

A DENSE SET OF OPERATORS WITH TINY COMMUTANTS

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ABSTRACT. For a (bounded linear) operator T on a complex, separable, infinite-dimensional Hilbert space \mathcal{H} , let $\mathcal{A}(T)$ and $\mathcal{A}^a(T)$ denote the weak closure of the polynomials in T and, respectively, the weak closure of the rational functions with poles outside the spectrum of T . Let $\mathcal{A}'(T)$ and $\mathcal{A}''(T)$ denote the commutant and, respectively, the double commutant of T . We say that T has a *tiny commutant* if $\mathcal{A}'(T) = \mathcal{A}^a(T)$. By constructing a large family of “models” and by using standard techniques of approximation, it is shown that $\{T \in \mathcal{L}(\mathcal{H}) : T \text{ has a tiny commutant}\}$ is norm-dense in the algebra $\mathcal{L}(\mathcal{H})$ of all operators acting on \mathcal{H} . Other related results: Let $\text{Lat } \mathcal{B}$ denote the invariant subspace lattice of a subalgebra \mathcal{B} of $\mathcal{L}(\mathcal{H})$. For a Jordan curve $\gamma \subset \mathbb{C}$, let $\hat{\gamma}$ denote the union of γ and its interior; for $T \in \mathcal{L}(\mathcal{H})$, let $\rho_{s-F}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is a semi-Fredholm operator}\}$, and let $\rho_{s-F}^+(T)(\rho_{s-F}^-(T)) = \{\lambda \in \rho_{s-F}(T) : \text{ind}(\lambda - T) > 0 \text{ } (< 0, \text{ resp.})\}$. With this notation in mind, it is shown that $\{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}(T) = \mathcal{A}^a(T)\}^- = \{T \in \mathcal{L}(\mathcal{H}) : \text{Lat } \mathcal{A}(T) = \text{Lat } \mathcal{A}^a(T)\}^- = \{A \in \mathcal{L}(\mathcal{H}) : \text{if } \gamma \text{ (Jordan curve)} \subset \rho_{s-F}^\pm(A), \text{ then } \hat{\gamma} \subset \sigma(A)\}$; moreover, $\{A \in \mathcal{L}(\mathcal{H}) : \text{if } \gamma \text{ (Jordan curve)} \subset \rho_{s-F}^\pm(A), \text{ then } \text{ind}(\lambda - A) \text{ is constant on } \hat{\gamma} \cap \rho_{s-F}(A) \subset \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}(T) = \mathcal{A}'(T)\}^- \subset \{T \in \mathcal{L}(\mathcal{H}) : \text{Lat } \mathcal{A}(T) = \text{Lat } \mathcal{A}'(T)\} \subset \{A \in \mathcal{L}(\mathcal{H}) : \text{if } \gamma \text{ (Jordan curve)} \subset \rho_{s-F}^\pm(A), \text{ then } \hat{\gamma} \cap \rho_{s-F}(A) \subset \rho_{s-F}^\pm(A)\} \subset \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}(T) = \mathcal{A}^a(T)\}$. (The first and the last inclusions are proper.) The results also include a partial analysis of $\text{Lat } \mathcal{A}''(T)$.

1. INTRODUCTION

To each operator T we can naturally associate four weakly closed subalgebras with identity; namely,

$$\mathcal{A}(T) = \text{the weak closure of the polynomials in } T \text{ and } 1,$$

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$\mathcal{A}^a(T)$ = the weak closure of the rational functions in T with poles outside the spectrum, $\sigma(T)$, of T ,

$\mathcal{A}'(T) = \{A \in \mathcal{L}(\mathcal{H}) : TA = AT\}$ = the *commutant* of T , and

$\mathcal{A}''(T) = \{B \in \mathcal{L}(\mathcal{H}) : AB = BA \text{ for all } A \in \mathcal{A}'(T)\}$ = the *double commutant* of T .

(Here $\mathcal{L}(\mathcal{H})$ denotes the algebra of all bounded linear operators acting on the complex, separable, infinite-dimensional Hilbert space \mathcal{H} .)

Clearly, $\mathcal{A}(T) \subset \mathcal{A}^a(T) \subset \mathcal{A}''(T) \subset \mathcal{A}'(T)$, and the corresponding invariant subspace lattices satisfy the reverse inclusions,

$$\text{Lat } \mathcal{A}(T) \supset \text{Lat } \mathcal{A}^a(T) \supset \text{Lat } \mathcal{A}''(T) \supset \text{Lat } \mathcal{A}'(T).$$

($\text{Lat } \mathcal{A}(T) = \text{Lat } T$ is just the invariant subspace lattice of T ; $\text{Lat } \mathcal{A}'(T)$ is the lattice of *hyperinvariant* subspaces of T , also denoted by $\text{Hyperlat } T$.)

The main result of this article says that operators with “tiny” commutants, in the sense that $\mathcal{A}'(T) = \mathcal{A}^a(T)$, form a norm-dense subset of $\mathcal{L}(\mathcal{H})$. Obviously, the same result is true if the condition “ $\mathcal{A}'(T) = \mathcal{A}^a(T)$ ” is replaced by “ $\mathcal{A}''(T) = \mathcal{A}^a(T)$,” or “ $\text{Lat } \mathcal{A}^a(T) = \text{Hyperlat } T$,” or any other weaker condition. (Indeed, the density of $\{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}''(T) = \mathcal{A}'(T)\}$ follows immediately from [10, Class (D), p. 109].)

On the other hand, an invariant subspace \mathcal{M} of T is invariant under $\mathcal{A}^a(T)$ if and only if $\sigma(T|_{\mathcal{M}}) \subset \sigma(T)$ [17]. (By *subspace* we always mean a closed linear manifold of \mathcal{H} ; $T|_{\mathcal{M}}$ denotes the restriction of T to \mathcal{M} .)

By using this observation, it is possible to show that $\{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}(T) = \mathcal{A}^a(T)\}$ and $\{T \in \mathcal{L}(\mathcal{H}) : \text{Lat } T = \text{Lat } \mathcal{A}^a(T)\}$ have exactly the same (norm) closure, and this closure admits a very simple spectral description and it is nowhere dense in $\mathcal{L}(\mathcal{H})$ (see Theorem 4.1 below).

This article sprouted out of a question of John B. Conway (personal communication):

What is the closure of the set $\{T \in \mathcal{L}(\mathcal{H}) : \text{Lat } T = \text{Hyperlat } T\}$?

Although unable to answer this question, the author shows that the closure of the set of operators T with $\text{Lat } T = \text{Hyperlat } T$ can be “sandwiched” between two nowhere dense closed subsets of $\mathcal{L}(\mathcal{H})$ (both having simple spectral descriptions). In particular, *the closure of this set is strictly included in* $\{T \in \mathcal{L}(\mathcal{H}) : \text{Lat } T = \text{Lat } \mathcal{A}^a(T)\}^-$.

All the subsets of $\mathcal{L}(\mathcal{H})$ considered here are similarity-invariant, and therefore we can apply the approximation machinery developed in the monographs [2] and [11] (see also [14], [15]).

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2. CONSTRUCTION OF MODELS

The core of the article is the following highly technical result on the existence of a peculiar type of operators with tiny commutants. The applications of these “models” will be developed in §§3, 4, and 5.

Proposition 2.1. *Let \mathbf{D} denote the open unit disk and let $\{\mathbf{D}(\alpha_j, r_j)\}_{j=1}^k$ be a finite collection of open disks with pairwise disjoint closures, such that $\mathbf{D}(\alpha_j, r_j)^-$ is included in \mathbf{D} ($k = 1, 2, \dots, k$; $\mathbf{D}(\alpha, r) = \{\lambda \in \mathbf{C}: |\lambda - \alpha| < r\}$, $r + |\alpha| < 1$). Let $\Omega = \mathbf{D} \setminus \bigcup_{j=1}^k \mathbf{D}(\alpha_j, r_j)^-$.*

Given an index m , $1 \leq m \leq \infty$, there exists T in $\mathcal{L}(\mathcal{H})$ such that

- (i) $\sigma(T) = \Omega^-$ and $\sigma_{le}(T) = \partial\Omega$;
- (ii) $\ker(\lambda - T) = \{0\}$ and $\dim \ker(\lambda - T)^* = m$ for all $\lambda \in \Omega$;
- (iii) $\lambda - T$ is a semi-Fredholm operator of index $-m$ for all $\lambda \in \Omega$, and
- (iv) $\mathcal{A}^a(T) = \mathcal{A}'(T)$ is isomorphic (but not necessarily isometric) with $H^\infty(\Omega)$.

Here $\sigma_{le}(T)$ and $\sigma_{re}(T)$ denote the left and, respectively, the right essential spectrum of T ; $\sigma_{lre}(T) = \sigma_{le}(T) \cap \sigma_{re}(T)$ is the complement in \mathbf{C} of the semi-Fredholm domain of T , $\rho_{s-F}(T) = \{\lambda \in \mathbf{C}: \lambda - T \text{ is a semi-Fredholm operator}\}$. The essential spectrum is $\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{re}(T)$.

Proposition 2.2. *Let $\{\alpha_j\}_{j=1}^k$ be a finite subset of the open unit disk, and let $\Omega = \mathbf{D} \setminus \{\alpha_j\}_{j=1}^k$.*

Given an index m , $1 \leq m \leq \infty$, there exists T in $\mathcal{L}(\mathcal{H})$ such that

- (i) $\sigma(T) = \mathbf{D}^-$ and $\sigma_{le}(T) = \partial\mathbf{D} \cup \{\alpha_j\}_{j=1}^k$ ($= \partial\Omega$);
- (ii) $\ker(\lambda - T) = \{0\}$ and $\dim \ker(\lambda - T)^* = m$ for all $\lambda \in \Omega$;
- (iii) $\lambda - T$ is a semi-Fredholm operator of index $-m$ for all $\lambda \in \Omega$, and
- (iv) $\mathcal{A}(T) = \mathcal{A}'(T)$ is isometrically isomorphic with $H^\infty(\mathbf{D})$.

The reader is referred to [4], [19] for definition and properties of the semi-Fredholm operators, and to treatise of B. Sz.-Nagy and C. Foiaş [23] for the definition and properties of the $H^\infty(\mathbf{D})$ functional calculus for contractions.

Our first result improves upon an example due to W. R. Wogen. In [25], Wogen proved that for each cardinal m , $1 \leq m \leq \infty$, there is an operator A in $\mathcal{L}(\mathcal{H})$ such that $\mathcal{A}'(A)$ has multiplicity m ; A is a particular type of unilateral operator weighted shift: Let \mathcal{R} be a Hilbert space of dimension m , and let $\{g_k\}_{k=1}^\infty$ be a denumerable dense subset of the unit sphere of \mathcal{R} .

Define $R_k = (1 - P_k) + (1/k)P_k$, where P_k denotes the orthogonal projection of \mathcal{R} onto the one-dimensional space spanned by g_k , $k = 1, 2, \dots$.

Wogen proved that if

$$T = \begin{pmatrix} 0 & & & & \\ T_0 & 0 & & & \\ & T_1 & 0 & & \mathbf{0} \\ & & T_2 & 0 & \\ & & & \cdot & \cdot \\ & \mathbf{0} & & \cdot & \cdot \\ & & & \cdot & \cdot \end{pmatrix} \begin{matrix} \mathcal{R} \\ \mathcal{R} \\ \mathcal{R} \\ \mathcal{R} \\ \cdot \\ \cdot \\ \cdot \end{matrix},$$

where $T_0 = R_1$, $T_k = (1/k)R_{k+1}R_k^{-1}$ ($k \geq 1$), then every operator A commuting with T has a lower triangular operator matrix (with respect to the above decomposition) with diagonal terms $A_{00} = A_{11} = A_{22} = \dots = \lambda_0 1_{\mathcal{R}}$ for some $\lambda_0 \in C$. (From this we can immediately deduce that $\mathcal{A}'(T)$ has multiplicity equal to m .)

A lot more can be said. (Beware, the calculations that follow are not for the feeble hearted reader!)

Lemma 2.3. *Let $\{T_n\}_{n=0}^\infty$ be a bounded sequence of invertible operators in \mathcal{R} such that for some subsequence $\{n_k\}_{k=1}^\infty$ and positive constants $\{\delta_k\}_{k=1}^\infty$ satisfies $T_{n_k} T_{n_k-1} \dots T_2 T_1 T_0 = \delta_k R_k$ ($k = 1, 2, \dots$; R_k is defined as above). Assume, moreover, that $\|T_n^{-1}\| \leq M$ for all $n = 0, 1, 2, \dots$.*

Let $T \in \mathcal{L}(\mathcal{H})$ be the unilateral operator weighted shift with weight sequence $\{T_n\}_{n=0}^\infty$ (defined by

$$T = \begin{pmatrix} 0 & & & & \\ T_0 & 0 & & & \\ & T_1 & 0 & & \mathbf{0} \\ & \cdot & T_2 & 0 & \\ & & & \cdot & \cdot \\ & \mathbf{0} & & \cdot & \cdot \\ & & & \cdot & \cdot \end{pmatrix} \begin{matrix} \mathcal{R} \\ \mathcal{R} \\ \mathcal{R} \\ \mathcal{R} \\ \cdot \\ \cdot \\ \cdot \end{matrix}$$

with respect to the orthogonal direct sum decomposition $\mathcal{H} = \sum \bigoplus_{n=0}^\infty \mathcal{R}_n$, $\mathcal{R}_n \simeq \mathcal{R}$ for all $n = 0, 1, 2, \dots$).

Then $\mathcal{A}'(T) = \mathcal{A}(T)$.

Proof. Let $A = (A_{ij})_{i,j=0}^\infty$ be the matrix of $A \in \mathcal{A}'(T)$ with respect to the given decomposition; then

$$\begin{aligned} O &= TA - AT \\ &= \begin{pmatrix} -A_{01}T_0 & -A_{02}T_1 & -A_{03}T_2 & \dots \\ T_0A_{00} - A_{11}T_0 & T_0A_{01} - A_{12}T_1 & T_0A_{02} - A_{13}T_2 & \dots \\ T_1A_{10} - A_{20}T_0 & T_1A_{11} - A_{22}T_1 & T_1A_{12} - A_{23}T_2 & \dots \\ T_2A_{20} - A_{30}T_0 & T_2A_{21} - A_{32}T_1 & T_2A_{22} - A_{33}T_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned}$$

Since the T_n 's are invertible, the first row indicates that $A_{0n} = 0$ for all $n \geq 1$. Now the second row indicates that $A_{1n} = 0$ for all $n \geq 2$, etc., so

that A has a lower triangular operator matrix. (This can also be deduced by observing that

$$\mathcal{R}_0 = \ker T^*, \quad \mathcal{R}_0 \oplus \mathcal{R}_1 = \ker T^{*2}, \quad \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 = \ker T^{*3}, \dots,$$

are hyperinvariant subspaces of T^* , and therefore their orthogonal complements, $\sum \bigoplus_{n=1}^{\infty} \mathcal{R}_n$, $\sum_{n=2}^{\infty} \bigoplus \mathcal{R}_n$, $\sum \bigoplus_{n=3}^{\infty} \mathcal{R}_n, \dots$ must be invariant under every A in $\mathcal{A}'(T)$.)

From the entries $(2, 1), (3, 2), \dots, (n + 1, n), \dots$, we inductively obtain

$$A_{nn} = (T_{n-1}T_{n-2} \cdots T_0)A_{00}(T_{n-1}T_{n-2} \cdots T_0)^{-1} \quad (n = 1, 2, \dots).$$

Thus (as in Wogen's article [25]),

$$\|A\| \geq \|A_{n_k+1, n_k+1}\| = \sup_{x \neq 0} \frac{\|(T_{n_k}T_{n_k-1} \cdots T_0)A_{00}x\|}{\|T_{n_k}T_{n_k-1} \cdots T_0x\|} = \sup_{x \neq 0} \frac{\|R_k(A_{00}x)\|}{\|R_kx\|}.$$

If A_{00} is not a multiple of the identity, then there is a unit vector x_0 such that $y_0 = A_{00}x_0$ is linearly independent of x_0 . By taking a subsequence $\{n_{k(i)}\}_{i=1}^{\infty}$ so that $\|x - g_{n_{k(i)}}\| \rightarrow 0$ ($i \rightarrow \infty$), we deduce that

$$\|A\| \geq \sup_i \|A_{n_{k(i)}+1, n_{k(i)}+1}x_0\| = \sup_i \frac{\|R_{k(i)}y_0\|}{\|R_{k(i)}x_0\|} = \infty,$$

a contradiction.

Hence, $A_{00} = \lambda_0 1_{\mathcal{H}}$, and therefore $A_{nn} = \lambda_0 1_{\mathcal{H}}$ for all $n \geq 0$ (for some $\lambda_0 \in \mathbf{C}$).

Now consider the $(k + i, k)$ -entries $T_{k+i-1}A_{k+i-1, k} - A_{k+i, k+1}T_k = 0$ ($i \geq 2, k \geq 0$); equivalently, $T_{k+i}A_{k+i, k} = A_{k+i+1, k+1}T_k$ ($i \geq 1, k \geq 0$).

Fix i . By induction over k , we obtain

$$\begin{aligned} A_{n+i, n} &= (T_{n+i-1}T_{n+i-2} \cdots T_i)A_{i0}(T_{n-1}T_{n-2} \cdots T_0)^{-1} \\ &= (T_{n+i-1}T_{n+i-2} \cdots T_0)S_{i0}(T_{n-1}T_{n-2} \cdots T_0)^{-1}, \end{aligned}$$

where $S_{i0} = (T_{i-1}T_{i-2} \cdots T_0)^{-1}A_{i0}$.

Thus,

$$\begin{aligned} \|A_{n+i, n}\| &= \sup_{x \neq 0} \frac{\|(T_{n+i-1}T_{n+i-2} \cdots T_0)S_{i0}x\|}{\|T_{n-1}T_{n-2} \cdots T_0x\|} \\ &= \sup_{x \neq 0} \frac{\|(T_{n+i-1}T_{n+i-2} \cdots T_n)(T_{n-1}T_{n-2} \cdots T_0)S_{i0}x\|}{\|T_{n-1}T_{n-2} \cdots T_0x\|} \\ &\geq M^{-i} \sup_{x \neq 0} \frac{\|(T_{n-1}T_{n-2} \cdots T_0)S_{i0}x\|}{\|T_{n-1}T_{n-2} \cdots T_0x\|}, \end{aligned}$$

whence we deduce (exactly as in the case of A_{00}) that $S_{i0} = \lambda_i 1_{\mathcal{H}}$ for some $\lambda_i \in \mathbf{C}$; equivalently, $A_{i0} = \lambda_i T_{i-1}T_{i-2} \cdots T_0$.

A fortiori, $A_{n+i, n} = \lambda_i T_{n+i-1}T_{n+i-2} \cdots T_n$ ($i, n \geq 0$).

Observe that $T^i = (L_{hk})_{h,k=0}^\infty$, where $L_{n+i,n} = T_{n+i-1}T_{n+i-2}\cdots T_n$ ($n \geq 0$) and $L_{hk} = 0$ if $h - k \neq i$. Hence, we can formally write $A = \sum_{i=0}^\infty \lambda_i T^i$.

Now it follows exactly as in the case of a unilateral scalar weighted shift that

$$A = (\text{strong}) \lim_{m \rightarrow \infty} \sum_{i=0}^m \left(1 - \frac{i}{m+1}\right) \lambda_i T^i$$

(see [6], [22]).

Therefore, $A \in \mathcal{A}(T)$. \square

Remarks 2.4. (i) As mentioned before, W. R. Wogen considered the sequence $T_0 = R_1$, $T_n = (1/n)R_{n+1}R_n^{-1}$ ($n \geq 1$). Since R_k is a positive invertible operator, for each $m_k \geq 1$, we can write $R_k = (L_k)^{m_k}$, where $L_k = (1 - P_k) + (1/k)^{1/m_k}P_k$ ($k \geq 1$). Clearly, if m_k is large, then $1_{\mathcal{R}} - L_k$ is an operator of very small norm and rank equal to m_k .

Given a normed ideal \mathcal{I} of compact operators, strictly larger than the trace class (see [19], [21]), we define

$$\begin{aligned} T_0 &= 1_{\mathcal{R}}, T_1 = T_2 = \cdots = T_{m_1} = L_1, T_{m_1+1} = T_{m_1+2} = \cdots = T_{3m_1} = L_1^{-1}, \\ T_{3m_1+1} &= T_{3m_1+2} = \cdots = T_{4m_1} = L_1, \\ &\dots\dots\dots \\ T_{4(m_1+\cdots+m_k)+1} &= \cdots = T_{4(m_1+\cdots+m_k)+m_{k+1}} = L_{k+1}, \\ T_{4(m_1+\cdots+m_k)+m_{k+1}+1} &= \cdots = T_{4(m_1+\cdots+m_k)+3m_{k+1}} = L_{k+1}^{-1}, \\ T_{4(m_1+\cdots+m_k)+3m_{k+1}+1} &= \cdots = T_{4(m_1+\cdots+m_{k+1})} = L_{k+1}, \\ &\dots\dots\dots \end{aligned}$$

Let T be defined as in Lemma 2.3 and let S be the similarly defined operator with weights equal to $1_{\mathcal{R}}$ for all $n = 0, 1, 2, \dots$. S is the unilateral shift of multiplicity equal to $\dim \mathcal{R}$. Given $\varepsilon > 0$, if $m_k \rightarrow \infty$ fast enough, then $S - T \in \mathcal{I}$ and $|S - T|_{\mathcal{I}} < \varepsilon$. In particular, $\sigma(T) = \mathbf{D}^-$, $\sigma_{le}(T) = \partial \mathbf{D}$, and for each λ in \mathbf{D} , $\lambda - T$ is a semi-Fredholm operator with index $-\dim \mathcal{R}$ and trivial kernel. Moreover, both $\{T_n\}_{n=0}^\infty$ and $\{T_n^{-1}\}_{n=0}^\infty$ have the property of the lemma. (This solves Proposition 2.1 for the case of a “nonperforated disk.”)

(ii) If we define T_n as above whenever $T_n = L_k$ (for some k) and $T_n = (1/\|L_k^{-1}\|)L_k$ whenever $T_n = L_k^{-1}$ (for some k), then $\|T\| = 1$. It is easy to see that $T^{*k} \rightarrow 0$ (strongly, $k \rightarrow \infty$) and a simple calculation (by using the facts that $L_k^{m_k} = R_k$ and $\{g_k\}_{k=1}^\infty$ is dense in the unit sphere of \mathcal{R}) shows that $T^k \rightarrow 0$ (strongly, $k \rightarrow \infty$).

Thus, in this case T is a C_{00} -contraction in the sense of B. Sz.-Nagy and C. Foias [23]; moreover, T is very close to the shift S of multiplicity $\dim \mathcal{R}$ (if \mathcal{R} is finite-dimensional, $|S - T|_{\mathcal{I}} < \varepsilon$), $\sigma(T) = \mathbf{D}$, $\sigma_{le}(T) = \mathbf{D}$, and for each $\lambda \in \mathbf{D}$, $\lambda - T$ is a semi-Fredholm operator with index equal to $-\dim \mathcal{R}$ and trivial kernel. For suitable subsequences, $T_{n_k}T_{n_k-1}\cdots T_1T_0 = \delta_k \mathcal{R}_k$ and $(T_{p_k}T_{p_k-1}\cdots T_1T_0)^{-1} = \eta_k R_k$ ($\delta_k, \eta_k > 0$). Furthermore, if $\varphi \in H^\infty(\mathbf{D})$,

$\varphi(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$, then the mapping

$$\varphi \rightarrow \varphi(T) := (\text{strong}) \lim_{m \rightarrow \infty} \varphi(rT) = (\text{strong}) \lim_{m \rightarrow \infty} \sum_{n=0}^{\infty} a_n r^n T^n$$

(these limits are well-defined) defines an isometric isomorphism from $H^{\infty}(\mathbf{D})$ onto $\mathcal{A}(T)$.

(iii) A close look of the proof of Lemma 2.3 shows that the condition “ $\|T_n\| \leq M$ for all $n \geq 0$ ” is superabundant. The only thing we must guarantee is that $\|(T_{n+i-1} T_{n+i-2} \cdots T_n)^{-1}\|$ does not grow too fast. For instance, if we take $T_0 = R_1$, $T_1 = R_2 R_1^{-1}$, and $T_n = (1/n \log n) R_{n+1} R_n^{-1}$ for $n \geq 2$, then the proof of Lemma 2.3 yields a *quasinilpotent* unilateral operator weighted shift T such that $\dim \ker T^* = \dim \mathcal{R}$ and $\mathcal{A}'(T) = \mathcal{A}(T)$; moreover, if \mathcal{R} is finite-dimensional, then T is a compact operator.

As a second step toward the proof of Proposition 2.1, we shall settle the case of a single hole.

Lemma 2.5. Let \mathbf{D} denote the open unit disk, let $\alpha \in \mathbf{D}$, and let $0 < r < 1 - |\alpha|$ (so that $\mathbf{D}(\alpha, r)^- \subset \mathbf{D}$).

Define $\{T_n\}_{n=0}^{\infty}$ as in Remark 2.4(ii), and let $T \in \mathcal{L}(\mathcal{H})$ ($\mathcal{H} = \sum \bigoplus_{n=-\infty}^{+\infty} \mathcal{R}_n$, $\mathcal{R}_n \simeq \mathcal{R}$) be the operator defined by

$$T = \begin{pmatrix} \cdot & & & & & & & & \cdot & & & & & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & \cdot & & & & & & & & & & & \cdot \\ & \cdot & \cdot & & & & & & \cdot & & & & & & & & & & & \cdot \\ & & \cdot & \alpha & & & & & \cdot & & & & & & & & & & & \cdot \\ & & & T_{-3} & \alpha & & & & \cdot & & & & & & & & & & & \cdot \\ & & & & T_{-2} & \alpha & & & \cdot & & & & & & & & & & & \cdot \\ & & & & & T_{-1} & (0) & & \cdot & & & & & & & & & & & \cdot \\ & & & & & & T_0 & 0 & & & & & & & & & & & & \cdot \\ & & & & & & & T_1 & 0 & & & & & & & & & & & \cdot \\ & & & & & & & & T_2 & 0 & & & & & & & & & & \cdot \\ & & & & & & & & & \cdot & \cdot & & & & & & & & & \cdot \\ & & & & & & & & & & \cdot & \cdot & & & & & & & & \cdot \\ & & & & & & & & & & & \cdot & \cdot & & & & & & & \cdot \\ & & & & & & & & & & & & \cdot & \cdot & & & & & & \cdot \\ & & & & & & & & & & & & & \cdot & \cdot & & & & & \cdot \\ & & & & & & & & & & & & & & \cdot & \cdot & & & & \cdot \\ & & & & & & & & & & & & & & & \cdot & \cdot & & & \cdot \\ & & & & & & & & & & & & & & & & \cdot & \cdot & & \cdot \\ & & & & & & & & & & & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & & & & & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & & & & & & & & & & & & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \end{pmatrix}$$

(the remaining entries of the matrix are 0's), where $T_{-n} = rT_n^{-1}$.
 Then $\sigma(T) = \mathbf{D}^- \setminus \mathbf{D}(\alpha, r)$, $T|_{\sum \bigoplus_{n=0}^{\infty} \mathcal{R}_n}$ and $r(T - \alpha 1)^{-1}|_{\sum \bigoplus_{n=1}^{\infty} \mathcal{R}_n}$ are C_{00} -contractions with $H^{\infty}(\mathbf{D})$ functional calculus,

$$\mathcal{A}'(T) = \mathcal{A}(T) = \{\varphi(T) + \psi(r(T - \alpha 1)^{-1}) : \varphi, \psi \in H^{\infty}(\mathbf{D}), \psi(0) = 0\}$$

is isomorphic (although not necessarily isometric) with $H^{\infty}(\mathbf{D} \setminus \mathbf{D}(\alpha, r)^-)$; moreover, $\sigma_e(T) = \partial \sigma(T)$ and, for each λ in $\sigma(T) \setminus \partial \sigma(T)$, $\lambda - T$ is a semi-Fredholm operator of index equal to $-\dim \mathcal{R}$ and trivial kernel.

Proof. Let $A = (A_{ij})_{-\infty}^{+\infty}$ be the matrix of $A \in \mathcal{A}'(T)$ with respect to the given decomposition; then

$$\begin{aligned}
 0 &= TA - AT \\
 &= \begin{pmatrix} (T_{i-1}A_{i-1,j} - A_{i,j+1}T_j)_{i,j < 0} & (T_{i-1}A_{i-1,j} - A_{i,j+1}T_j + \alpha A_{ij})_{i < 0, j \geq 0} \\ (T_{i-1}A_{i-1,j} - A_{i,j+1}T_j - \alpha A_{ij})_{i \geq 0, j < 0} & (T_{i-1}A_{i-1,j} - A_{i,j+1}T_j)_{i, j \geq 0} \end{pmatrix}
 \end{aligned}$$

with respect to the decomposition $\mathcal{H} = (\sum \bigoplus_{n < 0} \mathcal{R}_n) \oplus (\sum \bigoplus_{n \geq 0} \mathcal{R}_n)$ ($\mathcal{H}_1 = \sum \bigoplus_{n < 0} \mathcal{R}_n$, $\mathcal{H}_0 = \sum \bigoplus_{n \geq 0} \mathcal{R}_n$). By repeating four times the same arguments as in the proof of Lemma 2.3, from the (1, 1)-entry and the (0, 0)-entry of the above 2×2 operator matrix, we deduce that

- (1) $A_{ii} = a_0 1_{\mathcal{R}}$ for all $i \geq 0$,
- (2) if $i, j \geq 0$, $i \neq j$, then either $A_{ij} = a_{i-j} T_{i-1} T_{i-2} \cdots T_j$ (for $i > j$), or $A_{ij} = a_{-(j-i)} (T_{j-1} T_{j-2} \cdots T_i)^{-1}$ (for $i < j$),
- (3) $A_{ii} = b_0 1_{\mathcal{R}}$ for all $i < 0$,
- (4) if $i, j < 0$, $i \neq j$, then either $A_{ij} = b_{i-j} T_{i-1} T_{i-2} \cdots T_j$ (for $i > j$), or $A_{ij} = b_{-(j-i)} (T_{j-1} T_{j-2} \cdots T_i)^{-1}$ (for $i < j$), where $a_n, b_n \in \mathbf{C}$ ($n \in \mathbf{Z}$), $a_{-1} = b_{-1}$, and $\limsup_{n \rightarrow \infty} |a_n|^{1/n} \leq 1$, $\limsup_{n \rightarrow \infty} |b_{-n}|^{1/n} \leq r$ (because the entries of A are uniformly bounded by $\|A\|$); moreover, we also have $A_{i0} = b_i (T_{-1} T_{-2} \cdots T_{-i})^{-1}$ for all $i < 0$.

On the other hand, the analysis of the (0, 1)-entry of $TA - AT = 0$ indicates that

- (5) if $i \geq 0 > j$, then $A_{ij} = \alpha^{-1} (T_{i-1} A_{i-1,j} - A_{i,j+1} T_j)$ is uniquely determined by $A_{i-1,j}$ (=the entry exactly above A_{ij} in the matrix) and $A_{i,j+1}$ (=the entry to the right of A_{ij} in the matrix), and a straightforward computation shows that $A_{ij} = c_{ij} T_{i-1} T_{i-2} \cdots T_j$.

Similarly, the analysis of the (1, 0)-entry of $TA - AT = 0$ indicates that

- (6) if $i < 0 \leq j$, then $A_{i,j+1} = (T_{i-1} A_{i-1,j} + \alpha A_{ij}) T_j^{-1}$; equivalently, if $i < 0 < j$, then $A_{ij} = (T_{i-1} A_{i-1,j-1} + \alpha A_{i,j-1}) T_{j-1}^{-1}$ is uniquely determined by $A_{i-1,j-1}$ (=the entry above and to the left of A_{ij} in the matrix), and $A_{i,j-1}$ (=the entry to the left of A_{ij} in the matrix), and a straightforward computation shows that $A_{ij} = c_{ij} (T_{j-1} T_{j-2} \cdots T_i)^{-1}$.

In both (5) and (6), $c_{ij} \in \mathbf{C}$ is defined by certain linear combinations of finitely many a_n 's and b_n 's.

Since $T = \begin{pmatrix} T(1,1) & 0 \\ T(0,1) & T(0,0) \end{pmatrix}$, where $T(0, 0)$ has the form of Lemma 2.3 and $T(1, 1) - \alpha$ is the adjoint of an operator of the form of Lemma 2.3, $\sigma(T(0, 0)) = \mathbf{D}^-$ and $\sigma(T(1, 1) - \alpha) = r\mathbf{D}^-$, it is easy to see that $\sigma(T) \subset \mathbf{D}^-$ and $T - \lambda$ is a semi-Fredholm operator with index equal to $-\dim \mathcal{R}$ and trivial kernel for all $\lambda \in \mathbf{D} \setminus \mathbf{D}(\alpha, r)^-$.

On the other hand, by solving the equation $A(T - \alpha) = 1$ (A as above) we can check that $T - \alpha$ is invertible, with inverse

$$(T - \alpha)^{-1} = \begin{pmatrix} \cdot & \cdot & & & & & & & & & & & \\ & \cdot & & & & & & & & & & & \\ & & \cdot & & & & & & & & & & \\ & & & \cdot & & & & & & & & & \\ & & & & 0 & T_{-3}^{-1} & & & & & & & \\ & & & & & 0 & T_{-2}^{-1} & & & & & & \\ & & & & & & 0 & T_{-1}^{-1} & & & & & \\ & & & & & & & (0) & T_0^{-1} & B_{02} & B_{03} & \cdot & \cdot & \cdot \\ & & & & & & & & 0 & T_1^{-1} & B_{13} & \cdot & \cdot & \cdot \\ & & & & & & & & & 0 & T_2^{-1} & \cdot & \cdot & \cdot \\ & & & & & & & & & & 0 & \cdot & \cdot & \cdot \\ & & & & & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & & & & \cdot & \cdot & \cdot \end{pmatrix}$$

(the remaining entries of the matrix are 0's), where

$$B_{ij} = \alpha^{j-i-1} (T_{j-1} T_{j-2} \cdots T_i)^{-1} \quad (j - 1 > i \geq 0).$$

A straightforward computation shows that

$$\begin{aligned} \| (T - \alpha)^{-1} \| &\leq \max \left[r^{-1}, \sum_{n=0}^{\infty} ((1 + \eta)|\alpha|)^n \right] \\ &= \max[r^{-1}, (1 - (1 + \eta)|\alpha|)^{-1}] = r^{-1}, \end{aligned}$$

where $\eta = \max_{n \geq 0} (\| T_n^{-1} \| - 1)$ satisfies $1 - (1 + \eta)|\alpha| > r$, provided all the m_k 's of Remark 2.4 are sufficiently large.

Since we obviously have $\partial \mathbf{D}(\alpha, r) \subset \sigma(T)$, it follows that $\| (T - \alpha)^{-1} \|^{-1} = r = \text{dist}[\alpha, \partial \mathbf{D}(\alpha, r)]$. Therefore, $T - \lambda$ is invertible for all $\lambda \in \mathbf{D}(\alpha, r)$ ($\| (T - \lambda)^{-1} \| = (\text{dist}[\lambda, \partial \mathbf{D}(\alpha, r)])^{-1} = (1 - |\lambda - \alpha|)^{-1}$), and $\| r(T - \alpha)^{-1} \| = 1$; that is, $r(T - \alpha)^{-1}$ is a contraction.

Thus, $\sigma(T) = \mathbf{D}^- \setminus \mathbf{D}(\alpha, r)$ and $\sigma_{le}(T) = \partial \mathbf{D} \cup \partial \mathbf{D}(\alpha, r)$. By using their matricial representations, it is not difficult to check that both T and $r(T - \alpha)^{-1}$ are C_{00} -contractions, and therefore they admit $H^\infty(\mathbf{D})$ functional calculi; moreover, $\mathcal{A}(T)$ and $\mathcal{A}(r(T - \alpha)^{-1})$ are isometrically isomorphic with $H^\infty(\mathbf{D})$ (via their functional calculi; see [23]).

Define $\varphi(\lambda) = \sum_{n=0}^\infty a_n \lambda^n$ and $\psi(\lambda) = \sum_{n=1}^\infty (b_{-n} r^{-n}) \lambda^n$. By (4), φ and ψ are analytic on the open unit disk.

Claim. $\varphi, \psi \in H^\infty(\mathbf{D})$ and $A = \varphi(T) + \psi(r(T - \alpha)^{-1})$.

Observe that, if this claim is true, then an analysis of the matrix of $\varphi(T) + \psi(r(T - \alpha)^{-1})$ indicates that $b_n = (1/n!) \varphi^{(n)}(\alpha)$ for all $n = 0, 1, 2, \dots$

Write $b_n = (1/n!) \varphi^{(n)}(\alpha) + \nu_n$ (ν_n is “the error term”; $n \geq 0$). By induction, we obtain (use (5))

$$\begin{aligned} A_{0,-1} &= \alpha^{-1} [\varphi(\alpha) - a_0 + \nu_0] T_{-1}, \\ A_{1,-1} &= \alpha^{-2} [\varphi(\alpha) - a_0 - a_1 \alpha + \nu_0] T_0 T_{-1}, \\ &\dots\dots\dots \\ A_{n,-1} &= \alpha^{-(n+1)} \left[\varphi(\alpha) - \sum_{i=0}^n a_i \alpha^i + \nu_0 \right] T_{n-1} T_{n-2} \cdots T_0 T_{-1}, \\ &\dots\dots\dots \end{aligned}$$

The coefficient is the sum of

$$\alpha^{-(n+1)} \left[\varphi(\alpha) - \sum_{i=0}^n a_i \alpha^i \right] = \sum_{i=0}^{\infty} a_{i+n+1} \alpha^i$$

(and the absolute values of these numbers are obviously uniformly bounded by some constant $C > 0$) and $\nu_0 \alpha^{-(n+1)}$.

Thus, if $\nu_0 \neq 0$, then we arrive at the contradiction

$$\begin{aligned} \|A\| &\geq \sup_n \|A_{n,-1}\| \geq \sup_n |\nu_0 \alpha^{-(n+1)}| \cdot \|T_{n-1} T_{n-2} \cdots T_0 T_{-1}\| - C \\ &\geq \sup_n |\nu_0 [(1 + |\alpha|)/2]^{-(n+1)}| - C = \infty. \end{aligned}$$

Therefore $\nu_0 = 0$.

Another inductive argument shows that $\nu_n = 0$ ($n \geq 0$); that is, b_n is, indeed, equal to $(1/n!) \varphi^{(n)}(\alpha)$ for all $n \geq 0$.

It follows that (use (4))

$$\begin{aligned} \sum_{n=1}^{\infty} |b_n| \max_{j < -n} \|T_{j+n-1} T_{j+n-2} \cdots T_j\| \\ \leq \sum_{n=1}^{\infty} (1/n!) |\varphi^{(n)}(\alpha)| \max_{j < -n} \|T_{j+n-1} T_{j+n-2} \cdots T_j\| < \infty \end{aligned}$$

because $|\alpha| + r < 1$, $\max_{j < -n} \|T_{j+n-1} T_{j+n-2} \cdots T_j\| = O([r(1 + \varepsilon)]^n)$ for all $\varepsilon > 0$, and φ has radius of convergence at least 1.

This means that the operator

$$A(1, 1) := (A_{ij})_{i,j < 0} : \sum_{n < 0} \bigoplus \mathcal{R}_n \rightarrow \sum_{n < 0} \bigoplus \mathcal{R}_n$$

can be written as the sum of two operators, $A(1, 1)^+$ and $A(1, 1)^-$, where $A(1, 1)^+$ ($A(1, 1)^-$) has the same entries as $(A_{ij})_{i,j < 0}$ for $i \leq j$ (for $i > j$, resp.) and the other entries equal to 0.

Another cumbersome calculation shows that $a_{-n} = (1/n!) \psi^{(n)}(r(1 - \alpha)^{-1})$, so that

$$\begin{aligned} \sum_{n=1}^{\infty} |a_{-n}| \max_{i \geq 0} \|(T_{i+n-1} T_{i+n-2} \cdots T_i)^{-1}\| \\ \leq \sum_{n=1}^{\infty} (1/n!) |\psi^{(n)}(r(1 - \alpha)^{-1})| \max_{i \geq 0} \|(T_{i+n-1} T_{i+n-2} \cdots T_i)^{-1}\| < \infty \end{aligned}$$

because $|r(1 - \alpha)^{-1}| < 1$, $\max_{i \geq 0} \|(T_{i+n-1} T_{i+n-2} \cdots T_i)^{-1}\| = O((1 + \varepsilon)^n)$ for all $\varepsilon > 0$, and ψ has radius of convergence at least 1.

Thus, $A(0, 0) := (A_{ij})_{i, j \geq 0}$ can also be “split” along the main diagonal as $A(0, 0) = A(0, 0)^+ + A(0, 0)^-$, where $A(0, 0)^+(A(0, 0)^-)$ has the same entries as $A(0, 0)$ for $i \geq j$ (for $i < j$, resp.) and the other entries equal to 0.

Let $A(1, 0) = (A_{ij})_{i \geq 0, j < 0}$ and $A(0, 1) = (A_{ij})_{i < 0, j \geq 0}$. We deduce from the above results that

$$A = \begin{pmatrix} A(1, 1) & A(1, 0) \\ A(0, 1) & A(0, 0) \end{pmatrix} = \begin{pmatrix} A(1, 1)^+ + A(1, 1)^- & A(1, 0) \\ A(0, 1) & A(0, 0)^+ + A(0, 0)^- \end{pmatrix} = A^+ + A^-,$$

where

$$A^+ = \begin{pmatrix} A(1, 1)^+ & 0 \\ A(0, 1) & A(0, 0)^+ \end{pmatrix} \quad \text{and} \quad A^- = \begin{pmatrix} A(1, 1)^- & A(1, 0) \\ 0 & A(0, 0)^- \end{pmatrix}.$$

It is obvious that both A^+ and A^- commute with T . Since an element in the commutant of T is uniquely determined by its 0th column, a comparison of the corresponding matrices shows that $A^+ = \varphi(T)$ and $A^- = \psi(r(T - \alpha)^{-1})$, whence we deduce that $\varphi, \psi \in H^\infty(\mathbf{D})$ and $A = \varphi(T) + \psi(r(T - \alpha)^{-1}) \in \mathcal{A}^a(T)$.

Hence, $\mathcal{A}'(T) = \mathcal{A}^a(T)$. (It is a straightforward exercise that every function ω in $H^\infty(\mathbf{D})$ admits a unique decomposition of the form $\omega(\lambda) = \varphi(\lambda) + \psi(r(\lambda - \alpha)^{-1})$, where $\varphi, \psi \in H^\infty(\mathbf{D})$ and $\psi(0) = 0$.)

The proof of Lemma 2.5 is now complete. \square

Now we are in a position to prove Proposition 2.1. The case $k = 1$ is the contents of Lemma 2.5: $T = \begin{pmatrix} rL + \alpha & 0 \\ C & R \end{pmatrix}$ with respect to the decomposition $\mathcal{H} = (\sum \bigoplus_{n < 0} \mathcal{R}_n) \oplus (\sum \bigoplus_{n \geq 0} \mathcal{R}_n)$, where R is given by Lemma 2.3 and Remark 2.4(ii), L is the adjoint of the similarly defined operator with the weight T_n replaced by T_n^{-1} ($n \geq 2$) and

$$C = \begin{pmatrix} \cdots & 0 & rT_{-1} \\ & 0 & \\ & & \vdots \end{pmatrix}.$$

Assume that $k \geq 2$; then we define

$$T = \begin{pmatrix} r_k L + \alpha_k & & & & & & \\ & r_{k-1} L + \alpha_{k-1} & & & & & \\ & & \ddots & & & & \\ & & & r_2 L + \alpha_2 & & & \\ & & & & r_1 L + \alpha_1 & & \\ C_k & C_{k-1} & \cdots & C_2 & C_1 & R \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = \{(\sum \bigoplus_{n < 0} \mathcal{R}_n) \oplus \cdots \oplus (\sum \bigoplus_{n < 0} \mathcal{R}_n) \text{ (} k \text{ copies)}\} \oplus (\sum \bigoplus_{n \geq 0} \mathcal{R}_n)$ and C_j is defined exactly as C , with r replaced

by $r_j, j = 1, 2, \dots, k$. (Compare with the generalized Rota’s model defined in [8].)

For each $h, 1 \leq h \leq k$,

$$\sigma \left(\begin{pmatrix} r_h L + \alpha_h & 0 \\ C_h & R \end{pmatrix} \right) = \mathbf{D}^- \setminus \mathbf{D}(\alpha_h, r_h),$$

which includes $\bigcup_{j \neq h} \sigma(r_j L + \alpha_j) = \bigcup_{j \neq h} \mathbf{D}(\alpha_j, r_j)^-$.

Since the latter set does not disconnect the plane, it is not difficult to prove that $\sigma(T) = \mathbf{D}^- \setminus \bigcup_{j=1}^k \mathbf{D}(\alpha_j, r_j)$; furthermore, $\sigma_{le}(T) = \partial \sigma(T)$ and $\lambda - T$ is a semi-Fredholm operator with index equal to $-\dim \mathcal{R}$ and trivial kernel for all $\lambda \in \sigma(T) \setminus \sigma_{le}(T)$.

Let $A = (B_{mn})_{m,n=0}^k$ be the operator matrix of an element of $\mathcal{A}'(T)$ (for each pair (m, n) , B_{mn} can be written as an infinite operator matrix in the obvious way). We have $0 = TA - AT = (M_{mn})_{m,n=0}^k$, where

- (1) $M_{jj} = [r_j L, B_{jj}] - B_{j0} C_j, j = k, k - 1, \dots, 2, 1,$
- (2) $M_{00} = [R, B_{00}] + \sum_{j=1}^k C_j B_{j0},$
- (3) $M_{ij} = (\alpha_i + r_i L) B_{ij} - B_{ij} (\alpha_j + r_j L) - B_{i0} C_j, k \geq i, j \geq 1, i \neq j,$
- (4) $M_{i0} = (\alpha_i + r_i L) B_{i0} - B_{i0} R, k \geq i \geq 1, \text{ and}$
- (5) $M_{0j} = \sum_{i=1}^k C_i B_{ij} + R B_{0j} - B_{0j} (\alpha_j + r_j L) - R C_j, k \geq j \geq 1.$

Exactly as in the proof of Lemma 2.3 and the first part of the proof of Lemma 2.5, it can be shown that

$$B_{00} = \begin{pmatrix} a_0 1_{\mathcal{R}} & a_{-1} T_0^{-1} & a_{-2} (T_1 T_0)^{-1} & a_{-3} (T_2 T_1 T_0)^{-1} & & & \\ a_1 T_0 & a_0 1_{\mathcal{R}} & a_{-1} T_1^{-1} & a_{-2} (T_2 T_1)^{-1} & \cdot & & \\ a_2 T_1 T_0 & a_1 T_1 & a_0 1_{\mathcal{R}} & a_{-1} T_2^{-1} & \cdot & \cdot & \\ a_3 T_2 T_1 T_0 & a_2 T_2 T_1 & a_1 T_2 & a_0 1_{\mathcal{R}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and

$$B_{jj} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

($j = k, k - 1, \dots, 2, 1$; use (1) and (2)).

Some simple estimates on the sizes of $a_n (n \geq 0)$ and $b_{-n}^j (n > 0)$ indicate that, if we define

$$\varphi(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \quad \text{and} \quad \psi_j(\lambda) = \sum_{n=1}^{\infty} (b_{-n}^j r_j^{-n}) \lambda^n,$$

then φ and ψ_j ($j = 1, 2, \dots, k$) are analytic on the open unit disk.

Furthermore, if in addition to (1) and (2) we consider also (3), (4), and (5), then we infer (as in the proof of Lemma 2.5) that the structure of A is uniquely determined by the constants $\{a_n\}_{n=0}^\infty$ and $\{b_n^j\}_{n=0}^\infty$ ($j = 1, 2, \dots, k$); then, a double inductive argument shows that, indeed, this structure is uniquely determined only by $\{a_n\}_{n=0}^\infty$ and $\{b_{-n}^j\}_{n=1}^\infty$ ($j = 1, 2, \dots, k$).

Thus (formally at least), we have $A = \varphi(T) + \sum_{j=1}^k \psi_j(r_j(T - \alpha_j)^{-1})$.

The operator T is contractive and the analysis of the matrix of $(T - \alpha_j)^{-1}$ indicates that $r_j(T - \alpha_j)^{-1}$ is also contractive; ($j = 1, 2, \dots, k$); furthermore, T and $r_j(T - \alpha_j)^{-1}$ are actually C_{00} -contractions with (isometrically isomorphic) $H^\infty(\mathbf{D})$ functional calculi [23].

We conclude that A is equal to the strong limit of

$$A_r = \varphi(rT) + \sum_{j=1}^k \psi_j(rr_j(T - \alpha_j)^{-1}) \in \mathcal{A}^a(T), \quad \text{as } r \uparrow 1.$$

Hence, $\mathcal{A}'(T) = \mathcal{A}^a(T)$.

The actual computations are a lot more cumbersome than the corresponding ones in Lemma 2.5, but the basic structure is the same: In $\varphi(T)$ there is a "main part" $\varphi(R)$, and

$$\varphi(\alpha_j + r_jL) = \sum_{n=0}^\infty a_n(\alpha_j + r_jL)^n = \sum_{n=0}^\infty (1/n!) \varphi^{(n)}(\alpha_j)(r_jL)^n$$

is the "irrelevant part" (the series converges in the *norm-topology*, $j = 1, 2, \dots, k$). For each h , $1 \leq h \leq k$, in $\psi_h(r_h(T - \alpha_h)^{-1})$ the "main part" is

$$\psi_h(r_h "[r_hL + \alpha_h]^{-1}) = \psi_h("L^{-1}")$$

(the quotation marks mean that, although L is not invertible, we can define $\psi_h("L^{-1}")$, where " L^{-1} " is the unilateral operator weighted shift defined by

$$\begin{pmatrix} \cdot & \cdot & & & & & \cdot \\ & \cdot & \cdot & & & & \cdot \\ & & \cdot & \cdot & & & \cdot \\ & & & 0 & T_4 & & \mathcal{R}_{-4} \\ & & & & 0 & T_3 & \mathcal{R}_{-3} \\ & & & & & 0 & T_2 & \mathcal{R}_{-2} \\ & & & & & & 0 & \mathcal{R}_{-1} \end{pmatrix}$$

which "looks like the inverse" of the backward operator weighted shift L . In this case the "irrelevant part" is the sum of $\psi_h(r_h(R - \alpha_h)^{-1})$ and $\psi_h(r_h(r_jL - \alpha_j)^{-1})$; these series also converge in the norm-topology and, moreover, $\psi_h(r_h(r_jL - \alpha_j)^{-1})$ is actually a uniform limit of polynomials in L for all $j \neq h$. Thus, the entries of the matrix of B_{hh} above the main diagonal can

only come from $\psi_h("L^{-1}")$ ($h = 1, 2, \dots, k$). Similarly, the entries of the matrix B_{00} on and below the main diagonal can only come from $\varphi(R)$.

The details of the constructions are left to the really dedicated reader.

The proof of Proposition 2.1 is now complete. \square

Remark 2.6. In Proposition 2.1, the operator T is defined so that T and $r_j(T - \alpha_j)^{-1}$ ($j = 1, 2, \dots, k$) are C_{00} -contractions with $H^\infty(\mathbf{D})$ functional calculi in order to simplify the construction. But this is not strictly necessary. Instead of choosing $\{T_n\}_{n=0}^\infty$ as in Remark 2.4(ii), we can choose the sequence defined in Remark 2.4(i), and then justify that

$$\begin{aligned} \varphi(T) &= (\text{strong}) \lim_{r \uparrow 1} \sum_{n=0}^\infty a_n (rT)^n \\ &= (\text{strong}) \lim_{m \rightarrow \infty} \sum_{n=0}^m \left(1 - \frac{n}{m+1}\right) a_n T^n \in \mathcal{A}(T) \end{aligned}$$

and

$$\begin{aligned} \psi_j(r_j(T - \alpha_j)^{-1}) &= (\text{strong}) \lim_{r \uparrow 1} \sum_{n=1}^\infty b_{-n}^j (T - \alpha_j)^{-n} \\ &= (\text{strong}) \lim_{m \rightarrow \infty} \sum_{n=0}^m \left(1 - \frac{n}{m+1}\right) b_{-n}^j (T - \alpha_j)^{-n} \end{aligned}$$

($j = 1, 2, \dots, k$) by using the same kinds of arguments as in [6] or [22].

In this case, $\mathcal{A}(T)$ is isometrically isomorphic with a certain Banach algebra $H^\infty(\mathbf{D}; T)$ continuously embedded in $H^\infty(\mathbf{D})$ (norm-decreasing embedding). Since the family $\{T_{i+1}T_{i+2} \cdots T_{i+n}\}_{i,n=0}^\infty$ is not uniformly bounded, the embedding of $H^\infty(\mathbf{D}; T)$ is strictly included in $H^\infty(\mathbf{D})$.

Proof of Proposition 2.2. Define T as in the proof of Proposition 2.1, except that $\{T_{-n}\}_{n=2}^\infty$ is chosen as $\{T_n\}_{n=2}^\infty$ in Remark 2.4(iii). The entries of an operator A commuting with T are computed in exactly the same way as in the previous case, and these computations indicate that $b_{-n}^j = 0$ for all $n \geq 1$ and all $j = 1, 2, \dots, k$, because all the sequences $\{T_{-n}^{-1}\}_{n=2}^\infty$, $\{T_{-n}T_{-(n+1)}\}_{n=2}^\infty$, $\{T_{-n}T_{-(n+1)}T_{-(n+2)}\}_{n=2}^\infty, \dots$ are unbounded.

A fortiori, $a_{-n} = 0$ for all $n \geq 1$, etc. Now it is easily seen that $A = \varphi(A) \in \mathcal{A}(T)$, where $\varphi(\lambda) = \sum_{n=0}^\infty a_n \lambda^n$.

Hence, $\mathcal{A}'(T) = \mathcal{A}(T)$. \square

Of course, the arguments of the proof of Propositions 2.1 and 2.2 can also be applied to the following ‘‘mixed’’ situation, as well:

Proposition 2.7. *Let \mathbf{D} denote the open unit disk, let $\{\mathbf{D}(\alpha_j, r_j)\}_{j=1}^k$ be a finite collection of open disks with pairwise disjoint closures such that $\mathbf{D}(\alpha_j, r_j)^- \subset \mathbf{D}$ for all $j = 1, 2, \dots, k$, and let $\{\mu_1, \mu_2, \dots, \mu_n\}$ be a finite subset of $\mathbf{D} \setminus \bigcup_{j=1}^k \mathbf{D}(\alpha_j, r_j)^-$.*

Let $\Omega = \mathbf{D} \setminus [(\bigcup_{j=1}^k \mathbf{D}(\alpha_j, r_j)^-) \cup \{\mu_1, \mu_2, \dots, \mu_n\}]$. Given an index m , $1 \leq m \leq \infty$, there exists T in $\mathcal{L}(\mathcal{H})$ such that

- (i) $\sigma(T) = \Omega^-$ and $\sigma_{le}(T) = \partial\Omega$;
- (ii) $\ker(\lambda - T) = \{0\}$ and $\dim \ker(\lambda - T)^* = m$ for all $\lambda \in \Omega$;
- (iii) $\lambda - T$ is a semi-Fredholm operator of index $-m$ for all $\lambda \in \Omega$, and
- (iv) $\mathcal{A}^a(T) = \mathcal{A}'(T)$ is isomorphic (but not necessarily isometric) with H^∞ (interior $[\Omega^-]$).

3. DENSITY OF OPERATORS WITH TINY COMMUTANTS

Theorem 3.1. *The class $\{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}'(T) = \mathcal{A}^a(T)\}$ is uniformly dense in $\mathcal{L}(\mathcal{H})$.*

Proof. Recall that an analytic Cauchy domain is a nonempty, bounded, open subset of \mathbf{C} whose boundary consists of finitely many pairwise disjoint regular analytic Jordan curves. An analytic Cauchy region is a connected analytic Cauchy domain.

Given A in $\mathcal{L}(\mathcal{H})$ and $\varepsilon > 0$, we can find $A_1 \in \mathcal{L}(\mathcal{H})$ such that $\|A - A_1\| < \varepsilon$, $\sigma_{le}(A_1)$ is the closure of an analytic Cauchy domain Ω , and $\sigma(A_1)$ has only finitely many isolated points $\lambda_1, \lambda_2, \dots, \lambda_p$ (see [2], [11], or [15]). Let $\Omega_1^+, \Omega_2^+, \dots, \Omega_m^+$ ($\Omega_1^-, \Omega_2^-, \dots, \Omega_n^-$) be an enumeration of all those components of the semi-Fredholm domain $\rho_{s-F}(A_1)$ of A_1 such that $\text{ind}(\lambda - A_1)$ is positive (negative, resp.), and let $i_h(-i_k)$ be the index of $\lambda - A_1$ for λ in Ω_h^+ (in Ω_k^- , resp.).

Since $\rho_{s-F}(A_1) = \mathbf{C} \setminus \sigma_{le}(A_1) = \mathbf{C} \setminus \Omega^-$, and Ω is an analytic Cauchy domain, it readily follows that $\Omega_1^+, \Omega_2^+, \dots, \Omega_m^+$ and $\Omega_1^-, \Omega_2^-, \dots, \Omega_n^-$ are analytic Cauchy regions. According to [7, Theorem 2, p. 237] there exist “perforated disks” (as in Proposition 2.1) $\mathbf{D}_1^+, \mathbf{D}_2^+, \dots, \mathbf{D}_m^+$ and $\mathbf{D}_1^-, \mathbf{D}_2^-, \dots, \mathbf{D}_n^-$ and analytic functions $\varphi_1^+, \varphi_2^+, \dots, \varphi_m^+$ and $\varphi_1^-, \varphi_2^-, \dots, \varphi_n^-$ such that φ_h^+ maps (a neighborhood of) the closure of $(\mathbf{D}_h^+)^- := \{\lambda \in \mathbf{C} : \bar{\lambda} \in \mathbf{D}_h^+\}$ conformally onto (a neighborhood of) the closure of $(\Omega_h^+)^-$ ($h = 1, 2, \dots, m$) and φ_k^- maps (a neighborhood of) $(\mathbf{D}_k^-)^-$ conformally onto (a neighborhood of) the closure of Ω_k^- ($k = 1, 2, \dots, n$).

Let R_k^- be the operator constructed in Proposition 2.1 such that $\sigma(R_k^-) = (\mathbf{D}_k^-)^-$, $\sigma_{le}(R_k^-) = \partial\mathbf{D}_k^-$, and $\text{ind}(\lambda - R_k^-) = -i_k$ for all $\lambda \in \mathbf{D}_k^-$ ($k = 1, 2, \dots, m$). Similarly, let $R_h^+ = (R_h^-)^*$, where R_h^- is the operator given by Proposition 2.1 such that $\sigma(R_h^-) = (\mathbf{D}_h^+)^-$, $\sigma_{le}(R_h^-) = \partial\mathbf{D}_h^-$, and $\text{ind}(\lambda - R_h^-) = -i_h$ for all $\lambda \in \mathbf{D}_h^+$ ($h = 1, 2, \dots, m$).

The operators $\varphi_h^+(R_h^+)$ and $\varphi_k^-(R_k^-)$ are well-defined via Riesz-Dunford functional calculus, and it is straightforward to check that

$$\sigma(\varphi_h^+(R_h^+)) = (\Omega_h^+)^-, \quad \sigma_{re}(\varphi_h^+(R_h^+)) = \partial\Omega_h^+,$$

$$\text{ind}(\lambda - \varphi_h^+(R_h^+)) = i_h \quad \text{and} \quad \ker(\lambda - \varphi_h^+(R_h^+))^* = \{0\} \quad \text{for all } \lambda \in \Omega_h^+,$$

$$\mathcal{A}'(\varphi_h^+(R_h^+)) = \mathcal{A}'(R_h^+) = \mathcal{A}^a(R_h^+) = \mathcal{A}^a(\varphi_h^+(R_h^+)) \quad (h = 1, 2, \dots, m),$$

and

$$\sigma(\varphi_k^-(R_k^-)) = (\Omega_k^-)^-, \quad \sigma_{le}(\varphi_k^-(R_k^-)) = \partial\Omega_k^-,$$

$$\text{ind}(\lambda - \varphi_k^-(R_k^-)) = -i_k \quad \text{and} \quad \ker(\lambda - \varphi_k^-(R_k^-)) = \{0\} \quad \text{for all } \lambda \in \Omega_k^-,$$

$$\mathcal{A}'(\varphi_k^-(R_k^-)) = \mathcal{A}'(R_k^-) = \mathcal{A}^a(R_k^-) = \mathcal{A}^a(\varphi_k^-(R_k^-)) \quad (k = 1, 2, \dots, n).$$

Clearly, the Riesz spectral invariant subspace $\mathcal{L}(A_1; \lambda_l)$ (associated with the clopen subset $\{\lambda_l\}$ of $\sigma(A_1)$) is finite-dimensional: $1 \leq d_l := \dim \mathcal{L}(A_1; \lambda_l) < \infty$ ($l = 1, 2, \dots, p$). For each of the (finitely many) components $\Omega_1, \Omega_2, \dots, \Omega_t$ of Ω , let $\mathbf{D}(\beta_s, r_s)$ be a disk whose closure is contained in Ω_s ($s = 1, 2, \dots, t$).

Define

$$T = \left\{ \sum_{h=1}^m \bigoplus \varphi_h^+(R_h^+) \right\} \oplus \left\{ \sum_{k=1}^n \bigoplus \varphi_k^-(R_k^-) \right\}$$

$$\oplus \left\{ \sum_{l=1}^p \bigoplus (\lambda_l + q_{d_l}) \right\} \oplus \left\{ \sum_{s=1}^t \bigoplus (\beta_s + r_s S) \right\},$$

where q_d denotes the nilpotent Jordan cell of order d and S is the unilateral shift of multiplicity one. Since the sets $\sigma(\varphi_h^+(R_h^+)) = (\Omega_h^+)^-$, $\sigma(\varphi_k^-(R_k^-)) = (\Omega_k^-)^-$, $\sigma(\lambda_l + q_l) = \{\lambda_l\}$, and $\sigma(\beta_s + r_s S) = \mathbf{D}(\beta_s, r_s)^-$ are pairwise disjoint, it is not difficult to infer that

$$\mathcal{A}'(T) = \left\{ \sum_{h=1}^m \bigoplus \mathcal{A}'(R_h^+) \right\} \oplus \left\{ \sum_{k=1}^n \bigoplus \mathcal{A}'(R_k^-) \right\}$$

$$\oplus \left\{ \sum_{l=1}^p \bigoplus \mathcal{A}'(q_{d_l}) \right\} \oplus \left\{ \sum_{s=1}^t \bigoplus \mathcal{A}'(S) \right\},$$

which coincides with $\mathcal{A}^a(T)$.

By using the Similarity Orbit Theorem [2, Theorems 9.1 and 9.2], we see that A_1 can be uniformly approximated by operators similar to T .

Hence, there exists A_2 similar to T such that $\|A - A_2\| \leq \|A - A_1\| + \|A_1 - A_2\| < 2\varepsilon$.

Clearly, $\mathcal{A}'(A_2) = \mathcal{A}^a(A_2)$. Since ε can be chosen arbitrarily small, we conclude that A belongs to the closure of the set of operators with tiny commutants.

Hence, $\{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}'(T) = \mathcal{A}^a(T)\}$ is dense in $\mathcal{L}(\mathcal{H})$. \square

Remarks 3.2. (i) D. A. Herrero [8] and D. Voiculescu [24] have independently constructed extensions of the classical Rota's model for a linear operator [20]: given an analytic Cauchy Ω , there exists an operator M on a Hilbert space \mathcal{H} such that for each $T \in \mathcal{L}(\mathcal{H})$ with $\sigma(T) \subset \Omega$ there is an invariant subspace \mathcal{M}

of M so that

$$M = \begin{pmatrix} M(T) & * \\ 0 & T' \end{pmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{H} \oplus \mathcal{M} \end{matrix},$$

where $M(T) = M|_{\mathcal{M}}$ is similar to M and T' is similar to T .

The proofs given in [8], [24] (see also [11, Chapter 3]) indicate that M can be chosen to be $\varphi(R)$, where R is the operator given by Proposition 2.1 for a suitable perforated disk, with \mathcal{H} an infinite-dimensional space, and ϕ is a conformal mapping from the closure of this perforated disk onto Ω^- . This construction produces a model M with the additional property that $\mathcal{A}'(M) = \mathcal{A}^a(M)$.

(ii) For $1 \leq \dim \mathcal{H} \leq \infty$, the above construction shows that for each Cauchy region Ω and each index m ($1 \leq m \leq \infty$), there exists an operator $M = M(\Omega, -m)$ such that $\sigma(M) = \Omega^-$, $\sigma_{le}(M) = \partial\Omega$, $\text{ind}(\lambda - M) = -m$ and $\ker(\lambda - M) = \{0\}$ for all $\lambda \in \Omega$, and $\mathcal{A}'(M) = \mathcal{A}^a(M)$ is isomorphic with $H^\infty(\Omega)$. In particular, $\mathcal{A}'(M)$ does not contain any nontrivial idempotent. (By taking adjoints, we can construct analogous examples with $\text{ind}(\lambda - M) = m$ and $\ker(\lambda - M)^* = \{0\}$ for all $\lambda \in \Omega$. If m is finite, then M is essentially normal, that is, $M^*M - MM^*$ is a compact operator.)

These kinds of operators play a very important role in the proof of the “approximate inverse of the Riesz decomposition theorem” given in [16]. Indeed, the constructions of [16] also include operators T such that T does not commute with any nontrivial idempotent, $\sigma(T) = \Omega^-$, $\sigma_{le}(T) = \sigma_{re}(T) = \partial\Omega$, $\text{ind}(\lambda - T) = 0$ and $\dim \ker(\lambda - T) = \dim \ker(\lambda - T)^* = 1$ for all $\lambda \in \Omega$.

However (as we shall see later, in Corollary 5.4), given a Cauchy region Ω , it is impossible to construct an operator T such that $\sigma(T) = \Omega^-$, $\sigma_{le}(T) = \sigma_{re}(T) = \partial\Omega$, $\text{ind}(\lambda - T) = 0$ and $\dim \ker(\lambda - T) = \dim \ker(\lambda - T)^* = 1$ for all $\lambda \in \Omega$, and $\mathcal{A}'(T) = \mathcal{A}^a(T)$ because such an operator commutes with “too many” rank-one operators.

4. OPERATORS SATISFYING $\text{Lat } T = \text{Lat } \mathcal{A}^a(T)$

Theorem 4.1. *Let $\rho_{s-F}^\pm(T) = \{\lambda \in \rho_{s-F}(T) : \text{ind}(\lambda - T) \neq 0\}$. Then*

$$\begin{aligned} & \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}(T) = \mathcal{A}^a(T)\}^- \\ &= \{T \in \mathcal{L}(\mathcal{H}) : \text{Lat } T = \text{Lat } \mathcal{A}^a(T)\}^- \\ &= \{A \in \mathcal{L}(\mathcal{H}) : \text{if } \gamma \text{ (Jordan curve)} \subset \rho_{s-F}^\pm(A), \text{ then } \hat{\gamma} \subset \sigma(A)\}. \end{aligned}$$

(Here $\hat{\gamma}$ denotes the polynomial hull of γ , that is, the complement of the unbounded component of $\mathbb{C} \setminus \gamma$.)

Proof. Let \mathcal{W}_1 , \mathcal{W}_2 , and \mathcal{W}_3 denote the first, the second, and, respectively, the third subset of $\mathcal{L}(\mathcal{H})$ in the statement of the theorem. Since $\mathcal{A}(T) = \mathcal{A}^a(T)$ obviously imply $\text{Lat } T = \text{Lat } \mathcal{A}^a(T)$, the inclusion $\mathcal{W}_1 \subset \mathcal{W}_2$ is trivial.

The inclusion $\mathcal{W}_2 \subset \mathcal{W}_3$ is an observation of J. B. Conway (personal communication): Suppose $\text{Lat } T = \text{Lat } \mathcal{A}^a(T)$ and γ is a Jordan curve included in

$\rho_{s-F}^\pm(T)$. Obviously, $\text{ind}(\lambda - T)$ is independent of the particular λ in the curve γ . Assume this index is negative.

Let $\mathcal{M} = \vee\{T^k y\}_{k=0}^\infty$ be the cyclic invariant subspace of T generated by a nonzero vector $y \in \ker(\xi - T)^*$ (for some $\xi \in \gamma$); then $\lambda - T|_{\mathcal{M}}$ is a Fredholm operator of index -1 for all λ in γ . Since \mathcal{M} is a cyclic subspace, it follows from [9], [12] (or [2, Chapter 11]) that $\lambda - T|_{\mathcal{M}}$ is a Fredholm operator of index -1 for all $\lambda \in \hat{\gamma}$.

Therefore, $\hat{\gamma} \subset \sigma(T|_{\mathcal{M}})$. But $\mathcal{M} \in \text{Lat } T = \text{Lat } \mathcal{A}^a(T)$, so that $\sigma(T|_{\mathcal{M}}) \subset \sigma(T)$ [17].

Hence, $\hat{\gamma} \subset \sigma(T)$.

By taking adjoints, we deduce that $\hat{\gamma} \subset \sigma(T)$ for every Jordan curve $\gamma \subset \rho_{s-F}(T)$ such that $\text{ind}(\lambda - T) > 0$ for $\lambda \in \gamma$.

It only remains to show that $\mathcal{W}_3 \subset \mathcal{W}_1$. Assume that $A \in \mathcal{W}_3$ and let $\varepsilon > 0$ be given. First of all, we approximate A by an operator $A_1 \in \mathcal{W}_3$ ($\|A - A_1\| < \varepsilon$) such that $\sigma_{lre}(A_1)$ is the closure of an analytic Cauchy domain Ω , including $\sigma_{lre}(A)$, $\sigma(A_1) = \sigma(A) \cup \Omega^-$, $\text{ind}(\lambda - A_1) = \text{ind}(\lambda - A)$ and $\dim \ker(\lambda - A_1) = \dim \ker(\lambda - A)$ for all nonisolated points λ of $\rho_{s-F}(A_1) \cap \sigma(A_1)$, and the restriction of A_1 to $\mathcal{H}(A_1; \lambda)$ is similar to “ $\lambda +$ nilpotent Jordan cell of order $\dim \mathcal{H}(A_1; \lambda)$ ” for each of the (finitely many) isolated points λ of $\sigma(A_1)$ (see [11]).

Thus A_1 is similar to the direct sum of a cyclic operator F acting on a finite-dimensional space and an operator $B \in \mathcal{W}_3$ such that $\sigma(B)$ has no isolated points; moreover, $\sigma(F)$ is disjoint from $[\rho_{s-F}^\pm(B)]^\wedge$.

Let $\Omega_1, \Omega_2, \dots, \Omega_m$ be an enumeration of the components of Ω , and let $\Phi_1, \Phi_2, \dots, \Phi_p$ be an enumeration of the components of $\rho_{s-F}^\pm(B)$. For each $i = 1, 2, \dots, m$, let N_j be any operator such that $\sigma_{le}(N_j) = \sigma_{re}(N_j) = \sigma(N_j)$ is a closed disk included in Ω_j . For each $k = 1, 2, \dots, p$, let M_k be any operator such that $\sigma(M_k) = (\Phi_k)^-$, $\sigma_{lre}(M_k) = \partial\Phi_k$, $\text{ind}(\lambda - M_k) = \text{ind}(\lambda - B)$ and $\min\{\dim \ker(\lambda - M_k), \dim \ker(\lambda - M_k)^*\} = 0$ for all $\lambda \in \Phi_k$. (These operators can be constructed, e.g., as in the proof of Theorem 3.1, or by using the results of [11, Chapter 3].)

Finally, if $\Psi_1, \Psi_2, \dots, \Psi_t$ is an enumeration of the bounded components of $\mathbb{C} \setminus [\bigcup_{k=1}^p \Phi_k]^-$, then for each $h = 1, 2, \dots, t$ we choose an operator L_h such that $\sigma(L_h) = (\Psi_h)^-$, $\sigma_{le}(L_h) = \sigma_{re}(L_h) = \partial\Psi_h$, and $\text{ind}(\lambda - L_h) = 0$ and $\dim \ker(\lambda - L_h) = \dim \ker(\lambda - L_h)^* = 1$ for all $\lambda \in \Psi_h$.

Let

$$R = F \oplus \left\{ \sum_{j=1}^m \bigoplus N_j \right\} \oplus \left\{ \sum_{k=1}^p \bigoplus M_k \right\} \oplus \left\{ \sum_{h=1}^t \bigoplus L_h \right\}.$$

By construction, $\sigma(R) = \sigma(F) \cup \{(\bigcup_{j=1}^m \sigma(N_j)) \cup (\bigcup_{k=1}^p \Phi_k) \cup (\bigcup_{h=1}^t \Psi_h)\}^-$ does not disconnect the plane; therefore (by using Runge’s theorem [5]), $\mathcal{A}^a(R) = \mathcal{A}(R)$.

Moreover, $\sigma(R)$ is a subset of $\sigma(A_1)$, $\rho_{s-F}(R) \supset \rho_{s-F}(A_1)$, and $\text{ind}(\lambda - R) = \text{ind}(\lambda - A_1)$ and $\min\{\dim \ker(\lambda - R)^k, \dim \ker[(\lambda - R)^*]^k\} \leq \min\{\dim \ker(\lambda - A_1)^k, \dim \ker[(\lambda - A_1)^*]^k\}$ for all $\lambda \in \rho_{s-F}(A_1)$ and all $k \geq 1$ (to see this, use that $A_1 \in \mathcal{W}_3!$), every component of $\sigma_{le}(A_1) \cap \sigma_{re}(A_1)$ meets the perfect set $\sigma_{le}(R) \cap \sigma_{re}(R)$, $\sigma(A_1)$ and $\sigma(R)$ have exactly the same isolated points, and $\dim \mathcal{H}(A_1; \lambda) = \dim \mathcal{H}(R; \lambda)$ for each such point.

By the Similarity Orbit Theorem [2, Theorems 9.1 and 9.2], there exists A_2 similar to R such that $\|A - A_2\| \leq \|A - A_1\| + \|A_1 - A_2\| < 2\varepsilon$.

Since $\mathcal{A}^a(R) = \mathcal{A}(R)$ and ε can be chosen arbitrarily small, we conclude that $A \in \mathcal{W}_1$.

The proof of Theorem 4.1 is now complete. \square

Remarks 4.2. (i) An easy corollary of the results of [3] (see also [1 and 11, Chapter 4]) is that for each R in $\mathcal{L}(\mathcal{H})$ there is a compact operator K such that $\sigma(R - K)$ does not disconnect the plane, and therefore $\mathcal{A}^a(R - K) = \mathcal{A}(R - K)$. Thus, if $\mathcal{K}(\mathcal{H})$ denotes the ideal of all compact operators, then $\{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}^a(T) = \mathcal{A}(T)\} + \mathcal{K}(\mathcal{H}) = \mathcal{L}(\mathcal{H})$.

(That is, $\{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}^a(T) = \mathcal{A}(T)\}$ “essentially” coincides with $\mathcal{L}(\mathcal{H})$.)

(ii) Given R in $\mathcal{L}(\mathcal{H})$, we have

$$\begin{aligned} & \text{dist}[R, \{T : \mathcal{A}^a(T) = \mathcal{A}(T)\}] \\ &= \inf\{\|B\| : B \in \mathcal{L}(\mathcal{H}) \text{ and } \sigma(R - B) \text{ does not disconnect the plane}\} \\ &\leq \inf\{\|K\| : K \in \mathcal{K}(\mathcal{H}) \text{ and for each Jordan curve } \gamma \subset \rho_{s-F}(R) \\ &\quad \text{such that } \text{ind}(\lambda - R) \neq 0 \text{ for } \lambda \text{ in } \gamma, \hat{\gamma} \subset \sigma(R - K)\}. \end{aligned}$$

Both the equality and the inequality can be justified by using the proof of Theorem 4.1. (The details are left to the reader.)

The author conjectures that the inequality is actually an equality. An upper estimate for the second infimum can be easily derived from the results of [13].

(iii) By using the theorem on density of operators with “bad properties” [11, Theorem 3.51], it is not difficult to see that all the classes $\{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}(T) \neq \mathcal{A}^a(T)\}$ and $\{T \in \mathcal{L}(\mathcal{H}) : \text{Lat } T \neq \text{Lat } \mathcal{A}^a(T)\}$ (and the classes similarly defined with $\mathcal{A}(T)$ and $\mathcal{A}^a(T)$ replaced by any other pair of algebras) are dense in $\mathcal{L}(\mathcal{H})$.

Furthermore, if B is any operator such that $\sigma(B) = \{\lambda \in \mathbf{C} : 1 \leq |\lambda| \leq 2\}$ and $\lambda - B$ is a Fredholm operator of index 1 for all λ with $1 < |\lambda| < 2$, then $\mathcal{A}(B') \neq \mathcal{A}^a(B')$ for all B' close enough to B . By using this observation and the proof of [11, Theorem 3.51], we deduce that $\{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}(T) = \mathcal{A}^a(T)\}^-$ is nowhere dense in $\mathcal{L}(\mathcal{H})$.

A fortiori, so is $\{T \in \mathcal{L}(\mathcal{H}) : \text{Lat } T = \text{Hyperlat } T\}^-$ (and the closure of any other class between these two).

5. THE CASE $\text{Lat } T = \text{Hyperlat } T$

Proposition 5.1. $\{A \in \mathcal{L}(\mathcal{H}) : \text{if } \gamma \text{ (Jordan curve)} \subset \rho_{s-F}^\pm(A), \text{ then } \text{ind}(\lambda - A) \text{ is constant for } \lambda \in \hat{\gamma} \cap \rho_{s-F}(A)\}$

$$\begin{aligned} &\subset \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}(T) = \mathcal{A}'(T)\}^- \\ &\subset \{T \in \mathcal{L}(\mathcal{H}) : \text{Lat } T = \text{Hyperlat } T\}^- \\ &\subset \{A \in \mathcal{L}(\mathcal{H}) : \text{if } \gamma \text{ (Jordan curve)} \subset \rho_{s-F}^\pm(A), \\ &\qquad\qquad\qquad \text{then } \hat{\gamma} \cap \rho_{s-F}(A) \subset \rho_{s-F}^\pm(A)\} \\ &\subset \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}(T) = \mathcal{A}^a(T)\}^- \end{aligned}$$

Furthermore, the first and the last inclusions are proper.

We shall need an auxiliary result which has some interest in itself. Recall that $\mu \in \rho_{s-F}(T)$ is a *singular point* of the semi-Fredholm domain (in the sense of C. Apostol [1]) if the mapping $\lambda \rightarrow P_{\ker(\lambda-T)}$ (= orthogonal projection of \mathcal{H} onto $\ker(\lambda - T)$) is discontinuous at $\lambda = \mu$. In this case, $T = W[(\mu + Q) \oplus B]W^{-1}$, where W is invertible, Q is a Jordan nilpotent (acting on a finite-dimensional space \mathcal{R}), and μ is not singular for B (see [1], [18], or [11, Chapter 3]).

Proposition 5.2. (i) *If $T \in \mathcal{L}(\mathcal{H})$ and $\text{Lat } \mathcal{A}''(T) = \text{Hyperlat } T$, then $\lambda \rightarrow P_{\ker(\lambda-T)}$ is continuous at $\lambda = \mu$ for all interior points μ of $\sigma(T) \cap \rho_{s-F}(T)$; that is, the only singularities of the semi-Fredholm domain of T are the isolated points of $\sigma(T) \setminus \sigma_e(T)$.*

Furthermore, if $\lambda \in \sigma(T) \setminus \sigma_e(T)$ is isolated, then $T|_{\mathcal{H}(T; \lambda)}$ is a cyclic operator (on the finite-dimensional space $\mathcal{H}(T; \lambda)$).

(ii) *If $\text{Lat } \mathcal{A}^a(T) = \text{Hyperlat } T$, then $\min\{\dim \ker(\lambda - T), \dim \ker(\lambda - T)^*\} = 0$ for each interior point λ of $\sigma(T) \cap \rho_{s-F}(T)$. In particular, $\mu - T$ is invertible for each interior point μ of $\sigma(T) \cap \{\lambda \in \rho_{s-F}(T) : \text{ind}(\lambda - T) = 0\}$.*

Proof. (i) If μ is a singular point of $\rho_{s-F}(T)$ and μ belongs to the interior of $\sigma(T)$, then T has the above described form and $\lambda - B$ is a semi-Fredholm operator with nontrivial kernel (or nontrivial cokernel) for all λ in some neighborhood of μ . To simplify the notation, we can directly assume that $W = 1$ and $\mu = 0$, that is, $T = Q \oplus B$.

It is obvious that T commutes with the orthogonal projection $P_{\mathcal{R}}$ (of \mathcal{H} onto \mathcal{R}), and therefore $\mathcal{R}, \mathcal{H} \ominus \mathcal{R} \in \text{Lat } \mathcal{A}''(T)$. If f is a unit vector in $\ker Q^*$ and g is a unit vector in $\ker B$, and we define the rank-one operator $g \otimes f^* \in \mathcal{L}(\mathcal{H})$ by $g \otimes f^*(x) = \langle x, f \rangle g$, then

$$T(g \otimes f^*) = (Bg) \otimes f^* = 0 \quad \text{and} \quad (g \otimes f^*)T = g \otimes (Q^*f)^* = 0,$$

so that $g \otimes f^*$ commutes with T . But $g \otimes f^*$ does not commute with $P_{\mathcal{R}}$, and therefore $g \otimes f^* \notin \mathcal{A}''(T)$.

Observe that $g = g \otimes f^*(f) \in g \otimes f^*(\mathcal{R})$, but $g \perp \mathcal{R}$.

Therefore, $\mathcal{R} \in \text{Lat } \mathcal{A}''(T)$, but $\mathcal{R} \notin \text{Hyperlat } T$.

(If $\ker B = \{0\}$, then we apply the same argument to T^* , instead of T .)

An operator Q acting on a finite-dimensional space satisfies $\mathcal{A}''(Q) = \mathcal{A}'(Q)$ if and only if Q is cyclic, whence we obtain the second statement (see, e.g., [11, Chapter 2]).

(ii) Now assume that T is semi-Fredholm and 0 is an interior point of $\sigma(T)$. If 0 is singular, then $\text{Lat } \mathcal{A}^a(T) \neq \text{Hyperlat } T$, by (i).

If 0 is not singular and $\ker(\lambda - T) \neq \{0\} \neq \ker(\lambda - T)^*$ for all λ in the component Ω of $\rho_{s-F}(T)$ containing the origin; then we pick unit vectors $g \in \ker T$ and $f \in \ker T^*$. We can directly assume that $\ker T$ is finite-dimensional. (If not, replace T by T^* .) This guarantees that for each invariant subspace \mathcal{M} , $(T|_{\mathcal{M}})|[\ker(T|_{\mathcal{M}})^\perp]$ is bounded below.

Let $\mathcal{M} = \{Af: A \in \mathcal{A}^a(T)\}^- \in \text{Lat } \mathcal{A}^a(T)$. Since \mathcal{M} is a rationally cyclic invariant subspace, T is semi-Fredholm with $\text{ind } T > \infty$, and $f \in \ker T^* = (\text{ran } T)^\perp$, it follows from [9], [12], or [2, Chapter 11] that $\lambda - T|_{\mathcal{M}}$ is a Fredholm operator with trivial kernel and index equal to -1 for all $\lambda \in \Omega$. This means, in particular, that $g \notin \mathcal{M}$.

But $g \otimes f^* \in \mathcal{A}'(T)$ and $g = g \otimes f^*(f) \in g \otimes f^*(\mathcal{M})$.

Therefore, $\mathcal{M} \in \text{Lat } \mathcal{A}^a(T)$, but $\mathcal{M} \notin \text{Hyperlat } T$.

Now the result follows by applying the above argument to $\mu - T$ for each interior point μ of $\rho_{s-F}(T) \cap \sigma(T)$. \square

Let f and g be as in the second part of the proof. It is easy to construct examples where $\mathcal{N} = \{Bf: B \in \mathcal{A}''(T)\}^- \in \text{Lat } \mathcal{A}''(T)$ satisfies that $\text{ind}(T|_{\mathcal{N}}) = -m$ for some m , $2 \leq m \leq \infty$ (see, e.g., [10]). Thus, \mathcal{M} is a proper subspace of \mathcal{N} , in general.

Question 5.3. Let f , g , \mathcal{M} , and \mathcal{N} be as above. Does $g \otimes f^*$ always belong to $\mathcal{A}'(T) \setminus \mathcal{A}''(T)$? Does $\mathcal{N} \in \text{Lat } \mathcal{A}''(T) \setminus \text{Hyperlat } T$?

Corollary 5.4. *If $\ker(\lambda - T)$ and $\ker(\lambda - T)^*$ are nontrivial for all λ in some open disk included in $\rho_{s-F}(T)$, then $\mathcal{A}^a(T) \neq \mathcal{A}'(T)$ and $\text{Lat } \mathcal{A}^a(T) \neq \text{Hyperlat } T$.*

Now we are in a position to prove Proposition 5.1. The second and the fourth inclusion are obvious, and Theorem 4.1 indicates that the fourth inclusion is actually proper (for instance, if $\sigma(A) = \mathbf{D}^-$, $\sigma_e(A) = \{\lambda: |\lambda| = 1 \text{ or } |\lambda| = 1/2\}$, $\text{ind}(\lambda - A) = 0$ for $|\lambda| < 1/2$, and $\text{ind}(\lambda - A) = 1$ for $1/2 < |\lambda| < 1$, then $\mathcal{A}(A) = \mathcal{A}^a(A)$, and $\gamma = \{\lambda \in \mathbf{C}: |\lambda| = 3/2\} \subset \rho_{s-F}^\pm(A)$, but $\hat{\gamma} \cap \rho_{s-F}(A) \not\subset \rho_{s-F}^\pm(A)$).

The third inclusion follows by Corollary 5.4. If γ (Jordan curve) $\subset \rho_{s-F}^\pm(A)$ and $\text{ind}(\lambda - A) = 0$ for some $\lambda \in \hat{\gamma} \cap \rho_{s-F}(A)$, then every T close enough to A has the same properties [4], [11, Chapter 1]. If $\lambda - T$ (or $\lambda' - T$ for some λ' close enough to λ) is invertible, then $\text{Lat } T \neq \text{Lat } \mathcal{A}^a(T)$, by Theorem 4.1. If $\ker(\mu - T) \neq \{0\} \neq \ker(\mu - T)^*$ for all μ on some neighborhood of λ , then $\text{Lat } \mathcal{A}^a(T) \neq \text{Hyperlat } T$, by Corollary 5.4. Thus, A cannot be approximated by operators satisfying $\text{Lat } T = \text{Hyperlat } T$.

Let \mathscr{W} denote the first of the five sets described in the statement of the proposition. Let $A \in \mathscr{W}$ and let $\varepsilon > 0$ be given. First of all, we find $A_1 \in \mathscr{W}$, with $\|A_1 - A\| < \varepsilon$, such that $\sigma_{lre}(A_1)$ is the closure of an analytic Cauchy domain Ω , and $\sigma(A_1)$ has only finitely many isolated points (see [11]).

Since $A_1 \in \mathscr{W}$, we can find a finite collection $\Phi_1, \Phi_2, \dots, \Phi_m$ of simply connected analytic Cauchy regions, with pairwise disjoint closures, such that $\partial\Phi_j \subset \Omega$ and $\text{ind}(\lambda - A_1) \equiv m_j \neq 0$ for $\lambda \in \Phi_j \cap \rho_{s-F}(A_1)$, and $\text{ind}(\lambda - A_1) = 0$ for all $\lambda \in \rho_{s-F}(A_1) \setminus \bigcup_{j=1}^m \Phi_j$.

Let $\Omega_1, \Omega_2, \dots, \Omega_n$ be an enumeration of the components of Ω . If $\Omega_k \subset \Phi_j$ (for some j), then we choose a closed disk $\Delta_k \subset \Omega_k$. If Ω_k is not included in the union of the Φ_j 's, then we choose a closed disk $\Delta_k \subset \Omega_k \setminus (\bigcup_{j=1}^m \Phi_j)^-$. Without loss of generality, we can assume that the Ω_k 's are ordered so that $\Omega_k \subset \bigcup_{j=1}^m \Phi_j$ if and only if $1 \leq k \leq p$ ($\leq n$).

Let N_k be a normal operator such that $\sigma(N_k) = \Delta_k$ ($k = 1, 2, \dots, n$).

By construction, A_1 is similar to the direct sum of an operator F acting on a finite-dimensional space ($\sigma(F)$ is the set of isolated points of $\sigma(A)$), and an operator B such that $\sigma(B) = \sigma(A) \setminus \sigma(F)$. Without loss of generality, we can directly assume that F is cyclic, and therefore $\mathscr{A}(F) = \mathscr{A}'(F)$.

Let $L \in \mathscr{L}(\mathscr{H})$ be any operator of the form

$$L = F \oplus \left\{ \sum_{k=1}^n \bigoplus N_k \right\} \oplus \left\{ \sum_{j=1}^m \bigoplus L_j \right\},$$

where $\sigma(L_j) = (\Phi_j)^-$, $\sigma_{lre}(L_j) = \partial\Phi_j$, and $\text{ind}(\lambda - L_j) = m_j$ and

$$\min\{\dim \ker(\lambda - L_j), \dim \ker(\lambda - L_j)^*\} = 0$$

for all $\lambda \in \Phi_j$, $j = 1, 2, \dots, m$.

The Similarity Orbit Theorem implies that there exists A_2 similar to L such that $\|A_1 - A_2\| < \varepsilon$.

For each $t \geq 1$, let $\{\mu_r^t\}$ be a set of pt distinct points such that exactly t of these points lie in Δ_k ($k = 1, 2, \dots, p$).

For each $j = 1, 2, \dots, m$, let ϕ_j be a conformal mapping from \mathbf{D}^- onto $(\Phi_j)^-$, and let $\Delta_{j,t} = \phi_j^{-1}(\Phi_j \cap \{\mu_r^t\})$.

By Proposition 2.2, for each j we can construct $R_{j,t}$ such that $\sigma(R_{j,t}) = \mathbf{D}^-$, $\sigma_{lre}(R_{j,t}) = \partial\mathbf{D} \cup \Delta_{j,t}$, and $\text{ind}(\lambda - R_{j,t}) = m_j$ and

$$\min\{\dim \ker(\lambda - R_{j,t}), \dim \ker(\lambda - R_{j,t})^*\} = 0$$

for all $\lambda \in \mathbf{D} \setminus \Delta_{j,t}$. (This is obvious for $m_j < 0$; if $m_j > 0$, then we construct $R_{j,t}$ by taking suitable adjoints.)

Let

$$T_t = F \oplus \left\{ \sum_{k=p+1}^n \bigoplus N_k \right\} \oplus \left\{ \sum_{j=1}^m \bigoplus \phi_j(R_{j,t}) \right\}.$$

It is immediate that $\sigma(T_t) = \sigma(F) \cup (\bigcup_{k=p+1}^n \Delta_k) \cup (\bigcup_{j=1}^m \Phi_j)^-$, does not disconnect the plane,

$$\begin{aligned} \sigma_{lre}(T_t) &= \left(\bigcup_{k=p+1}^n \Delta_k \right) \cup \left\{ \bigcup_{j=1}^m [\partial\Phi_j \cup (\Phi_j \cap \{\mu_r^t\})] \right\} \\ &\subseteq \left(\bigcup_{k=1}^n \Delta_k \right) \cup \left(\bigcup_{j=1}^m \partial\Phi_j \right) \subset \Omega, \end{aligned}$$

and $\text{ind}(\lambda - T_t) = m_j$ and $\min\{\dim \ker(\lambda - T_t), \dim \ker(\lambda - T_t)^*\} = 0$ for all $\lambda \in \Phi_j \setminus \{\mu_r^t\}$; moreover,

$$\mathcal{A}'(T_t) = \mathcal{A}'(F) \oplus \left\{ \sum_{k=p+1}^n \bigoplus \mathcal{A}'(N_k) \right\} \oplus \left\{ \sum_{j=1}^m \bigoplus \mathcal{A}'(R_{j,t}) \right\} = \mathcal{A}(T_t).$$

The Similarity Orbit Theorem indicates that if M_t is a diagonal normal operator of uniform infinite multiplicity such that $\sigma(M_t) = \sigma_e(M_t) = \{\mu_r^t\}$, then

$$L_t = F \oplus \left\{ \sum_{k=p+1}^n \bigoplus N_k \right\} \oplus M_t \oplus \left\{ \sum_{j=1}^m \bigoplus L_j \right\}$$

can be uniformly approximated by operators similar to T_t . Furthermore, for t large, we can also approximate $\sum \bigoplus_{k=1}^p N_k$ by operators similar to M_t (see [11, Chapter 5]).

Thus, if t is sufficiently large, then we can find A_3 similar to T_t such that $\|A_2 - A_3\| < \varepsilon$.

Since $\|A - A_3\| < 3\varepsilon$, $\mathcal{A}'(A_3) = \mathcal{A}(A_3)$, and ε can be chosen arbitrarily small, we conclude that A is the uniform limit of a sequence of operators $\{A_t\}_{t=1}^\infty$ satisfying the condition $\mathcal{A}'(A_t) = \mathcal{A}(A_t)$ for all $t = 1, 2, \dots$.

Hence, $\mathcal{W} \subset \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{A}(T) = \mathcal{A}'(T)\}^-$.

It only remains to show that the first inclusion is also proper. This follows immediately from the following.

Example 5.5. Let B denote the bilateral weighted shift defined by

$$Be_n = \begin{cases} e_{n+1} & (n \geq 0), \\ (1/|n|)e_{n+1} & (n < 0), \end{cases}$$

with respect to the orthonormal basis $\{e_n\}_{-\infty}^{+\infty}$ of \mathcal{H} , and let $\mathcal{H}_+ = \vee\{e_n\}_{n \geq 0}$ and $\mathcal{H}_- = \vee\{e_n\}_{n < 0}$.

Define $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ by

$$T = \begin{pmatrix} (1/2)B_- & 0 & 1/2 & 0 \\ (1/2)Z & (1/2)B_+ & 0 & 0 \\ 0 & 0 & B_- & 0 \\ 0 & 0 & Z & B_+ \end{pmatrix} \begin{matrix} \mathcal{H}_- \\ \mathcal{H}_+ \\ \mathcal{H}_- \\ \mathcal{H}_+ \end{matrix},$$

where

$$B = \begin{pmatrix} B_- & 0 \\ Z & B_+ \end{pmatrix} \begin{matrix} \mathcal{H}_- \\ \mathcal{H}_+ \end{matrix}.$$

It is immediate that $\sigma(T) = \mathbf{D}^-$, $\sigma_e(T) = \{\lambda \in \mathbf{C}: |\lambda| = 0, 1/2, \text{ or } 1\}$, $\text{ind}(\lambda - T) = -2$ for $0 < |\lambda| < 1/2$, and $\text{ind}(\lambda - T) = -1$ for $1/2 < |\lambda| < 1$. (So that $T \notin \mathcal{W}^-$!)

A cumbersome calculation shows that T is a C_{00} -construction with (isometrically isomorphic) $H^\infty(\mathbf{D})$ functional calculus, and $\mathcal{A}'(T) = \mathcal{A}(T)$.

The proof of Proposition 5.1 is now complete. \square

Example 5.5 is a particular case of a large family of operators satisfying the condition $\mathcal{A}'(T) = \mathcal{A}(T)$, to be described in a sequel article. Unfortunately, this family is not large enough to completely characterize $\{T \in \mathcal{L}(\mathcal{H}): \mathcal{A}'(T) = \mathcal{A}(T)\}^-$.

REFERENCES

1. C. Apostol, *The correction by compact perturbations of the singular behavior of operators*, Rev. Roumaine Math. Pures Appl. **21** (1976), 155–175.
2. C. Apostol, L. A. Fialkow, D. A. Herrero, and D. Voiculescu, *Approximation of Hilbert space operators*, Volume II, Research Notes in Math., vol. 102, Pitman, Boston-London-Melbourne, 1984.
3. C. Apostol, C. M. Pearcy, and N. Salinas, *Spectra of compact perturbations of operators*, Indiana Univ. Math. J. **26** (1977), 345–350.
4. S. R. Caradus, W. E. Pfaffenberger and B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Lecture Notes in Pure and Appl. Math., vol. 9, Marcel Dekker, New York, 1974.
5. T. W. Gamelin, *Uniform algebras*, Prentice Hall, Englewood Cliffs, N.J., 1969.
6. R. Gellar and D. A. Herrero, *Hyperinvariant subspaces of bilateral weighted shifts*, Indiana Univ. Math. J. **23** (1974), 771–790.
7. G. M. Goluzin, *Geometric theory of functions of a complex variable*, Amer. Math. Soc., Providence, R.I., 1969. (Translated from the Russian)
8. D. A. Herrero, *A Rota universal model for operators with multiply connected spectrum*, Rev. Roumaine Math. Pures Appl. **21** (1976), 15–23.
9. —, *On multicyclic operators*, Integral Equations Operator Theory **1** (1978), 57–102.
10. —, *Quasimilar operators with different spectra*, Acta Sci. Math. (Szeged) **41** (1979).
11. —, *Approximation of Hilbert space operators*, Volume I, Research Notes in Math., vol. 72, Pitman, Boston-London-Melbourne, 1982.
12. —, *The Fredholm structure of an n -multicyclic operator*, Indiana Univ. Math. J. **36** (1987), 549–566.
13. —, *Economical compact perturbations. II: Filling in the holes*, J. Operator Theory **19** (1988), 25–42.
14. —, *Similarity and approximation of operators*, Proc. Sympos. Pure Math., vol. 51, part 1, Amer. Math. Soc., Providence, R.I., 1990, pp. 225–237.
15. —, *A metatheorem on similarity and approximation of operators*, J. London Math. Soc. (to appear).
16. D. A. Herrero and C.-L. Jiang, *Limits of strongly irreducible operators and the Riesz decomposition theorem*, Michigan J. Math. **37** (1990), 283–291.

17. D. A. Herrero and N. Salinas, *Analytically invariant and bi-invariant subspaces*, Trans. Amer. Math. Soc. **173** (1972), 117–136.
18. T. Kato, *Perturbation theory for nullity, deficiency, and other quantities of linear operators*, J. Analyse Math. **6** (1958), 261–322.
19. ———, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1966.
20. G.-C. Rota, *On models for linear operators*, Comm. Pure Appl. Math. **13** (1960), 469–472.
21. R. Schatten, *Norm ideals of completely continuous operators*, Ergeb. Math. Grenzgeb., vol. 27, 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York, 1970.
22. A. L. Shields and L. J. Wallen, *The commutant of certain Hilbert space operators*, Indiana Univ. Math. J. **20** (1971), 777–788.
23. B. Sz.-Nagy and C. Foiaş, *Analyse, harmonique des opérateurs de l'espace de Hilbert*, Akademiai Kiadó, Budapest and Masson, Paris, 1967.
24. D. Voiculescu, *Norm-limits of algebraic operators*, Rev. Roumaine Math. Pures Appl. **19** (1974), 371–378.
25. W. R. Wogen, *On cyclicity of commutants*, Integral Equations Operator Theory **5** (1982), 141–143.

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