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STABLE RANK AND APPROXIMATION THEOREMS IN H^∞

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ABSTRACT. It is conjectured that for H^∞ the Bass stable rank (bsr) is 1 and the topological stable rank (tsr) is 2. $\mathrm{bsr}(H^\infty) = 1$ if and only if for every $(f_1, f_2) \in H^\infty \times H^\infty$ which is a corona pair (i.e., there exist $g_1, g_2 \in H^\infty$ such that $f_1g_1 + f_2g_2 = 1$) there exists a $g \in H^\infty$ such that $f_1 + f_2g \in (H^\infty)^{-1}$, the invertibles in H^∞; however, it suffices to consider corona pairs (f_1, f_2) where f_1 is a Blaschke product. It is also shown that there exists a $g \in H^\infty$ such that $f_1 + f_2g \in \exp(H^\infty)$ if and only if $\log f_1$ can be boundedly, analytically defined on $\{z \in \mathbb{D}: |f_2(z)| < \delta\}$, for some $\delta > 0$. $\mathrm{tsr}(H^\infty) = 2$ if and only if the corona pairs are uniformly dense in $H^\infty \times H^\infty$; however, it suffices to show that the corona pairs are uniformly dense in pairs of Blaschke products. This condition would be satisfied if the interpolating Blaschke products were uniformly dense in the Blaschke products.

For b an inner function, $K = H^2 \ominus bH^2$ is an H^∞-module via the compressed Toeplitz operators $C_f = P_K T_f|_K$, for $f \in H^\infty$, where T_f is the Toeplitz operator $T_f g = fg$, for $g \in H^2$. Some stable rank questions can be recast as lifting questions: for $(f_i)_1^n \subset H^\infty$, there exist $(g_i)_1^n$, $(h_i)_1^n \subset H^\infty$ such that $\sum_{i=1}^n (f_i + bg_i)h_i = 1$ if and only if the compressed Toeplitz operators $(C_{f_i})_1^n$ may be lifted to Toeplitz operators $(T_{F_i})_1^n$ which generate $B(H^2)$ as an ideal.

INTRODUCTION

In [1], Bass introduced the notion called the Bass stable rank. This is an algebraic invariant of a ring which he discovered was useful in studying the algebraic K-theory of a ring. In [16], Rieffel introduced the notion called the topological stable rank. This is a functional analytic invariant of a Banach algebra, i.e., it is defined using both the analytic and the algebraic structure. Rieffel noted that the topological stable rank dominates the Bass stable rank. This fact and the analytic definition of the topological stable rank made the application of K-theoretical methods to Banach algebras more tractable. In particular, Rieffel found it useful in studying the K-theory of C^*-algebras.

In his introduction of the topological stable rank, Rieffel made many initial observations and raised many questions concerning its relationship to the Bass stable rank. One of these questions was answered in [11], where Herman and

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Vaserstein showed that the two stable ranks are equal for C^*-algebras. Another of these questions was answered independently in [8] by Corach and Suárez and in [12] by Jones, Marshall, and Wolff, who demonstrated that in general the two stable ranks can be different. They did this by showing that the Bass stable rank of the disk algebra is 1 while noting that the topological stable rank is 2.

The work by Corach et al, was part of continuing work [5–9] in the study of stable rank questions about Banach algebras, especially uniform algebras. One question which they raise and leave unanswered is: what is the Bass stable rank of the algebra of bounded analytic functions in the unit disk. In this article I will present work towards an answer to this question and the related topological stable rank question.

In §1 an approximate inner-outer factorization theorem is proved: the products of Blaschke products and invertibles form a dense subset of H^∞. This is used in later sections to reduce questions about H^∞ to ones about Blaschke products. Finite products of interpolating Blaschke products are shown to be uniformly approximable by interpolating Blaschke products and to be characterized by the sizes of the sets where they are small.

In §2 it is shown that the corona pairs are dense in pairs containing an interpolating Blaschke product. Using the approximate inner-outer factorization, a necessary and sufficient condition for $\mathrm{tsr}(H^\infty) = 2$ to hold is that corona pairs be dense in pairs of Blaschke products. Together, these imply: if the uniform closure of the interpolating Blaschke products contains the Blaschke products, then $\mathrm{tsr}(H^\infty) = 2$.

In §3, a necessary and sufficient condition is found for a corona pair (f_1, f_2) to be reducible to $\exp(H^\infty)$, the connected component of the identity in the invertibles. The condition is a natural one—$\log f_1$ must be boundedly analytically definable for f_2 sufficiently small. Some special cases of corona pairs are shown to satisfy this condition, e.g., for f_2 a finite product of interpolating Blaschke products. Not all corona pairs are reducible to $\exp(H^\infty)$, but to show that $\mathrm{bsr}(H^\infty) = 1$ it suffices to show that corona pairs of the form (b, f), where b is a Blaschke product, are reducible.

In §4, an alternate view of reducibility questions is looked at. For b an inner function, the compressed analytic Toeplitz operators on $K = H^2 \ominus bH^2$ make K into an H^∞-module. It is shown that a row $(f_1, \dots, f_n, b)$ of functions in H^∞ generate H^∞ as an ideal if and only if the compressed row $(C_{f_1}, \dots, C_{f_n})$ of operators on K generate $B(K)$ as an ideal. The following statement thus makes sense, and is proved: a unimodular row $(f_1, \dots, f_n, b)$ is reducible if and only if the unimodular row of compressed analytic Toeplitz operators $(C_{f_1}, \dots, C_{f_n})$ may be lifted to a unimodular row of analytic Toeplitz operators.

0. Preliminaries

0.1. Stable rank definitions and propositions. Let R be a ring with identity. R^n, the n-tuples from R, forms and R-bimodule under the usual operations.

The *left unimodular rows* in R^n are the n-tuples which generate R as a left ideal, i.e., $\mathrm{LU}_n(R) = \{(r_i)_1^n \in R^n : Rr_1 + \cdots + Rr_n = R\}$. The equality $Rr_1 + \cdots + Rr_n = R$ is equivalent to the existence of an n-tuple $(s_i)_1^n \in R^n$ such that $s_1 r_1 + \cdots + s_n r_n = 1_R$. The *right unimodular rows* in R^n are similarly defined as $\mathrm{RU}_n(R) = \{(r_i)_1^n \in R^n : r_1 R + \cdots + r_n R = R\}$, and the *unimodular rows* are $\mathrm{U}_n(R) = \mathrm{LU}_n(R) \cap \mathrm{RU}_n(R)$. If R is a commutative ring, then of course $\mathrm{U}_n(R) = \mathrm{LU}_n(R) = \mathrm{RU}_n(R)$. Clearly $(R)^{-1}$, the set of invertible elements of R, is just $\mathrm{U}_1(R)$.

If $(r_i)_1^{n+1}$ is an $(n+1)$-tuple, then it is *left reducible* if there exists an n-tuple $(b_i)_1^n$ such that $(r_i + b_i r_{n+1})$ is in $\mathrm{LU}_n(R)$. In this situation it is said that $(r_i)_1^{n+1}$ is *left reducible to* $(r_i + b_i r_{n+1})_1^n$. Note that if $(r_i)_1^{n+1}$ is left reducible, then in fact $(r_i)_1^{n+1}$ is already in $\mathrm{LU}_{n+1}(R)$; this concept is only applied to left unimodular rows. *Right reducibility* is similarly defined, and a unimodular row is *reducible* if it is both left and right reducible.

For a positive integer n, R is said to be *left n-stable* if every $(n+1)$-tuple in $\mathrm{LU}_{n+1}(R)$ is left reducible. The *left Bass stable rank* of R, $\mathrm{lbsr}(R)$, is the smallest positive integer such that R is left n-stable, if such an n exists; otherwise it is said to be infinite. The *right Bass stable rank* of R, $\mathrm{rbsr}(R)$ may be analogously defined. However, it is a theorem of ring theory [17] that $\mathrm{rbsr}(R) = \mathrm{lbsr}(R)$. Hereafter, this quantity will be referred to as just the *Bass stable rank* of R, $\mathrm{bsr}(R)$.

Let A be a Banach algebra. The A-bimodule A^n is a Banach algebra under coordinate-wise operations with the norm defined by $\|(a_i)_1^n\|_{A^n} = \max_i \|a_i\|_A$.

For a positive integer n, A is said to be *left n-topologically stable* if $\mathrm{LU}_n(A)$ is dense in A^n. The *left topological stable rank* of A, $\mathrm{ltsr}(A)$, is the smallest positive integer n such that A is left n-topologically stable, if such an n exists; otherwise it is said to be infinite. The *right topological stable rank* of A, $\mathrm{rtsr}(A)$ is analogously defined. It is an open question as to whether $\mathrm{ltsr}(A)$ and $\mathrm{rtsr}(A)$ may be different. When A is a commutative Banach algebra, then they are equal and this quantity will be referred to as just the *topological stable rank* of A, $\mathrm{tsr}(A)$.

The following theorems should introduce the reader to the type of results that are known about stable rank problems. The first is the basic result that the Bass stable rank is dominated by the topological stable rank.

Theorem 0.1 [16]. *Let A be a Banach algebra with identity and n a positive integer. If A is left n-topologically stable, then A is right n-stable. Similarly, if A is right n-topologically stable, then A is left n-stable. Thus,* $\mathrm{bsr}(A) \leq \min\{\mathrm{ltsr}(A), \mathrm{rtsr}(A)\}$.

Often, topological methods come strongly into play. For X a compact space, let $\dim X$ be the *covering dimension* of X (see [14]). The "floor" function is $\lfloor r \rfloor =$ the greatest integer $\leq r$.

Theorem 0.2 [17]. $\mathrm{bsr}(C_{\mathbb{R}}(X)) = \dim X + 1$ *and* $\mathrm{bsr}(C_{\mathbb{C}}(X)) = \lfloor \frac{1}{2} \dim X \rfloor + 1$.

In proving this theorem, Vaserstein implicitly used the ideas that Rieffel makes explicit in his definition of the topological stable rank. So Vaserstein's proof also shows that $\mathrm{bsr}(C_{\mathbb{R}}(X)) = \mathrm{tsr}(C_{\mathbb{R}}(X))$ and $\mathrm{bsr}(C_{\mathbb{C}}(X)) = \mathrm{tsr}(C_{\mathbb{C}}(X))$.

Two generalizations of this theorem follow. First, for communtative Banach algebras:

Theorem 0.3 [5]. *Let A be a complex commutative Banach algebra with identity, and let M_A be the maximal ideal space of A with the Gelfand topology. Then* $\mathrm{bsr}(A) \leq \lfloor \frac{1}{2} \dim M_A \rfloor + 1$.

If A is also a regular Banach algebra (the hull-kernel topology coincides with the Gelfand topology on M_A) then equality holds, $\mathrm{bsr}(A) = \lfloor \frac{1}{2} \dim M_A \rfloor + 1$.

Second, for some noncommutative Banach algebras:

Theorem 0.4 [11]. *If A is a C^*-algebra with identity, then* $\mathrm{bsr}(A) = \mathrm{tsr}(A)$.

But, in general, the two stable ranks may differ:

Theorem 0.5 [8, 12]. $\mathrm{bsr}(A(\mathbb{D})) = 1$ *and* $\mathrm{tsr}(A(\mathbb{D})) = 2$.

Remark 0.6. My main reference for this section is Rieffel [16], wherein a general introduction is given to these ideas. Another good reference for these ideas are [5–9] by Corach, Larotonda, and Suárez, wherein they exploit the use of topological methods.

0.2 Analytic definitions. Except for §4, the main calculations in this article will be done in the Banach algebra H^∞ of bounded analytic functions on the unit disk. The more specialized methods available in Hardy spaces will be exploited to obtain information about the topological and Bass stable ranks of H^∞. My main reference is Garnett [10].

For later convenience, let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ denote the unit disk in $\mathbb{C}$, let $\partial\mathbb{D} = \{z \in \mathbb{C} \colon |z| = 1\}$ denote its boundary, and define

$$\begin{aligned}
H^\infty &= \{f \colon f \text{ is an analytic function on } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}} |f(z)| < \infty\}, \\
A(\mathbb{D}) &= \{f \in H^\infty \colon f \text{ has a continuous extension to } \mathbb{D} \cup \partial\mathbb{D}\}, \\
\mathscr{B} &= \{b \in H^\infty \colon b \text{ is a Blaschke product }\}, \\
\mathscr{I} &= \{b \in H^\infty \colon b \text{ is an interpolating Blaschke product }\}, \\
Z_b &= \text{ the zero sequence, counting multiplicity, of a Blaschke product } b,
\end{aligned}$$

$$\rho(z, w) = \left|\frac{z-w}{1-\overline{w}z}\right|, \text{ the pseudohyperbolic metric on } \mathbb{D},$$

$$D(z; r) = \{w \in \mathbb{D} \colon \rho(z, w) < r\}, \text{ a pseudohyperbolic disk.}$$

The terms *corona pairs* and *corona data* are used for elements of $\mathrm{U}_2(H^\infty)$ and $\mathrm{U}_n(H^\infty)$ because of the following:

Theorem 0.7 (corona theorem [3]). *An n-tuple $(f_i)_1^n \in (H^\infty)^n$ is a unimodular row if and only if* $\inf_{z\in\mathbb{D}} \max_i |f_i(z)| = \delta_0 > 0$.

1. Some approximation and interpolation theorems

Some basic tools for the calculations in the following sections will be approximation and interpolation theorems. It is because H^∞ has such a rich structure, as seen in [10], that the following calculations are possible.

Theorem 1.1. *The set* $\mathscr{B}\cdot(H^\infty)^{-1}=\{bh\colon b$ *is a Blaschke product,* h *is invertible*$\}$ *is dense in* H^∞.

Note. This may be viewed as an approximate inner-outer factorization, as $(H^\infty)^{-1}$ is contained in the outer functions.

Proof. Let $f\in H^\infty$ and $\varepsilon>0$ be given. Using the identification of H^∞ as a subset of $L^\infty(\partial\mathbb{D})$, via boundary values, define η as follows:

$$\eta(e^{i\theta})=\begin{cases}\varepsilon/4, & \text{if } |f(e^{i\theta})|>\varepsilon/2,\\ 3\varepsilon/4, & \text{if } |f(e^{i\theta})|\le\varepsilon/2.\end{cases}$$

Then

$$k(z)=\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\eta\, d\theta\right)$$

is an outer function in H^∞, with $|k(e^{i\theta})|=\eta(e^{i\theta})$ on the boundary $\partial\mathbb{D}$. Hence $\|k\|\le 3\varepsilon/4$ and $|f(e^{i\theta})-k(e^{i\theta})|\ge \||f(e^{i\theta})|-\eta(e^{i\theta})|\ge\varepsilon/4$ on $\partial\mathbb{D}$.

In the inner-outer factorization $f-k=Ih$, the outer factor h is bounded below on $\partial\mathbb{D}$, and thus is invertible. By Frostman's theorem [10, p. 79] there is a Blaschke product b such that $\|I-b\|\le\varepsilon/(4\|h\|)$; hence

$$\begin{aligned}\|f-bh\|&\le\|f-k-Ih\|+\|k\|+\|Ih-bh\|\\ &\le 0+3\varepsilon/4+\varepsilon/4=\varepsilon.\quad\square\end{aligned}$$

This theorem can be used to give a standard form for corona solutions to given corona data.

Corollary 1.2. *If* $(f_i)_1^n\in \mathrm{U}_n(H^\infty)$, *then there exists an* n*-tuple* $(b_ih_i)_1^n$ *with each* $b_ih_i\in\mathscr{B}\cdot(H^\infty)^{-1}$, *such that* $f_1b_1h_1+\cdots+f_nb_nh_n=1$.

Proof. Let $(g_i)_1^n$ be a corona solution, i.e., $f_1g_1+\cdots+f_ng_n=1$. Approximate each g_i by a $b_ik_i\in\mathscr{B}\cdot(H^\infty)^{-1}$ such that $\|(g_i-b_ik_i)_1^n\|<1/(2n\|(f_i)_1^n\|)$. Thus

$$\left\|1-\sum f_ib_ik_i\right\|=\left\|\sum f_ig_i-\sum f_ib_ik_i\right\|<n\|(f_i)_1^n\|\,\|(g_i-b_ik_i)_1^n\|<1/2.$$

So $k=\sum f_ib_ik_i$ is invertible and $(b_i(k_ik^{-1}))_1^n$ is a corona solution of the desired form. $\square$

The usefulness of this theorem is apparent from the fact that multiplying a unimodular row by an invertible leaves it unimodular. In later sections this will reduce questions involving general functions in H^∞ to questions involving just Blaschke products.

Recall that b is an interpolating Blaschke product if and only if the interpolation map $\Phi_b: H^\infty \to l^\infty(Z_b)$, defined by $\Phi_b f = f|_{Z_b} = (f(z_n): z_n \in Z_b)$, is onto.

Proposition 1.3. *If b is an interpolating Blaschke product, then the restriction*

$$\Phi_b|_{(H^\infty)^{-1}}: (H^\infty)^{-1} \to (l^\infty(Z_b))^{-1}$$

of the interpolation map is onto. In fact $\Phi_b(\exp(H^\infty)) = (l^\infty(Z_b))^{-1}$.

Proof. $\Phi_b|_{(H^\infty)^{-1}}$ maps into $(l^\infty(Z_b))^{-1}$ since $f \in (H^\infty)^{-1}$ if and only if $|f(z)|$ is bounded below on $\mathbb{D}$, which implies that $|f(z)|$ is bounded below on Z_b.

Let (w_n) be an invertible sequence. Choose a bounded sequence (α_n) such that $\exp(\alpha_n) = w_n$ for all n. This is possible since $(\log|w_n|) = (\operatorname{Re}\alpha_n)$ is a bounded sequence, and $(\operatorname{Im}\alpha_n)$ may be chosen so that $-\pi < \arg w_n = \operatorname{Im}\alpha_n \leq \pi$ for all n.

As Z_b is an interpolating sequence, there is an $h \in H^\infty$ such that $h|_{Z_b} = (\alpha_n: z_n \in Z_b)$. Then $\exp(h)|_{Z_b} = \exp(h|_{Z_b}) = \exp((\alpha_n)) = (w_n)$. □

An alternate proof of this proposition may be found by using that Φ_b has a continuous section—using the open mapping theorem—, and that $(l^\infty(Z_b))^{-1} = \exp(l^\infty(Z_b))$—the invertibles in l^∞ are a connected set. See [8, Corollary 2.5].

This proposition can be used to prove the following result about reducibility in H^∞.

Corollary 1.4. *If (f, b) is a corona pair and b is an interpolating Blaschke product, then (f, b) is reducible to an element of* $\exp(H^\infty)$.

Proof. If the equation

$$fe^h + bg = 1 \tag{1}$$

can be solved for some g, $h \in H^\infty$ then (f, b) is reducible to $f + bge^{-h} = e^{-h}$. Solving (1) is equivalent to finding an $h \in H^\infty$ such that $g = (1 - fe^h)/b$ is in H^∞.

Since (f, b) is a corona pair, the sequence $(|f(z_n)|: z_n \in Z_b)$ is bounded away from zero, hence $f|_{Z_b} \in (l^\infty(Z_b))^{-1}$. By the previous proposition, let $h \in H^\infty$ be such that $\exp(h)|_{Z_b} = (f|_{Z_b})^{-1}$. Then $1 - fe^h$ vanishes on Z_b, so b, as a Blaschke product, divides it, i.e., $(1 - fe^h)/b \in H^\infty$. □

This result will be strengthened in a later section, but its proof demonstrates the appeal of working with interpolating Blaschke products. Other instances of noticing this appeal has led to the following open question due to Garnett [10, p. 430].

Question 1.5. *Can every Blaschke product be uniformly approximated by interpolating Blaschke products?*

The following theorems are work towards answering this question. As with many results on interpolating Blaschke products, these theorems may already be known, but may not be in the literature. The first result has been stated without proof in [4].

Proposition 1.6. *The interpolating Blaschke products are uniformly dense in the finite products of interpolating Blaschke products.*

A product of two interpolating Blaschke products, say $B_1 B_2$, can be approximated by carefully factoring $B_2 = B_a B_b$, so that there exists a small $\delta > 0$ such that $B_1 B_a (B_b - \delta)/(1 - \delta B_b)$ is an interpolating Blaschke product. The details which show that this can be done follow from Carleson's characterization of interpolating Blaschke products, [2] or [10, p. 287].

The next proposition extends results of Kerr-Lawson [13].

Proposition 1.7. *Let b be an inner function. Then the following are equivalent:*

1. *b is a finite product of interpolating Blaschke products.*
2. *Given any r, $d > 0$, there exist a δ, $d > \delta > 0$, an ε, $r > \varepsilon > 0$, and a sequence $(z_n) \subset \mathbb{D}$ such that each component of $\{z \in \mathbb{D}: |b(z)| < \delta\}$ is contained within a pseudo-hyperbolic disk $D(z_n; \varepsilon)$, for some n.*

Kerr-Lawson's proof of [13, Lemma 3] can be modified slightly to show that 2 implies 1.

To see that 1 implies 2, let $(z_n) = Z_b$ be the zero sequence of b and factor $b = b_1 b_2 \dots b_N$ into a product of N interpolating Blaschke products. Choose ε such that

$$0 < \varepsilon < \min\{1, r, (1/2)\inf\{\rho(z, w): z, w \in Z_{b_i}, 1 \le i \le N\}\},$$

using that each zero sequence, Z_{b_i}, is separated. For $1 \le i \le N$, use [13, Lemma 1] to choose $\delta_i > 0$ such that

$$\{|b_i| < \delta_i\} \subset \bigcup_{z \in Z_{b_i}} D(z; \varepsilon/(2N)) \subset \{|b_i| < \varepsilon/(2N)\}.$$

Let $\delta = \delta_1 \delta_2 \cdots \delta_N$. These choices of δ, ε, and (z_n) can be shown to satisfy condition 2.

2. A SUFFICIENT CONDITION FOR $\mathrm{tsr}(H^\infty) = 2$

In raising the question of whether the topological and Bass stable ranks could be different, Rieffel suggested the disk algebra $A(\mathbb{D})$ as a possible example. $\mathrm{bsr}(A(\mathbb{D}))$ will be discussed in the next section. It is easy to see that $\mathrm{tsr}(A(\mathbb{D})) = 2$. The function $z \mapsto z$ is of distance 1 from the invertibles by Rouché's theorem, so $\mathrm{tsr}(A(\mathbb{D})) > 1$. But given two functions in $A(\mathbb{D})$, approximate them by two polynomials with disjoint sets of roots. The two polynomials generate $A(\mathbb{D})$ as an ideal. Thus $\mathrm{tsr}(A(\mathbb{D})) = 2$.

The situation is more difficult in H^∞. By the same argument as above, $\mathrm{tsr}(H^\infty) > 1$. The work I have done suggests the plausibility of $\mathrm{tsr}(H^\infty) = 2$, that $H^\infty \times H^\infty$ is contained in the uniform closure of $\mathrm{U}_2(H^\infty)$. The tools needed are the approximation techniques from §1.

Theorem 2.1. *$H^\infty \times \mathcal{I}$ is contained in the uniform closure of* $\mathrm{U}_2(H^\infty)$.

Proof. Let $(f, b) \in H^\infty \times \mathcal{I}$, and $\varepsilon > 0$ be given. Let

$$M = \sup_{\|(w_n)\| \le 1, (w_n) \in l^\infty(Z_b)} \inf\{\|f\| : f_{Z_b} = (w_n), f \in H^\infty\}.$$

From the theory of interpolating sequences, the constant M is finite [10, p. 285].

Let $\eta \in H^\infty$, with $\|\eta\| \le \varepsilon$, interpolate

$$w_n = \begin{cases} \varepsilon/(2M), & \text{if } |f(z_n)| \le \varepsilon/(4M), \\ 0, & \text{if } |f(z_n)| > \varepsilon/(4M). \end{cases}$$

So $\|(f, b) - (f + \eta, b)\| \le \varepsilon$ and $(f + \eta, b) \in \mathrm{U}_2(H^\infty)$, this being an easy case of the corona theorem, as b is an interpolating Blaschke product and $\inf_{z_n \in Z_b} |f(z_n) + \eta(z_n)| \ge \varepsilon/(4M) > 0$. □

Theorem 2.2. *If the uniform closure of* $\mathrm{U}_2(H^\infty)$ *contains* $\mathcal{B} \times \mathcal{B}$, *then* $\mathrm{tsr}(H^\infty) = 2$.

Proof. As mentioned above, the example $z \mapsto z$ shows that $\mathrm{tsr}(H^\infty) > 1$. Now suppose that $\mathcal{B} \times \mathcal{B}$ is contained in the uniform closure of $\mathrm{U}_2(H^\infty)$.

Let $(f_1, f_2) \in H^\infty \times H^\infty$ and $\varepsilon > 0$. By Theorem 1.1 approximate f_i by $b_i h_i$, $b_i \in \mathcal{B}$, $h_i \in (H^\infty)^{-1}$, for $i = 1, 2$, such that $\|(f_1, f_2) - (b_1 h_1, b_2 h_2)\| < \varepsilon/2$. Now approximate (b_1, b_2) by $(k_1, k_2) \in \mathrm{U}_2(H^\infty)$ such that

$$\|(b_1 h_1, b_2 h_2) - (k_1 h_1, k_2 h_2)\| < \varepsilon/2.$$

So $\|(f_1, f_2) - (k_1 h_1, k_2 h_2)\| < \varepsilon$ and $(k_1 h_1, k_2 h_2) \in \mathrm{U}_2(H^\infty)$. □

Corollary 2.3. *If every Blaschke product can be uniformly approximated by interpolating Blaschke products, then* $\mathrm{tsr}(H^\infty) = 2$.

Note. The hypothesis is Garnett's open question (1.5). It is not known whether $\mathrm{tsr}(H^\infty) = 2$ is equivalent to the hypothesis.

Proof. The hypothesis implies $\mathcal{B} \times \mathcal{B} \subset \overline{\mathcal{B} \times \mathcal{I}}$. But $\overline{\mathcal{B} \times \mathcal{I}} \subset \overline{H^\infty \times \mathcal{I}} \subset \overline{\mathrm{U}_2(H^\infty)}$ is true, the last containment being Theorem 2.1. Hence we would have $\mathcal{B} \times \mathcal{B} \subset \overline{\mathrm{U}_2(H^\infty)}$, which is the hypothesis for the previous theorem. □

3. Reducibility of corona pairs

In asking that $\mathrm{bsr}(A(\mathbb{D}))$ be calculated, Rieffel was looking for an example of a Banach algebra A such that $\mathrm{bsr}(A) \ne \mathrm{tsr}(A)$. When Corach and Suárez [8] and Jones, Marshall and Wolff [12] independently showed that $\mathrm{bsr}(A(\mathbb{D})) = 1$, Rieffel's hunch was confirmed, as $\mathrm{tsr}(A(\mathbb{D})) = 2$. That the disk algebra $A(\mathbb{D})$ should provide such an example is suggested by the fact that both the algebra of complex polynomials and the algebra of entire functions have Bass stable rank of 1. Corach and Suárez have conjectured that $\operatorname{bsr}(H^\infty) = 1$ also, thus adding this complex algebra to the list. This section represents my work towards this calculation.

The difficulty in calculating $\operatorname{bsr}(H^\infty)$ seems to be tied to the complexity of the set $(H^\infty)^{-1}$. For a commutative Banach algebra with identity, A, the connected component of the identity in A^{-1} is just $\exp(A)$. In the case of the disk algebra, $A(\mathbb{D})^{-1} = \exp(A(\mathbb{D}))$, because for $f \in A(\mathbb{D})^{-1}$ the logarithm $\log f$ may be continuously, and thus boundedly, defined on $\overline{\mathbb{D}}$. For $f \in (H^\infty)^{-1}$ the logarithm is definable, but may be unbounded (a specific example of this is given in §3.2). So $(H^\infty)^{-1}$ is disconnected.

The use of bounded logarithms is implicit in proofs of $\operatorname{bsr}(A(\mathbb{D})) = 1$. In §3.1 these ideas are generalized to reducibility to the connected component of identity in $(H^\infty)^{-1}$. In §3.2, reducibility to other components is studied.

3.1 Reducibility to $\exp(H^\infty)$.

Theorem 3.1. *Let (f_1, f_2) be a corona pair. Then (f_1, f_2) is reducible to an element of the connected component of the identity in $(H^\infty)^{-1}$ if and only if there exists a $\delta > 0$ such that $\log f_1$ can be boundedly, analytically defined on $\{z \in \mathbb{D} : |f_2(z)| < \delta\}$.*

Proof. The reducibility referred to is the existence of an $e^h \in \exp(H^\infty)$, and a $g \in H^\infty$ so that $f_1 + f_2 g = e^h$. If $f_2 = 0$, then $f_1 = e^h \in \exp(H^\infty)$ if and only if $\log f_1 = h$ works as a bounded choice for the logarithm. In the case $f_2 \neq 0$, this existence simplifies to the existence of an $h \in H^\infty$ such that $(e^h - f_1)/f_2 \in H^\infty$.

Suppose that (f_1, f_2) is reducible to $f_1 + f_2 g = e^h$. Let

$$\delta_0 = \inf_{z \in \mathbb{D}} \max\{|f_1(z)|, |f_2(z)|\}.$$

Because (f_1, f_2) is a corona pair, $\delta_0 > 0$. Let $M = \|(e^h - f_1)/f_2\|$, so

$$|e^{h(z)} - f_1(z)| \le M|f_2(z)| \quad \text{for all } z \in \mathbb{D}.$$

Let $\delta = \min\{\delta_0, \delta_0/(2M)\}$. By the maximum principle, the connected components of $\{z \in \mathbb{D} : |f_2(z)| < \delta\}$ are simply connected. So $\log f_1$ may be analytically defined on $\{|f_2| < \delta\}$ as f_1 never vanishes on the set. In the following calculations, each logarithm is analytic on $\{|f_2| < \delta\}$, and well defined up to $2\pi i k$, $k \in \mathbb{Z}$, k constant on each component of $\{|f_2| < \delta\}$.

$$h - \log f_1 = \log\left(\frac{e^h}{f_1}\right) + 2\pi i k = \log\left(1 + \frac{e^h - f_1}{f_1}\right) + 2\pi i k.$$

But $|f_2(z)| < \delta$ implies that $|(e^{h(z)} - f_1(z))/f_1(z)| \le M\delta/\delta_0 \le 1/2$. Looking at the disk of radius $1/2$ centered at 1, one finds that

$$\left|\log\left(1 + \frac{e^h - f_1}{f_1}\right) + 2\pi i k\right| \le \left|\log\frac{1}{2} + i\frac{\pi}{6}\right|,$$

if k is chosen on each component so as to minimize the left side. So, as h is bounded, $\log f_1$ can be boundedly analytically defined on $\{z \in \mathbb{D}: |f_2(z)| < \delta\}$.

For the converse, first assume that f_1 and f_2 are analytic in a neighborhood of $\overline{\mathbb{D}}$, then conclude with a normal families argument. Define $\|\log f_1\|_{\{|f_2|<\delta\}} = \sup\{|\log f_1(z)|: z \in \mathbb{D}, |f_2(z)| < \delta\}$, using the assumed bounded logarithm. The motivation for what follows is to try to find Ψ and u such that $h = \Psi \log f_1 - uf_2$ and $(e^h - f_1)/f_2$ are in H^∞. Doing this involves solving the $\overline{\partial}$-equation

$$0 = \frac{\partial h}{\partial \overline{z}} = \frac{\partial \Psi}{\partial \overline{z}} \log f_1 - \frac{\partial u}{\partial \overline{z}} f_2 \tag{2}$$

(which makes h analytic) with nice bounds.

From [10, Theorem VIII.5.1], there is a function Ψ of class C^∞ in a neighborhood of $\overline{\mathbb{D}}$ (using the analyticity of f_2 across the boundary of $\overline{\mathbb{D}}$) such that

$$0 \leq \Psi(z) \leq 1 \text{ on } \mathbb{D}, \quad \Psi(z) = 0 \text{ on } \{|f_2| \geq \delta\}, \quad \Psi(z) = 1 \text{ on } \{|f_2| < \varepsilon\},$$

and

$$|\partial\Psi/\partial\overline{z}|\,dx\,dy \text{ is a Carleson measure on } \mathbb{D}, \text{ with Carleson constant } A,$$

where ε and A depend only on δ and $\|f_2\|$. Since $\partial\Psi/\partial\overline{z} = 0$ on $\{|f_2| \geq \delta\} \cup \{|f_2| < \varepsilon\}$,

$$\left|\frac{\log f_1}{f_2}\frac{\partial\Psi}{\partial\overline{z}}\right| \leq \frac{\|\log f_1\|_{\{|f_2|<\delta\}}}{\varepsilon}\left|\frac{\partial\Psi}{\partial\overline{z}}\right|.$$

Thus $(\log f_1/f_2)\partial\Psi/\partial\overline{z}$ is bounded on $\mathbb{D}$, and $|(\log f_1/f_2)\partial\Psi/\partial\overline{z}|\,dx\,dy$ is a Carleson measure on $\mathbb{D}$, with Carleson constant $\leq A\|\log f_1\|_{\{|f_2|<\delta\}}/\varepsilon$.

From [10, Theorem VIII.1.1], there exists a function $u \in C^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\frac{\partial u}{\partial \overline{z}} = \frac{\log f_1}{f_2}\frac{\partial \Psi}{\partial \overline{z}} \quad \text{on } \mathbb{D},$$

and

$$\|u\|_\infty = \sup_{|z|=1} |u(z)| \leq cA\|\log f_1\|_{\{|f_2|<\delta\}}/\varepsilon$$

(c an absolute constant).

These choices of Ψ and u make $h = \Psi \log f_1 - uf_2$ analytic on D by satisfying the $\overline{\partial}$-equation (2), and thus $h \in H^\infty$ because

$$\|h\| = \lim_{r \nearrow 1} \sup_{|z|=r} |(\Psi \log f_1 - uf_2)(z)| \leq (1 + cA\|f_2\|/\varepsilon)\|\log f_1\|_{\{|f_2|<\delta\}}. \tag{3}$$

The function $(e^h - f_1)/f_2$ is analytic on $\mathbb{D}$, except where $f_2(z) = 0$. On $\{|f_2| < \varepsilon\}$ the function u is analytic and $h(z) = 1 \cdot \log f_1(z) - u(z)f_2(z)$, thus

$$\frac{e^h - f_1}{f_2}(z) = \frac{(e^{-uf_2} - 1)f_1}{f_2}(z),$$

which has removable singularities where $f_2(z) = 0$. Also,

$$\lim_{r\nearrow 1} \sup_{|z|=r,\, |f_2(z)|<\varepsilon} \left| \frac{(e^{-uf_2}-1)f_1}{f_2}(z) \right| \le \|f_1\| \sup_{|\xi|=\|u\|_\infty,\, |w|=\varepsilon} \left| \frac{e^{\xi w}-1}{w} \right|$$
$$\le \|f_1\|\, \|u\|_\infty e^{\|u\|_\infty \varepsilon},$$

by the maximum principle and $|(e^a-1)/a| \le e^{|a|}$. Since

$$|(e^{h(z)} - f_1(z))/f_2(z)| \le (e^{\|h\|} + \|f_1\|)/\varepsilon$$

when $|f_2(z)| \ge \varepsilon$, we have $(e^h - f_1)/f_2 \in H^\infty$ with the norm estimate

$$\left\| \frac{e^h - f_1}{f_2} \right\| \le \max\left\{ \|f_1\|\, \|u\|_\infty e^{\|u\|_\infty \varepsilon},\ \frac{e^{\|h\|} + \|f_1\|}{\varepsilon} \right\}. \tag{4}$$

The general case follows from a normal families argument: given f_1, $f_2 \in H^\infty$ with $\|\log f_1\|_{\{|f_2|<\delta\}}$ finite, the functions $f_1(rz)$ and $f_2(rz)$ are analytic in a neighborhood of $\overline{\mathbb{D}}$, for $r < 1$. For such r, take $h^{(r)} \in H^\infty$ such that $(e^{h^{(r)}(z)} - f_1(rz))/f_2(rz) \in H^\infty$, with both collections of functions having uniform bounds depending only on δ, $\|\log f_1\|_{\{|f_2|<\delta\}}$, $\|f_1\|$, and $\|f_2\|$, using (3) and (4). For some sequence $r_n \nearrow 1$, the functions $h(z) = \lim_n h^{(r_n)}(z)$ and $(e^h - f_1)/f_2$ are in H^∞, with the same respective bounds. □

The above proof is reminiscent of the proof in [12] that $\operatorname{bsr}(A(\mathbb{D})) = 1$. However, the method of using Carleson measures is needed to pass from $A(\mathbb{D})$ to H^∞. The theorem is motivated by the following three corollaries, each of which had been proved by different methods, and each of which has the curious similarity that its original proof involved reducing a corona pair to $\exp(H^\infty)$. The theorem demonstrates that this was not accidental, and indeed classifies its occurrences.

Corollary 3.2 [9]. *Let (f_1, f_2) be a corona pair with $f_1 \in A(\mathbb{D})$. Then (f_1, f_2) is reducible to* $\exp(H^\infty)$.

Proof. Let $\delta_0 = \inf_{z\in\mathbb{D}} \max\{|f_1(z)|, |f_2(z)|\}$, which is positive. Approximate f_1 by a polynomial $p(z) = \alpha \prod_{j=1}^N (z - z_j)$ such that $\{z_j\} \subset \mathbb{D}$ and $\|p - f_1\| < \delta_0/2$. So

$$\{z \in \mathbb{D}\colon |f_2(z)| < \delta_0\} \subset \{z \in \mathbb{D}\colon |f_1(z)| \ge \delta_0\} \subset \{z \in \mathbb{D}\colon |p(z)| \ge \delta_0/2\}.$$

The components of $\{|f_2| < \delta_0\}$ are simply connected sets on which $\log f_1$ and $\log p$ may be analytically defined such that

$$|\log p - \log f_1| = \left| \log\left(1 + \frac{p - f_1}{f_1}\right) \right| \le \left| \log\frac{1}{2} + i\frac{\pi}{6} \right|$$

(similar to the first part of the proof of Theorem 3.1). To show that $\log f_1$ is bounded on a subset of $\{|f_2| < \delta_0\}$, it thus suffices to show that $\log p$ is bounded there.

Since $f_2 \in H^\infty$, a nonzero nontangential limit of $f_2(z)$ exists at a.e. $e^{i\theta} \in \partial\mathbb{D}$. For all but countably many $e^{i\theta} \in \partial\mathbb{D}$ and all j, the function f_2 does not vanish on any of the line segments $[z_j, e^{i\theta})$, because f_2 has countably many zeroes on $\mathbb{D}$ and $|f_2(z_j)| \neq 0$. Take $e^{i\theta_0} \in \partial\mathbb{D}$ satisfying both of these conditions, so that

$$\inf\{|f_2(z)|: z \in [z_j, e^{i\theta_0}),\ 1 \le j \le N\} = \delta_1 > 0.$$

Let $\delta = \min\{\delta_0, \delta_1\}$ and $S = \mathbb{D}\setminus\bigcup_{j=1}^N [z_j, e^{i\theta_0})$, which is simply connected and contains $\{|f_2| < \delta_1\}$.

The polynomial p does not vanish on S, so $\log p$ may be analytically defined on S. Take analytic branches of the logarithm so that $\log p(z) = \log\alpha + \sum_{j=1}^N \log(z - z_j)$. But $\arg(z - z_j) = \operatorname{Im}\log(z - z_j)$ has variation 2π on S, as a line segment from z_j to $\partial\mathbb{D}$ has been removed. So $\log p$ may be defined so that $|\arg p(z)| \le \pi + 2\pi N$.

Thus, one may analytically define $\log p$ on $\{|f_2| < \delta\}$, so that

$$\begin{aligned} |\log p(z)| &\le |\log|p(z)|| + |\arg p(z)| \\ &\le \max\{\log\|p\|, -\log(\delta_0/2)\} + \pi + 2\pi N. \end{aligned}$$

Thus $\log f_1$ may also be boundedly analytically defined on $\{|f_2| < \delta\}$. Apply Theorem 3.1. □

Corollary 3.3 [8, 12]. $\operatorname{bsr}(A(\mathbb{D})) = 1$.

Proof. Let $(f_1, f_2) \in U_2(A(\mathbb{D})) = U_2(H^\infty) \cap (A(\mathbb{D}) \times A(\mathbb{D}))$. By the previous corollary, (f_1, f_2) is reducible to $\exp(H^\infty)$. What is wanted is for the pair to be reducible in $A(\mathbb{D})$ to $A(\mathbb{D})^{-1}$.

Let $g, h \in H^\infty$ be so that $f_1 + f_2 g = e^h$. So $f_1 e^{-h} + f_2 g e^{-h} = 1$. Let $f_{(r)}(z) = f(rz)$ for $0 < r < 1$, $z \in \mathbb{D}$. So for $f \in H^\infty$, $f_{(r)} \in A(\mathbb{D})$, $\|f_{(r)}\| \le \|f\|$. The equation $f_{1(r)}(e^{-h})_{(r)} + f_{2(r)}(ge^{-h})_{(r)} = 1_{(r)}$ involves only functions from $A(\mathbb{D})$.

$$\begin{aligned} &\|1 - [f_1(e^{-h})_{(r)} + f_2(ge^{-h})_{(r)}]\| \\ &\quad = \|[f_{1(r)} - f_1](e^{-h})_{(r)} + [f_{2(r)} - f_2](ge^{-h})_{(r)}\| \\ &\quad \le \|f_{1(r)} - f_1\|\,\|e^{-h}\| + \|f_{2(r)} - f_2\|\,\|ge^{-h}\|, \end{aligned}$$

which converges to 0 as r goes to 1, because the disk algebra functions f_1, f_2 are the uniform limits of $f_{1(r)}$, $f_{2(r)}$ respectively.

So, for r sufficiently close to 1, $k = f_1(e^{-h})_{(r)} + f_2(ge^{-h})_{(r)} \in A(\mathbb{D})$ is close to 1 and is thus invertible in $A(\mathbb{D})^{-1}$. So $f_1 + f_2 g_{(r)} = k(e^{-h})_{(r)}^{-1} = ke^{h_{(r)}}$ demonstrates the reducibility of (f_1, f_2) to $A(\mathbb{D})^{-1}$, reducing in $A(\mathbb{D})$ using $g_{(r)} \in A(\mathbb{D})$. □

Corollary 3.4. *If* (f, b) *is a corona pair and* b *is an interpolating Blaschke product, then* (f, b) *is reducible to an element of* $\exp(H^\infty)$.

Proof. See the next corollary, or Corollary 1.4. □

My earlier proof of Corollary 3.4 involves solving an interpolation problem, and does not seem to generalize to the following corollary. But changing the question from an interpolation problem to a geometric one involving sets of the form " $\{|b| < \delta\}$ "—as Theorem 3.1 and some work of Kerr-Lawson do—produces the following.

Corollary 3.5. *If* (f, b) *is a corona pair and* b *is a finite product of interpolating Blaschke products, then* (f, b) *is reducible to an element of* $\exp(H^\infty)$.

Proof. As usual, let $\delta_0 = \inf_{z\in\mathbb{D}} \max\{|f(z)|, |b(z)|\}$. Find ε, δ as in Proposition 1.7: $\min\{\delta_0, 1/2\} > \varepsilon > \delta > 0$ such that each component of $\{z \in \mathbb{D}: |b(z)| < \delta\}$ is wholly contained within one $D(z_n; \varepsilon)$, for some $z_n \in Z_b$. For $z \in D(z_n; \varepsilon)$, since $b(z_n) = 0$ and $\rho(z, z_n) < \varepsilon < \delta_0$, it follows that $|b(z)| < \delta_0$ and thus that $|f(z)| \geq \delta_0$. Fix a component V and z_n, with $V \subset D(z_n; \varepsilon)$, in the following discussion of boundedly analytically defining $\log f$ on a single component.

Take $|\operatorname{Im} \log f(z_n)| \leq \pi$ and note that

$$|\operatorname{Re} \log f(z_n)| \leq \max\{\log \|f\|, -\log \delta_0\},$$

so as to boundedly define $\log f(z_n)$. For $z \in V$, define

$$\log f(z) = \log f(z_n) + \int_{z_n}^{z} \frac{f'(\zeta)}{f(\zeta)} d\zeta,$$

where the integral is along a path contained in $D(z_n; \varepsilon)$. The integral is path independent as f does not vanish on the disk $D(z_n; \varepsilon)$, so $\log f$ has been defined analytically. A bound on $\log f$ is gotten by an estimate of the integral along a specified path.

Let $\zeta = \zeta(w) = (w + z_n)/(1 + \overline{z}_n w)$, so $w(\zeta) = (\zeta - z_n)/(1 - \overline{z}_n \zeta)$ and $f \circ \zeta$ is bounded by $\|f \circ \zeta\| = \|f\|$ on a Euclidean disk of radius $1 - |w|$, with center w. By a Cauchy estimate, $|(f \circ \zeta)'(w)| \leq \|f\|/(1 - |w|)$. Thus

$$\begin{aligned} |\log f(z) - \log f(z_n)| &= \left| \int_{z_n}^{z} \frac{f'(\zeta)}{f(\zeta)} d\zeta \right| = \left| \int_0^{w(z)} \frac{(f \circ \zeta)'(w)}{(f \circ \zeta)(w)} dw \right| \\ &\leq \left| \frac{z - z_n}{1 - \overline{z}_n z} \right| \frac{\|f\|/(1 - |(z - z_n)/(1 - \overline{z}_n z)|)}{\delta_0} \\ &\leq \frac{\varepsilon}{1 - \varepsilon} \|f\| \frac{1}{\delta_0}, \end{aligned}$$

by integrating along the line segment $[0, w(z)]$ (on an arc in $D(z_n; \varepsilon)$) and using $|(z - z_n)/(1 - \overline{z}_n z)| = \rho(z, z_n) < \varepsilon < 1$.

By varying the z_n, $|\log f(z)|$ is bounded on $\{|b| < \delta\}$ by

$$\sup_n \left\{ |\log f(z_n)| + \frac{\varepsilon}{1 - \varepsilon} \|f\| \frac{1}{\delta_0} \right\} \leq \max\{\log \|f\|, -\log \delta_0\} + \pi + \frac{\varepsilon}{1 - \varepsilon} \|f\| \frac{1}{\delta_0}.$$

Apply Theorem 3.1. □

3.2. Reducibility to $(H^\infty)^{-1}$**.** It is not always possible to reduce a corona pair in H^∞ to $\exp(H^\infty)$. A trivial example of this is $(f, 0)$ where $f \in (H^\infty)^{-1} \setminus \exp(H^\infty)$.

If $g: \mathbb{D} \to \{z \in \mathbb{C}: -1 < \operatorname{Re} z < 0\}$ is a conformal map such that $\operatorname{Im} g(z) \to \infty$ as $z \to 1$ in $\mathbb{D}$, then the function e^g is invertible in H^∞ as the real part of g is bounded. It is not in $\exp(H^\infty)$ as $\operatorname{Im} g$, the argument of e^g, is unbounded. Theorem 3.1 shows that $(e^{g+(z-1)/(z+1)}, e^{(z+1)/(z-1)})$ is a less trivial example of a corona pair which is not reducible to an element of $\exp(H^\infty)$. But $(e^{g+(z-1)/(z+1)}, e^{(z+1)/(z-1)})$ is reducible to an element of the coset

$$e^g \exp(H^\infty) \subset (H^\infty)^{-1},$$

as $(e^{(z-1)/(z+1)}, e^{(z+1)/(z-1)})$ is reducible to $\exp(H^\infty)$ by Theorem 3.1.

Theorem 1.1 gave a method of generating invertible functions which were not necessarily in $\exp(H^\infty)$, so the following theorem is not bothered by the disconnectedness of $(H^\infty)^{-1}$.

Theorem 3.6. *If for all* $(b, f) \in \mathrm{U}_2(H^\infty) \cap (\mathscr{B} \times H^\infty)$ *it is true that* (b, f) *is reducible, then* $\operatorname{bsr}(H^\infty) = 1$.

Proof. Let (f_1, f_2) be a general corona pair to be shown reducible. By Corollary 1.2 find $b_i \in \mathscr{B}$, $h_i \in (H^\infty)^{-1}$, $i = 1, 2$, such that

$$f_1 b_1 h_1 + f_2 b_2 h_2 = 1. \tag{5}$$

So (b_1, f_2) is a corona pair, which is reducible by assumption. So $b_1 + f_2 u = v \in (H^\infty)^{-1}$, for some $u \in H^\infty$. Using $b_1 = v - f_2 u$ to rewrite (5),

$$1 = f_1(v - f_2 u)h_1 + f_2 b_2 h_2 = f_1 v h_1 + f_2(b_2 h_2 - f_1 u h_1).$$

So (f_1, f_2) is reducible to $f_1 + f_2(b_2 h_2 - f_1 u h_1)(v h_1)^{-1} = (v h_1)^{-1} \in (H^\infty)^{-1}$. □

So the question of what $\operatorname{bsr}(H^\infty)$ is is reduced to one involving Blaschke products. It would be nice to combine Theorem 3.1 and Theorem 3.6 to show $\operatorname{bsr}(H^\infty) = 1$ by reducing every (b, f) to $\exp(H^\infty)$.

Question 3.7. *Is* (b, f) *reducible to* $\exp(H^\infty)$ *for every corona pair in* $\mathscr{B} \times H^\infty$?

4. An application of reducibility to compressed Toeplitz operators

Definition 4.1. The Hilbert space $L^2 = L^2(\partial\mathbb{D})$ is the usual Lebesque space. The Hardy subspace is

$$H^2 = \{f \in L^2: \text{ the Poisson integral of } f \text{ is analytic on } \mathbb{D}\}$$

For $\varphi \in L^\infty$, the *Toeplitz operator* $T_\varphi: H^2 \to H^2$ is defined by

$$T_\varphi f = P_{H^2}(\varphi f).$$

This view of functions in H^2 as functions on $\partial\mathbb{D}$ as well as analytic functions on $\mathbb{D}$ is analogous to the view that $H^\infty \subset L^\infty$. For $\varphi \in H^\infty$ and $f \in H^2$, viewing these as analytic functions on $\mathbb{D}$, one sees that $T_\varphi f = P_{H^2}(\varphi f) = \varphi f$, as φf is analytic. Thus, the map

$$T_{(\cdot)}\colon H^\infty \to B(H^2), \qquad \varphi \mapsto T_\varphi$$

is an algebra homomorphism.

Beurling's theorem [10, p. 82] classifies the subspaces of H^2 invariant under the unilateral shift T_z as spaces of the form bH^2, for some inner function b. Clearly, each Beurling space bH^2 is also T_φ-invariant, for $\varphi \in H^\infty$.

Definition 4.2. For bH^2 a Beurling space, let

$$K = H^2 \ominus bH^2.$$

For $\varphi \in L^\infty$, the *compressed Toeplitz operator* $C_\varphi\colon K \to K$ is defined by

$$C_\varphi f = P_K(\varphi f) = P_K T_\varphi f,$$

i.e., $C_\varphi = P_K T_\varphi|_K$.

It follows from T being a homomorphism and the T-invariance of bH^2 that the map

$$C_{(\cdot)}\colon H^\infty \in B(K), \qquad \varphi \mapsto C_\varphi$$

is also an algebra homomorphism.

The following list of standard results will be used in the calculations of Theorem 4.4.

Lemma 4.3.

1. $k_\lambda = (1-\bar{\lambda}z)^{-1}$ *is the reproducing kernel for* H^2:

$$\langle f, k_\lambda\rangle_{H^2} = f(\lambda), \quad \textit{for } f \in H^2,\ \lambda \in \mathbb{D},$$

and $\|k_\lambda\|_2^2 = (1-|\lambda|^2)^{-1}$.

2. *For* $\varphi \in H^\infty$, $C_\varphi^* = P_K T_{\bar{\varphi}}|_K = T_{\bar{\varphi}}|_K$.
3. $P_{bH^2} = T_b T_{\bar{b}}$, *hence* $\underline{P_K = I_{H^2}} - T_b T_{\bar{b}}$.
4. *For* $\varphi \in H^\infty$, $T_{\bar{\varphi}} k_\lambda = \overline{\varphi(\lambda)} k_\lambda$.
5. *For* $\varphi \in H^\infty$, $C_\varphi = \mathbf{0}$, *the zero operator, if and only if* $\varphi \in bH^\infty$.

Theorem 4.4. *Let* $f_1, \dots, f_n \in H^\infty$. *Then the following are equivalent*:

1. $(f_1, \dots, f_n, b) \in \mathrm{U}_{n+1}(H^\infty)$, *i.e., the* $(n+1)$*-tuple is corona data.*
2. $(C_{f_1}, \dots, C_{f_n}) \in \mathrm{U}_n(B(K)) = \mathrm{RU}_n \cap \mathrm{LU}_n$.
3. $(C_{f_1}, \dots, C_{f_n}) \in \mathrm{RU}_n(B(K))$.

Note. Condition 3 is that there exist $A_i \in B(K)$ such that $C_{f_1}A_1 + \dots + C_{f_n}A_n = I_K$, while condition 2 adds that there exist $B_i \in B(K)$ such that $B_1 C_{f_1} + \dots + B_n C_{f_n} = I_K$. However, $(C_{f_1}, \dots, C_{f_n}) \in \mathrm{LU}_n(B(K))$ need not imply conditions 1–3.

Proof. 1 implies 2. Let $(g_1, \dots, g_n, g_{n+1}) \in \mathrm{U}_{n+1}(H^\infty)$ such that $f_1g_1+\cdots+f_ng_n + bg_{n+1} = 1$. The map $\varphi \mapsto C_\varphi$ is a homomorphism of $H^\infty \to B(K)$, so

$$\begin{aligned} I_K &= C_{f_1g_1+\cdots+f_ng_n+bf_{n+1}} = C_{f_1}C_{g_1} + \cdots + C_{f_n}C_{g_n} + 0 \\ &= C_{g_1}C_{f_1} + \cdots + C_{g_n}C_{f_n}. \end{aligned}$$

2 implies 3. Clear.

3 implies 1. Let $A_i \in B(K)$ such that $C_{f_1}A_1 + \cdots + C_{f_n}A_n = I_K$. So $A_1^*C_{f_1}^* + \cdots + A_n^*C_{f_n}^* = I_K^* = I_K$. Then

$$\begin{aligned} \left(\sum A_i^*C_{f_i}^*\right)(P_Kk_\lambda) &= \sum A_i^*T_{\overline{f}_i}(I_{H^2} - T_bT_{\overline{b}})(k_\lambda) \\ &= \sum A_i^*(\overline{f_i(\lambda)}k_\lambda - \overline{b(\lambda)}T_{\overline{f}_i}T_b(k_\lambda)) \\ &= \sum A_i^*(\overline{f_i(\lambda)}k_\lambda - \overline{b(\lambda)}T_{\overline{f}_i}(bk_\lambda)). \end{aligned}$$

So $||P_Kk_\lambda||_2 = ||(\sum A_i^*C_{f_i}^*)(P_Kk_\lambda)||_2 \le \sum ||A_i^*||(|f_i(\lambda)| + |b(\lambda)|\,||f_i||)||k_\lambda||_2$. But

$$\begin{aligned} ||P_Kk_\lambda||_2^2 &= ||k_\lambda||_2^2 - ||P_{K^\perp}k_\lambda||_2^2 = ||k_\lambda||_2^2 - ||T_bT_{\overline{b}}k_\lambda||_2^2 \\ &= ||k_\lambda||_2^2 - |\overline{b(\lambda)}|^2||bk_\lambda||_2^2 = (1 - |b(\lambda)|^2)||k_\lambda||_2^2. \end{aligned}$$

So $\sum ||A_i^*||(|f_i(\lambda)| + |b(\lambda)|\,||f_i||) \ge (1 - |b(\lambda)|^2)^{1/2}$ for all $\lambda \in \mathbb{D}$.

Suppose that $(f_1, \dots, f_n, b) \notin \mathrm{U}_{n+1}(H^\infty)$. Then by the corona theorem, there exists a sequence $(\lambda_j) \subset \mathbb{D}$ such that $(f_1, \dots, f_n, b)(\lambda_j) \to (0)$ as $j \to \infty$. Evaluate the previous inequality at (λ_j), take $j \to \infty$, and get $0 \ge 1$. Contradiction. □

The above proof is a minor extension of that of Nikol′skiĭ [15, Lecture III.3] wherein the spectrum of a single operator, C_f, is computed. In fact, if one wished to refer to a "joint spectrum" in this noncommutative Banach algebra, then the theorem which Nikol′skiĭ gives is the case $n = 1$ of the following

Corollary 4.5. *For $f_1, \dots, f_n \in H^\infty$, the joint spectrum of $(C_{f_1}, \dots, C_{f_n})$ is*

$$\begin{aligned} &\sigma(C_{f_1}, \dots, C_{f_n}) \\ &\quad \stackrel{\text{def}}{=} \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : (C_{f_1} - \lambda_1I_K, \dots, C_{f_n} - \lambda_nI_K) \notin U_n(B(K))\} \\ &\quad = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : \inf_{z\in\mathbb{D}}(|f_1(z) - \lambda_1| + \cdots + |f_n(z) - \lambda_n| + |b(z)|) = 0\}. \end{aligned}$$

Proof. $C_f - \lambda I_K = C_{f-\lambda}$, the equivalence of conditions 1 and 2 of Theorem 4.4, and the corona theorem. □

In the situation of Theorem 4.4, there is the following application of results on reducibility in H^∞. The theorem recasts a large class of reducibility problems in H^∞ as lifting problems from $B(K)$ to $B(H^2)$.

Theorem 4.6. *Let $f_1, \dots, f_n \in H^\infty$ be such that $(C_{f_1}, \dots, C_{f_n}) \in \mathrm{U}_n(B(K))$. Then the following are equivalent*:

1. $(f_1, \ldots, f_n, b)$ *is reducible.*

2. *The unimodular row of compressed analytic Toeplitz operators,* $(C_{f_i})_1^n$ *may be lifted to a unimodular row of analytic Toeplitz operators,* $(T_{F_i})_1^n$. *That is, there exist* $F_1, \ldots, F_n \in H^\infty$ *such that* $(T_{F_i}, \ldots, T_{F_n}) \in U_n(B(H^2))$ *and* $C_{f_i} = P_K T_{F_i}|_K = C_{F_i}$, *for* $1 \le i \le n$.

Hence, if $\operatorname{bsr}(H^\infty) \le n$, *then all unimodular rows, of length* n, *of compressed analytic Toeplitz operators may be lifted to unimodular rows of analytic Toeplitz operators. In particular, if* $\operatorname{bsr}(H^\infty) = 1$ *then invertibles may be lifted to invertibles.*

Proof. Suppose that $(f_1, \ldots, f_n, b)$ is reducible to $(f_1 - bh_1, \ldots, f_n - bh_n) \in \mathrm{U}_n(H^\infty)$, with $\sum (f_i - bh_i)g_i = 1$. Then $\sum T_{f_i - bh_i} T_{g_i} = \sum T_{g_i} T_{f_i - bh_i} = I_{H^2}$. Hence $(T_{f_i - bh_i})_1^n \in \mathrm{U}_n(B(H^2))$ and $C_{f_i - bh_i} = C_{f_i}$, for $1 \le i \le n$.

Conversely, suppose that condition 2 holds. Similarly to the proof of the previous theorem, the equation $T_{F_i}^* k_\lambda = \overline{F_i(\lambda)} k_\lambda$ and the corona theorem can be used to show that $(T_{F_i})_1^n \in \mathrm{U}_n(B(H^2))$ implies that $(F_i)_1^n \in \mathrm{U}_n(H^\infty)$. As $\mathbf{0} = C_{f_i} - C_{F_i} = C_{f_i - F_i}$, then for some $h_i \in H^\infty$, $f_i - F_i = bh_i$, for $1 \le i \le n$. Thus $(f_1, \ldots, f_n, b)$ is reducible to $(f_i - bh_i)_1^n = (F_i)_1^n \in \mathrm{U}_n(H^\infty)$. □

Remark 4.7. The facts used about Toeplitz operators are well known, for example see [15, Appendix 4]. The main reference for compressed Toeplitz operators is [15].

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