LINEAR SERIES WITH AN $N$-FOLD POINT ON A GENERAL CURVE

DAVID SCHUBERT

Abstract. A linear series $(V, L)$ on a curve $X$ has an $N$-fold point along a divisor $D$ of degree $N$ if $\dim(V \cap H^0(X, L(-D))) > \dim V - 1$. The dimensions of the families of linear series with an $N$-fold point are determined for general curves.

We work over the field of complex numbers $\mathbb{C}$.

Let $X$ be a smooth projective curve. A $g^r_d$ on $X$ is a linear series of dimension $r$ and degree $d$ on $X$, i.e., a pair $(V, L)$ consisting of a line bundle $L$ of degree $d$ and an $r + 1$ dimensional subspace $V \subset H^0(X, L)$. The $g^r_d$'s on $X$ are parameterized by a projective scheme $G^r_d(X)$. If $X$ is general in moduli, then $\dim G^r_d(X) = \rho(g, r, d) = g - (r + 1)(g + r - d)$ [ACGH].

Definition. We say that a $g^r_d(V, L)$ has an $N$-fold point along a divisor $D$ of degree $N \geq 2$ in $X$ if $\dim(V \cap H^0(X, L(-D))) > r$.

If $\pi : X \rightarrow V$ is a flat proper irreducible family of smooth curves, there exists a scheme $G^r_d(X/V)$ which is projective over $V$ whose fiber over each $v \in V$ is $G^r_d(X_v)$ where $X_v = \pi^{-1}(v)$ [EH-2]. It is easy to see that, by considering the construction of $G^r_d(X \times_v X^N/X^N)$ as the degeneracy locus of a vector bundle map, there is a closed subscheme $N(X/V) \subset G^r_d(X \times_v X^N/X^N)$ such that the fiber over each point $(P_1, \ldots, P_N) \in X^N_v$ consists of the $g^r_d$'s on $X_v$ with an $N$-fold point along $\sum P_j$. We will write $N(X)$ if $V$ is a point.

Let $\mathcal{M}_g$ denote the moduli space of smooth curves of genus $g \geq 3$, and let $U$ be the open subset of $\mathcal{M}_g$ corresponding to curves without nontrivial automorphisms. Let $Z \rightarrow U$ be the universal curve over $U$. We will prove the following theorem.

Theorem. With $g \geq N \geq 2$, $g \geq 3$, $r \geq 2$ and the notation as above, $\dim N(Z_u) \leq \rho(g, r, d) - r(N - 1) + N$ when $u$ is a sufficiently general point of $U$.

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This extends a result of Marc Coppens who proved the case when \( N = 2 \) \([C]\). It is easy to see that \( \rho(g, r, d) - r(N - 1) + N \) is a lower bound for the dimension of \( N(Z_u) \) when \( N(Z_u) \neq \emptyset \). If \( \rho(g, r, d) - r(N - 1) + N \geq 0 \) and \( \rho(g, r - 1, d - N) \geq 0 \) then \( N(Z_u) \neq \emptyset \) \([S]\).

Let \( X \) be a connected complete curve. We say that \( X \) is of compact type if and only singularities of \( X \) are ordinary double points, and the dual graph is a tree. We say a connected closed subcurve \( Y \subset X \) is a tail if it meets \( X - Y \) at at most one point. We say that a curve \( X \) of genus \( g \) is of special type if: it is of compact type; each irreducible component is a nonsingular rational or nonsingular elliptic curve; and each irreducible elliptic component is a tail.

We say that a sequence \( a = (a_0, a_1, \ldots, a_r) \) is of type \((r, d)\) if \( 0 \leq a_0 < a_1 < \cdots < a_r \leq d \). If \( X \) is a smooth curve containing a point \( P \), and \( L = (V, \mathcal{L}) \) is a \( g_d \) on \( X \), then the orders of vanishing of the sections of \( V \) determine a sequence \( a \) of type \((r, d)\). We call \( a \) the vanishing sequence of \( L \) at \( P \). We denote by \( W(a) = \sum (a_i - i) \) the weight of the sequence \( a \).

If \( b = (b_0, \ldots, b_r) \) is a sequence of type \((r, d)\), and a \( g_d \) \( L \) has vanishing sequence \( a = (a_0, \ldots, a_r) \) at \( P \), we say that \( L \) satisfies the vanishing condition \( b \) at \( P \) if \( a_i \geq b_i \) for \( i = 0, \ldots, r \).

In \( \S 2 \) we show the existence of a family of smooth curves \( X_{T-\{0\}} \rightarrow T - \{0\} \) which specialize to a curve \( X_0 \) of special type and a family \( A' \) of \( g_d' \)'s with \( N \)-fold points on \( X_{T-\{0\}} \rightarrow T - \{0\} \). The \( N \)-fold points of \( A' \) specialize to a genus \( M \) tail of \( X_0 \). The \( g_d' \)'s of \( A' \) also satisfy a vanishing condition \( a \) at a point. The relative dimension of \( A' \) over \( T - \{0\} \) is \( 0 \), and the codimension of \( A' \) in \( N(X_{T-\{0\}}/T - \{0\}) \) is \( \leq N - M + W(a) \).

In \( \S 3 \) we use the theory of limit linear series as developed by Eisenbud and Harris \([EH-2]\) to show that the crude limit linear series on \( X_0 \) induced by \( A' \) forces \( \rho(g, r, d) - r(N - 1) + M - W(a) \geq 0 \).

The products involving \( P^1 \) in the proof of Lemma 1 are taken over \( \text{Spec } C \). All other products are fibered over \( \mathcal{M}_g \) unless specified otherwise.

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We will make use of Knudsen's results concerning stable \( n \)-pointed curves \([K1, K2]\). A stable \( n \)-pointed curve is a connected projective curve \( X \) and \( n \) distinct nonsingular points \( P_1, \ldots, P_n \) of \( X \) such that: the only singularities of \( X \) are ordinary double points; and on every smooth rational component \( Y \subset X, \# \{ P_i \mid P_i \in Y \} + \# \{ Y \cap X - Y \} \geq 3 \). For each \( g \) and \( n \), there exists a coarse moduli space for \( n \)-pointed stable curves of genus \( g \) which we denote by \( \mathcal{M}_{g,n} \). For each \( g \) and \( n \), \( \mathcal{M}_{g,n} \) is a projective variety.

The functors of relative stable \( n \)-pointed curves and relative stable \( n - 1 \)-pointed curves with an additional section are isomorphic. For each stable \( n \)-
pointed curve \((X, P_1, \ldots, P_n)\) there is a curve \(X_c\) and morphism \(c: X \to X_c\) such that: \((X_c, c(P_1), \ldots, c(P_{n-1}))\) is a stable \(n-1\)-pointed curve; and either \(c\) is an isomorphism, or \(P_n\) lies on a rational component \(Y \subset X\) whose image in \(X_c\) is a point and \(c|_{X-Y}\) is an isomorphism of \(X-Y\) with \(X_c-c(Y)\).

When we wish to consider \(\mathcal{M}_{g,n}\) as coarsely representing the functor of stable \(n-1\)-pointed curves with an additional section, we will write \(\mathcal{M}_{g,n} \simeq \mathcal{Z}_{g,n-1}\). When \(n = 0\) we will write \(\mathcal{M}_g\) instead of \(\mathcal{M}_{g,0}\) and \(\mathcal{Z}_g\) instead of \(\mathcal{Z}_{g,0}\).

For each \(m < n\) and each subset \(\{i_1 < \cdots < i_m\}\) of \(\{1, \ldots, n\}\), we have a contraction morphism \(\pi: \mathcal{M}_{g,n} \to \mathcal{M}_{g,m}\) obtained by forgetting the points not indexed by \(\{i_1 < \cdots < i_m\}\) and collapsing certain rational subcurves, if necessary.

There is a natural clutching morphism
\[
\gamma: \mathcal{M}_{g_1,n+1} \times_c \mathcal{M}_{g_2,m+1} \to \mathcal{M}_{g_1+g_2,n+m}.
\]
If \((X, P_1, \ldots, P_{n+1})\) and \((Y, Q_1, \ldots, Q_{m+1})\) correspond to a point \((x, y) \in \mathcal{M}_{g_1,n+1} \times_c \mathcal{M}_{g_2,m+1}\), then \(\gamma(x, y)\) corresponds to the \(n+m\)-pointed curve obtained by joining \(X\) and \(Y\) at the points \(P_{n+1}\) and \(Q_{m+1}\).

Fix \(g \geq 3\), \(2 \leq N \leq g\), \(r \geq 2\), and \(d\). As before, we let \(U \subset \mathcal{M}_g\) be the open subset corresponding to smooth curves without nontrivial automorphisms, and we let \(\pi: Z \to U\) be the universal curve over \(U\). Let \(H \subset N(Z/U)\) be a component of the scheme of \(g_d\)'s with \(N\)-fold points whose dimension is maximal with respect to the property that \(\pi(H) = U\).

We have a natural morphism \(\alpha: H \to (\mathcal{Z}_g)^N\). Let \(B\) be the closure of \(\alpha(H)\) in \((\mathcal{Z}_g)^N\), and let \(\beta: B \to \mathcal{M}_g\) be the natural morphism. Note that \(\beta(B) = \mathcal{M}_g\), because \(\pi(H) = U\). Let \(M = N - \min\{\dim \beta^{-1}(x) \mid x \in \mathcal{M}_g\}\).

Note that \(0 \leq M \leq N\). If \(M = 0\), then the theorem holds, because the \(g_d\)'s on a general curve with an \(N\)-fold point along a divisor whose support is a general point have codimension \(r(N-1)\) by Theorem 4.5 of [EH-2]. We will henceforth assume \(M \geq 1\).

**Lemma 1.** There exists a point \(b \in B\) corresponding to a curve \(X_b\) which is of special type and points \(P_1, \ldots, P_N \in X_b\) which lie on a tail of genus \(\leq M\) or \(P_1 = P_2 = \cdots = P_N\) is a nonsingular point on a rational component of \(X_b\).

**Proof.** All products involving \(P^1\) in this proof are fibered over \(\text{Spec } \mathbb{C}\). All other products are fibered over \(\mathcal{M}_g\).

There is a set-theoretic map on closed points \(\delta: \mathcal{M}_{0,g} \times (\mathcal{Z}_g)^N \to (\mathcal{Z}_{0,g})^N\) which we will now describe. A point \(w \in \mathcal{M}_{0,g} \times (\mathcal{Z}_g)^N\) corresponds to a \(g\)-pointed rational curve \((X, Q_1, \ldots, Q_g)\) and \(N\) points \(P_1, \ldots, P_N\) on the stable genus \(g\) curve \(\tilde{X}_{Q_1, \ldots, Q_g}\) obtained by attaching a fixed elliptic curve \(E\) to \(X\) at each point \(Q_1, \ldots, Q_g\). Let \(\eta: \tilde{X}_{Q_1, \ldots, Q_g} \to X\) be the natural map which collapses the elliptic tails. The point \(\delta(w) \in (\mathcal{Z}_{0,g})^N\) corresponds to \((X, Q_1, \ldots, Q_g)\) and the \(N\) points \(\eta(P_1), \ldots, \eta(P_N)\).
For each \( I = (i_1, \ldots, i_N) \) such that \( 0 \leq i_j \leq g \) for \( j = 1, \ldots, N \) there is a scheme \( D_I \) which parameterizes \( g \)-pointed rational curves \( (X, Q_1, \ldots, Q_g) \) and points \( P_1, \ldots, P_N \) on \( \tilde{X}_{Q_1, \ldots, Q_g} \) such that each \( P_j \) lies on the elliptic tail attached to \( X \) at the point \( Q_j \) if \( i_j > 0 \), or \( P_j \) lies on \( X \) if \( i_j = 0 \). Each \( D_I \) is easily shown to exist by considering contraction and clutching morphisms.

Furthermore, there are morphisms \( \phi_I: D_I \rightarrow (\mathbb{Z}_0)^g \) and \( \Psi_I: D_I \rightarrow \mathcal{M}_{0,g} \times (\mathbb{Z}_g)^N \) such that \( \delta \circ \Psi_I = \phi_I \). It follows that \( \delta(p_Z^{-1}(B)) \) is a closed subset of \( (\mathbb{Z}_0, g)^N \) where \( p_Z \) is the projection of \( \mathcal{M}_{0,g} \times (\mathbb{Z}_g)^N \) to \( (\mathbb{Z}_g)^N \).

There are morphisms

\[
\xi_N: (\mathbb{Z}_0, g)^N \rightarrow (\mathbb{Z}_0, 3)^N \simeq (\mathbb{P}^1)^N \quad \text{and} \quad \xi_g: (\mathbb{Z}_0, g)^g \rightarrow (\mathbb{Z}_0, 3)^g \simeq (\mathbb{P}^1)^g
\]

which correspond to forgetting about the last \( g - 3 \) points on a \( g \)-pointed rational curve and designating the first three points as 0, 1 and \( \infty \). The morphisms \( \xi_N \) and \( \xi_g \) are products of contraction morphisms. There is a morphism \( \epsilon: \mathcal{M}_{0,g} \rightarrow (\mathbb{Z}_0, g)^g \) which corresponds to associating the \( g \)-pointed rational curve \( (X, Q_1, \ldots, Q_g) \) to itself and the points \( Q_1, \ldots, Q_g \) on \( X \). The morphism \( \epsilon \) exists because of the functorial properties \( \mathcal{M}_{0,g} \) and \( \mathbb{Z}_0, g \).

We get a set-theoretic map

\[
f = (\xi_g \circ \epsilon \circ p_M) \times (\xi_N \circ \delta): \mathcal{M}_{0,g} \times (\mathbb{Z}_g)^N \rightarrow (\mathbb{P}^1)^g \times (\mathbb{P}^1)^N
\]

where \( p_M \) is the projection of \( \mathcal{M}_{0,g} \times (\mathbb{Z}_g)^N \) onto \( \mathcal{M}_{0,g} \). Furthermore, \( f(p_Z^{-1}(B)) \) is closed in \( (\mathbb{P}^1)^g \times (\mathbb{P}^1)^N \).

We will prove the lemma by showing that there exists a point

\[
(Q_1, \ldots, Q_g, P_1, \ldots, P_N) \in f(p_Z^{-1}(B)) \subset (\mathbb{P}^1)^g \times (\mathbb{P}^1)^N
\]

such that either: (i) \( P_1 = \cdots = P_N \) and at most \( M \) of the \( Q_i \) are equal to \( P_1 \); or (ii) there exists a point \( Q \in \mathbb{P}^1 \) such that all of the \( P_i \) are distinct from \( Q \) and at least \( g - M \) of the \( Q_i \) are equal to \( Q \). Suppose that \( (X, Q_1, \ldots, Q_g) \) is a stable \( g \)-pointed rational curve. Let \( \tilde{X}_{Q_1, \ldots, Q_g} \) be the curve obtained by attaching the elliptic curve \( E \) at each \( Q_i \), and let \( \eta: \tilde{X}_{Q_1, \ldots, Q_g} \rightarrow X \) be the map which collapses the elliptic tails as before. The map \( \tilde{Z}_{0,g} \rightarrow \tilde{Z}_{0,3} \simeq \mathbb{P}^1 \) corresponding to forgetting about the last \( g - 3 \) points of a \( g \)-pointed rational curve and designating the first three points as 0, 1 and \( \infty \) induces a morphism \( X \rightarrow \mathbb{P}^1 \) and hence a morphism \( \tilde{X}_{Q_1, \ldots, Q_g} \rightarrow \mathbb{P}^1 \). Note that if \( P \) is a point in \( \mathbb{P}^1 \) and \( k \) of the points \( Q_1, \ldots, Q_g \) map to \( P \), then the pre-image of \( P \) in \( \tilde{X}_{Q_1, \ldots, Q_g} \) is a tail of genus \( k \). Hence condition (i) above implies that there exists a point in \( B \) corresponding to a curve and \( N \) points which lie on a tail of genus \( \leq M \). Note that the preimage of \( \mathbb{P}^1 \{ \{P \} \} \) in \( \tilde{X}_{Q_1, \ldots, Q_g} \) is a tail of genus \( g - k \). Thus condition (ii) will also imply the lemma.
Let $D_k = \{(P_1, \ldots, P_N) \in (\mathbb{P}^1)^N \mid \text{at least } k \text{ points coincide}\}$. We will use the fact that if $Y$ is a closed subset of $(\mathbb{P}^1)^N$ and $\dim(Y) \geq k - 1$, then $D_k \cap Y \neq \emptyset$. This fact follows from: the diagonal is ample in $\mathbb{P}^1 \times \mathbb{P}^1$; thus $\dim(D_k \cap Y) \geq 1$ implies $\dim(D_{k+1} \cap Y) \neq \emptyset$; and codim $D_k = k - 1$.

Let $p_g$ and $p_N$ denote the projections of $(\mathbb{P}^1)^g \times (\mathbb{P}^1)^N$ to $(\mathbb{P}^1)^g$ and $(\mathbb{P}^1)^N$, respectively.

The dimension of the fiber of $f(p_Z^{-1}(B))$ over each point of $(0, 1, \infty) \times (\mathbb{P}^1)^{g-3}$ is $\geq N - M$, because the dimension of the fiber of $B$ over each point of $\mathbb{M}_g$ is $\geq N - M$. Hence

$$p_g(f(p_Z^{-1}(B)) \cap (\mathbb{P}^1)^g \times D_{N-M+1}) = (0, 1, \infty) \times (\mathbb{P}^1)^{g-3}.$$ 

Let $\kappa = \max\{k \mid (\mathbb{P}^1)^g \times D_k \cap f(p_Z^{-1}(B)) \neq \emptyset\}$. We will prove the lemma by showing that condition (i) holds if $\kappa = N$ and condition (ii) holds if $\kappa \leq N - 1$.

Suppose $\kappa = N$. Then $p_g(f(p_Z^{-1}(B)) \cap (\mathbb{P}^1)^g \times D_N)$ has codimension $\leq N - (N - M + 1) = M - 1$ in $(0, 1, \infty) \times (\mathbb{P}^1)^{g-3}$. Hence

$$\dim(p_g(f(p_Z^{-1}(B)) \cap (\mathbb{P}^1)^g \times D_N)) \geq g - 2 - M.$$ 

Let $E_{M+1} = \{(Q_1, \ldots, Q_g) \in (0, 1, \infty) \times (\mathbb{P}^1)^{g-3} \mid \text{at least } M+1 \text{ of the points } Q_1, \ldots, Q_g \text{ coincide}\}$. We have

$$\dim(E_{M+1}) = g - 3 - M < \dim(p_g(f(p_Z^{-1}(B)) \cap (\mathbb{P}^1)^g \times D_N)).$$ 

Thus we can find $(Q_1, \ldots, Q_g, P_1, \ldots, P_N) \in f(p_Z^{-1}(B))$ so that $P_1 = \cdots = P_N$ and at most $M$ of the $Q_i$'s are equal to $P_i$.

Suppose $\kappa \leq N - 1$. Let $W$ be a component of $f(p_Z^{-1}(B)) \cap (\mathbb{P}^1)^g \times D_k$. Then $p_N(W)$ is a point $(P_1, \ldots, P_N) \in (\mathbb{P}^1)^N$, because otherwise $(\mathbb{P}^1)^g \times D_{k+1} \cap f(p_Z^{-1}(B)) \neq \emptyset$. Choose a point $R \in \mathbb{P}^1$ so that $R \neq P_i$ for $i = 1, \ldots, N$. If $\kappa = N - 1$, we choose $R \in \{0, 1, \infty\}$. Now $W$ has codimension $\leq \kappa - (N - M + 1)$ in $f(p_Z^{-1}(B)) \cap (\mathbb{P}^1)^g \times D_{N-M+1}$ so $p_g(W)$ has codimension $\leq \kappa - (N - M + 1)$ in $(0, 1, \infty) \times (\mathbb{P}^1)^{g-3}$. Thus $\dim p_g(W) \geq g - M$ if $\kappa \leq N - 2$, and $\dim p_g(W) \geq g - M - 1$ if $\kappa = N - 1$. Since $p_g(W)$ is closed in $(0, 1, \infty) \times (\mathbb{P}^1)^{g-3}$ there are $\dim(p_g(W))$ factors of $(\mathbb{P}^1)^{g-3}$ so that the projection of $p_g(W)$ to the product of these factors is onto. Thus there is a point $(Q_1, \ldots, Q_g) \in p_g(W) \subset (0, 1, \infty) \times (\mathbb{P}^1)^{g-3}$ so that at least $g - M$ of the $Q_i$'s are equal to $R$. Thus condition (ii) holds and the lemma follows.

We can find a smooth curve $T$ containing a point $0$ and a morphism $\phi: T \to B \subset (\mathbb{Z}_g)^N$ such that $\phi(0)$ is the point $b \in B$ described in Lemma 1, and the induced map $T \to \mathbb{M}_g$ sends $T - \{0\}$ to the subset of $U$ where the fibers of $\beta: B \to \mathbb{M}_g$ have dimension $N - M$.

After replacing $T$ with a base extension, if necessary, there is a family of $0$-pointed stable curves $X \to T$ which corresponds to the morphism $T \to \mathbb{M}_g$. 

Note that if $T$ is replaced with a base extension and the singularities of $X$ are resolved by blowing up, the curve $X_0$ will change by inserting chains of rational curves at the nodes of $X_0$. Thus we may assume (by replacing $T$ with a base extension and blowing up the singularities of $X$ if necessary) that there is a family of curves $X \to T$ which extends $Z \times_{U} (T - \{0\})$ and has the following properties: (1) $X_0$ is of special type; (2) there is a tail $Y$ of $X_0$ of genus $\leq M$ so that the sections $s_i : T \to X$ induces by the map $\phi : T \to B = (\mathbb{P}_k)_{\text{rat}}^N$ are such that each $s_i(0)$ is smooth point of $X_0$ which lies in $Y$ (if $P_1 = P_2 = \cdots = P_N$ is a point on a rational component of $X_0$, then, after blowing up, there will exist $s_i$ such that the $s_i(0)$ lie on a tail of genus 0 in $X_0$); (3) there is a section $s : T \to X$ such that $s(0)$ is a smooth point of $X_0$ which lies in a rational component of $X_0 - Y$; and (4) $X$ is smooth. Note that these properties are unchanged if $T$ is replaced with a base extension which sends one point to 0 and the singularities of the new $X$ are resolved by blowing up.

**Theorem 2.** There is a sequence $a$ of type $(r, d)$ and a closed subscheme $A$ of $H \times_B (T - \{0\})$ such that: $A$ consists of all linear series in $H \times_B (T - \{0\})$ which satisfy vanishing condition $a$ along $s(T - \{0\})$; the fibers $A_t$ are nonempty for each $t \in T - \{0\}$; and $\dim A_t = 0$ for all but finitely many $t \in T - \{0\}$.

**Proof.** Note that $H \times_B (T - \{0\})$ is proper over $T - \{0\}$, and the subset of linear series satisfying a particular vanishing condition along $s(T - \{0\})$ is closed. Since there are only finitely many sequences of type $(r, d)$, the lemma is a consequence of the following.

**Lemma 2a.** Let $X$ be a smooth curve. Let $A$ be a closed subset of $Gr^d(X)$, and let $a$ be a sequence of type $(r, d)$. If $\dim A \geq 1$ and every linear series in $A$ satisfies vanishing condition $a$ at a point $P \in X$, then there exists a sequence $a'$ with $a'_i \geq a_i$ for $i = 0, \ldots, r$ and $a'_k > a_k$ for some $k$ such that $A$ contains a linear series which satisfies vanishing condition $a'$ at $P$.

**Proof.** We may assume $A$ is irreducible. Consider the natural map $\Phi : A \to \text{Pic}^d(X)$. Suppose for some $x \in \text{Pic}^d(X)$, the fiber $\Phi^{-1}(x)$ has dimension $\geq 1$, and let $\mathcal{L}$ be the line bundle corresponding to $x$. The set of $(r + 1)$ dimensional vector spaces of $H^0(X, \mathcal{L})$ which have vanishing sequence $a$ at $P$ is a Schubert variety, and hence is affine. It follows that $A$ contains a $g^f_d$ which does not have vanishing sequence $a$ at $P$, because $\Phi^{-1}(x)$ is closed and $\dim \Phi^{-1}(x) \geq 1$. So we may assume $\Phi(A)$ contains a closed curve in $\text{Pic}^d(x)$.

If the lemma were false, then there would exist a family $F$ of $g^0_{d-a}$'s obtained from $A$ by taking $(H^0(Y, \mathcal{L}(-a)) \cap V, \mathcal{L}(-a))$ for every $(V, \mathcal{L})$ of $A$. But a family of $g^0_{d-a}$'s is a family of divisors of degree $d - a$. Since $\dim(F) \geq 1$, it must contain a divisor with $P$ in its support. But the linear series in $A$ associated with this divisor satisfies vanishing condition $(a_0, \ldots, a_{r-1}, a_r + 1)$.
If $T$ is replaced with an appropriate base extension, and the singularities of $X$ blown up, we may also assume that $A \to T - \{0\}$ gives an isomorphism $A' \to T - \{0\}$ for some component $A'$ of $A$.

The codimension of $H \times_B T$ in $H \times_U T$ is $N - M$, and $A'$ has codimension $\leq W(a)$ in $H \times_B T$. Thus the theorem will follow when we show that $W(a) + N - M \leq \rho(g, r, d) - r(N - 1) + N$.

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The family of curves $X \to T$ and $A'$ determine a crude limit linear series on $X_0$ [EH-2]. This crude limit linear series is a collection consisting of a $g^r_d$ for each of the components of $X_0$. If $C$ is a component of $X_0$, the $g^r_d(V_C, \mathcal{L}_C)$ on $C$ is determined in the following manner. Let $R$ be the local ring of $T$ at 0, and let $\eta$ be the generic point of Spec $R$. Then $A'$ determines a unique line bundle $\mathcal{L}_C$ on $X_{\text{Spec}R}$ such that the restriction of $\mathcal{L}_C$ to $X_0$ has degree 0 on every component of $X_0$ except $C$. Also, $A'$ determines a subspace $V \subset H^0(X_\eta, \mathcal{L}_C|_{X_\eta})$. Let $\tilde{V}_C = V \cap H^0(X_{\text{Spec}R}, \mathcal{L}_C)$. Then $V_C = \tilde{V}_C \otimes k(0) \subset H^0(X_0, \mathcal{L}_C|_{X_0})$ and $\mathcal{L}_C = \mathcal{L}_C|_C$ where $k(0)$ the residue field of $R$.

Let $Q = Y \cap (\overline{X_0})$ and let $Y'$ be the component of $Y$ containing $Q$. Let $Y_1, \ldots, Y_k$ be the remaining components of $Y$. In addition to parameterizing a family of $g^r_d$'s on $X_{T - \{0\}} \to T - \{0\}$, $A'$ also determines a family of $g^r_{d - N}$'s corresponding to the sections which vanish along $\sum s_i(T - \{0\})$.

Let $(W_{Y'}, \eta_{Y'})$ be the $g^r_{d - N}$ on $Y'$ of the limit $g^r_{d - N}$ on $X_0$. Then

$$\eta_{Y'} = \mathcal{L}_{Y'} \left( - \sum s_i(T) - \sum n_i Y_i \right)$$

where $n_i \geq 0$,

so $\eta_{Y'} = \eta_{Y'}|_{Y'} = \mathcal{L}_{Y'}(-\sum Q_i)$ where the $Q_i$ are points not equal to $Q$. Also,

$$W_{Y'} = (\tilde{V}_{Y'} \cap H^0(X_{\text{Spec}R}, \mathcal{L}_{Y'})) \otimes k(0) \subset V_{Y'} \cap H^0(\eta_{Y'}).$$

So the vanishing sequence $b$ of $(W_{Y'}, \eta_{Y'})$ at $Q$ is a subsequence of the vanishing sequence $c$ of $(V_{Y'}, \mathcal{L}_{Y'})$ at $Q$.

The following lemma is an immediate consequence of Theorem 4.5 of [EH-2] and Theorem 2.3 of [EH-1].

**Lemma 3.** If $C$ is a genus $g$ curve of special type and $P$ and $Q$ are two smooth points in rational components of $C$, then the existence of a crude limit $g^r_d$ on $C$, which satisfies vanishing conditions $a$ and $b$ at points $P$ and $Q$, respectively, implies that $\rho(g, r, d) - W(a) - W(b) \geq 0$.

Now the crude limit $g^r_{d - N}$ on $X_0$ determined by $A'$ restricts to a limit $g^r_{d - N}$ on the genus $M$ tail $Y$ which satisfies vanishing condition $b$ at $Q$. Hence $\rho(M, r - 1, d - N) - W(b) \geq 0$. Since $b$ is a subsequence of $c$ we have $W(c) \leq W(b) + d - r$. So $W(c) \leq \rho(M, r - 1, d - N) + d - r$. 


Let $F$ be the component of $X_0 - Y$ which contains $Q$. If $c'$ is the vanishing sequence of $V_F$ at $Q$, then the definition of crude limit linear series requires

$$W(c') \geq (r + 1)(d - r) - W(c)$$

$$\geq r(d - r) - p(M, r - 1, d - N).$$

Now the crude limit $g'_d$ on $X_0$ determined by $A'$ restricts to a crude limit $g'_d$ on $X - Y$ satisfying vanishing conditions $c$ at $Q$ and $a$ at $s(0)$. Thus,

$$0 \leq \rho(g - M, r, d) - W(a) - W(c')$$

$$\leq \rho(g - M, r, d) - W(a) + \rho(M, r - 1, d - N) - r(d - r)$$

$$= (r + 1)(d - r) - r(g - M) - w(a)$$

$$+ r(d - N - r + 1) - (r - 1)M - r(d - r)$$

$$= (r + 1)(d - r) - rg - W(a) - r(N - 1) + N - (N - M)$$

$$= \rho(g, r, d) - W(a) - r(N - 1) + N - (N - M).$$

Thus $W(a) + N - M \leq \rho(g, r, d) - r(N - 1) + N$, and the theorem follows.

REFERENCES


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Department of Mathematics, State University of New York, College at Geneseo, Geneseo, New York 14454