, td^2)$$

and

$$(3.6) \quad \|\Delta g_t\| \leq Ct^{-2}\tilde{\omega}(f, td^2).$$

In (3.6) Δg_t is taken as a sum of strong derivatives for the estimate of $\tilde{K}(f, t^2)_B$ and a weak distributional derivative for the estimate of $\tilde{K}_*(f, t^2)_B$.

We prove (3.5) first. For $1 \leq l \leq d$ we denote

$$(3.7) \quad \begin{aligned} I(f, l) &\equiv I(f, l, e_{i_1}, \dots, e_{i_d}) \\ &\equiv t^{-2d+2l-2} \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} f \left(x + \sum_{j=1}^d (u_{2j-1} + u_{2j})e_{i_j} \right) du_{2l-1} \cdots du_{2d}, \end{aligned}$$

$$I(f, d+1) \equiv f(x).$$

We may now write

$$g_t(x) - f(x) = \frac{1}{d!} \sum \sum_{l=1}^d [I(f, l) - I(f, l + 1)]$$

where the first sum indicates summation on all permutations of $e_1 \cdots e_d$. Therefore, it is sufficient to estimate

$$\sum (I(f, l) - I(f, l + 1))$$

for a fixed l , $1 \leq l \leq d$, where the sum is taken on all permutations of $e_1 \cdots e_d$. We can write

$$\begin{aligned} & I(f, l) - I(f, l + 1) \\ &= t^{-2} \int_0^{t/2} \int_0^{t/2} \Delta_{(u_{2l-1}+u_{2l})e_{i_l}}^2 I(f, l + 1) du_{2l-1} du_{2l} \\ &= t^{-2} \int_0^{t/2} \int_0^{t/2} \Delta_{(u_{2l-1}+u_{2l})e_{i_l}}^2 f(x) du_{2l-1} du_{2l} \\ &\quad + \sum_{k=l+1}^d t^{-2} \int_0^{t/2} \int_0^{t/2} \Delta_{(u_{2l-1}+u_{2l})e_{i_l}}^2 (I(f, k) - I(f, k + 1)) du_{2l-1} du_{2l} \\ &= \sum_{\{l_i\}_{i=1}^m \subset \{j\}_l^d, l_i=l} t^{-2m} \\ &\quad \times \int_0^{t/2} \cdots \int_0^{t/2} \left\{ \prod_{k=1}^m \Delta_{(u_{2k-1}+u_{2k})e_{i_k}}^2 \right\} f(x) du_1 \cdots du_{2m}. \end{aligned}$$

To summarize,

$$\begin{aligned} g_t(x) - f(x) &= \frac{1}{d} t^{-2} \int_0^{t/2} \int_0^{t/2} \tilde{\Delta}_{(u_1+u_2)} f(x) du_1 du_2 \\ &\quad + \frac{1}{d!} \sum^* \sum_{m=2}^d t^{-2m} \int_0^{t/2} \cdots \int_0^{t/2} \\ &\quad \times \left\{ \sum^{**} \Delta_{(u_1+u_2)e_{j_1}}^2 \cdots \Delta_{(u_{2m-1}+u_{2m})e_{j_m}}^2 f(x) \right\} du_1 \cdots du_{2m} \\ &\equiv I + J, \end{aligned}$$

where \sum^* is the sum on all permutations of $\{e_i\}$, and \sum^{**} is the sum on all possible collections $e_{j_1} \cdots e_{j_m}$ such that $e_{j_r} \in \{e_j^*\}$, (where $\{e_j^*\}$ is the given permutation) and $e_{j_r} \neq e_{j_s}$ for $r \neq s$. The estimate of J is obvious. For the estimate of I we will need the results in Theorems 2.2 and 2.3.

To estimate I it is clear that we have only to estimate terms of the type

$$(3.8) \quad \sum \Delta_{(u_1+u_2)e_{j_1}}^2 \cdots \Delta_{(u_{2m-1}+u_{2m})e_{j_m}}^2 f(x)$$

where the sum is taken on all sequences of m different vectors $e_{j_1} \cdots e_{j_m}$ such that $e_{j_i} \in \{e_i\}_{i=1}^d$ and u_i satisfy $u_i \geq 0$ and $u_{2j-1} + u_{2j} \leq t$. We can write the

expression in (3.8) as

$$\sum \Delta_{v_1 e_{j_1}}^2 \cdots \Delta_{v_m e_{j_m}}^2 f(x)$$

with $0 < v_i \leq t$. Following Theorems 2.3 and 2.4, we have

$$\begin{aligned} & \sum \Delta_{v_1 e_{j_1}}^2 \cdots \Delta_{v_m e_{j_m}}^2 f(x) \\ &= (-1)^m \sum \prod_{l=1}^m \left(\sum_{i=1}^{d-1} \alpha_i \left(\frac{v_l}{t} \right) \Delta_{te_{j_l}}^{2i} - \Delta_{v_l e_{j_l}}^2 \right) f(x) \\ & \quad + (-1)^{m+1} \sum \prod_{l=1}^m \sum_{i=1}^{d-1} \alpha_i \left(\frac{v_l}{t} \right) \Delta_{te_{j_l}}^{2i} f(x) + R \\ & \equiv S + L + R \end{aligned}$$

where R is a finite sum of translations of

$$\sum_{i=1}^{d-1} \alpha_i \left(\frac{v_l}{t} \right) \Delta_{te_{j_l}}^{2i} - \Delta_{v_l e_{j_l}}^2 f(x)$$

multiplied by finite products of $\alpha_i(v_j/t)$, and $\alpha_i(v_j/t)$ are those given in Theorems 2.3 and 2.4, and therefore, bounded for $0 \leq v_j \leq t$. We now use Theorems 2.3, 2.4 and 2.2 to obtain

$$\begin{aligned} (3.9) \quad & \left\| \left\{ \sum_{i=1}^{d_1} \alpha_i \left(\frac{v_l}{t} \right) \Delta_{te_{j_l}}^{2i} - \Delta_{v_l e_{j_l}}^2 \right\} f \right\| \leq M \sup_{0 < h \leq t} \|\Delta_{he_{j_l}}^{2d} f\| \\ & \leq M_1 \sup_{0 < h \leq t} \sum_{k=1}^d \|\tilde{\Delta}_{kh} f\| \leq M_2 \tilde{\omega}(f, t, d). \end{aligned}$$

The inequality (3.9) yields the appropriate estimate for S and R .

To estimate L we have to estimate terms like

$$\sum \Delta_{te_{j_1}}^{2i_1} \cdots \Delta_{te_{j_m}}^{2i_m} f(x)$$

where the sum is taken to all subcollections of different $e_{j_1} \cdots e_{j_m}$, and therefore, using Theorem 2.1,

$$\begin{aligned} \left\| \sum \Delta_{te_{j_1}}^{2i_1} \cdots \Delta_{te_{j_m}}^{2i_m} f \right\| & \leq M_2 \sum_{k=1}^{i_1 + \cdots + i_m} \|\tilde{\Delta}_{kt} f\| \\ & \leq M_3 \tilde{\omega}(f, (d-1)mt) \\ & \leq M_3 \tilde{\omega}(f, d^2 t). \end{aligned}$$

To prove the second crucial inequality, i.e. the estimate (3.6), we write (3.10)

$$\begin{aligned} \Delta g_t &= t^{-2} \sum_{i=1}^d \Delta_{te_i} t^{-2d+2} \\ &\times \left\{ \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} f \left(x + \sum_{l \neq i} (u_{2l-1} + u_{2l}) e_l \right) du_1 \cdots du_{2i-2} \right. \\ &\quad \left. \times du_{2i+1} \cdots du_{2d} \right\}. \end{aligned}$$

The expression (3.10) for Δg_t follows as a strong derivative of g_t in case B is strongly continuous and as a weak (distributional) derivative for $B \subset \mathcal{D}'$.

We now follow earlier considerations to obtain

$$\begin{aligned} \Delta g_t &= t^{-2} \sum_{i=1}^d \Delta_{te_i}^2 f(x) + \frac{1}{d!} t^{-2} \sum_{i=1}^d \Delta_{te_i}^2 \\ &\times \left\{ \sum_{e_{j_l} \neq e_i}^* t^{-2d+2} \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} f \left(x + \sum_{r=1}^{d-1} (u_{2r-1} + u_{2r}) e_{j_r} \right) \right. \\ &\quad \left. \times du_1 \cdots du_{2d-2} - f(x) \right\} \end{aligned}$$

where the sum \sum^* is taken on all permutations of e_j with $j \neq i$. The estimate (3.6) now follows the estimate used for (3.5) with the change that here we have to estimate

$$t^{-2} \sum \Delta_{te_{j_1}}^2 \Delta_{(u_1+u_2)e_{j_2}}^2 \cdots \Delta_{(u_{2m-3}+u_{2m-2})e_{j_m}}^2$$

for $m > 1$ and the sum is on all $j_1 \cdots j_m$ such that $e_{j_k} \in \{e_i\}_{i=1}^d$ and $e_{j_r} \neq e_{j_s}$ for $r \neq s$ rather than (3.8). The fact that the first term is already $\Delta_{te_{j_1}}^2$ does not make much difference to the proof which follows that of (3.8) but leads to somewhat smaller bounds. \square

4. THE LOWER ESTIMATE

We now estimate $\tilde{\omega}$ by \tilde{K} or \tilde{K}_* to complete the proof of the equivalence of $\tilde{\omega}$ with these K -functionals. We first prove the result for $B = C(R^d)$.

Theorem 4.1. For $f \in C(R^d)$

$$(4.1) \quad \tilde{\omega}(f, t)_{C(R^d)} \leq C \tilde{K}(f, t^2)_{C(R^d)}.$$

Remark. For the sake of Theorem 4.1 the infimum in the definition of \tilde{K} (see (1.4)) is on all g such that Δg exists in the strong sense and is in $C(R^d)$.

However, as we will see later, existence can be taken in the weak (Sobolev) sense or weak* sense and in $L_\infty(\mathbb{R}^d)$ with no effect on (4.1).

Proof. From the definition of $\tilde{K}(f, t^2)_{C(\mathbb{R}^d)}$ it follows that a function $g_t \in C(\mathbb{R}^d)$ exists satisfying

$$(4.2) \quad \|f - g_t\|_B \leq 2\tilde{K}(f, t^2)_B$$

and

$$(4.3) \quad t^2 \|\Delta g_t\|_B \leq 2\tilde{K}(f, t^2)_B$$

for $B = C(\mathbb{R}^d)$ where Δg_t is the strong derivative in $C(\mathbb{R}^d)$. Obviously,

$$(4.4) \quad \tilde{\omega}(f, t)_B \leq \tilde{\omega}(f - g_t, t)_B + \tilde{\omega}(g_t, t)_B$$

for $B = C(\mathbb{R}^d)$ as well as for other B . For any g_t satisfying (4.2) we have

$$(4.5) \quad \omega(f - g_t, t)_B \leq 4d\|f - g_t\|_B \leq 8d\tilde{K}(f, t^2)_B.$$

To estimate $\omega(g_t, t)_{C(\mathbb{R}^d)}$ we use the result of [3, p. 112] and write

$$(4.6) \quad \begin{aligned} \tilde{\omega}(g_t, t)_{C(\mathbb{R}^d)} &\leq \sup_{0 < h \leq t} \|\tilde{\Delta}_h g_t\|_{C(\mathbb{R}^d)} \\ &\leq t^2 \sup_{0 \leq h \leq t} \|h^{-2} \tilde{\Delta}_h g_t\|_{C(\mathbb{R}^d)} \leq M t^2 \|\Delta g_t\|_{C(\mathbb{R}^d)}, \end{aligned}$$

which concludes the proof of our theorem. \square

We can now deduce the lower estimate, for other spaces B , of $\tilde{K}(f, t^2)_B$ as well as of $\tilde{K}_*(f, t^2)_B$.

Theorem 4.2. *Suppose $f \in B$, translation on B are isometries and the expressions $\tilde{K}(f, t^2)_B$ and $\tilde{\omega}(f, t)_B$ are given by (1.4) and (1.5) respectively. Then for a Banach space for which translations are strongly continuous*

$$(4.8) \quad \tilde{\omega}(f, t)_B \leq A\tilde{K}(f, t^2)_B;$$

and for a Banach space B , $B \subset \mathcal{D}'$, for which translations are weakly continuous and \mathcal{D} dense in B^ or weakly* continuous and \mathcal{D} dense in X ($X^* = B$)*

$$(4.9) \quad \tilde{\omega}(f, t)_B \leq A\tilde{K}_*(f, t^2)_B.$$

Proof. We follow the proof of Theorem 4.1. From the definitions of \tilde{K} and \tilde{K}_* we have a function g_t satisfying

$$\|f - g_t\|_B \leq 2\tilde{K}(f, t^2)_B \quad \text{or} \quad \|f - g_t\|_B \leq 2\tilde{K}_*(f, t^2)_B$$

and

$$t^2 \|\Delta g_t\|_B \leq 2\tilde{K}(f, t^2)_B \quad \text{or} \quad \|\Delta g_t\|_B \leq 2\tilde{K}_*(f, t^2)_B.$$

This implies (4.5) for the first step in the proof of (4.8) or with $\tilde{K}_*(f, t^2)_B$ replacing $\tilde{K}(f, t^2)_B$ for the corresponding step in the proof of (4.9). To continue with the proof of (4.8) we write

$$G_t(x) \equiv \langle g_t(\cdot + x), \varphi(\cdot) \rangle, \quad \varphi \in B^*, \quad \|\varphi\|_{B^*} \leq 1,$$

and therefore, ΔG_t exists in the sense of strong derivatives in $C(R^d)$ and it satisfies

$$\|\Delta G_t\|_{C(R^d)} \leq \|\Delta g_t\|_B.$$

This implies for $0 < h \leq t$

$$\|\tilde{\Delta}_h G_t\|_{C(R^d)} \leq Mt^2 \|\Delta G_t\|_{C(R^d)} \leq Mt^2 \|\Delta g_t\|_B$$

with M that is independent of t or φ . Therefore, for $0 < h \leq t$

$$\|\tilde{\Delta}_h g_t\| \leq Mt^2 \|\Delta g_t\|_B \leq 2Mt^2 \tilde{K}(f, t^2)_B,$$

which completes the proof of (4.8).

To continue with the proof of (4.9) we define again

$$G_t(x) \equiv \langle g_t(\cdot + x), \varphi(\cdot) \rangle$$

with $\varphi \in B^*$ when translations are weakly continuous and \mathcal{D} is dense in B^* and with $\varphi \in X$ when translations are weakly* continuous on $B = X^*$ and \mathcal{D} is dense in X . We now complete the proof of (4.9) following the steps used earlier for the proof of (4.8). \square

5. REMARKS

Remark 5.1. We obtain the K -functional $\tilde{K}^{**}(f, t^2)_B$ by replacing S of $\tilde{K}(f, t^2)_B$ by S^{**} where S^{**} is the collection of all the functions g for which Δg exists in the weak or weak* sense and B is such that translations are weakly or weakly* continuous respectively. This K -functional is also equivalent to $\tilde{\omega}(f, t)_B$.

Remark 5.2. If for $L_p(R^d)$, $1 \leq p \leq \infty$, we define S^{***} by

$$S^{***} = \left\{ g \in L_p(R^d); \frac{\partial}{\partial u} g(x + he_i) \in \text{A.C.}_{\text{loc}} \text{ and } \Delta g \in L_p(R^d) \right\},$$

the corresponding K -functional, $K_{***}(f, t^2)_p$, is equivalent to $\tilde{\omega}(f, t)_p$.

Remark 5.3. From the above results and those in earlier sections we see that for some spaces the K -functionals $\tilde{K}(f, t^2)$, $\tilde{K}_*(f, t^2)$, $\tilde{K}_{**}(f, t^2)$ and $\tilde{K}_{***}(f, t^2)$ are defined and are equivalent. In fact in $L_p(R^d)$ and in other situations they are equal.

Remark 5.4. The K -functionals $\tilde{K}(f, t^2)_B$ and $\tilde{K}_*(f, t^2)_B$ are independent of the orthogonal system $\{e_i\}$, and therefore, so is the corresponding $\tilde{\omega}(f, t)_B$.

Remark 5.5. For $1 \leq p \leq \infty$ one usually investigates the K -functional

$$K_2(f, t^2)_p = \inf_{g \in \mathcal{A}} \left(\|f - g\| + t^2 \sum_{i,j} \left\| \frac{\partial^2}{\partial x_i \partial x_j} g \right\| \right)_p$$

where A is the class for which $\partial^2 g / \partial x_i \partial x_j \in L_p$ (and the derivatives are taken in the Sobolev sense). For $1 < p < \infty$, $\tilde{K}(f, t^2)_p$ and $K_2(f, t^2)_p$ are equivalent as

$$\left\| \frac{\partial^2}{\partial x_i \partial x_j} g \right\|_p \leq C(p) \|\Delta g\|_p \quad (\text{for } 1 < p < \infty).$$

Remark 5.6. For $p = \infty$ and $p = 1$, $K_2(f, t^2)_p$ is no longer equivalent to $\tilde{K}(f, t^2)_p$; otherwise the inequality

$$\left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_p \leq C \|\Delta f\|_p,$$

which is valid only for $1 < p < \infty$, would be valid for $p = 1$ and $p = \infty$ as well, and this is not true. For $p = \infty$ an example can be given that shows that $K_2(f, t^2)_\infty$ is not equivalent to $\tilde{K}(f, t^2)_\infty$ (see for instance [2]).

Remark 5.7. In the next section we will compare Besov space induced by $\tilde{K}(f, t^2)_B$ and $K_2(f, t^2)_B$ and find them equivalent, and hence only classes of functions with optimal or near optimal behaviour are different.

6. BESOV TYPE SPACES

For a given K -functional the related Besov space is given by its norm

$$(6.1) \quad \|f\|_{B_q^\alpha} = \left\{ \int_0^1 (t^{-\alpha} K(f, t))^q \frac{dt}{t} \right\}^{1/q}, \quad 1 \leq q < \infty.$$

$$\|f\|_{B_\infty^\alpha} = \sup_t t^{-\alpha} K(f, t).$$

Generally, an additional index p is given to indicate that the K functional is between the space L_p and a subspace of L_p . However, we allow here a general class of spaces B rather than L_p to be the underlying space for our K -functional.

We define the spaces \tilde{B}_q^α and \bar{B}_q^α as the Besov spaces introduced by (6.1) with the K -functionals $\tilde{K}(f, t^2)_B$ and $K_2(f, t^2)_B$ respectively.

We can now state and prove the following equivalence theorem.

Theorem 6.1. *For $1 \leq q \leq \infty$ and $0 < \alpha < 2$ the spaces \tilde{B}_q^α and \bar{B}_q^α are equivalent.*

Proof. For $\alpha < 2$, \bar{B}_q^α is clearly equivalent to B_q^α given by

$$(6.2) \quad \|f\|_{B_q^\alpha} = \begin{cases} \left\{ \int_0^1 (t^{-\alpha} \omega^{-2d}(f, t)_B)^q \frac{dt}{t} \right\}^{1/q}, & 1 \leq q < \infty, \\ \sup_t t^{-\alpha} \omega^{2d}(f, t)_B, & q = \infty, \end{cases}$$

where

$$\omega^{2d}(f, t)_B = \sup_{\substack{0 < h \leq t \\ e \in \bar{S}}} \|\Delta_{he}^{2d} f\|_B$$

as it is valid also for any integer r replacing $2d$ in (6.2).

Using Theorem 2.2, it follows that

$$(6.3) \quad \omega^{2d}(f, t)_B \leq C\tilde{K}(f, d^2t)$$

since the direction e_i given in that theorem is immaterial because the Laplacian can be expressed in different coordinates. Therefore,

$$(6.4) \quad \|f\|_{B_q^\alpha} \leq C\|f\|_{\tilde{B}_q^\alpha} \leq C_1\|f\|_{\tilde{B}_q^\alpha} \leq C_2\|f\|_{B_q^\alpha}. \quad \square$$

7. ON AN A PRIORI ESTIMATE

A priori estimates are discussed in many articles (see for instance [4]) and we would like to note that the present result makes a small contribution in this direction.

Theorem 7.1. *Suppose B is a Banach space for which translations are isometries on R^d , and $B \subset \mathcal{D}'$ (where \mathcal{D}' is the space of distributions) translations are strongly continuous on B and \mathcal{D} is dense in the dual to B or in the predual of B . Then Δf exists in \mathcal{D}' and $\Delta f \in \text{Lip}(\alpha, B)$, $0 < \alpha < 1$, implies $f \in \text{Lip}^*(2 + \alpha, B)$.*

Remark 7.2. There are other analogues of Theorem 7.1, but I believe the present form will be of use. The condition $0 < \alpha < 1$ in Theorem 7.1 can be replaced by any $\beta > 0$ such that $[\beta] \neq \beta$. Recall that $g \in \text{Lip}^*(\beta, B)$ means

$$(7.1) \quad \|\Delta_h^r g\| \leq Mh^\beta \quad \text{for } r > \beta.$$

Proof. We can write

$$\|\Delta\Delta_{he}f\|_B = \|\Delta_{he}\Delta f\|_B \leq Ch^\alpha.$$

Using [3], we have

$$\|\eta^{-2}\tilde{\Delta}_\eta g\| \leq M\|\Delta g\|,$$

and therefore,

$$\|\eta^{-2}\tilde{\Delta}_\eta\Delta_{he}f\| \leq \|\Delta_{he}\Delta f\|.$$

Using Theorem 2.2, we have

$$\begin{aligned} \|\Delta_{he}^{2d}\Delta_{he}f\| &\leq C \sum_{j=1}^d \|\tilde{\Delta}_{jh}\Delta_{he}f\| \\ &\leq C_1 h^2 \sum_{j=1}^d \|(jh)^{-2}\tilde{\Delta}_{jh}\Delta_{he}f\| \\ &\leq C_2 h^2 \sup_\eta \|\eta^{-2}\tilde{\Delta}_\eta\Delta_{he}f\| \\ &\leq C_2 h^{2+\alpha}, \end{aligned}$$

which concludes the proof of our theorem. \square

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