

## SEPARATING POINTS FROM CLOSED CONVEX SETS OVER ORDERED FIELDS AND A METRIC FOR $\tilde{R}^n$

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**ABSTRACT.** Let  $R$  be an arbitrary ordered field, let  $\bar{R}$  be a real closure, and let  $\tilde{R}$  and  $\tilde{R}^n$  denote the real spectra of  $\bar{R}[X]$  and  $\bar{R}[X_1, \dots, X_n]$ . We prove that a closed convex subset in  $R^n$  may be separated from a point not in it via a continuous “linear” functional taking values in  $\tilde{R}$  and that there is a  $\tilde{R}$ -valued metric on  $\tilde{R}^n$ . The methods rely on the ultrafilter interpretation of points in  $\tilde{R}^n$  and on the existence of suprema and infima of sets in  $\tilde{R}$ .

### INTRODUCTION

A basic result which uses the completeness and order relation on  $\mathbb{R}$  is

**Theorem I** (for  $\mathbb{R}$ ). *Let  $C \subset \mathbb{R}^n$  be a closed convex set and let  $\mathbf{p} \in \mathbb{R}^n \setminus C$ . Then there exists a linear function*

$$L(\mathbf{x}) = \mathbf{a} \cdot (\mathbf{x} - \mathbf{b})$$

such that  $L(\mathbf{p}) < 0$  and  $L(\mathbf{q}) > 0$  for all  $\mathbf{q} \in C$ .  $\square$

Theorem I remains true if  $\mathbb{R}^n$  is replaced by a locally convex real vector space and  $\mathbf{p}$  is replaced by an arbitrary compact set [5, 3.3, Satz 4] but we are more interested in the remark that the *statement* of Theorem I makes sense if  $\mathbb{R}$  is replaced by any ordered field  $R$ . It is false as stated for every  $R \neq \mathbb{R}$ , but if we replace  $L$  by an  $\tilde{R}$ -valued linear function where  $\tilde{R} := \text{Sper } R[X]$  is the real spectrum of  $R[X]$ , we obtain

**Theorem I.** *Let  $R$  be any ordered field, let  $C \subset R^n$  be a closed convex set, and let  $\mathbf{p} \in R^n \setminus C$ . Then there exists a continuous  $\tilde{R}$ -valued linear function  $L : R^n \rightarrow \tilde{R}$  such that  $L(\mathbf{p}) < 0$  and  $L(\mathbf{q}) > 0$  for all  $\mathbf{q} \in C$ .*

In order to prove Theorem I for general  $R$ , we let  $\bar{R}$  be a real closure of  $R$  and extend the Euclidean norm to the space

$$\tilde{R}^n := \text{Sper } \bar{R}[X_1, \dots, X_n],$$

which may be thought of as consisting of those points in  $\text{Sper } R[X_1, \dots, X_n]$  which induce the given order on  $R$ . This extended norm does not give us

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a metric on  $\tilde{R}^n$  because  $\tilde{R}^n$  is not equipped with an addition, and we cannot define an inner product on  $\tilde{R}^n$  as is discussed after Definition 2 in §3. However, with virtually no fudging of the definitions, we are able to prove

**Theorem II.** *Let  $R$  be any ordered field. Then there is a metric*

$$\mu : \tilde{R}^n \times \tilde{R}^n \rightarrow \tilde{R}$$

*which extends the Euclidean distance function.*

The point of this paper, then, is that the real spectrum provides the correct tool for both compactifying and completing general ordered fields with a view towards preserving at least some of the structure familiar from linear analysis over  $\mathbb{R}$ . We assume a certain degree of familiarity with the spaces  $\tilde{R}$  and  $\tilde{R}^n$ . In particular we assume that the reader is familiar with

- (1) the various types of points in  $\tilde{R}$  for general real closed  $R$ ,
- (2) the description given in [7 §2]
- (3) the “ultrafilter theorem” [3] and ultrafilter arguments, and
- (4) semialgebraic maps  $f : R^n \rightarrow R$  and their extensions  $\tilde{f} : \tilde{R}^n \rightarrow \tilde{R}$ .

All of the necessary background is available in [2, Chapter 7], [1], and [7, §2].

The paper is organized as follows: In §1 we present a simple counterexample showing why Theorem I is false if we only allow  $R$ -valued linear functions, and in §2 we present an elementary proof of Theorem I for  $\mathbb{R}$ . In §3 we explain what we mean by a  $\tilde{R}$ -valued linear function and prove a continuity result for these functions. In §4 we isolate two results needed to modify the proof given in §2, and in §5 we prove Theorem I for general  $R$ . In §6 we generalize the notion of “slices” from [7] to one called “rips”. Rips allow us to add points in  $\tilde{R}^n$  and to compare the resulting sums so that we may interpret the triangle inequality. In §7 we define  $\mu$  and prove Theorem II.

*We point out right away that, for the purposes of our proofs, we will assume that  $R$  is real closed.* This is no loss of generality since if  $\bar{R}$  is a real closure of the ordered field  $R$  and  $n \geq 1$ , then  $\tilde{R}^n = \tilde{\bar{R}}^n$  as we have defined it.

The motivation for this paper came from Bruce Reznick who conjectured that Theorem I (as stated for  $\mathbb{R}$ ) failed for nonarchimedean  $R$  during a talk on blenders in May, 1989. A few weeks later Tom Craven pointed out the counterexample in §1 for the case  $R = \mathbb{Q}$ . The author wishes to thank both Reznick and Craven for their inspiration and an anonymous referee for suggesting some clarifications and simplifications which have been incorporated.

## 1. A COUNTEREXAMPLE

Suppose  $R \neq \mathbb{R}$  and that  $\alpha$  is a (finite) Dedekind cut of  $R$  which is not represented by an element of  $R$ . In keeping with notation from [7, §2], we will represent  $\alpha$  by its *left set*  $\alpha_l$  and its *right set*  $\alpha_r$ . Thus  $\alpha_l$  and  $\alpha_r$  are

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We use the word “interval” to include singletons, half-infinite, half-open, closed, infinite, and half-infinite intervals—in short, any semialgebraically connected subset of  $R$ .

two nonempty open half-infinite intervals<sup>1</sup> with the properties that  $\alpha_l \leq \alpha_r$ , meaning that  $x \in \alpha_l, y \in \alpha_r \Rightarrow x \leq y$ , and that  $\alpha_l \cup \alpha_r = \mathbb{R}$ . In the case at hand,  $\alpha_l < \alpha_r$ .

Now consider the upper half-plane  $H \subset \mathbb{R}^2$  whose boundary is the non-existent line  $x_2 = \alpha x_1$ . Thus  $H$  consists of all points  $(0, x_2)$  with  $x_2 \geq 0$  plus all  $(x_1, tx_1) \in \mathbb{R}^2$  such that

$$(1) \quad x_1 \geq 0 \text{ and } t \in \alpha_r \quad \text{or} \quad x_1 \leq 0 \text{ and } t \in \alpha_l.$$

$H$  is closed and convex, but there is no way to separate  $H$  from any  $\mathbf{p} \notin H$  using a line. To see this, suppose  $t \in \alpha_r$  and consider the line  $x_2 = tx_1 + b$ . Using the fact that  $\alpha_r$  has no left endpoint, we may choose  $x_1$  positive and large enough so that  $tx_1 + b = t'x_1$  with  $t'$  still in  $\alpha_r$ . Thus  $(x_1, tx_1 + b) = (x_1, t'x_1) \in H$ . The same idea works for  $t \in \alpha_l$  and the lines  $x_1 = c$ . In other words, every line intersects  $H$ . This shows that Theorem I as stated for  $\mathbb{R}$  needs some modification for other  $\mathbb{R}$ .

### 2. A PROOF OF THEOREM I FOR $\mathbb{R}$

This section contains a very simple proof of Theorem I for  $\mathbb{R}$  based upon the following lemma:

**Lemma 1.** *Let  $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$  and suppose that  $\mathbf{p} \cdot \mathbf{q} < \|\mathbf{p}\|^2$ . Then for all sufficiently small  $t > 0$  we have*

$$(2) \quad \|t\mathbf{q} + (1-t)\mathbf{p}\|^2 < \|\mathbf{p}\|^2.$$

*Proof.* Use the fact that

$$\|t\mathbf{q} + (1-t)\mathbf{p}\|^2 - \|\mathbf{p}\|^2 = t\{t(\|\mathbf{q}\|^2 + \|\mathbf{p}\|^2 - 2\mathbf{q} \cdot \mathbf{p}) + 2(\mathbf{q} \cdot \mathbf{p} - \|\mathbf{p}\|^2)\}. \quad \square$$

*Proof of Theorem I over  $\mathbb{R}$ .* We may assume that  $\mathbf{p} = \mathbf{0}$ . Let  $\mathbf{a} \in C$  be a point with minimal distance to  $\mathbf{0}$ , i.e., with minimal Euclidean norm. The existence of  $\mathbf{a}$  is implied by the completeness of  $\mathbb{R}^n$ . Lemma 1 and convexity imply that

$$(3) \quad \mathbf{a} \cdot \mathbf{x} \geq \|\mathbf{a}\|^2$$

for all  $\mathbf{x} \in C$ . Thus  $L(\mathbf{x}) = \mathbf{a} \cdot (\mathbf{x} - \mathbf{a}/2)$  is the sought after function.  $\square$

In §5 we carry out this proof for general  $\mathbb{R}$ , but first we need to compactify the domain  $\mathbb{R}^n$ , complete the range  $\mathbb{R}$ , and define new linear functions. This is done via the real spectrum, starting in the next section.

### 3. $\tilde{\mathbb{R}}$ -VALUED LINEAR FUNCTIONS

The semialgebraic subsets of  $\mathbb{R}^n$  form a boolean algebra and  $\tilde{\mathbb{R}}^n$  may be defined as the set of ultrafilters of this algebra. (Remember, we are assuming that  $\mathbb{R}$  is real closed.) We will use this definition exclusively and denote points of  $\tilde{\mathbb{R}}^n$  by lower case Greek letters. The topology on  $\tilde{\mathbb{R}}^n$  is that generated by

the sets

$$\tilde{U} := \{ \alpha \in \tilde{R}^n \mid U \in \alpha \}$$

for open semialgebraic sets  $U \subset R^n$ .  $\tilde{R}^n$  is quasicompact but not Hausdorff. If  $\beta$  is in the closure of  $\alpha$  we say that  $\alpha$  *specializes* to  $\beta$  and write  $\alpha \rightarrow \beta$ . The concepts of closed and bounded for points are defined by

**Definition 1.** A point  $\alpha \in \tilde{R}^n$  is *closed* if it has no proper specializations and *bounded* if it contains a bounded set.

If  $A \subseteq R^n$  and  $B \subseteq R$  are semialgebraic sets, and  $f$  is a map with semialgebraic graph, then  $f(A)$  and  $f^{-1}(B)$  are also semialgebraic [2]. It is easy to verify that

$$(4) \quad \{ f(A) \mid A \in \alpha \}$$

generates an ultrafilter if  $\alpha \in \tilde{R}^n$  is an ultrafilter. Thus  $f$  induces a function  $\tilde{f}: \tilde{R}^n \rightarrow \tilde{R}$  where  $\tilde{f}(\alpha)$  is defined to be the ultrafilter in  $\tilde{R}$  generated by (4). If  $f$  is also continuous, then  $\tilde{f}$  is continuous as a function from  $\tilde{R}^n$  to  $\tilde{R}$ . Details appear in [1], [2, 7.2.8], and [2, 7.3].

As a particular example of this, fix  $\mathbf{x} \in R^n$  and consider the dual linear function

$$(5) \quad L_{\mathbf{x}}(\mathbf{y}) := \mathbf{x} \cdot \mathbf{y}.$$

Then  $L_{\mathbf{x}}$  is a continuous semialgebraic function, so we may extend it to a function

$$(6) \quad \tilde{L}_{\mathbf{x}}: \tilde{R}^n \rightarrow \tilde{R}$$

whose value at the ultrafilter  $\alpha$  is computed by considering each semialgebraic subset  $A \in \alpha$  and dotting each point in  $A$  with  $\mathbf{x}$ . This produces a semialgebraic subset of  $R$ , and the set of these sets forms an ultrafilter  $\tilde{L}_{\mathbf{x}}(\alpha)$  in  $\tilde{R}$ .

This process may be dualized to produce a function

$$(7) \quad L_{\alpha}: R^n \rightarrow \tilde{R}, \quad \mathbf{x} \mapsto L_{\mathbf{x}}(\alpha).$$

**Definition 2.** An  $\tilde{R}$ -valued linear function is a map  $L: R^n \rightarrow \tilde{R}$  of the form  $L(\mathbf{x}) = L_{\alpha}(\mathbf{x} - \mathbf{b})$  with  $L_{\alpha}$  as in (7) and  $\mathbf{b} \in R^n$ .

We remark that extending  $L_{\alpha}$  to all of  $\tilde{R}^n$ , i.e., defining an inner product on  $\tilde{R}^n$ , is too much to ask for. Indeed, for  $n = 1$  this would define a multiplication on  $\tilde{R}$ , and this is hopeless if  $R$  is nonarchimedean. To see why, let  $\eta$  be the ultrafilter of semialgebraic sets which span the gap between the positive infinitesimals and the positive noninfinitesimal elements (with respect to  $\mathbb{Q}$ ), and let  $\zeta$  be the ultrafilter spanning the gap between the positive finite and the positive infinite elements. The product of any set from  $\eta$  with any set from  $\zeta$  contains all positive, finite, noninfinitesimal elements. Thus “ $\eta \cdot \zeta$ ” is contained in an infinite number of ultrafilters.

Also, the function  $L_\eta(x)$  jumps from  $\eta$  to  $\zeta$  as the argument  $x \in R$  crosses the gap  $\zeta$ .  $L_\eta$  is therefore *not a slice* [7] and certainly cannot be extended to a continuous function  $\tilde{R}^n \rightarrow \tilde{R}$ . In general,  $L_\alpha$  need not even be continuous as function from  $R^n \rightarrow \tilde{R}$  as can be seen by taking  $n = 1$  and  $\alpha = +\infty$ . But we do have

**Proposition 1.** *If  $\alpha$  is closed and bounded, then the function  $L$  in Definition 2 is continuous. For any  $\alpha$  we have the following “sublinearity” property: If  $s_1 \leq L(\mathbf{x}) \leq s_2$  and  $t_1 \leq L(\mathbf{y}) \leq t_2$  for  $s_1, s_2, t_1, t_2 \in R$ , then  $s_1 + s_2 \leq L(\mathbf{x} + \mathbf{y}) \leq t_1 + t_2$ .*

*Proof.* We may assume that  $L = L_\alpha$ . A subbasic open set in  $\tilde{R}$  consists of the set  $\tilde{I}$  of ultrafilters containing an open interval  $I = (a, b) \subset R$ . We have

$$(8) \quad L_\alpha(\mathbf{x}) \in \tilde{I} \text{ if and only if } \mathbf{x} \cdot A \subseteq I \text{ for some } A \in \alpha.$$

Now, if  $L_\alpha(\mathbf{x}) \in \tilde{I}$ , there must be some closed bounded  $B \in \alpha$  such that  $\mathbf{x} \cdot B$  is a finite union of (necessarily closed and bounded) subintervals of  $(a, b)$ . For otherwise every closed bounded set  $B \in \alpha$  would contain points in the semialgebraic set

$$K := \{ \mathbf{y} \mid \mathbf{x} \cdot \mathbf{y} \leq a \text{ or } \mathbf{x} \cdot \mathbf{y} \geq b \}.$$

Since every closed semialgebraic set in  $\alpha$  contains a closed bounded semialgebraic set in  $\alpha$ , the set of  $B \cap K$  with  $B \in \alpha$  and  $B$  closed would then have the finite intersection property, from which the existence of a specialization  $\beta$  of  $\alpha$  with  $L_\beta(\mathbf{x}) \notin (a, b)$  would follow. Since  $L_\beta(\mathbf{x}) \neq L_\alpha(\mathbf{x})$ , we see  $\beta \neq \alpha$ , contradicting the assumption that  $\alpha$  is closed.

We have seen that there is a closed bounded  $B \in \alpha$  such that  $\mathbf{x} \cdot B \subseteq [c, d]$  with

$$(9) \quad a < c < d < b.$$

Using (9) and a bound on the norm of points in  $B$  it is straightforward to find an  $\varepsilon > 0$  so that  $\mathbf{z} \cdot B \subseteq I$  whenever  $\|\mathbf{x} - \mathbf{z}\| < \varepsilon$ . This implies  $L_\alpha(\mathbf{z}) \in \tilde{I}$  whenever  $\|\mathbf{x} - \mathbf{z}\| < \varepsilon$  and establishes continuity.

For the last statement, we note that there is a set  $A \in \alpha$  such that for every  $\mathbf{a} \in A$  we have  $s_1 \leq \mathbf{a} \cdot \mathbf{x} \leq s_2$  and  $t_1 \leq \mathbf{a} \cdot \mathbf{y} \leq t_2$  and hence  $s_1 + s_2 \leq \mathbf{a} \cdot (\mathbf{x} + \mathbf{y}) \leq t_1 + t_2$ .  $\square$

Finally, we point out that if  $R$  is an arbitrary ordered field and  $\alpha = \mathbf{x}$  a rational point in  $\overline{R}^n$  over a real closure of  $R$ , then  $L_\alpha$  is just the restriction of the dot product with  $\mathbf{x}$  to  $R^n$ .

#### 4. SOME GROUNDWORK

The purpose of this section is to recall some key results on  $\tilde{R}$  from [7] and to isolate a slightly technical but trivial lemma on abstract functions.

In [7, §2] it is shown that the points  $\alpha \in \tilde{R}$  for  $R$  real closed may be represented as pairs  $(\alpha_l, \alpha_r)$  of subsets of  $R$ , called *slices*, satisfying  $\alpha_l \leq \alpha_r$

and  $\alpha_l \cup \alpha_r = R$ . Specifically, for a point  $x \in R$  we have

$$(10) \quad \begin{aligned} x \in \alpha_l & \text{ if and only if } (-\infty, x] \in \alpha, \\ x \in \alpha_r & \text{ if and only if } [x, +\infty) \in \alpha. \end{aligned}$$

The crucial result for us is

**Proposition 2.**  $\tilde{R}$  is totally ordered with

$$(11) \quad \alpha \leq \beta \text{ if and only if } \alpha_l \subseteq \beta_l \text{ and } \beta_r \subseteq \alpha_r.$$

Every subset of  $\tilde{R}$  has both a supremum and an infimum in  $\tilde{R}$ .

*Proof.* [7, §2].  $\square$

The technical result we need, which we state in more generality than necessary, is a souped-up version of the result that the extension of a continuous semialgebraic function to  $\tilde{R}^n$  assumes a minimum on a closed subset: Let  $K \subseteq \tilde{R}^n$  be closed and let  $f : R^n \rightarrow R$  be a continuous semialgebraic function. Let

$$\lambda := \inf_{\alpha \in K} \tilde{f}(\alpha).$$

In addition, let  $\{g_s\}_{s \in \mathcal{S}}$  be a family of continuous semialgebraic functions from  $R^n$  to  $R$  and let  $\{\kappa_s\}_{s \in \mathcal{S}}$  be a family of closed points in  $\tilde{R}$  such that for any finite subset  $s_1, \dots, s_m \in \mathcal{S}$  and any  $\varepsilon \in R$  with  $\varepsilon \geq \lambda$  there are points  $\alpha \in K$  such that simultaneously  $\tilde{f}(\alpha) \leq \varepsilon$  and  $\tilde{g}_{s_i}(\alpha) \geq \kappa_{s_i}$  for  $i = 1, \dots, m$ .

**Lemma 2.** Under the hypotheses just given there is a closed point  $\zeta \in K$  with  $\tilde{f}(\zeta) = \lambda$  and  $g_s(\zeta) \geq \kappa_s$  for all  $s \in \mathcal{S}$ .

*Proof.* The sets

$$K_{\varepsilon, s_1, \dots, s_m} := \{ \alpha \in K \mid \tilde{f}(\alpha) \leq \varepsilon \text{ and } \tilde{g}_{s_i}(\alpha) \geq \kappa_{s_i} \text{ for } i = 1, \dots, m \}$$

are closed and have the finite intersection property. Therefore there is a closed point  $\zeta$  in their intersection.  $\square$

### 5. PROOF OF THEOREM I

Let  $C \subset R^n$  be closed and convex and assume that  $0 \notin C$  with the intention of proving Theorem I. Let

$$N^2(\mathbf{x}) = \|\mathbf{x}\|^2 = \sum_{i=1}^n x_i^2$$

be the square of the Euclidean norm, which is a continuous semialgebraic function. The following is a version of Lemma 1 from §2:

**Lemma 3.** Let  $\mathbf{p}$  and  $\mathbf{q}_1, \dots, \mathbf{q}_m$  be finitely many points in  $C$ . Then there is a point  $\mathbf{a} \in C$  such that  $N(\mathbf{a}) \leq N(\mathbf{p})$  and such that for  $i = 1, \dots, m$  we have

$$(12) \quad \mathbf{q}_i \cdot \mathbf{a} \geq N^2(\mathbf{a}).$$

*Proof.* We might as well assume that  $C$  is the convex hull of the  $\mathbf{q}_i$ . This is a closed bounded semialgebraic set, so there is a point  $\mathbf{a}$  in it of minimal norm [2, 2.5.8]. The calculation used to prove Lemma 1 is valid over any  $R$ , from which (12) follows.  $\square$

The same argument works in the case of an arbitrary closed bounded semialgebraic set  $C$  and (using the fact that a semialgebraic function achieves a maximum on  $C$  as well) yields

**Lemma 4.** *If  $C$  is a closed bounded semialgebraic set in  $R^n$  which does not contain  $0$ , then there is a vector  $\mathbf{x} \in R^n$  such that  $\mathbf{x} \cdot C$  is contained in a closed interval  $[a, b]$  with  $0 < a \leq b < \infty$ .  $\square$*

*Proof of Theorem I.* Let  $K$  be the closure of  $C$  in  $\tilde{R}^n$  and let

$$\lambda := \inf_{\mathbf{q} \in C} N(\mathbf{q}).$$

Since  $C$  is closed, we have  $0^+ < \lambda$ , so we may choose a  $z \in R$  with  $0 < z \leq \lambda$ . We now consider the semialgebraic function  $N$ , the family  $\{L_{\mathbf{q}}\}_{\mathbf{q} \in C}$  of semialgebraic functions, and the constant family  $\kappa_{\mathbf{q}} = z^2$ . If  $\varepsilon \geq \lambda$ , there is a point  $\mathbf{p} \in C$  with  $N(\mathbf{p}) \leq \varepsilon$ . Lemma 3 says that this setup satisfies the hypotheses of Lemma 2, so we find a closed  $\zeta \in K$  such that

$$(13) \quad \tilde{N}(\zeta) = \lambda \quad \text{and} \quad \tilde{L}_{\mathbf{q}}(\zeta) \geq z^2 \quad \text{for all } \mathbf{q} \in C.$$

Let  $\tilde{L}_{\zeta}$  be defined as in (7). From (13) we see both that  $\zeta$  is bounded and that  $\tilde{L}_{\zeta}(\mathbf{q}) \geq z^2$  for  $\mathbf{q} \in C$ . By Proposition 1 the function

$$L(\mathbf{x}) = \tilde{L}_{\zeta}(\mathbf{x} - \mathbf{b})$$

fulfills the requirements of Theorem I for any  $\mathbf{b}$  with  $-z^2/2 < \tilde{L}_{\zeta}(-\mathbf{b}) < 0$ .

To see that such a  $\mathbf{b}$  exists, note that  $\zeta$  contains a closed bounded  $B$  which does not contain  $0$ . We now apply Lemma 4 to  $B$  and let  $\mathbf{b} = -\frac{z^2}{2b}\mathbf{x}$  with  $\mathbf{x}$  and  $b$  as in the lemma.  $\square$

### 6. SLICES AND RIPS

In the description of  $\tilde{R}$  from [7], the set  $\alpha_l$  is called the *left set* of  $\alpha$  and consists of all  $x \in R$  with  $x \leq \alpha$ , while  $\alpha_r$  is called the *right set* of  $\alpha$  and consists of all  $x$  with  $\alpha_r \leq x$ . If we try to add two points  $\alpha$  and  $\beta$  by adding their left and right sets, we see that

$$\alpha_l + \beta_l \leq \alpha_r + \beta_r$$

and also that

$$(14) \quad \begin{aligned} x \in \alpha_l + \beta_l \text{ and } y < x &\Rightarrow y \in \alpha_l + \beta_l, \\ x \in \alpha_r + \beta_r \text{ and } y > x &\Rightarrow y \in \alpha_r + \beta_r, \end{aligned}$$

but in general there is a gap between the set sums in (14). For example, the point  $2$  is in neither the sum of the left sets nor in the sum of the right sets of “ $1^- + 1^+$ ”. In the nonarchimedean case entire intervals may be left out.

For this reason we are led to consider more general pairs

$$(15) \quad \zeta = (\zeta_l, \zeta_r)$$

of left and right sets and define

**Definition 3.** A *rip* is a pair (15) satisfying

$$(17) \quad \begin{aligned} &\zeta_l \leq \zeta_r, \\ &x \in \zeta_l \text{ and } y < x \Rightarrow y \in \zeta_l, \text{ and} \\ &x \in \zeta_r \text{ and } y > x \Rightarrow y \in \zeta_r. \end{aligned}$$

Thus points in  $\tilde{R}$  are special types of rips. If  $\zeta$  and  $\xi$  are rips, we define

$$\zeta + \xi = (\zeta_l + \xi_l, \zeta_r + \xi_r),$$

which is again a rip. Next, we borrow from John Conway’s philosophy (used to define the ordering on the surreal numbers in [JC, Chapter 1]) and define  $\zeta \leq \xi$  unless there is an obstruction to this inequality. An obstruction is a point  $x \in R$  with either  $\xi < x \leq \zeta$  or  $\xi \leq x < \zeta$ . We interpret  $\xi \leq x$  to mean that  $x \in \xi_r$  and  $\xi < x$  to mean that  $x \in \xi_r \setminus \xi_l$ . There is no such obstruction  $x$  if

$$\zeta_l \cap \xi_r \setminus \xi_l = \zeta_l \cap \xi_r \setminus \zeta_r = \emptyset,$$

which may be rephrased as

**Definition 4.** Let  $\zeta$  and  $\xi$  be two rips. We define  $\zeta \leq \xi$  if and only if and  $(\zeta_l \cap \xi_r) \subseteq (\zeta_r \cap \xi_l)$ .

**Proposition 3.** The “ $\leq$ ”-relation on rips extends the total ordering on  $\tilde{R}$ . Given two rips  $\zeta$  and  $\xi$ , either  $\zeta \leq \xi$ , or  $\xi \leq \zeta$  or both.

*Proof.* The first statement follows from the development of Definition 4 and can be checked by looking at cases. To prove the second statement, we need to exclude the possibility that there are  $x, y \in R$  with

$$(18) \quad \begin{aligned} &x \in \zeta_l \cap \xi_r \text{ and } x \notin \zeta_r \cap \xi_l, \\ &y \in \zeta_r \cap \xi_l \text{ and } y \notin \zeta_l \cap \xi_r. \end{aligned}$$

But if (18) holds, then  $x \in \zeta_l$  and  $y \in \zeta_r$  imply that  $x \leq y$ , while  $x \in \xi_r$  and  $y \in \xi_l$  imply that  $x \geq y$ . Hence  $x = y$ , but now (18) is clearly contradictory.  $\square$

### 7. THE METRIC $\mu$

To define  $\mu$ , consider  $\alpha, \beta \in \tilde{R}^n$ . If  $A \in \alpha$  and  $B \in \beta$ , we define

$$(19) \quad d(A, B) := \inf_{\substack{\mathbf{a} \in A \\ \mathbf{b} \in B}} \|\mathbf{a} - \mathbf{b}\|.$$

Note that the infimum in (19) is to be taken not in  $R$  (where it need not exist) but in  $\tilde{R}$ . We then go on to define

$$(20) \quad \mu(\alpha, \beta) := \sup_{\substack{A \in \alpha \\ B \in \beta}} d(A, B)$$

where the supremum is taken in  $\tilde{R}$  as well. Interpreting the values of  $\mu$  as rips for the triangle inequality, we have:

**Theorem II.** *The function  $\mu: \tilde{R}^n \times \tilde{R}^n \rightarrow \tilde{R}$  is a positive definite symmetric function satisfying the triangle inequality. Moreover, if  $\mathbf{a}, \mathbf{b} \in R^n \subset \tilde{R}^n$ , then  $\mu(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|$ .*

*Proof.* Symmetry is obvious from the definition as is the fact that  $\mu(\alpha, \beta) \geq 0$ . For the rest of the proof we let  $\alpha, \beta, \gamma \in \tilde{R}^n$ .

Suppose  $\mu(\alpha, \beta) = 0$ . Since  $\{x \in R \mid x > 0\}$  has the infimum  $0^+$  in  $\tilde{R}$ , every set  $A \in \alpha$  must have zero distance to every set  $B \in \beta$ . In other words,  $A \cap B \neq \emptyset$  for every  $A \in \alpha$  and  $B \in \beta$ . Thus  $\alpha \cup \beta$  is a filter, which implies  $\alpha = \beta$  since both are ultrafilters. Thus  $\mu$  is positive definite.

The triangle inequality states that

$$\mu(\alpha, \gamma) \leq \mu(\alpha, \beta) + \mu(\beta, \gamma).$$

To verify this we need to show that

$$(21) \quad (\mu(\alpha, \gamma)_l \cap (\mu(\alpha, \beta)_r + \mu(\beta, \gamma)_r)) \subseteq (\mu(\alpha, \gamma)_r \cap (\mu(\alpha, \beta)_l + \mu(\beta, \gamma)_l)).$$

So suppose there is a point  $z$  in the left-hand set given in (21). Since  $z \in \mu(\alpha, \beta)_r + \mu(\beta, \gamma)_r$ , there are points  $x, y \in R$  such that  $x + y = z$  and such that given any  $A \in \alpha$ ,  $B \in \beta$ , and  $C \in \gamma$  there are points  $\mathbf{a} \in A$ ,  $\mathbf{b} \in B$  and  $\mathbf{b}' \in B$ ,  $\mathbf{c} \in C$  with

$$(22) \quad \|\mathbf{a} - \mathbf{b}\| \leq x \quad \text{and} \quad \|\mathbf{b}' - \mathbf{c}\| \leq y.$$

Fixing  $A$ ,  $B$ , and  $C$ , consider the set

$$B' := \{\mathbf{b} \in B \mid d(A, \{\mathbf{b}\}) \leq x\}.$$

We must have  $B' \in \beta$ , for otherwise its complement would be in  $\beta$  and (22) could not hold for any  $\mathbf{b} \in B'$ . Now consider the set

$$(23) \quad B'' := \{\mathbf{b}' \in B' \mid d(\{\mathbf{b}'\}, C) \leq y\}.$$

Again, we must have  $B'' \in \beta$ . But this implies that  $d(A, C) \leq z$  for any  $A \in \alpha$  and  $C \in \gamma$ . Thus we conclude that  $z \in \mu(\alpha, \gamma)_r$ .

By assumption,  $z \in \mu(\alpha, \gamma)_l$ . Thus there are sets  $A \in \alpha$  and  $C \in \gamma$  with  $d(A, C) \geq z$ . Fix these sets. Suppose there were an  $E \in \beta$  with  $d(A, \{\mathbf{e}\}) < x$  for all  $\mathbf{e} \in E$ . By intersecting  $E$  with  $B''$  from (23) we would obtain  $d(A, C) < z$ . Thus the set

$$E := \{\mathbf{b} \in B' \mid d(A, \{\mathbf{b}\}) = x\}$$

is in  $\beta$ , from which we see that  $x \in \mu(\alpha, \beta)_l$ . Similarly,  $y \in \mu(\beta, \gamma)_l$ , and so  $z = x + y \in \mu(\alpha, \beta)_l + \mu(\beta, \gamma)_l$ , establishing (21).  $\square$

**Some examples.** In order to illustrate  $\mu$ , we state (without proof) some values in a few simple cases. First, a table of values for pairs of points in  $\tilde{R}$  with the first point specializing to 1 and the second to 2:

$$\begin{aligned} \mu(1^-, 2^-) &= 1^-, & \mu(1, 2^-) &= 1^-, & \mu(1^+, 2^-) &= 1^+, \\ \mu(1^-, 2) &= 1^+, & \mu(1, 2) &= 1, & \mu(1^+, 2) &= 1^-, \\ \mu(1^-, 2^+) &= 1^+, & \mu(1, 2^+) &= 1^+, & \mu(1^+, 2^+) &= 1^-. \end{aligned}$$

For a slightly more complicated case, consider the two-dimensional point  $\alpha \in \tilde{\mathbb{R}}^2$  consisting of all semialgebraic subsets  $A$  containing a set of the form  $\{(x, e^x) \mid 0 < x < \varepsilon\}$  for some  $\varepsilon > 0$ . Let  $\beta$  be the one-dimensional point consisting of all semialgebraic sets containing some piece of the algebraic half-branch  $y = 0, x > 0$  at  $(0, 0)$ . Then  $\mu(\alpha, \beta) = 1^+$ .

In order to clarify the nature of  $\mu$ , we point out

**Proposition 4.** *Suppose  $\alpha, \beta \in \tilde{R}^n$  and  $\alpha$  is bounded. Then  $\mu(\alpha, \beta) = 0^+$  if and only if  $\alpha$  and  $\beta$  are distinct but have a common specialization.*

*Proof.* Suppose  $\mu(\alpha, \beta) = 0^+$ . Let  $A \in \alpha$  be bounded, let  $B \in \beta$  be arbitrary, and let  $\bar{A}$  and  $\bar{B}$  be their closures in  $R^n$ . If we fix  $\mathbf{x}$ , then the function of  $\mathbf{x}$  which measures the distance to  $\bar{B}$  is a continuous semialgebraic function [2 2.5.8]. This takes a minimum on the closed bounded set  $\bar{A}$ . If  $\mu(\alpha, \beta) = 0^+$ , this minimum cannot be strictly positive, so  $\bar{A} \cap \bar{B} \neq \emptyset$ . We conclude that

$$(24) \quad \{\bar{A} \cap \bar{B} \mid A \in \alpha, B \in \beta\}$$

has the finite intersection property and is contained in at least one ultrafilter  $\gamma$ . Since any closed set in  $\alpha$  is in  $\gamma$ ,  $\alpha \rightarrow \gamma$ . Similarly,  $\beta \rightarrow \gamma$ .

If  $\alpha$  and  $\beta$  have no common specialization, we can find disjoint closed sets  $A \in \alpha$  and  $B \in \beta$  [7], and since we may choose  $A$  to be bounded, it is immediate that  $\mu(\alpha, \beta) \geq d(A, B) = x > 0$  for some  $x \in R$ , so  $\mu(\alpha, \beta) > 0^+$ .  $\square$

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