FIRST STEPS IN DESCRIPTIVE THEORY OF LOCALES

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ABSTRACT. F. Hausdorff and D. Montgomery showed that a subspace of a completely metrizable space is developable if and only if it is F_σ and G_δ. This extends to arbitrary metrizable locales when "F_σ" and "G_δ" are taken in the localic sense (countable join of closed, resp. meet of open, sublocales). In any locale, the developable sublocales are exactly the complemented elements of the lattice of sublocales. The main further results of this paper concern the strictly pointless relative theory, which exists because—always in metrizable locales—there exist nonzero pointless-absolute G_δ's, G_δ in every pointless extension. For instance, the pointless part $\mathrm{pl}(\mathbf{R})$ of the real line is characterized as the only nonzero zero-dimensional separable metrizable pointless-absolute G_δ. There is no nonzero pointless-absolute F_σ. The pointless part of any metrizable space is, if not zero, second category, i.e. not a countable join of nowhere dense sublocales.

INTRODUCTION

This paper initiates, not a generalization of descriptive set theory, but a reformulation of it admitting "pointless" subsets or *sublocales* of a metrizable space. An open, or closed, or F_σ sublocale is a subspace—open, closed, or F_σ respectively. However, the countable intersections of open sublocales, in the lattice of sublocales, are not in general subspaces. We call them O_δ's. In completely metrizable spaces, O_δ is the same as G_δ. In general, any O_δ which is a subspace is a G_δ. One may think of the O_δ's as (1) those which are subspaces: the nice G_δ's, and (2) the others. The set of points of an O_δ is a G_δ, but two different O_δ's can have the same set of points.

This first paper does not go much beyond a study of O_δ's. Hausdorff and Montgomery showed [K] that in a completely metrizable space, the subspaces which are both F_σ and G_δ coincide with those which can be expanded in an alternating series of closed sets,

$$\cdots(((\cdots(((F_0\backslash F_1)\cup F_2)\backslash F_3)\cup\cdots)\cup F_\omega)\backslash F_{\omega+1})\cup\cdots,$$

with $\{F_\alpha\}$ decreasing (nonstrictly). It turns out that completeness is unnecessary, indeed spatiality is unnecessary for the Hausdorff-Montgomery theorem; it holds in metrizable locales with "O_δ" in place of "G_δ".

In arbitrary locales, the complemented elements of the lattice of sublocales are exactly the sublocales which can be expanded in an alternating series of closed sublocales. Thus

In metrizable locales, the complemented sublocales are the same as the sublocales which are at once F_σ and O_δ.

That is much the best result of the paper. The most interesting of the further results are (1) in metrizable locales, pointless absolute O_δ's exist: a pointless metrizable locale is O_δ in every pointless metrizable extension if and only if it is the pointless part (= largest pointless sublocale) of some completely metrizable space. But (1') this does not extend to dense embeddings in pointless completely regular locales, as "absolute G_δ" does for spaces. (2) There are no nonzero pointless absolute F_σ's. (3) The pointless part of the real line is characterized as the only nonzero zero-dimensional pointless metrizable absolute O_δ with a countable base. (4) The pointless part of any nonzero metrizable space (complete or not) is second category, i.e. not a countable join of nowhere dense sublocales.

1. COMPLEMENTS

In metrizable locales there seems to be a working descriptive theory; in more generality, fragments. But the fragments are numerous, so we consider general locales, as far as is easily possible. The needed background is almost all in $[J_1]$, if you can stand the point of view: Johnstone's locales keep intruding their frames into innocent conversation, rather as if people were continually showing you their skeletons—

> "*Webster was much possessed by death*
> *And saw the skull beneath the skin.*"

For me, a locale A has a frame $T(A)$ of open parts (sublocales) and a lattice $S(A)$ of all sublocales. $S(A)$, upside down, is the frame $T(A_d)$ of the dissolution locale A_d, which has a distinguished monomorphism to A representing the sublocales of A by their pullbacks (intersections) in A_d, which are precisely the closed sublocales of A_d $[I_1, J_2]$. $S(A)^{op}$ is not just a frame but a zero-dimensional frame; more fully, if $x \neq y$ in $S(A)$, there is a locally closed sublocale $U \Delta V$ of A (symmetric difference of two open parts) which has zero meet with one of x and y but not with the other. (This holds just because x and y are determined by the respective relations $U \wedge x = V \wedge x$, $U \wedge y = V \wedge y$, on $T(A)$.) Zero-dimensionality follows, for locally closed sublocales have complements.

There are three properties equivalent to complementedness in any locale. A *developable* sublocale is a sublocale

$$(*) \qquad\qquad F_0 - F_1 \vee F_2 - F_3 \vee \cdots \vee F_\omega - F_{\omega+1} \vee \ldots \, ,$$

where $\{F_\alpha\}$ is a transfinite descending sequence of closed sublocales. This notation (standard for spaces [K]) is interpreted in the obvious way *from the*

left, $(((F_0 - F_1) \vee F_2) - F_3) \vee \ldots$, and at a limit ordinal, intersect the descending initial segments F_0, $(F_0 - F_1) \vee F_2$, ... ("–" means "meet with the complement of".) For a development (∗) one also requires continuity at limit ordinals λ : $F_\lambda = \bigwedge_{\kappa < \lambda} F_\kappa$. (If it were omitted, it could be restored by increasing the length by 2, interpolating '$\vee M - M$' at each limit position, where M is the required term.) Second, we need the operation $'$, where S' is the meet of S with the closure of the join of all sublocales of S^- disjoint from S (telegraphically, $S' = S \wedge (S^- \backslash S)^-$). Put $S_0 = S$, $S_{\alpha+1} = S'_\alpha$, $S_\lambda = \bigwedge_{\kappa < \lambda} S_\kappa$ at limit ordinals λ. Third, say that S *splits* T *inseparably* if $(S \wedge T)^- \wedge (T \backslash S)^- \supset T$. (Precisely, no closed proper sublocale of T contains $S \wedge T$ or contains all sublocales of T disjoint from S.)

1.1. Theorem. *For a sublocale S of a locale A, the following are equivalent:*
  (1) S *is a complemented sublocale.*
  (2) S *splits no nonzero* (*closed*) *sublocale inseparably.*
  (3) S *is a developable sublocale.*
  (4) $S_\alpha = 0$ *for some ordinal α.*

The proof depends on a lemma about developments, 1.2. For classical developments in subspaces, the operations $\cup$ and $\backslash$ are performed in a lattice of all subsets and the lemma is evident. Here, recall that limit ordinals are even. (2α is not $\alpha 2$, but is the order type of α blocks of 2.) Recall also that every locale A has a smallest dense sublocale $D(A)$, $[I_1$ or $J_1]$.

1.2. Lemma. *For a development in complemented sublocales,*

$$F_0 - F_1 \vee F_2 - F_3 \vee \cdots \vee F_\omega - F_{\omega+1} \vee \ldots ,$$

indexed by the ordinals less than a limit ordinal λ, let M be the meet of the even initial segments, F_0, $F_0 - F_1 \vee F_2$, ... ; let J be the join of the odd initial segments $F_0 - F_1$, $F_0 - F_1 \vee F_2 - F_3$, ... Then $J \vee (\bigwedge_{\kappa < \lambda} F_\kappa) = M$.

Proof. Just as for alternating series of real numbers, each odd initial segment is $\leq$ each even initial segment, so $J \leq M$; and evidently $(\bigwedge_{\kappa < \lambda} F_\kappa) \leq M$. So we need only show that if $z \wedge M \neq 0$ then z meets J or $\bigwedge_{\kappa < \lambda} F_\kappa$. But if $z \wedge M$ does not meet J, it never meets $F_0 - F_1 \vee \cdots \vee F_{2\alpha} - F_{2\alpha+1}$. It is, however, $\leq M \leq (F_0 - F_1 \vee \cdots \vee F_{2\alpha} - F_{2\alpha+1}) \vee F_{2\alpha+2}$. Hence $z \wedge M \leq F_{2\alpha+2} \leq F_{2\alpha+1}$; $z \wedge M \leq F_\kappa$ for all $\kappa < \lambda$, so $z \wedge M \leq \bigwedge_{\kappa < \lambda} F_\kappa$.

Proof of 1.1. If (1) S has a complementary sublocale C, and T is a nonzero sublocale, $D(T)$ is nonzero. Not both S and C contain $D(T)$; hence $S \wedge T$ or $C \wedge T$ is not dense in T, so S does not split it inseparably.

Assume (2). Then every nonzero closed sublocale T has a closed proper part T^* which contains either $S \wedge T$ or $(T \backslash S)^-$ [all parts of T disjoint from S]. Put $E_0 = A$, $E_{\alpha+1} = E_\alpha^*$, $E_\lambda = \bigwedge_{\kappa < \lambda} E_\kappa$ at limit ordinals. Evidently the descending sequence $\{E_\alpha\}$ must reach 0, at some limit ordinal.

We now construct a subsequence of $\{E_\alpha\}$ which will be a development of S (discontinuous. As noted above, a continuous development can be obtained by

interpolating suitable pairs of terms $\vee M - M$). $E_0 = A$ contains S. Let F_0 be the last E_α which contains S; since $\{E_\alpha\}$ is continuous at limit ordinals and has a last term, F_0 exists. If $F_0 = 0$, stop. Otherwise F_0 is some E_β, and $E_{\beta+1} = F_0^*$ does not contain $S \wedge F_0 = S$, so it does contain $(F_0\backslash S)^-$. Let F_1 be the last E_α which contains $(F_0\backslash S)^-$. [At this point the reader may prefer to amplify "and so on" in his own notation.] Inductively, stopping whenever we are out of E's ($F_\alpha = 0$), with each $F_{2\alpha+2}$ being the last E_β which contains $S \wedge F_{2\alpha+1}$ and $F_{2\alpha+1}$ the last E_β which contains $(F_{2\alpha}\backslash S)^-$, the successor $E_{\beta+1}$ contains the other (i.e., respectively $(F_{2\alpha+2}\backslash S)^-$ or $S \wedge F_{2\alpha+1}$). Let (respectively) $F_{2\alpha+3}$ be the last E_γ containing $(F_{2\alpha+2}\backslash S)^-$, and $F_{2\alpha+2}$ the last E_γ containing $S \wedge F_{2\alpha+1}$. Having—as we do for F_0 and $F_0 - F_1$—every even initial segment $I_{2\delta}$ of the development containing S and every odd one $I_{2\delta+1}$ contained in S, these extensions preserve those inclusions, as follows. $I_{2\alpha+3} = I_{2\alpha+1} \vee F_{2\alpha+2} - F_{2\alpha+3}$, removing $F_{2\alpha+3}$ and thus $(F_{2\alpha+2}\backslash S)^-$, so still $I_{2\alpha+3} \leq S$. And $I_{2\alpha+2} = I_{2\alpha} - F_{2\alpha+1} \vee F_{2\alpha+2}$, putting back $F_{2\alpha+2} \geq S \wedge F_{2\alpha+1}$, so still $I_{2\alpha+2} \geq S$.

At a limit ordinal λ, observe that vacuously (as for $\lambda = 0$) the next E, which is $\bigwedge_{\kappa<\lambda} F_\kappa$, contains $S \wedge (\bigwedge_{\kappa<\lambda} F_\kappa)$. Let F_λ be the last E_γ which does so, and let $F_{\lambda+1}$ be the last E_γ which contains $(F_\lambda\backslash S)^-$. As before, $I_\lambda \geq S$ and $I_{\lambda+1} \leq S$. The induction runs. It can only terminate with a last term 0 or with no last term but 0 intersection. In either case (in view of 1.2) we have a development of S.

For $(3) \Rightarrow (4)$, if S has a (continuous) development $F_0 - F_1 \vee F_2 - F_3 \vee \cdots$ of a certain length α (i.e. $F_\alpha = 0$ or the indices are the predecessors of α), we shall see that $S_\alpha = 0$. First, $S_1 = S \wedge (S^-\backslash S)^-$ has a development $G_0 - G_1 \vee G_2 - G_3 \vee \cdots$ where $G_\alpha = F_\alpha \wedge (S^-\backslash S)^-$. But further, the first two terms can be omitted, because $(F_0 - F_1) \wedge (S^-\backslash S)^- = 0$; that follows from $S^- \leq F_0$, $F_0 - F_1 \leq S$, and thence $(S^-\backslash S)^- \leq F_1$. Now by an evident induction each S_β has a development $H_{2\beta} - H_{2\beta+1} \vee \cdots$, where $H_\gamma = F_\gamma \wedge K_\beta$, K_β being $(S_{\beta-1}^-\backslash S_{\beta-1})^-$ if β is a successor or $\bigwedge_{\gamma<\beta} K_\gamma$ if β is a limit ordinal. So $S_\alpha = 0$.

Assume $S_\alpha = 0$. Then for S to be complemented it will suffice that each $S - S_\beta$ is complemented. (S_β is complemented in S, being relatively closed.) Now $S - S_1 = S - (S^-\backslash S)^- = S^- - (S^-\backslash S)^-$, open in a closed sublocale of A, thus complemented. Putting S_β for S, $S_\beta - S_{\beta+1}$ is complemented; so $S - S_{\beta+1}$ is complemented if $S - S_\beta$ is. For a limit ordinal λ, the complements C_κ of $S - S_\kappa$ ($\kappa < \lambda$) are determined by $C_\kappa \wedge S = S_\kappa$, $C_\kappa \vee S = 1$ in $S(A)$. Put $C_\lambda = \bigwedge_{\kappa<\lambda} C_\kappa$. We have $C_\lambda \vee S = 1$ since $S(A)$ is anti-isomorphic with a frame. And $C_\lambda \wedge S = (\bigwedge C_\kappa) \wedge S = \bigwedge(C_\kappa \wedge S) = \bigwedge S_\kappa = S_\lambda$, so C_λ is the complement of $S - S_\lambda$. The induction runs; the proof is finished.

The equivalence of (2), (3), and (4) for subspaces of a space is classical [K].

1.3. *A complemented sublocale of a space is a subspace.*

Proof. If B and C are complements in $S(X)$, every point is in B or in C; the join B_- of the points in B satisfies $B_- \vee C = X$ since $B_- \vee C$ contains all the points; since $B_- \wedge C = 0$, $B_- = B$, a subspace.

1.4. *Every complemented sublocale of a regular locale is a join of closed sublocales; in fact, in any locale A, when a meet m of complemented sublocales x_α is complemented, $1 - m = \bigvee(1 - x_\alpha)$.*

Proof. The second clause implies the first, since every sublocale of a regular locale is a meet of open sublocales [I_1]. To prove it, let $c = \bigvee(1 - x_\alpha)$. Each $c \vee x_\alpha = \bigvee(1 - x_\beta) \vee x_\alpha \geq 1 - x_\alpha \vee x_\alpha = 1$. Then, since $S(A)$ is anti-isomorphic with a frame, the join $c \vee m = c \vee (\bigwedge x_\alpha) = \bigwedge(c \vee x_\alpha) = \bigwedge 1 = 1$. Given the existence of $1 - m$, from $c \vee m = 1$ we have $c \geq 1 - m$ and from $m \leq x_\alpha$ we have $1 - m \geq 1 - x_\alpha$, so $1 - m \geq \bigvee(1 - x_\alpha) = c$. Thus $1 - m = c$.

Of course 1.4 fails for frames; in fact, it fails in frames anti-isomorphic with sublocale lattices $S(A)$. That is, in $S(A)$ itself, a join j of complemented elements x_α can have a complement which is not $\bigwedge(1 - x_\alpha)$. Let A be a dense-in-itself Hausdorff space and the x_α the singletons; then $\bigvee x_\alpha = 1$ has complement 0, but $\bigwedge(1 - x_\alpha) \supset D(A) \neq 0$.

The main application of the formula in 1.4 is that the complement of a "G_δ", if it exists, is an F_σ. Which brings up F_σ and all that. We call a countable meet, in $S(A)$, of open sublocales an O_δ sublocale—because when A is a space, "G_δ" has an established meaning. However, a countable join of closed sublocales is simply an F_σ sublocale. In a space, closed sublocales are subspaces (= joins of their points), so an F_σ sublocale is also a subspace; moreover, clearly, the F_σ sublocales are exactly the F_σ subspaces. As for G_δ and O_δ, the relationship is controlled by

1.5. *A sublocale B of a regular locale A is the meet of sublocales S_i containing it if and only if for every nonzero closed part J disjoint from B, not all $S_i \wedge J$ are dense in J.*

Proof. Clearly if all $S_i \wedge J$ are dense, they and $\bigwedge S_i$ contain $D(J)$, which is nonzero and contained in complemented J, disjoint from B. Conversely, suppose $C = \bigwedge S_i \neq B$. Since $C > B$, there is a nonzero symmetric difference L of open parts of A which meets C but not B. Now L is locally closed; thus there are closed parts L_α of A contained in L whose relative interiors I_α cover L. $L \wedge C = \bigvee(I_\alpha \wedge C)$ (by 1.0 of [I_3]: a sublocale is the join of a relatively open cover), so some $I_\alpha \wedge C \neq 0 \neq L_\alpha \wedge C$. Consider $J = (L_\alpha \wedge C)^-$: nonzero closed, contained in L_α and thus disjoint from B, with $J \wedge C$ dense in J. Then every $J \wedge S_i$ is dense in J.

The further Borel types will be called $F_{\sigma\delta}$, $O_{\delta\sigma}$, and so on. An $F_{\sigma\delta}$ is a countable sublocale meet of F_σ's, so not the same in spaces as a classical $F_{\sigma\delta}$; so one must say "classical $F_{\sigma\delta}$" or "$F_{\sigma\delta}$ sublocale".

f^*, i.e. every open part of $A \subset B$ is of the indicated form. Since f^{-1} preserves meets and complements, the rest follows.

From 1.9, a zero set of a real-valued continuous function is an O_δ, which generalizes 1.7. A closed O_δ in a completely regular space need not be a zero set, and there is a painfully cute example. The familiar Tychonoff plank has a long side L and a short side S, disjoint but without disjoint neighborhoods; but since S *is* short, L is a zero set. So wedge two planks, S to S. Each copy of L is still a closed O_δ but no longer a zero set.

1.10. *If A is a sublocale of B, every complemented sublocale of A is $C \wedge A$ for some complemented sublocale C of B.*

Proof. A complemented sublocale D of A is developable,

$$D = F_0 - F_1 \vee F_2 - F_3 \vee \cdots$$

for some descending transfinite sequence $\{F_\alpha\}$ of closed sublocales. Let $G_\alpha = F_\alpha^-$ in B. The development given by $\{G_\alpha\}$ expresses a complemented sublocale C of B. The initial segments I_0, J_0 of the two developments ($I_0 = F_0$, $J_0 = G_0$) satisfy $J_0 \wedge A = I_0$; when this is true for I_α and J_α it is so for $\alpha + 1$ too, and similarly at limit ordinals. Hence $C \wedge A = D$.

Now a metrizable locale A can be embedded in a complete metric space B, where developable subspaces C, $B - C$ are F_σ and G_δ (Hausdorff, Montgomery [K, M]). So they are O_δ; $C \wedge A$, $A - C$ are O_δ, and by 1.4 they are F_σ.

1.11. *Complemented sublocales of a metrizable locale are F_σ and O_δ.*

The converse of 1.11 is also true. Of course it is known in complete spaces; this seems [K] to be due to Hausdorff alone. (For 1.11, the nonseparable case essentially depends on paracompactness, and Montgomery's proof [M] before paracompactness was thought of is fairly formidable. It is given also in [K].) The key to finishing this is

1.12. *A nonzero metrizable locale cannot be split inseparably by a sublocale which is F_σ and O_δ.*

Proof. First let X be a metric space and F, $G = X \backslash F$ two dense F_σ's: $F = \bigcup F_i$, $G = \bigcup G_j$, F_i and G_j closed. Since F and G are dense, each G_j and each F_i are nowhere dense.

We may assume $F_{i+1} \supset F_i$, $G_{j+1} \supset G_j$. We construct a nonzero closed subspace C of X contained in F, with each $C \backslash F_i$ dense in C. If this succeeds, it will show that G is not O_δ; for any sequence of open neighborhoods U_i of G with intersection G gives us a representation of F as $\bigcup (X \backslash U_i)$, and we shall always have a violation of 1.5.

Let p_1 be a point of F. Then p_1 is at positive distance $2\varepsilon_1$ from G_1; *all* later choices of points p_j, p_{jk} will be outside the ε_1-neighborhood of G_1. Let p_2 be a point of $F \backslash F_2$ within distance 2^{-1} of p_1. It is at positive distance $2\varepsilon_2$

from G_2, and all later p's will be outside the ε_2-neighborhood of G_2. Having a finite set of points p_{jk} of $F \backslash F_j$, they are at positive minimum distance $2\varepsilon_j$ from G_j, and we shall stay (hereafter) outside the ε_j-neighborhood of G_j. In $F \backslash F_{j+1}$ (which is dense), choose a finite set of points $p_{j+1,k}$ including one within distance 2^{-j} of each preceding p_{im}. Completing the induction, we have a countable subset $C_0 = \{p_{jk}\}$ of F. Any convergent sequence in C_0 that is not finally constant is finally out of each finite subset of C_0, and therefore the limit is not in any G_j; so $C = C_0^- \subset F$. By construction, each $C_0 \backslash F_i$ is dense in C.

Now observe that one can do nearly the same thing in a metric locale A, replacing points p_{jk} with smallish closed sublocales π_{jk}. For instance, first choose points p_{jk} of the completion outside F_j and the proscribed neighborhoods of G_i's; then let N_{jk} be the trace on F of a suitably small closed neighborhood of p_{jk}, closed in F and nonzero; and let π_{jk} be the first nonzero $F_i \wedge N_{jk}$. (Since N_{jk} is closed, $0 \neq N_{jk} \wedge (\bigvee F_i) = \bigvee (N_{jk} \wedge F_i)$.) Finitely many indices k for each j will no longer suffice, but $\{p_{jk}\}_k$ can be a set coming within 2^{-j} of all of $\bigvee_{i<j} \pi_{im}$, with no two p_{jk} twice that close to each other. Joins of discrete families of closed sublocales are closed. Again we find that G is not O_δ in A.

1.13. **Theorem.** *A sublocale of a metrizable locale is complemented if and only if it is F_σ and O_δ.*

Proof. Necessity is in 1.11. If $B \subset A$ is F_σ and O_δ, so is $B \wedge C$ for each closed sublocale C, so by 1.12 B splits no such C inseparably, and B is complemented.

2. Pointless parts

It seems highly unlikely that locales without points are less diverse than topological spaces; but our descriptive vocabulary for them is much poorer. The pointless locales which are best known are the 'discrete' or *Boolean* ones, whose frame is a complete Boolean algebra. (Thus they can be defined by "Every open part is closed". In fact every part, i.e. sublocale, is open [and closed] [I₁].) They occur in every locale A, as its (smallest) *dense part* $D(A)$. As the heading indicates, we are going to look at *pointless parts*, meaning largest pointless parts of spaces. But first we look at the simple and definitive results on dense parts.

2.1. *The Boolean sublocales of a space X (or of a locale A) are precisely the sublocales $D(H)$ where H is a closed subspace of X (respectively a closed sublocale of A). If $H \neq H'$ then $D(H) \neq D(H')$.*

Proof. Every sublocale B is dense in its closure H. If B is Boolean, $D(H) \subset B$, and $D(H)$ is dense in B, so $D(H) = B$. The correspondence is bijective since $H = B^-$, $B = D(H)$.

2.2. *The real line* $\mathbf{R}$ *has* ω_0 *nonisomorphic Boolean sublocales, namely the discrete spaces of cardinal* n *for* $0 \le n \le \omega_0$, *and* $D(\mathbf{R})$ *plus a discrete space of cardinal* n *for those* n.

Proof. Evidently $\mathbf{R}$ can be replaced by $[0, 1]$. Each of its closed subspaces consists of the closure of the countable set E of isolated points and an open set U which is empty or locally compact dense-in-itself. If $U^- \ne 0$ it is an irreducible quotient of the Cantor set $\mathbf{C}$, and $D(U) \approx D(\mathbf{C}) \approx D(\mathbf{R})$ [I_4]; and $D(E) = E$, countable and discrete.

2.3. $D(\mathbf{R})$ *is characterized, up to isomorphism, as a nonzero Boolean pointless separable metrizable locale.*

Proof. If A is such a locale, a metric completion of A is separable, dense in itself, and without isolated points. Being separable, it has a metric compactification F. F is an irreducible quotient of $\mathbf{C}$, and $A = D(F) \approx D(\mathbf{C}) \approx D(\mathbf{R})$.

The first thing about the largest pointless part is that it need not exist: the join of all pointless parts of a space may have points. Describing this (in order to avoid it) involves the *weakly scattered* spaces of Niefield and Rosenthal [NR], one definition for which is "Every nonzero sublocale has a point". There are various characterizations [NR].

2.4. *The join* $\mathrm{pl}^+(X)$ *of the pointless sublocales of a sober space* X *is the meet of the sublocales (subspaces) which have weakly scattered complements. Two open sets* U, V *have the same trace on* $\mathrm{pl}^+(X)$ *if and only if the symmetric difference* $U \Delta V$ *is weakly scattered.*

Proof. If $U \Delta V$ (which is open in its closure, and thus is a complemented sublocale) is weakly scattered, all pointless sublocales of X and therefore their join are contained in the complement, so U and V have the same trace on $\mathrm{pl}^+(X)$. If $U \Delta V$ is not weakly scattered, not both of its parts $U - V$, $V - U$ (in the distributive lattice of sublocales) lack pointless parts; so U and V have different traces on some pointless part and therefore on $\mathrm{pl}^+(X)$. This shows that the join $\mathrm{pl}^+(X)$ is the meet of the complements of certain weakly scattered sublocales. But any sublocale with weakly scattered complement must contain $\mathrm{pl}^+(X)$, so it is the meet of all of them.

2.5. *A sober space* X *has a largest pointless sublocale* $\mathrm{pl}(X)$ *if and only if each point* x *has a weakly scattered relative neighborhood in* $\{x\}^-$.

Proof. The question is when no point x is in $\mathrm{pl}^+(X)$. Now if x has a weakly scattered relative neighborhood N in $\{x\}^-$, x is in the relative interior of N which is disjoint from $\mathrm{pl}^+(X)$. If all relative neighborhoods of x in $\{x\}^-$ fail to be weakly scattered, they all have nonzero pointless sublocales. The meet of $\{x\}^-$ and $\mathrm{pl}^+(X)$ is dense in the irreducible space $\{x\}^-$, so it contains $\{x\}^-$ and x.

In particular, if each point is open in its closure—e.g., closed— $\mathrm{pl}(X)$ exists.

Remark. "Weakly scattered" is "without pointless part": pointlessnessless. So the meet of complements of weakly scattered subspaces is pointlessnesslessnessless. 2.5 tells us when it is simply pointless. Clearly in any case pointlessnesslessnesslessnessless = pointlessnessless, or weakly scattered.

In particular, for $X \subset \mathbf{R}$ (in fact for X sober T_1) we have pl(X). I have no idea how to characterize the locales which are pl (something) (like the Boolean locales which are D (something)). It is *not* true (as it is for $D(\ )$) that a sublocale of X which is isomorphic with some pl(Y) is pl(Z) for some $Z \subset X$, even for complete separable metric X (2.17.b below). We can, however, imitate 2.1 so far as to describe the pairs of subspaces Y, Z, for which pl$(Y) =$ pl$(Z) \subset X$.

One would hope to characterize locales A of the form pl(X) by constructing a suitable X from A. (For $D(\ )$, this is easily done.) In any case, what can we say about X, given pl(X)? One such result is

2.6. *For X regular and dense in itself, pl(X) has the same weight as X.*

Proof. Of course any sublocale S has no greater weight, since the trace of a basis on S is a basis for S. Conversely, if X is dense in itself and T_1, pl(X) contains the smallest dense part; but in a regular space, that has the same weight.

Probably there is no closer relation between the number of points of X and the size of pl(X), in any considerable generality, than the inequalities implied by 2.6. (For weight w and power p, $w \leq 2^p$ and $p \leq 2^w$.) For X metrizable and dense in itself, I do not know if pl(X) determines whether X is countable, i.e. homeomorphic with Q. (Using 2.8, it is simple Polish routine to prove that X is countable if an isomorphism of pl(X) and pl(Q) is induced by embeddings $X \subset Y$, $Q \subset Y$.) What I know about pl(Q) is in 2.17 and 2.18.

A metrizable space is called *totally imperfect* if every Čech-complete subspace is scattered. Equivalently, it contains no Cantor set—for it is easy to construct a Cantor set in a complete dense-in-itself metric space. (In fact, this is done in 2.15 below, a bit more generally.) Countable complete metric spaces are scattered; scattered separable metric spaces are countable; so separable metrizable X is totally imperfect iff all its Čech-complete subspaces are countable.

2.7. *For a subspace A of a space X, if pl$(X\backslash A) \subset X$ is pl(X) then every subspace of A closed in X is weakly scattered, and the converse is true in regular spaces. Thus in completely metrizable X, pl$(X\backslash A) =$ pl(X) iff A is totally imperfect.*

Proof. From 2.4, pl$(X\backslash A) =$ pl(X) iff every locally closed $D \subset X$ with $D\backslash A$ weakly scattered is weakly scattered, which certainly implies that subsets of A closed in X are weakly scattered. If weakly scattered subsets of X are scattered (e.g., if they are regular), and pl$(X\backslash A) \neq$ pl(X), we have locally closed $D \subset X$ with $D\backslash A$ scattered but D not scattered. The dense-in-itself kernel K is closed in D. $K\backslash A$ is scattered, so $K\backslash A$ is not dense in K. In the regular space K, this means there is a regular closed set $L \neq 0$ disjoint from $K\backslash A$. L is dense in itself, contained in A, closed in D. So L is locally closed in X: there are closed sets L_α of X contained in L whose interiors relative to L cover L.

But any of those interiors that is nonempty is dense in itself, so its closure is dense in itself, closed in X, and contained in A.

In regular spaces, 2.7 gives the criterion for the general relationship:

2.8. *For subspaces Y, Z of a regular space X, $\mathrm{pl}(Y) \subset Z$ if and only if every subspace of $Y \backslash Z$ closed in $Y \cup Z$ is scattered.*

Proof. Observe

(†) *For a (sober) subspace S of a space T having a pointless part $\mathrm{pl}(T)$, $\mathrm{pl}(S) = S \wedge \mathrm{pl}(T)$.*

Proof. Concerning "sober", $\mathrm{pl}(\ )$ is defined in terms of sublocales, so only sober subspaces count. [If you prefer, define $\mathrm{pl}(\ )$ on spaces via sobrification.] Then $S \wedge \mathrm{pl}(T)$ is pointless and contained in S, so it is contained in $\mathrm{pl}(S)$. But $\mathrm{pl}(S)$ is contained in S and in $\mathrm{pl}(T)$.

(††) *In spaces having pl, $\mathrm{pl}(Y \cup Z) = \mathrm{pl}(Y) \vee \mathrm{pl}(Z)$.*

Proof. Note $Y \cup Z = Y \vee Z$, since it is the join of its points. So $\mathrm{pl}(Y \cup Z) = \mathrm{pl}(Y \cup Z) \wedge (Y \vee Z) = [\mathrm{pl}(Y \cup Z) \wedge Y] \vee [\mathrm{pl}(Y \cup Z) \wedge Z] = \mathrm{pl}(Y) \vee \mathrm{pl}(Z)$.

But now $\mathrm{pl}(Y) \subset Z$ iff $\mathrm{pl}(Y \cup Z) = \mathrm{pl}(Z)$, and 2.5 applies (with $A = Y \backslash Z$, $X = Y \cup Z$).

Remark. Paralleling (††), $\mathrm{pl}(Y \wedge Z) = \mathrm{pl}(Y) \wedge \mathrm{pl}(Z)$—trivially. But $Y \wedge Z \neq Y \cap Z$ in general.

2.9. $\mathbf{R}$ *has $2^{\mathbf{c}}$ nonisomorphic sublocales $\mathrm{pl}(X)$ $(X \subset \mathbf{R})$.*

Proof. By 2.7, $\mathrm{pl}(\mathbf{R}) \approx \mathrm{pl}((0, 1)) \approx \mathrm{pl}([0, 1])$. There is a map of the ternary Cantor set $\mathbf{C}$ onto $[0, 1]$ whose restriction off the endpoints of intervals of the complement is a homeomorphism to $[0, 1]$ minus the dyadic rationals. The exceptional sets are countable, so totally imperfect; so $\mathrm{pl}(\mathbf{R}) \approx \mathrm{pl}(\mathbf{C})$. This is $\mathrm{pl}(\mathbf{C} \times \mathbf{C})$ by homeomorphism. The subspaces of $\mathbf{C} \times \mathbf{C}$ of the form $\mathbf{C} \times S$, $S \subset \mathbf{C}$, are $2^{\mathbf{c}}$ in number and by 2.8, they all have different pointless parts in $\mathbf{C} \times \mathbf{C}$. Since a separable metrizable locale has at most $\mathbf{c}$ embeddings in $\mathbf{C} \times \mathbf{C}$ (at most $\mathbf{c}$ morphisms; for the frames are countably generated and of power $\mathbf{c}$), the result follows.

There are partial results to the effect that $\mathrm{pl}(Y) \subset X$ is $\mathrm{pl}(Z \subset X)$; at least 2.10 here. It depends on the Lavrentiev theorem:

Any isomorphism between sublocales A, B of completely metrizable locales A^{-}, B^{-}, extends to a homeomorphism between G_{δ} subspaces $C \supset A$ of A^{-}, $D \supset B$ of B^{-}.

This was proved by Hager [H] more generally (but he notes that if his spaces are paracompact, they must be completely metrizable).

2.10. Theorem. *For spaces, X completely metrizable and E metrizable, each sublocale P of E isomorphic with $\mathrm{pl}(X)$ is $\mathrm{pl}(S)$ for a subspace S of E homeomorphic with a subspace of X whose complement is totally imperfect. If $P = \mathrm{pl}(E)$, $E \backslash S$ is also totally imperfect and $E \backslash S^{-}$ is scattered.*

Proof. Let F be a metric completion of E and consider an isomorphism i from $\mathrm{pl}(X)$ to a sublocale P of $E \subset F$. There is a Lavrentiev extension $h : C \leftrightarrow D$, C a G_δ of X containing $\mathrm{pl}(X)$, D a G_δ of F containing P. Since $\mathrm{pl}(\;)$ takes the largest pointless part, $\mathrm{pl}(X) = \mathrm{pl}(C)$. Therefore the homeomorphism h takes $\mathrm{pl}(X)$ to $\mathrm{pl}(D) = P$. But every subspace of $D \backslash E$ closed in D is scattered, as it is completely metrizable and if not scattered it would contain a Cantor set H, closed in F; this would mean that the two open sets F and $F \backslash H$ have the same trace on E but not on $\mathrm{pl}(D) \subset E$, a contradiction. Then by 2.7, $\mathrm{pl}(D \cap E) = \mathrm{pl}(D)$. h^{-1} takes $D \cap E$ to a subspace of X which contains $\mathrm{pl}(X)$, so its complement is totally imperfect. If P is also $\mathrm{pl}(E)$, $E \backslash D$ is similarly totally imperfect; and since $\mathrm{pl}(D \cap E) = P = \mathrm{pl}(E)$, $(D \cap E)^-$ contains the dense-in-itself kernel of E.

2.11. Corollary. *For separable completely metrizable spaces, $\mathrm{pl}(X) \approx \mathrm{pl}(Y)$ if and only if X and Y have homeomorphic subspaces with countable complements.*

Proof. Sufficiency is evident. Conversely, if $\mathrm{pl}(X) \approx \mathrm{pl}(Y)$, this is the situation of 2.10 with E also an absolute G_δ. So $D \cap E$ is G_δ, and the totally imperfect difference sets are relative F_σ's, absolute $G_{\delta\sigma}$'s. Separable totally imperfect G_δ's are countable, so $G_{\delta\sigma}$'s are too.

2.12. Corollary. *For X completely metrizable and E metrizable, if $\mathrm{pl}(X) \approx \mathrm{pl}(E) \neq 0$ then E is of the second category in itself.*

Proof. The proof of 2.10 gives completely metrizable D with $D \cap E$ dense in E and $D \backslash E$ totally imperfect (and $D \cap E \neq 0$). Any countable union of nowhere dense sets in $D \cap E$ is contained in a countable union of closed nowhere dense sets in D, whose complement is then a dense G_δ—thus not contained in $D \backslash E$. So $D \cap E$ itself is not such a union, but is second category. Since E has a dense second category subspace, E is second category.

On the other hand, $\mathrm{pl}(X)$ for X complete can be isomorphic with $\mathrm{pl}(E)$ where E is totally imperfect; in fact, it **is** (for every separable completely metrizable X, at least). For every separable metric space X has at most $\mathbf{c}$ points. If fewer, X is totally imperfect. If $\mathbf{c}$, then since X contains at most $\mathbf{c}$ Cantor sets, each of power $\mathbf{c}$, one can partition X into two subsets A, B, each of which meets every Cantor set, so that each is totally imperfect; and by 2.7, if X is complete, $\mathrm{pl}(A) = \mathrm{pl}(X) = \mathrm{pl}(B) \subset X$.

The countable complements in 2.11 are zero-dimensional, so they add 0 or 1 to the dimension of the subspace. But this generalizes, to perfectly normal spaces and a bit further to Dowker's *totally normal* spaces, or to any other class of normal spaces X whose subspaces have covering dimension no greater than $\dim X$. [D]

2.13. *For totally normal spaces X, $\dim \mathrm{pl}(X)$ is $\dim X$ or $\dim X - 1$.*

Proof. Since dim is monotone on subspaces it is monotone on sublocales, as follows. If A is a sublocale of X, it is dense in A^-, which is a subspace.

Any finite open covering $\{U_i\}$ of A extends to $\{A^-\backslash(A\backslash U_i)^-\}$, a finite open (in A^-) covering of whatever it covers: a relatively open set A^* between A and A^-. So there is a finite open refinement covering A^* of order at most $1 + \dim A^* \leq 1 + \dim X$, and its trace on A refines $\{U_i\}$.

For the converse, observe that $\mathrm{pl}(X)^-$ is the complement of a scattered open set. Hence $\dim X = \dim \mathrm{pl}(X)^-$ (unless $\mathrm{pl}(X)$ is empty, which makes $\dim X$ zero). Put $X' = \mathrm{pl}(X)^-$. Now for any finite open covering $\{V_j\}$ of X', by normality there is a shrinking $\{W_j\}$ with $W_j^- \subset V_j$. The trace of $\{W_j\}$ on $\mathrm{pl}(X)$ has a finite open refinement $\{U_k\}$ of order at most $1 + \dim \mathrm{pl}(X)$. As before, $\{X'\backslash(\mathrm{pl}(X)\backslash U_k)^-\}$ is a finite open covering $\mathscr{C}$ of an open set P between $\mathrm{pl}(X)$ and X'. Since $\mathrm{pl}(X)$ is dense in X', any subset of $\mathscr{C}$ having a nonempty intersection has a nonempty intersection in $\mathrm{pl}(X)$, i.e. those U_k's meet; so we have not increased the order. $\mathscr{C}$ refines the trace of $\{V_j\}$ on P. $X'\backslash P$ is scattered, so the trace of $\{V_j\}$ on it can be refined to a partition into relatively open-closed sets Z_m. Since $X'\backslash P$ is closed, the Z_m are closed and can be extended to disjoint open sets still each contained in one V_j. This gives an open refinement $\{V_j\}$ of order at most $2 + \dim \mathrm{pl}(X)$, completing the proof.

We speak, above, of the dimension of the ghostly space $\mathrm{pl}(X)$, but not of its category: only, as in 2.12, of the category of another space E with $\mathrm{pl}(E) \approx \mathrm{pl}(X)$. Why? It seems clear how *second category* locales A should be defined: in the complete lattice of sublocales, A is not a countable join of nowhere dense sublocales. This is easily seen to agree with Baire category for sober T_1 spaces. A non-T_1 space with a dense point is always a second category space but sometimes a first category locale, since nonsober spaces are not sublocales. This imperfection is not a serious drawback (how much Baire category theory is there outside T_1 spaces?), and I shall use this definition of category. However, for all I know, every nonzero $\mathrm{pl}(X)$ may be second category. At least:

2.14. *If X is metrizable and $\mathrm{pl}(X) \neq 0$ then $\mathrm{pl}(X)$ is second category.*

Proof. Since $\mathrm{pl}(X)$ is contained in the dense-in-itself kernel D and is $\mathrm{pl}(D)$, we may assume X is dense in itself. Given countably many nowhere dense parts K_i ($i \geq 1$) of $\mathrm{pl}(X)$, we have closed nowhere dense subspaces $L_i = K_i^-$ of X. We now prove

(‡) *For any sequence of closed nowhere dense sets L_i in a dense-in-itself metric space X and any dense subset P of X, there is a closed dense-in-itself subset H of X such that each $H \cap L_i$ is finite and $H \cap P$ is dense in H.*

The dense set P has nothing to do with 2.14; we want it for 2.18.b.

Proof of (‡). Choose $x_0 \in P$. Choose a surjective ∞-to-1 function $f : \omega\backslash\{0\} \to \omega$ satisfying $f(n) < n$. We shall successively choose x_n within a small distance ε_n of $x_{f(n)}$, in (dense) P but not in (nowhere dense) $L_1 \cup \cdots \cup L_n$. In fact we require (1) $\varepsilon_n \leq d(x_{n-1}, x_{f(n-1)})/3$. ($\varepsilon_1 = 1$, say.) Also, for each

$i \le f(n)$, (2) $\varepsilon_n \le d(x_{f(n)}, L_i)/3$. (That is > 0, as $x_{f(n)}$ was chosen outside these L's.) Finally S is the set of x_i, $H = S^-$. Clearly H is closed dense-in-itself, $H \cap P \supset S$ is dense in it, and each $S \cap L_i$ is finite.

I claim $S^- \cap L_i = S \cap L_i$. Consider the natural numbers partially ordered by f: the *ancestors* of n are $f(n)$, $f^2(n)$, ... , 0, and it is their *descendant*. Consider any convergent sequence of distinct elements x_{n_k}. Infinitely many n_k, in fact all of them, are descendants of 0. Either there is a genetic sequence $0 \to m_1 \to m_2 \to \cdots$ of natural numbers all of which have infinitely many descendants n_k, or there is a maximal such m. In the latter case, the next generation $f^{-1}(m)$ has infinitely many members r_i such that r_i or one of its descendants is an n_k. But observe, first, $d(x_{r_i}, x_m) \to 0$; and because of (1), any descendants y_i of x_{r_i} also converge to x_m. For the other case, first, for each i, almost all m_j are $\ge i$, so x_{m_j} is not in L_i; and because of (2), when $m > i$, all descendants of x_m are at least half as far from L_i. Thus in neither case is there a limit in $L_i \backslash S$. So (‡) holds, and 2.14 holds since $\bigvee K_i \le \mathrm{pl}(X - H)$, which is not $\mathrm{pl}(X)$ since $\mathrm{pl}(H) \ne 0$.

The result 2.16 below seems feeble compared with 2.14; but metrizability is hard to do without, and 2.16 seems worth mentioning. First:

2.15. *A space X which is a G_δ in a compact Hausdorff space and is not scattered has a compact dense-in-itself subspace.*

Proof. Let $X \subset Y$ be the countable intersection of open sets U_i in compact Hausdorff Y. Let $A(0)$ and $A(1)$ be disjoint regular closed sets contained in U_1, whose interiors meet the dense-in-itself part of X. Inductively, having 2^n disjoint regular closed sets $A(\sigma)$, indexed by all sequences σ of n terms 0 or 1, whose interiors meet the dense-in-itself part of X, take two such sets $A(\sigma, 0)$ and $A(\sigma, 1)$ interior to $A(\sigma)$ and contained in U_{n+1} (for each σ); this is always possible and yields nested sets $A(\sigma)$ for all finite $0 - 1$ sequences σ. For every point $\pi \in 2^\omega$, the intersection $A(\pi)$ of $A(\sigma)$ over all initial segments σ of π is nonempty by compactness and contained in the G_δ set X. The union A of all $A(\pi)$ is compact (as an intersection of compact unions $A(0) \cup A(1)$ etc.). The map $f : A \to 2^\omega$ taking each $A(\pi)$ to π is continuous by construction. Therefore [R] A is not scattered, and its dense-in-itself part proves the assertion.

2.16. *If X is a dense-in-itself G_δ in a compact Hausdorff space, $\mathrm{pl}(X)$ is second category.*

Proof. X is a dense G_δ in a compact dense-in-itself space Y (take its closure). So $Y \backslash X$ is a countable union of nowhere dense closed sets. Suppose $\mathrm{pl}(X)$ is the join of nowhere dense $A_i \subset \mathrm{pl}(X)$; then $(Y \backslash X) \cup (\bigcup A_i^-)$ is an F_σ subspace S of Y containing $Y \backslash X$ and $\mathrm{pl}(X)$. Moreover, the subspace intersection I of S and X contains $\mathrm{pl}(X)$, since each $A_i \subset X \cap A_i^-$, which is closed in X and therefore a subspace. Then by 2.7, every subspace of $X \backslash I$ closed in X

is scattered. But $X \backslash I = Y \backslash S$, a G_δ, so by 2.15 it is scattered. In particular, $Y \backslash S$ is empty or has an isolated point. $Y \backslash S$ empty would mean Y is first category; an isolated point p of $Y \backslash S$ would have an open neighborhood U containing no other point of $Y \backslash S$, so the locally compact dense-in-itself space $U \backslash \{p\}$ would be first category; both impossible.

Now, all I know about $\mathrm{pl}(Q)$, the pointless part of the rationals. First,

2.17.a. *If X is metrizable and $\mathrm{pl}(X) \approx \mathrm{pl}(Q)$ then X is totally imperfect.*

Proof. If X is not totally imperfect it contains a Cantor set $\mathbf{C}$, $\mathrm{pl}(\mathbf{C}) \subset \mathrm{pl}(X)$ is isomorphic with a sublocale P of $\mathrm{pl}(Q)$, and by 2.10, P is $\mathrm{pl}(T)$ for some subspace T of Q. But $\mathrm{pl}(T)$ is 0 or isomorphic with $\mathrm{pl}(Q)$, unlike $\mathrm{pl}(\mathbf{C})$ (2.12).

Second:

2.17.b. *A (complete) metric space X can have a sublocale $P \approx \mathrm{pl}(Q)$ which is not $\mathrm{pl}(S)$ for any subspace S of X.*

Proof. As noted in $[\mathrm{I}_1]$ (and easily seen from 2.4), in the real line $\mathbf{R}$ the space of irrationals $\mathbf{R} \backslash Q$ contains the sublocale $\mathrm{pl}(Q)$. But $\mathbf{R}$ has no subspace S disjoint from Q with $\mathrm{pl}(S) = \mathrm{pl}(Q)$. For, if $S \supset \mathrm{pl}(Q)$ then S is dense in $\mathbf{R}$. But we can list the elements x_i of Q and enclose them in open intervals I_i of length ε_i, whose union will be an open set U of measure < 1; let us see what else the construction requires. $\mathbf{R} \backslash U$ will be closed and $(\mathbf{R} \backslash U) \cap Q$ scattered, in fact empty; if $(\mathbf{R} \backslash U) \cap S$ is dense in itself, this will show Q does not contain $\mathrm{pl}(S)$. For this, clearly, no two I_i can abut, leaving an isolated point of S between them. Further, we list the elements V_j of a basis of open intervals of $\mathbf{R}$ and, in attending alternately to x_1, V_1, x_2, V_2, ... , when attending to V_j, if it is not already covered by U, mark *two* points of $S \cap V_j$ to be left out of U. The finite union of nonabutting intervals already put in U has regular closed complement, so the construction is possible.

Recall (from several pages ago) the first pullback lemma, 1.9. It is about open/closed/locally closed/ O_δ sublocales. Nothing like it holds for F_σ's. Q is F_σ in $\mathbf{R}$ (or anywhere); but $\mathrm{pl}(Q)$ is not only not a countable join of closed parts of $\mathrm{pl}(\mathbf{R})$, it is not a join of them at all. Nor of O_δ parts; moreover,

2.18.a. *If Q' has pl and $\mathrm{pl}(\mathbf{R})$ cannot be embedded in $\mathrm{pl}(Q')$, then for any absolute G_δ space X, a sublocale of $\mathrm{pl}(X)$ isomorphic with $\mathrm{pl}(Q')$ contains no nonzero O_δ.*

For the O_δ's of $\mathrm{pl}(X)$ are $\mathrm{pl}(S)$, S a G_δ in X; if $\mathrm{pl}(S)$ is nonzero, S contains $\mathbf{C}$ and $\mathrm{pl}(S)$ contains $\mathrm{pl}(\mathbf{C}) \approx \mathrm{pl}(\mathbf{R})$.

Turning from non-$O_{\delta\sigma}$'s to non-$F_{\sigma\delta}$'s, we have

2.18.b. *If T is a dense subspace of a completely metrizable space X, then $\mathrm{pl}(T) \subset \mathrm{pl}(X)$ either contains a dense O_δ or is not $F_{\sigma\delta}$.*

Proof. The core is

(‡‡) *A dense embedding of metrizable T in completely metrizable X induces an embedding of* $\mathrm{pl}(T)$ *in* $\mathrm{pl}(X)$ *such that every* F_σ *sublocale containing* $\mathrm{pl}(T)$ *contains a dense open sublocale.*

Proof. We may assume X is dense in itself. For any F_σ sublocale $\bigvee \mathrm{pl}(J_i)$ of $\mathrm{pl}(X)$ that contains $\mathrm{pl}(T)$, $X \backslash \bigcup J_i$ is nowhere dense; for if it were dense in nonempty open U, the closed sets $L_i = J_i \cap U^-$ would be nowhere dense and subject to (‡): U^- would have a closed dense-in-itself subset $H = (H \cap T)^-$ with all $H \cap L_i$ finite. This means $\mathrm{pl}(X \backslash H)$ contains all $\mathrm{pl}(J_i)$ but not $\mathrm{pl}(T)$, a contradiction. So the closure of $X \backslash \bigcup J_i$ has void interior, i.e. $W = (\bigcup J_i)^\circ$ is dense. Next we show that a symmetric difference $U \Delta V$ of two open sets of X which meets $\mathrm{pl}(W)$ meets some $\mathrm{pl}(J_i)$. For if the absolute G_δ set $U \Delta V$ has nonscattered intersection with absolute G_δ W, the intersection contains a Cantor set C, which is not a countable union of scattered (hence nowhere dense) subsets $C \cap J_i$; some $C \cap J_i$ has nonzero $\mathrm{pl}(C \cap J_i) \subset (U \Delta V) \cap \mathrm{pl}(J_i)$. But this means $\bigvee \mathrm{pl}(J_i)$ contains $\mathrm{pl}(W)$, as claimed.

2.18.b is immediate; F_σ's containing $\mathrm{pl}(T)$ contain dense opens, so $F_{\sigma\delta}$'s containing it contain dense O_δ's $\mathrm{pl}(\bigcap W_i)$.

Remark. (‡‡) can be restated: *in these* $\mathrm{pl}(X)$, *an* F_σ *containing a dense* $\mathrm{pl}(T)$ *has dense interior.* These $\mathrm{pl}(X)$ (to anticipate 2.19) are the pointless metrizable absolute O_δ's. $\mathrm{pl}(T)$ is essential; the F_σ cannot be merely dense. (Proof, imitate a countable dense set in X using a little Cantor set for each point.) But I do not know if $\mathrm{pl}(T)$ ends up in the interior.

2.18.a, or 2.18.b, shows that the first sentence of 1.9 fails for F_σ's. As for the second:

2.18.c. *There exists an* F_σ *sublocale of* $\mathrm{pl}(Q)$ *which is not* $\mathrm{pl}(S)$ *for any subspace* (= F_σ *sublocale*) S *of* Q.

Proof. Observe, Q is homeomorphic with $Q \times Q$. That's a union of ω vertical sections, which are closed. The pointless parts of the vertical sections are closed in $\mathrm{pl}(Q \times Q)$—which is just $\mathrm{pl}(Q)$; take their union, an F_σ in $\mathrm{pl}(Q)$. If it were the trace of an F_σ in Q (= $Q \times Q$), it would be second category in itself by 2.14, which it plainly is not.

A further detail:

2.18.d. *There is an* F_σ *in* $\mathrm{pl}(\mathbf{R})$ *which is not* $\mathrm{pl}(S)$ *for any subspace* S (F_σ *or not*) *of* $\mathbf{R}$.

For there is an ascending sequence of closed parts whose join is first category in itself.

2.19.   **Theorem.** *For a completely metrizable space* X, $\mathrm{pl}(X)$ *is an* O_δ *sublocale of every pointless metrizable locale containing it. A pointless metrizable*

locale which is an O_δ in every pointless metrizable extension is $\mathrm{pl}(X)$ *for some completely metrizable space* X.

Proof. If $\mathrm{pl}(X)$ is isomorphic with $P \subset E$, E pointless metrizable, let F be a metric completion of E. There is a Lavrentiev extension of $i : \mathrm{pl}(X) \approx P$ to a homeomorphism h of containing G_δ's C in X, D in F. $P = \mathrm{pl}(X) = \mathrm{pl}(C)$, so h takes P isomorphically to $\mathrm{pl}(D)$. $\mathrm{pl}(D)$ is O_δ in $\mathrm{pl}(F)$, hence also in $E \subset \mathrm{pl}(F)$. Now suppose merely that P is an O_δ in $\mathrm{pl}(F)$ for a metric completion F; then by 1.9, $P = \mathrm{pl}(D)$ for some O_δ sublocale D of F, but D is spatial and completely metrizable.

2.20. *For every G_δ part X of* **R**, $\mathrm{pl}(X)$ *is* 0 *or isomorphic with* $\mathrm{pl}(\mathbf{R})$.

Proof. Since the dense-in-itself kernel K of X is a G_δ, it suffices to show that every dense-in-itself G_δ part K has $\mathrm{pl}(K) \approx \mathrm{pl}(\mathbf{R})$, or equivalently, $\approx \mathrm{pl}(\mathbf{R}\backslash Q)$. But K minus a countable dense subset S of K is homeomorphic with $\mathbf{R}\backslash Q$. This will suffice, since $\mathrm{pl}(K\backslash S) = \mathrm{pl}(K)$ by 2.7. It is true by the Alexandroff-Urysohn characterization [AU] of $\mathbf{R}\backslash Q$: a separable metric 0-dimensional absolute G_δ (obvious for $K\backslash S$), which is nowhere locally compact. $K\backslash S$ is nowhere locally compact because S is dense in K.

2.21. $\mathrm{pl}(\mathbf{R})$ *is characterized, up to isomorphism, as a nonzero pointless zero-dimensional separable metrizable locale which is O_δ in every pointless metrizable extension.*

Proof. Given such a locale F, it has a zero-dimensional metrizable compactification [I$_3$], so it is embeddable in **R**. Being pointless, it goes into $\mathrm{pl}(\mathbf{R})$, where it is O_δ; so it is $\mathrm{pl}(X)$ for some G_δ subspace X of **R**, and by 2.20 it is $\approx \mathrm{pl}(\mathbf{R})$.

Among spaces, a metrizable absolute G_δ is a G_δ in any completely regular space in which it is dense. Not among pointless locales.

(§) *Every nonzero pointless metrizable locale is dense but not O_δ in some completely regular pointless locale.*

Proof. We may assume the given locale A is $\mathrm{pl}(M)$ for some dense-in-itself completely metrizable space M. For each finite subset F of M, let M_F be the subset of $\beta(M\backslash F)$ which maps to M under the natural map $\beta(M\backslash F) \to \beta M$. Let L be the inverse limit of the M_F (mapping $M_G \to M_F$, for $G \supset F$, by restriction of β of $M\backslash G \subset M\backslash F$); let $B = \mathrm{pl}(\beta L)$. $A \subset M$ is contained in each $M\backslash F$, so naturally embedded in M_F and in $L \subset \beta L$. Since A is pointless it is contained in B. If it were O_δ there, it would be the trace on B of a G_δ set D of L. Then $\beta L\backslash D$ would be a countable union of closed sets H_i, all disjoint from A. But if closed $H \subset \beta L$ is disjoint from A, the natural map $f : \beta L \to \beta M$ cannot take H to a set containing a Cantor set $C \subset M$. For $K = H \cap f^{-1}(C)$ would be compact with C as a quotient, so nonscattered [R]; and in the inverse mapping system we have $\mathrm{pl}(C) \subset \mathrm{pl}(M) = A$ mapping

isomorphically throughout, and $\mathrm{pl}(K)$ projecting into it at every stage, so into $\mathrm{pl}(C)$ in the limit, i.e. $0 \neq \mathrm{pl}(K) \subset A$, a contradiction. We conclude that each $f(H_i) \cap M$ is scattered. But then they are nowhere dense; the union misses $f^{-1}(p)$ for some point p of M, D contains $f^{-1}(p)$, and $D \wedge B = \mathrm{pl}(D) \supset \mathrm{pl}f^{-1}(p))$, so $D \wedge B \neq A$.

2.22. *There are* **c** *nonisomorphic connected dense* O_δ *sublocales* $\mathrm{pl}(X_\alpha)$ *of* $\mathrm{pl}(\mathbf{R}^2)$.

Proof. The indices α will be the subsets of $\omega_0 \backslash \{0\}$. We first describe non-homeomorphic compact sets B_α. B_0 (0 for the null set) is $(\omega^\omega + 1) \times I$, where $\omega = \omega_0$, $\omega^\omega + 1$ has the order topology, and I is a closed interval. $\omega^\omega + 1$ can be embedded in $\mathbf{R}$, and B_0 can be embedded in $\mathbf{R}^2$. For a general set α of positive integers, modify B_0 to B_α by lengthening its ω^nth interval $\{\omega^n\} \times I$ for each $n \in \alpha$. (So the limit of $\{\beta\} \times I$ as $\beta \to \omega^n$ is a proper subinterval. For compactness, either do not lengthen them very much or lengthen $\{\omega^\omega\} \times I$ too.) Evidently the B_α are topologically distinct and embeddable in $\mathbf{R}^2$.

Let V_α be a countable discrete set in $\mathbf{R}^2 \backslash B_\alpha$ whose derived set is B_α; but that is just to mark where to put W_α, which is a topological sum of ω Cantor sets in $\mathbf{R}^2 \backslash B_\alpha$ with $W_\alpha^- \backslash W_\alpha = B_\alpha$. This makes $X_\alpha = \mathbf{R}^2 \backslash W_\alpha$ a dense G_δ, so $\mathrm{pl}(X_\alpha)$ is dense G_δ in $\mathrm{pl}(\mathbf{R}^2)$. If we also choose W_α to approach B_α (locally) only from one side, then X_α will be a union of closed 2-cells and $\mathrm{pl}(X_\alpha)$ connected. ($\mathrm{pl}(X_\alpha)$ would be connected without this precaution, but it is harder to see.)

By 2.11, $\mathrm{pl}(X_\alpha) \approx \mathrm{pl}(X_\beta)$ iff they have homeomorphic subspaces S_α, S_β with countable complements. Now S_α and S_β are each S^2 minus a zero-dimensional subspace. So their Freudenthal compactifications are S^2; and S_γ ($\gamma = \alpha, \beta$) determines the remainder $R_\gamma = S^2 \backslash S_\gamma$ as a subspace of S^2. But the set of accumulation points of R_γ (points at which R_γ is not locally countable) is W_γ^-, the nondegenerate components constitute B_γ, so S_α and S_β are **not** homeomorphic.
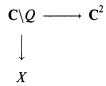
2.23. *For each connected dense open subset* X *of* $\mathbf{R}^2$, $\mathrm{pl}(X)$ *is isomorphic with* $\mathrm{pl}(\mathbf{R}^2)$ *or with* $\mathrm{pl}(\mathbf{R}^2 \backslash \mathbf{C})$.

Proof sketch. The Freudenthal compactification of X, or of X minus a countable set, is S^2. For clearly the countable set does not affect it, and the point is that $S^2 \backslash X$ (closed nowhere dense nonseparating) is the intersection of a shrinking sequence of nice neighborhoods: all X knows is how many components $S^2 \backslash X$ has. With the floating countable set, all $\mathrm{pl}(X)$ knows is whether $S^2 \backslash X$ has uncountably many components.

There is nothing like 2.23 in $\mathbf{R}^3$; one can remove a sequence of different knots in **c** ways.

2.24. *No nonzero pointless metrizable locale is an* F_σ *in every pointless metrizable extension.*

Proof. Consider $A \neq 0$ embedded in a completely metrizable space X. If A is F_σ in $\mathrm{pl}(X)$, it contains $\mathrm{pl}(H)$ for some nonscattered H closed in X; hence A has a closed sublocale $B = \mathrm{pl}(\mathbf{C})$, $\mathbf{C} \subset X$. Now $\mathrm{pl}(\mathbf{C}) = \mathrm{pl}(\mathbf{C} \backslash Q)$, and $\mathbf{C} \backslash Q$ is homeomorphic with its square $S \subset \mathbf{C}^2$. Let Y be the metrizable space $X \backslash Q$—the copy of Q in $\mathbf{C}$ that we are using—and let Z be the union of Y and $\mathbf{C}^2$ with the two copies of $\mathbf{C} \backslash Q$ identified. This is a metrizable space; it is easy to construct it via an embedding of $\mathbf{C}^2$ in an absolute retract and

$$\mathbf{C} \backslash Q \longrightarrow \mathbf{C}^2$$
$$\downarrow$$
$$X$$

The closure of B in Z is $\mathbf{C}^2$. B is not an F_σ in $\mathrm{pl}(\mathbf{C}^2)$; for B is second category, and a closed set of $\mathbf{C}^2$ which is not nowhere dense has a perfect subset disjoint from the dense subset S. Hence A is not F_σ in $\mathrm{pl}(Z)$.

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