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ISOTOPY INVARIANTS OF GRAPHS

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ABSTRACT. The development of oriented and semioriented algebraic invariants associated to a class of embeddings of regular four valent graphs is given. These generalize the analogous invariants for classical knots and links, can be determined from them by means of a weighted averaging process, and define them by means of a new state model. This development includes the elucidation of the elementary spatial equivalences (generalizations of the classical Reidemeister moves), and the extension of fundamental concepts in classical knot theory, such as the linking number, to this class spatial graphs.

0. INTRODUCTION

The purpose of the article is to give a description of some of the results of our effort, during the spring of 1986, to extend the various polynomial invariants associated to classical knots and links, the Alexander/Conway polynomial [1, 2, 5], the Jones polynomial [11, 12], and their generalizations to the oriented and semioriented polynomials [3, 7, 8, 9, 15, 16, 19, 23, 24], to embeddings of graphs in three dimensional Euclidean space. Some applications of these invariants have been discussed elsewhere, [13, 22], and a related approach to invariants of rigid vertex graphs was worked out by Kauffman and Vogel [18].

The approach which we employ was inspired by the methods employed by Kauffman and others to give a state model for the Jones polynomial, [17, 20], and to which method it owes fundamental aspects of structure in our proofs. In order to develop these invariants we have restricted ourselves to a rather particular category of structure which we have called "chimerical graphs": these are the oriented regular 4-valence rigid vertex graphs (in which the orientations of the edges at each vertex are consistent) and spatial equivalences which preserve these structures. In the course of this study it is necessary to extend many of the standard facts and definitions from the setting of classical knots and links to that of the chimerical graphs, e.g. components become circuits, linking of components becomes linking of circuits, Reidemeister moves become chimerical Reidemeister moves, etc. The associated algebraic structures, subject to quite natural additional equivalence relations, which define the invariants are shown to be freely generated by three "geometric" elements over the field of rational

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functions in variables derived from the corresponding invariants associated to the classical knots and links.

We show how the fundamental aspects of these chimerical graph invariants incorporate two important theoretical dimensions. First, they provide a completely combinatorial definition of the classical knot and link invariants as a special case, i.e., they provide a "state model" for the oriented and semioriented polynomials. The specializations to the Alexander/Conway and Jones polynomials give interesting states models for these polynomials as well. The second dimension is the vision of the chimerical graph invariants as weighted averages of an associated collection of classical knots and links. This later perspective allows one to easily demonstrate the existence, invariance, and uniqueness of these chimerical invariants based upon the corresponding existence, invariance, and uniqueness of the appropriate invariants associated to the classical knots and links. In addition, this expression of the invariants allows for easier calculations by means of the invariants calculated for the classical knots and links.

We describe the analogs of some of the fundamental facts for the oriented and semioriented invariants and show how these are reflected in several examples in which the "chirality" of the chimerical graph, i.e. its invariance under mirror reflection, is the object.

In the first section we shall develop the fundamental aspects of the category of chimerical graphs and include a discussion (worked out with Keith Wolcott) of the "Reidemeister moves" which must be employed in the study of topological equivalence of graphs as well as chimerical graphs. The second section introduces the oriented chimerical graph invariant whose state model is presented in the third section. There the proof of its existence, invariance, and uniqueness from the weighted average perspective is presented. In the fourth section, the algebraic aspects of the state model for the oriented invariant are developed and a proof of its existence, invariance, and uniqueness from the algebraic/combinatorial perspective is presented. These lead to state models for the Alexander/Conway and Jones polynomials. There appear to be two natural directions in which to develop such models, one which emphasizes the combinatorial structure of the invariant and another, the "interaction" model, cf. [28], which emphasizes the algebraic structure of the invariant. Examples of both types of models are presented in the fifth section. The development of the corresponding aspects of the semioriented invariant are described in §6, basic properties; §7, the state model; §8, algebraic aspects; §9, additional models for the Jones polynomial. Finally, in §10, we give some examples of the application of these invariants to the study of the spatial properties of some interesting embeddings of graphs.

1. Chimerical graphs

The development of algebraic topological methods applicable to certain aspects of the spatial placement of graphs in 3-space is given by means of combinatorial structures associated to appropriate representations of their placements

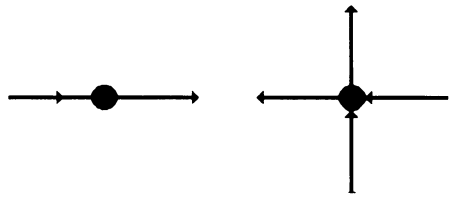

FIGURE 1.1. Oriented graph templates

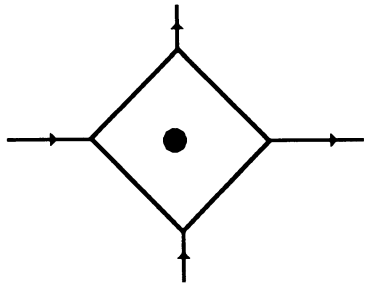

FIGURE 1.2. Rectangular template structure

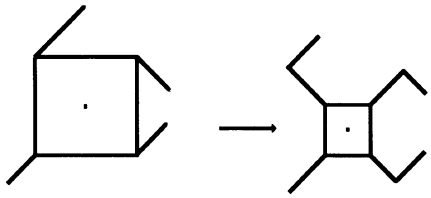

FIGURE 1.3

in R^3 in much the same way that invariants are associated to classical knots and links. We shall restrict ourselves to topologically 4-valent graphs, i.e. graphs in which each vertex meets exactly two or four edges, cf. Figure 1.1. Second, a fundamental restriction is the provision of an appropriate rigidity in the vicinity of each vertex having valence 4. This is accomplished by means of the notion of a chimerical graph. The essential content of this definition is the choice of a fixed model, or template, for the position of the edges in the vicinity of each vertex.

Thus, *a chimerical graph*, Γ, is a topologically 4-valent graph which, near each vertex, is endowed with a standard rigid planar template determining the local relative position of the adjacent edges, i.e. having a *planar template with equal angles between edges at each vertex*, exactly as shown in Figure 1.1.

We shall be concerned with piecewise linear embeddings of these graphs which preserve the template structure at the vertices. One means by which this can be accomplished is by the replacement of each of the 4-valent vertices by planar squares, as shown in Figure 1.2, and to consider only piecewise linear embeddings in R^3 which do not require the subdivision of these squares to

achieve a simplicial realization. The preservation of the local template structure is achieved by requiring that each pair of squares project bijectively to a plane and that the 1-skeleton project generically. These squares are defined only up to scale, i.e. two templates will be considered to be equivalent if they differ only by a local radical expansion or contraction of the square as illustrated in Figure 1.3. With this definition, a presentation of the placement of the chimerical graph enjoying much the same structure as that employed for classical knots and links can be defined, i.e. both classical and the chimerical vertices (and their associated square templates) project bijectively and are disjoint from the images of the edges, the projections of edges meet in at most single interior points, and the intersection of three or more edges is empty. A *restricted generic projection* giving such a presentation, can be accomplished by an arbitrarily small perturbation of the vertices of the square templates at each of the 4-valent vertices until they project homeomorphically followed by local change of scale at the chimerical vertices and an arbitrarily small perturbation of the given plane of projection in R^3 so as to achieve the remaining requirements for generic representation. Thus, the resulting restricted generic projections have the properties that templates project one-to-one, that the multiple points (i.e. the "crossing points" of the projection) in the image are finite in number, are only double points, and the segments cross transversely at each such double point of the image exactly as in the case of the classical knots and links considered earlier. Following the tradition of classical knot theory, a restricted generic projection is represented by a planar diagram representing the shadow of the projection but broken at the singular points of the projection so as to indicate under and over crossings. By a homothetical change of the local scale at the vertices, one may assume that the templates at the 4-valent vertices project disjointly from each other and contain no singular points. An example of a chimerical embedding of the K_5 graph is shown in Figure 1.4.

We define an *orientation* of a chimerical graph as a choice of direction on each edge of the graph subject to the requirement that the directions at 4-valent vertices are consistent for opposite edges at a vertex, as shown in Figure 1.1.

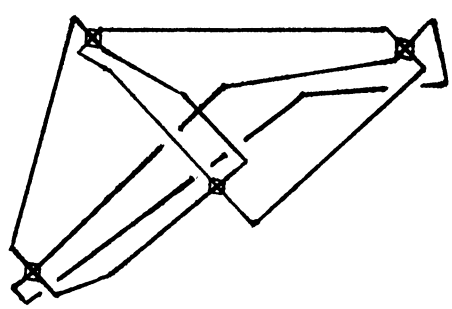

FIGURE 1.4. A chimerical embedding of the K_5 graph

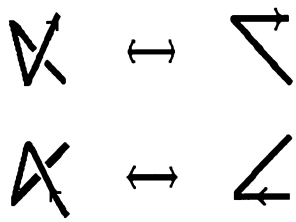

FIGURE 1.5. Type I Reidemeister moves

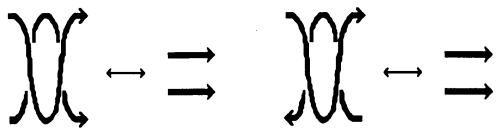

FIGURE 1.6. Type II Reidemeister moves

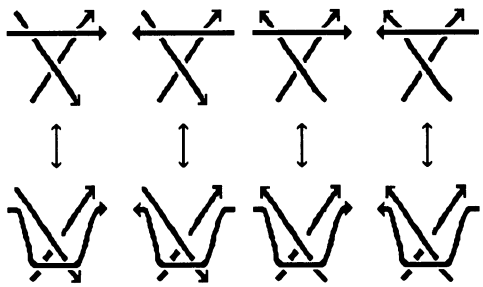

FIGURE 1.7. Type III Reidemeister moves

Recall that a circuit in a classical graph is a sequence of edges, $e_1, e_2, \ldots, e_m$, joining distinct vertices, $\{v_0, v_1\}$, $\{v_1, v_2\}, \ldots, \{v_{m-1}, v_0\}$. A *circuit in a chimerical graph* is a circuit which "goes straight ahead" at every vertex. There are three distinct circuits shown in Figure 1.4. This concept of a circuit is well defined and invariant because of the local template structure at the vertices. Two circuits in a chimerical graph are either identical or are transverse, in the sense that their intersection consists only of vertices.

Two chimerical graphs are equivalent if there is a piecewise linear isotopy taking one embedding to another, preserving the template structure. Although the spatial movements of chimerical graphs can, in general, be quite complicated, we shall give a sketch (worked out with Keith Wolcott) of the "folk theorem" that any such movement can be described in terms of a finite sequence of rather simple movements. We call these movements "chimerical Reidemeister moves" after the analogous primitive movements which generate the isotopy equivalence relation in the setting of classical knots and links, [4, 25, 26]. The first three moves are exactly those which occur for the classical knots and links as they do not involve the special structure of the chimerical graph. They are shown in Figures 1.5, 1.6, and 1.7. Because we are concerned with the issue of orientation in

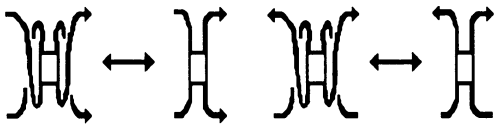

FIGURE 1.8. Type II chimerical Reidemeister moves

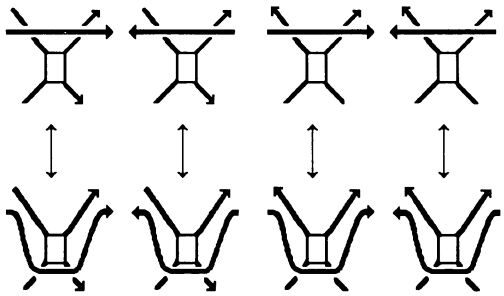

FIGURE 1.9. Type III chimerical Reidemeister moves

a fundamental way it is necessary to indicate several versions of essentially the same spatial movement. In Figures 1.8 and 1.9 are shown the typical examples of the additional moves that are required to represent the spatial movements of chimerical graphs. The squares have been added to indicate the rigid template structure at the vertex of the graph. Note that this rigid template structure has been preserved in each of these cases although it has been "turned over" in the case of the generalized type II move shown in Figure 1.8.

We shall first give a sketch of the proof of two "folk theorems" giving analogs of the Reidemeister moves of classical knot and link theory for the case of piecewise linear ambient isotopy of finite graphs embedded in R^3 before sketching the required theorem for chimerical graphs. As we are concerned with the theory of piecewise linear (pl) ambient isotopy type of pl embeddings of finite graphs, i.e. finite 1-dimensional simplicial complexes, in R^3, the basic references for the concepts and methods which we shall employ are Hudson, [10], or Rourke-Sanderson, [27]. Recall that a pl map between two simplicial complexes is a simplicial map between appropriate subdivisions of the range and domain spaces. Two pl embeddings $f_0, f_1 \colon K \to R^3$ are pl ambient isotopic if there exists a pl homeomorphism $H \colon R^3 \times I \to R^3 \times I$, commuting with projection on the second factor, such that $H(f_0, 0) = (f_1, 1)$. In order to work with these pl embeddings and the ambient isotopy relation it is useful to project the embeddings and isotopies to a convenient (generic) hyperplane and attempt to describe the pl isotopy relation by means of sequence of elementary moves on the generic representations of the initial and terminal embeddings. By the method of general position one can select a 2-dimensional hyperplane, P, such that vertical projection to the plane, π, has the property that the image under π of any set of three vertices of the range of H determine a triangle in P.

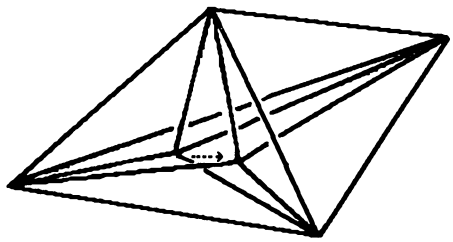

FIGURE 1.10

Recall that if $f: L \to L$ is a pl homeomorphism of a pl space, the *support of* h is the closure of the set of ponts where f is *not* the identity. Usually a move is defined to be a pl homeomorphism supported in a pl ball. We shall want to demand a bit more: a *move* is a simplicial homeomorphism which is the identity except in the star of an exceptional vertex in the two triangulations of L (which are identical except in the star of these vertices) as the range and the domain and which takes the one vertex to the other. An example is shown in Figure 1.10. We shall say that two pl homeomorphisms of L, f and g, are *isotopic by moves* if there is a finite sequence $h_1, h_2, \dots, h_k$ of moves of L such that $h_1 h_2 \cdots h_k f = g$. In the same manner one would say that two embeddings of a pl space into L, denoted f and g, are isotopic by moves if such a sequence of moves exists providing the same relation between f and g. Since a move defines an ambient isotopy of L, any two embeddings which are isotopic by moves are ambient isotopic. The first result is that the converse is also true.

Folk Theorem 1.1. *If L is a compact pl manifold and $H: L \times I \to L \times I$ is an ambient isotopy, then there exists a sequence of moves, $h_1, h_2, \dots, h_k$ of L such that $H(x, 1) = (h_k h_{k-1} \cdots h_2 h_1(x), 1)$.*

Sketch of proof. The sketch is an extension of the proof of Theorem 6.2 in Hudson [10]. Let $L \times I \subset R^n \times R$ be triangulated as a linear subcomplex. Any pl map $\varphi: L \to I$ determines a pl embedding $(1, \varphi): L \to L \times I$ and a pl map $\varphi^* = \pi_1 H(1, \varphi)$, where π_1 denotes the projection to the first factor.

Let $l \subset \sigma$ be a vertical line segment ($\pi_1(l) =$ point) contained in a simplex in the triangulation of the range of H for which H is simplicial. $H^{-1}(l)$ is a line in the simplex $H^{-1}(\sigma)$ making an angle less than $\pi/2$ since H is level preserving. Note that this angle depends only on the simplex so that, since L can be triangulated as a finite complex, there exists an angle $\alpha < \pi/2$ such that each $H^{-1}(l)$, for any σ, makes an angle no larger than α with the vertical. On the other hand, there exists a $\delta > 0$ (depending on the triangulation of L), so that if the diameter of $\varphi(L)$ in I is less than δ and $\sigma \in L$, then a vertical $l \subset \sigma$ makes an angle greater than α with the vertical.

If φ is a pl map having diameter less than δ, then φ^* is a pl automorphism of L. Connectivity of $x \times I$, for each $x \in L$, shows that φ^* is onto. If we

can show that the intersection of $(1, \varphi)(L)$ with $H^{-1}(x \times I)$, for any x, is no larger than one point then φ^* will be a pl homeomorphism. If $(y, \varphi(y) = t_0) = H^{-1}(x, t_0)$ denotes the smallest value of "t" for the intersection, then the set $H^{-1}(x \times [t_0 1])$ must lie within the solid cone of rays from (y, t_0) making an angle no larger than α with the vertical. The graph of φ meets this cone only at the point (y, t_0).

There exists a finite sequence of pl maps, $\varphi_1, \varphi_2, \ldots, \varphi_k$, of L into I such that $\varphi(L) = 0$ and $\varphi_k(L) = 1$; the diameter of $\varphi_i(L)$ is less than δ for all $1 \leq i \leq k$; and φ_i and φ_{i+1} agree on all but one vertex of L. As a consequence, $\varphi_0^* =$ identity and $(\varphi_k^*(x), 1) = H(x, 1)$. Define $h_i = \varphi_i^*(\varphi_{i-1}^*)^{-1}$. The support of h_i is the star of the single vertex $v \in L$ at which φ_i and φ_{i-1} have different values. Therefore h_i is move. Since $\varphi_0^* =$ Identity and $(\varphi_k^*(x), 1) = H(x, 1)$, $H(x, 1) = (h_k h_{k-1} \cdots h_2 h_1(x), 1)$ as required.

In order to achieve the desired factorization of ambient isotopies of embeddings of graphs (or finite 1-dimensional complexes), K, in R^3, we shall want to study the nature of the moves and their restrictions to 1-dimensional subcomplexes. Suppose that $h: R^3 \to R^3$ is a move supported in the star of a vertex, v, determining an interval, I_v, whose other endpoint is $h(v)$. The closure of the intersection of the interior of the support of h with K is join of v with a finite set of vertices, X, contains in the boundary of the star. If X is empty there is nothing to prove since K does change as the result of the move. Suppose that X is non empty. We subdivide and perturb the interval, I_v, by adding a vertex, $\tilde{v}$, at the midpoint but in general position with respect to the vertices of the domain of h and $h(v)$. The reason for this additional complication is that $h(v)$ may not be in general position with respect to the vertices of the triangulation of the domain of h. This determines two intervals, $I_1 = [v, \tilde{v}]$ and $I_2 = [\tilde{v}, h(v)]$, having the property that their joins with X are each embedded in the star of v.

By the principle of general position there is a 2-dimensional hyperplane, P, in R^3 so that the vertical projection of R^3 to P, π_P, takes the vertices of the triangulation of the domain of h together with $\tilde{v}$ and $h(v)$ into general position. As a consequence, all the 2-dimensional simplices project homeomorphically, the images of the interiors of the 1-dimensional simplices do not contain the images of any vertices, there are no triple point intersections of 1-dimensional simplices, and the vertices project bijectively. Consider the join $X^* I_1$ containing the initial configuration, $X^*(v)$, and the intermediate configuration, $X^*(\tilde{v})$. We shall study the sequence of operations that are required to change the initial configuration to the intermediate configuration. Consider $\pi_P^{-1}(\pi_P(K)) \cap X^* I_1$ and note that $Y \cap I_1$ consists of a finite number of points projecting corresponding to double points of the projection. By a planar isotopy corresponding to triangle moves in R^3, with support arbitrarily near I_1, we can push these double points off the end of the image of I_1.

The next step is to use a planar pl isotopy to move the vertex v to $\tilde{v}$ via a planar isotopy respecting the double points of the projected image of the initial and final configurations. The result is a finite sequence of triangle moves which take the modified initial configuration to the final position. As each of these triangles are embedded in R^3, they may be taken in any order. We shall, however be somewhat more careful to factor each of these triangle moves, supported on a triangle σ, into "elementary triangle moves" corresponding to the structure of $\pi_P^{-1}(\pi_P(\tilde{K})) \cap \sigma$, where $\tilde{K}$ denotes a subdivision of K containing the simplices determined by the previous two moves. We shall want to subdivide σ so that each 2-simplex of the subdivision meets the double point set only in the interiors of the 2-simplices and with at most one double point in a 2-simplex. Furthermore, the double point set will meet the 1-skeleton of the subdivision only in a single arc from a vertex, a single arc meeting two sides of the triangle, or (in the case in which the simplex contains a double point) in a pair of arcs meeting at an interior double point. We then use "elementary triangle moves" to factor the move associated to σ.

Once we have done all these moves, one must replace the arcs meeting I_1. This is accomplished by the final type of elementary move in which a triangle gives a move of an arc over or under a vertex of the graph. This completes the list of elementary moves and, therefore, the sketch of a proof of the following folk theorem for graphs.

Folk Theorem 1.2. *Two generic projections of finite graphs piecewise linearly embedded in R^3 represent ambient isotopic embeddings if and only if the projections*

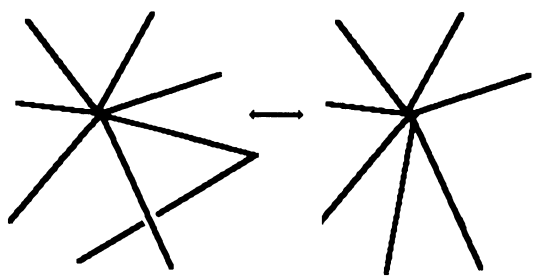

FIGURE 1.11. Generalized type I Reidemeister move

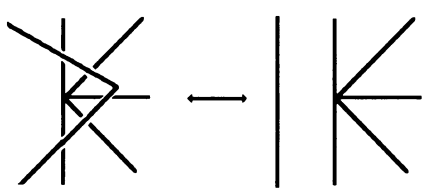

FIGURE 1.12. Generalized type II Reidemeister move

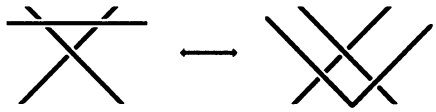

FIGURE 1.13. Type III Reidemeister move

are related by a finite sequence of generalized Reidemeister moves of types I, II *and* III.

The case of the ambient isotopies of chimerical graphs is proved in a manner analogous to this result for graphs. By way of explanation, the proof follows from an analysis of the previous proof in which one includes the squares which determine the template structure and prohibits subdivisions of these squares during any simplicial realization of the piecewise linear ambient isotopy relating two restricted generic projections of the chimerical graphs. The only change in the proof occurs with the analysis of the consequences of the individual moves. Because of the prohibition of subdivision of the squares giving the template structure, any move which changes a vertex of one of the rectangles is supported in the simplicial neighborhood of the square and changes the qualitative property of the projection only in the case, generically, where the local orientation of the image is reversed thereby giving a chimerical Reidemeister move of type II. A chimerical Reidemeister move of type III arises by virtue of a simplicial move which does not change the position of the rectangle. Employing a local change of scale, illustrated in Figure 1.2, one can control their occurrence so that they arise only when an interval passes the center of a rectangle and so that rectangles always have disjoint images, as required for restricted generic projections.

Folk Theorem 1.3. *Two restricted generic projections of chimerical graphs piecewise linearly embedded in* R^3 *represent ambient isotopic embeddings if and only if the projections are related by a finite sequence of classical and chimerical Reidemeister moves of types* I (*Figure* 1.5), II (*Figures* 1.6 *and* 1.8) *and,* III (*Figures* 1.7 *and* 1.9).

The goal of the remaining portion of this section is the description of the method by which one may define linking numbers for oriented chimerical graphs which are simultaneously easily calculable and yet sufficiently sensitive to the spatial nature of the placement to often, if not always, successfully distinguish between spatially distinct placements of the same chimerical graph. In particular, we shall wish to be able to determine the chirality of several specific embeddings of oriented chimerical graphs. These concepts will be employed in the development of the more complex polynomial invariants in the next section.

The *linking number of two oriented circuits* is calculated from a generic projection of the link by taking one-half the sum of the algebraic crossing numbers assigned according to the convention defined in Figure 1.14. By way of generalization, we define the algebraic contribution of chimerical "vertex crossing" to

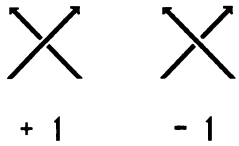

FIGURE 1.14. The algebraic crossing convention

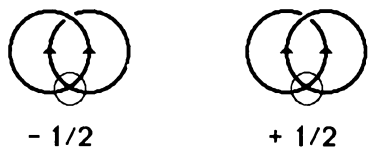

FIGURE 1.15. Fractional linking in chimerical graphs

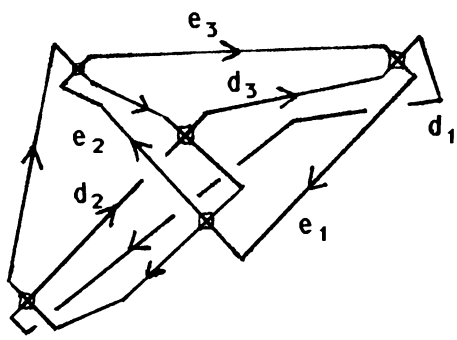

FIGURE 1.16. Two circuits $\{e_i\}$ and $\{d_j\}$

be equal to zero, being the average of the 1 and -1 contribution of the other crossings. This definition of the linking number clearly extends the classical concept of linking number for orinted links and introduces a "fractional linking" for chimerical graphs such as those shown in Figure 1.15. In the case of the two circuits shown in the K_5 chimerical graph, one finds the linking to be equal to $-1/2$.

A fundamental fact is the proposition that the linking number of two circuits in a chimerical graph is an invariant of the spatial placement of the chimerical graph, i.e. any spatial deformation which leaves invariant the template structures at the vertices must leave the linking number unchanged. This is demonstrated by observing that each of the chimerical Reidemeister moves, as described above, leaves the linking number of the circuits unchanged.

Proposition 1.4. *The linking number associated to two circuits in a chimerical graph is unchanged by chimerical graph isotopy, i.e. linking number is a chimerical graph invariant.*

An elementary application of this proposition is the demonstration that the K_5 chimerical graph and its mirror image are spatially inequivalent, i.e. it is a

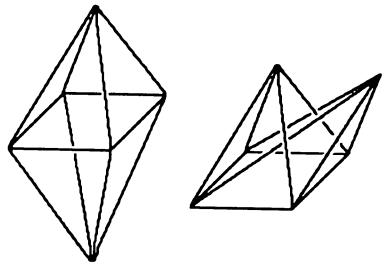

FIGURE 1.17. Achiral and chiral octahedrons

chiral graph. To prove this one need only calculate the linking number between the two circuits, indicated in Figure 1.16, to find that they are $-1/2$ and $1/2$ for the K_5 chimerical graph and its mirror reflection, respectively. Therefore the placement shown in Figure 1.16 and its mirror image are spatially distinct.

In the context of 4-valent graphs, the regular octahedron, shown on the left in Figure 1.17, is sufficiently complex to again illustrate the application of linking numbers to questions of chirality. Although we have not, for the sake of simplicity in the figure, preserved this rigid vertex structure at the central vertices we have shown a spatially distinct placement of the octahedron to its right. The linking method can be employed to show that the right hand placement is chiral while, by a simple spatial rotation, one can show that the left-hand placement is, in fact, achiral. It is useful to note that any placement equivalent to a planar placement of a graph is automatically achiral because the plane of reflection of the mirror can always be taken to be precisely the plane of the graph and, therefore, the graph is unchanged by the reflection. This is the case for the first placement of the octahedron.

2. Oriented algebraic invariants for chimerical graphs

The extension of the oriented and semioriented polynomials defined for classical knots and links to the case of oriented chimerical graphs can be accomplished in two manners: directly, by the creation of a "state model" for these polynomials which provides for the inclusion chimerical graph vertices, or indirectly, by assuming the existence of the oriented or semioriented polynomials for classical knots and links. The later method is a generalization of the averaging method which lead from the linking numbers of classical links to linking numbers for the chimerical graphs. A special case of the later approach is presented in Kauffman and Vogel, [18], whose perspective has demonstrated the essential simplicity of the existence of these chimerical polynomials. The general averaging approach, which we present here, has the advantage of not requiring a proof from first principles that these classical invariants are well defined and is, therefore, the quickest route to the development of the chimerical graph invariants and their properties. To show how this is accomplished and how the two approaches are related we shall first describe the fundamental conceptual issues involved in the creation of an averaging model for the polynomials.

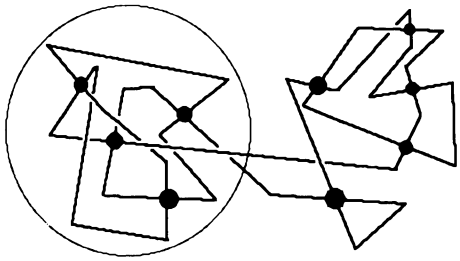

FIGURE 2.1. The connected sum of two graphs, Γ_1 and Γ_2

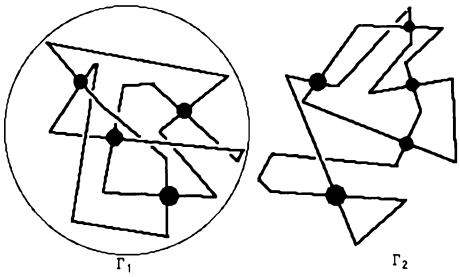

FIGURE 2.2. The summands, Γ_1 and Γ_2

An important algebraic structural simplification is obtained if one imposes an assumption upon the class of potential invariants that are to be studied. This assumption is the *connected sum axiom or localization axiom.* The axiom states that if there is a circle in the plane of the projection which transversely meets the image of a generic projection of an oriented chimerical graph in exactly 2 points, such as illustrated in Figure 2.1, then the invariant associated to the object is required to be the product of the invariants associated to the two pieces of the graph gotten by breaking the graph at the two points and connecting the inside graph along the circle to get one graph and, similarly, connecting the outside piece along the circle to get the second graph, such as shown in Figure 2.2. This axiom can be understood as an analog of the connected sum property of the oriented and semioriented polynomials or as an axiom of the states models. As a consequence, we have the following proposition:

Proposition 2.1. *Every planar* 4*-valent graph can be expressed as a unique monomial* (*up to order of terms*) *in irreducible planar* 4*-valent graphs.*

Sketch of proof. The proof of the proposition is an elementary application of standard techniques employed to analyze families of circles in the plane. Without loss of generality, we may assume that the graph in question is connected. Since there are only a finite number of vertices, any reduction is finite and there are only a finite number of possible reductions. Two circles occurring in a reduction are said to be parallel if they are disjoint and the annular region they bound (in S^2) contains no vertices. One first shows that the circles in any max-

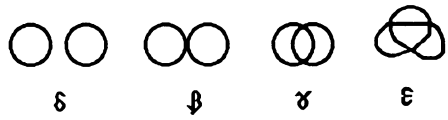

FIGURE 2.3. Chimerical graph variables

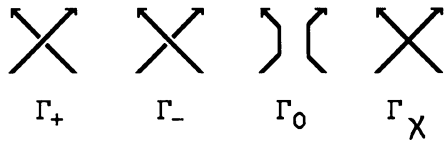

FIGURE 2.4. The chimerical graph state

imal family of nonparallel reducing circles can be taken to be pairwise disjoint. This is accomplished by putting them in general position with respect to each other and by considering innermost regions in the complement of the union of the circles. If the region is empty, the circles can be moved so as to eliminate the intersections. Since each of the circles intersects the graph in at most two edge points (they are reducing circles) one can employ a case-by-case analysis to show that the intersections can be removed (either by isotopy or, equivalently, by "cutting and pasting") or cannot occur by virtue of the maximality of the family.

To show uniqueness one considers two such families of reducing circles and employs these same methods to remove intersections between reducing circles until the two families consist of parallel circles and, therefore, give identical reductions of the graph.

This representation is unique but not not faithful, i.e. two nonisomorphic planar graphs can easily give the same irreducible decomposition but any two such decompositions differ only in the order in which the symbols are written.

A second basic aspect of such theories which one might wish to exploit is the existence of a recursion formula to facilitate calculations in the spirit of the earlier theories. We will shall show that these new oriented and semioriented polynomials, for oriented chimerical 4-valent graphs, satisfy recursion relations which are exactly the same as those which occurred for oriented knots and links in space. Formally, the recursion relations will allow one to express the invariant of a specific placement of an oriented chimerical 4-valent graph in terms of the invariants of simpler placements.

Theorem 2.2 (Oriented invariants for chimerical graphs). *There is a unique way of associating to each oriented chimerical graph in* 3*-dimensional Euclidean space,* Γ*, an algebraic function,* $PG(\Gamma)$*, in the algebraic variables* l *and* m*, and elementary oriented chimerical graph variables* β *and* γ*, satisfying the connected sum axiom, such that spatially equivalent oriented chimerical graphs have the same associated algebraic function and such that*

(i) $PG(U) = 1$, $PG(\beta) = \beta$, $PG(\gamma) = \gamma$ (*see Figure* 2.3).

(ii) *if* Γ_+, Γ_-, *and* Γ_0 *are any three oriented chimerical* 4*-valent graphs that are identical except near one point where they are as shown in Figure* 2.4, *then*

$$lPG(\Gamma_+) + l^{-1}PG(\Gamma_-) + mPG(\Gamma_0) = 0 \quad (\textit{see Figure } 2.4).$$

(iii) *if* Γ_+, Γ_- *and,* Γ_χ *are any three oriented chimerical graphs that are identical except near one point where they are as in Figure* 2.4, *then*

$$\begin{aligned} PG(\Gamma_\chi) = {} & [\beta l - i(\beta^2 - \gamma\delta)^{1/2}(\delta^2 - 1)^{-1/2}](l^{-1} + l)^{-1}PG(\Gamma_+) \\ & + [\beta l^{-1} + i(\beta^2 - \gamma\delta)^{1/2}(\delta^2 - 1)^{-1/2}](l^{-1} + l)^{-1}PG(\Gamma_-). \end{aligned}$$

The chimerical graph variables are shown in Figure 2.3 without orientations in as much as they are spatially independent of these choices when considered as chimerical graphs in S^2. The second recursion relation derives from the goal of providing an extension of the oriented polynomial invariant while the third recursion relation allows for the expression of the influence of the chimerical graph vertices in terms of the algebraic variables and the two graph variables, β and γ. Note that, as in the case of the knot and link invariants, δ can be expressed in terms of the algebraic variables.

The precise relationship is determined by the requirement of invariance under spatial movements of the algebraic function to be associated to the chimerical graph. As a first step towards this result one may invoke only the first recursion relation in order to calculate the invariant in terms of elementary oriented chemirical graphs, i.e. one does not attempt to reduce the number of vertices of the graph but only simplify, as much as possible, the spatial placement of the graph by using crossing changes in the manner employed for classical knots and links.

The concept of an elementary oriented chimerical graph involves making arbitrary choices in a manner analogous to the choice of basis of a vector space. For example, exactly one of the two placements shown in Figure 2.6 must be selected to be the elementary placement.

In addition to using the theorem to give the recursive formula for calculation one also employs several elementary consequences of the theory so as to simplify some of the calculations. It is convenient to give a specific symbol, δ, to the invariant associated to the distant union of two unknotted circles shown as the "0" state in Figure 2.5.

The recursion identity gives the equation $l + l^{-1} + m\delta = 0$ since the "+" and "−" cases are both presentations of the trivial knot. Thus $\delta = -m^{-1}(l^{-1} + l)$.

FIGURE 2.5. Determination of δ

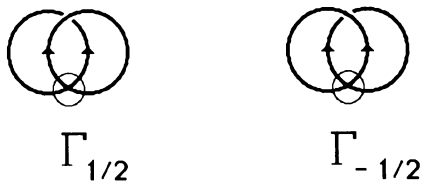

FIGURE 2.6

The following formulae are direct consequences of the recursion relations given in the theorem: The first and second follow from the multiplicative structure of finite Laurent polynomials and the connected sum axiom for planar graphs, while the third uses the asymmetry in the l variable and the mirror reflection (which changes $+1$ crossings to -1 crossings) and recursion on the number of crossings in the presentation and chimerical vertices.

Proposition 2.3. *Suppose that Γ_1 and Γ_2 are two chimerical graphs in S^3, then*

(i) $PG(\Gamma_1 \# \Gamma_2) = PG(\Gamma_1)PG(\Gamma_2)$, *where $\Gamma_1 \# \Gamma_2$ denotes any "connected sum" of Γ_1 and Γ_2.*

(ii) $PG(\Gamma_1 \cup \Gamma_2) = \delta PG(\Gamma_1)PG(\Gamma_2)$, *where $\Gamma_1 \cup \Gamma_2$ is the "distant union" of Γ_1 and Γ.*

(iii) $\overline{P}G(\Gamma_1)(l, m) = PG(\overline{\Gamma}_1)(l, m)$, *where $\overline{\Gamma}_1$ is the mirror image of Γ_1, i.e. reverses all the crossings in a generic projection of Γ_1, and the conjugation in the algebra takes l to l^{-1}, i to i^{-1}, and leaves m unchanged.*

The last property is of particular interest in the development of stereotopological indices for graphs since it is this change in the algebraic function of the variables which allows one to often distinguish one placement of a chimerical graph in space from its mirror image in a way analogous to the linking number discussed in the previous section. Specifically, if the two placements are to be topologically equivalent via the allowed chimerical spacial movements then the associated invariant must be unchanged when l is replaced by l^{-1} and i is replaced by i^{-1}, i.e. by complex conjugation. In order to illustrate the use of the recursion formula and these associated properties, we will consider the graphs, shown in Figure 2.6, and describe how the calculational methods would apply to this representative case. First one can apply Theorem 2.2(ii) to show that

$$lPG(\Gamma_{1/2}) + l^{-1}PG(\Gamma_{-1/2}) + mPG(\Gamma_0) = 0.$$

Since Γ_0 is β, this shows that

$$PG(\Gamma_{-1/2}) = -l^2 PG(\Gamma_{1/2}) - ml\beta.$$

Thus, it is sufficient to calculate $PG(\Gamma_{1/2})$, which is accomplished by application of Theorem 2.2(iii) as follows:

$$\begin{aligned} PG(\Gamma_{1/2}) = {} & [\beta l - i(\beta^2 - \gamma\delta)^{1/2}(\delta^2 - 1)^{-1/2}](l^{-1} + l)^{-1}PG(\Gamma_+) \\ & + [\beta l^{-1} + i(\beta^2 - \gamma\delta)^{1/2}(\delta^2 - 1)^{-1/2}](l^{-1} + l)^{-1}PG(\Gamma_-), \end{aligned}$$

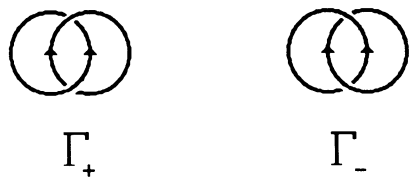

FIGURE 2.7

where Γ_+ and Γ_- are shown in Figure 2.7. Since these are classical links, the polynomial invariant of the Γ_+ can be computed by recursion via part (ii) to find that $PG(\Gamma_+) = (l^{-3} + l^{-1})m^{-1} - l^{-1}m$, while the second is equivalent to the unlink of two components and give $\delta = -(l^{-1} + l)m^{-1}$. Substitution in the formula gives

$$\begin{aligned} PG(\Gamma_{1/2}) &= -\beta(l^{-1} + l)^{-1}m \\ &\quad + il^{-1}(\beta^2 - \gamma\delta)^{1/2}(\delta^2 - 1)^{-1/2}(l^{-1} + l)^{-1}((l^{-1} + l)\delta + m) \\ &= \beta/\delta + il^{-1}(\beta^2 - \gamma\delta)^{1/2}(\delta^2 - 1)^{-1/2}(\delta - \delta^{-1}) \end{aligned}$$

and therefore,

$$\begin{aligned} PG(\Gamma_{-1/2}) &= -\beta(l^{-1} + l)^{-1}m \\ &\quad - il(\beta^2 - \gamma\delta)^{1/2}(\delta^2 - 1)^{-1/2}(l^{-1} + l)^{-1}((l^{-1} + l)\delta + m) \\ &= \beta/\delta - il(\beta^2 - \gamma\delta)^{1/2}(\delta^2 - 1)^{-1/2}(\delta - \delta^{-1}). \end{aligned}$$

Expressing these invariants by means of expressions which are invariant under mirror reflection assists in the recognition of important relationships. Thus we use δ, β and, γ as they represent planar graphs which are, therefore, unchanged under mirror reflection. From the two calculations, one notes that the graphs are achiral as their invariants are changed under complex conjugation and sending l to l^{-1} and that their invariants are interchanged as they are mirror images of each other, as required by Proposition 2.3(iii).

3. A STATE MODEL FOR THE ORIENTED CHIMERICAL GRAPH INVARIANT

The state model approach to the development of the polynomial invariant begins with the definition of the elementary states. Let Γ denote a generic presentation of an oriented chimerical graph. At each crossing of the presentation one makes replacements according to the following table.

The result is a formal algebraic sum of products of the formal variables A, B, C, and D times S^2 equivalence classes of oriented planar 4-valent graphs, denoted $[\![\Gamma]\!]$, i.e. elements of the free module over the equivalence classes of the graphs with coefficients in the ring of finite integral polynomials in the algebraic variables.

For each generic projection of an oriented special 4-valent graph one defines the algebraic crossing number, $\omega(\Gamma)$, to be the sum over all crossings of the

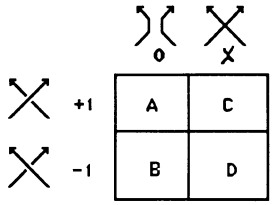

FIGURE 3.1. The fundamental oriented state relations

± 1 associated to the crossings of Γ. In order to complete the definition, we introduce a normalization factor, a power of a variable denoted "α", to provide a compensation for the type I Reidemeister moves. We *define* the P_G (for chimerical graph) invariant associated to the generic projection, Γ, of an oriented 4-valent graph to be $P_G(\Gamma) = \alpha^{-\omega(\Gamma)}[\![\Gamma]\!]$, where α will be defined below. We shall see later that the natural home for this invariant is $Z[l^{\pm 1}, m^{\pm 1}, \beta, \gamma]$. This occurs because although one begins in a free module, the requirement of the connected sum axiom and the requirement that the resulting algebraic expression be a spatial invariant of chimerical graphs forces the introduction of relations on the algebra which lead to the representation of the invariants as elements of $Z[l^{\pm 1}, m^{\pm 1}, \beta, \gamma]$.

Restricting discussion to a single crossing of a given presentation of a chimerical graph, let σ_+ and σ_- denote the $+1$ and -1 versions of the crossing, let 0 denote the removed crossing and χ denote the chimerical vertex form of the crossing. From the fundamental state model formulae one derives the following formulae, with notation restricted to the fixed crossing under consideration:

$$D\sigma_+ - C\sigma_- = (AD - BC)0, \tag{3.2}$$

$$B\sigma_+ - A\sigma_- = (BC - AD)\chi. \tag{3.3}$$

The first formula provides the recursion formula associated to the oriented polynomial of a classical knot or link thereby providing the connection of the polynomial to the oriented polynomial while the second formula provides the weighted averaging interpretation which can be exploited to prove the existence of the chimerical graph polynomial assuming the existence of the oriented polynomial for classical knots and links.

We shall first show the existence of the recursion formula. The initial step is the relation required in order to insure invariance under type I Reidemeister moves. The two cases are shown in Figure 3.4. The connected sum axiom gives the following relations:

$$[\![\Gamma_+]\!] = (A\delta + C\beta)[\![\Gamma]\!] = \alpha[\![\Gamma]\!], \quad [\![\Gamma_-]\!] = (B\delta + D\beta)[\![\Gamma]\!] = \alpha^{-1}[\![\Gamma]\!],$$

where we *define* $\alpha = (A\delta + C\beta)$ and $\alpha^{-1} = (B\delta + D\beta)$. The choice of notation for α and α^{-1} is a result of the first application of the desired invariance under type I implied by the simple two crossing projection shown in Figure 3.5. By

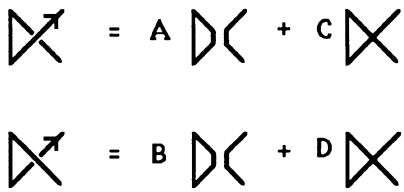

FIGURE 3.4. Type I Reidemeister move relations

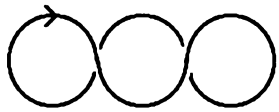

FIGURE 3.5

application of the first and second type I moves one finds that $(A\delta + C\beta) \times (B\delta + D\beta)1$ is equal to the configuration shown in the figure. This is isotopic to the standard circle to which we assign the value 1. Thus we shall require the relation:

$$(A\delta + C\beta)(B\delta + D\beta) = 1\,.$$

The definition of α completes the determination of the defining formula for the polynomial invariant, it does not prove that it is well defined, however. In order to do this we return to the first of the two relations, (3.2), arising out of the states formula which, after normalization, gives:

$$i[\alpha(D/C)^{1/2}[\alpha^{-1}\sigma_+] - i\alpha^{-1}(C/D)^{1/2}[\alpha\sigma_-] + i((BC - AD)/(CD)^{1/2})[0] = 0\,,$$

which, if we make the following definitions,

$$l = i\alpha(D/C)^{1/2} \quad \text{and} \quad m = i(BC - AD)/(CD)^{1/2}$$

proves, upon multiplication by $\alpha^{-\omega(\Gamma_0)}$, the recursion formula for the oriented polynomial given in Theorem 2.2(ii):

$$lP_G(\Gamma_+) + l^{-1}P_G(\Gamma_-) + mP_G(\Gamma_0) = 0\,.$$

Thus, modulo proving that the polynomial is well defined, one has a generalization of the oriented polynomial by means of the state model applied to oriented chimerical graphs. One may reverse the argument to prove that, assuming the existence of the oriented polynomial for oriented classical knots and links, [ ], one may define the chimerical graph polynomial by an averaging process growing out of the second formula. This will be the next step in the development of the theory.

Enumerate the vertices of the oriented chimerical graph, Γ, by $1, 2, 3, \ldots, k$ and let $\sigma_+(j)$ and $\sigma_-(j)$ denote operators on these vertices which replace the jth vertex by the $+1$ or -1 crossing, respectively, according to the standard convention in Figure 1.13. Let S denote the set of all functions, ε, from $\{1, 2, \ldots, k\}$ into $\{-1, +1\}$. These are called link states since the result of applying the operator $\sigma_{\varepsilon(j)}(j)$ to the jth chimerical vertex, for each the

vertices, results in an oriented knot or link, denoted $\varepsilon\Gamma$. Let $\varepsilon+$ and $\varepsilon-$ denote the number of $+1$ values and -1 values, respectively, taken on by the link state ε. The second formula, (3.3), implies that $[\alpha B/(BC-AD)][\alpha^{-1}\sigma_+]+[-\alpha^{-1}A/(BC-AD)][\alpha\sigma_-]=\chi$ and, as a consequence,

$$P_G(\Gamma)=\sum_{\varepsilon\in S}(\alpha B/(BC-AD))^{\varepsilon+}(-\alpha^{-1}A/(BC-AD))^{\varepsilon-}P(\varepsilon\Gamma)$$

where $P(\varepsilon\Gamma)$ denotes the oriented polynomial of the classical oriented knot or link, $\varepsilon\Gamma$.

One may use this formula to define P_G by verifying that the quantity so defined is unchanged under the generalized Reidemeister moves associated to the chimerical graphs. This and the verification of the evaluations giving the formulae for the graph evaluations give the first proof of Theorem 2.2. We shall first consider the question of invariance under the Reidemeister moves. The invariance under the classical Reidemeister moves follows directly from the existence of the oriented polynomial, [19], since the defining formula is local in character and the oriented polynomials satisfy these formulae. Invariance under the other two moves follows by means of an argument similar to that employed to show that the Jones polynomial is well defined when developed via the Kauffman state model, [17]. Specifically, one invokes the averaging at the crossing under consideration to create a classical knot or link projection within the region in question, employs invariance under the classical Reidemeister moves and reassembles the chimerical crossing following the move to achieve the result that the quantity associated to the result of the generalized move is the same as that of the initial situation.

First consider the type II chimerical Reidemester moves and their expansions, shown in Figure 3.6. There are two cases, depending on the relative orientation at the chimerical vertex with respect to the move. The equality of the two invariants is demonstrated by taking the weighted average expansion at each of the chimerical vertices, via the state model, and employing the invariance of the classical oriented knot and link invariants under the type II Reidemeister moves.

One employs the same technique for the type III chimerical Reidemeister moves, a typical cases of which is shown in Figure 3.7. Again the equality of the two invariants is demonstrated by taking the weighted expansion at each of the chimerical vertices and then applying the invariance of the classical oriented invariants under the type III Reidemeister moves.

We have now demonstrated the existence of a Laurent polynomial in variables l, m, A, B, C and D which satisfies the second part of the statement of Theorem 2.2, i.e. the classical recursion relation, and which is invariant under the classical and the chimerical Reidemeister moves. In addition, by its very definition, the invariant satisfies a version of the third portion of the theorem. In order to complete the demonstration of Theorem 2.2 we need only verify the formulae in the first and third parts while eliminating the undesired vari-

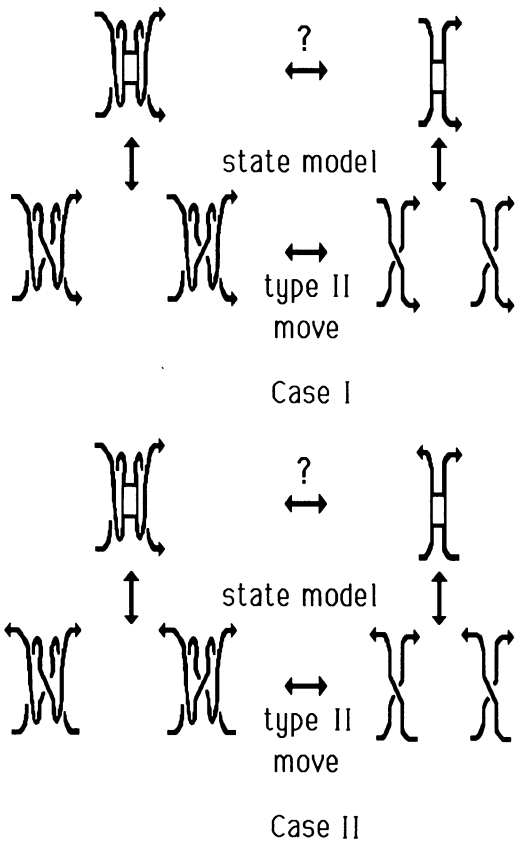

FIGURE 3.6. Type II chimerical Reidemeister moves

ables. Indeed, these relations are employed to determine the following algebraic substitutions which accomplish these purposes:

$$A = \frac{il^{3/2}m^{1/2}(lm - l^2 - 1)^{1/2}(lm + l^2 + 1)^{1/2}}{D(l^2+1)(\beta^2 lm + \gamma l^2 + \gamma)^{1/2}} + \frac{\beta(lm - l^2 - 1)(lm + l^2 + 1)}{D(l^2+1)(\beta^2 lm + \gamma l^2 + \gamma)},$$

$$B = \frac{D\beta lm}{l^2+1} - \frac{Dil^{1/2}m^{3/2}(\beta^2 lm + \gamma l^2 + \gamma)^{1/2}}{(l^2+1)(lm - l^2 - 1)^{1/2}(lm + l^2 + 1)^{1/2}},$$

$$C = \frac{(lm - l^2 - 1)(lm + l^2 + 1)}{Dlm(\beta^2 lm + \gamma l^2 + \gamma)}.$$

Because of the homogeneous role of the varaible "D" in the calculation, this variable does not appear in the final polynomial, nor does the introduced variable "α" play any role in the end result. This can be observed directly from

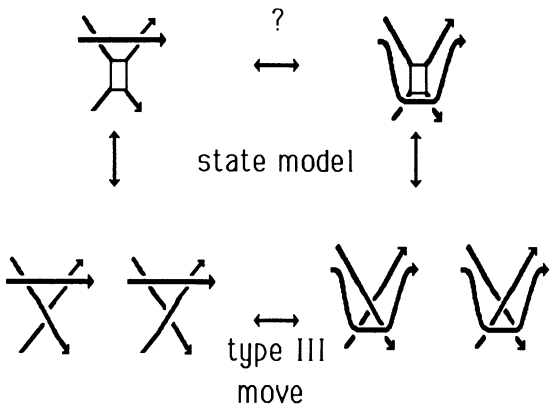

FIGURE 3.7. A type III chimerical Reidemeister move

the weighted average formula where the two factors are, respectively:

$$\begin{aligned}(-\alpha^{-1}A/(BC-AD)) &= \beta l^{-1}(l^{-1}+l)^{-1} + im^{1/2}(\beta^2 m + \gamma(l^{-1}+l))^{1/2} \\ &\quad \cdot (m^2 - (l^{-1}+l)^2)^{-1/2}(l^{-1}+l)^{-1} \\ &= [\beta l^{-1} + i(\beta^2 - \gamma\delta)^{1/2}(\delta^2-1)^{-1/2}](l^{-1}+l)^{-1},\end{aligned}$$

and

$$\begin{aligned}(\alpha B/(BC-AD)) &= \beta l(l^{-1}+l)^{-1} - im^{1/2}(\beta^2 m + \gamma(l^{-1}+l))^{1/2} \\ &\quad \cdot (m^2 - (l^{-1}+l)^2)^{-1/2}(l^{-1}+l)^{-1} \\ &= [\beta l - i(\beta^2 - \gamma\delta)^{1/2}(\delta^2-1)^{-1/2}](l^{-1}+l)^{-1}.\end{aligned}$$

We note that taking l to l^{-1} and taking complex conjugate exchanges the two terms thereby determining an involution on the polynomials corresonding to that associated to mirror reflection. The l to l^{-1} operation exchanges the real parts, the imaginary parts being exchanged by taking the complex conjugate. By changing the representation so as to visibly reflect the influence of δ, one has the following formulation of the averaging formula:

$$\begin{aligned}P_G(\Gamma_\chi) &= [\beta l - i(\beta^2 - \gamma\delta)^{1/2}(\delta^2-1)^{-1/2}](l^{-1}+l)^{-1}P_G(\Gamma_+) \\ &\quad + [\beta l^{-1} + i(\beta^2 - \gamma\delta)^{1/2}(\delta^2-1)^{-1/2}](l^{-1}+l)^{-1}P_G(\Gamma_-),\end{aligned}$$

proving part (iii) of Theorem 2.2. The proof of Theorem 2.2 is completed by application of the weighted averaging formula to the chimerical graphs β and γ to verify the normalizations indicated in part (i) of the theorem.

The first two parts of Proposition 2.2 are a consequence of the weighted average expansion for the chimerical graph polynomial and the application of the corresponding results for the oriented polynomial, i.e. one first employs the weighted average formula to calculate the invariant in terms of oriented polynomials of knots and links, uses the corresponding property for the oriented polynomial applied to each of the terms in the summation, and reassociates the

terms of the expansion to give the corresponding chimerical graph invariant formula.

4. Algebraic aspects of the state model for the oriented chimerical graph invariant

In addition to providing an extension of the oriented polynomial from the category of classical knots and links to the category of chimerical graphs the state model provides a completely independent method of defining the oriented polynomial, even in the category of classical knots and links.

Enumerate the crossings in a restricted generic presentation, Γ, of the chimerical graph by $\{1, 2, \dots, n\}$ and let σ denote a function from this set to $\{0, 1\}$. We shall interpret σ as an operator acting on the chimerical graph presentation which replaces the jth crossing by the oriented "0" connection, if $\sigma(j) = 0$, and by the 4-valent graph "χ" connection, if $\sigma(j) = 1$. The set of such functions, called states, will be denoted by S. We also define auxiliary functions as follows: $\varepsilon(\sigma, 0[\chi], +[-])$ is the number of times $\sigma(j)$ is 0 [respectively, χ] and j is a positive [respectively, negative] crossing. We, finally, define the state expansion of Γ as follows:

$$\alpha^{-\omega(\Gamma)} \sum_{\sigma \in S} A^{\varepsilon(\sigma, 0, +)} B^{\varepsilon(\sigma, 0, -)} C^{\varepsilon(\sigma, \chi, +)} D^{\varepsilon(\sigma, \chi, -)} [\![\sigma\Gamma]\!]. \tag{4.1}$$

Recall that α was defined to be $A\delta + C\beta$ and that α^{-1} was defined to be $B\delta + D\beta$, where δ and β denoted the variables associated to, respectively, the distant union of two planar circles and the connected 4-valent planar graph with one vertex, β. Thus, the state expansion of a chimerical graph can be considered as an element of the free module of S^2 equivalence classes of oriented 4-valent planar graphs, denoted $[\![\sigma\Gamma]\!]$, over the field of rational functions in variables A, B, C, D, δ, and β. In order to use this state expansion to determine chimerical isotopy invariants for chimerical graphs we shall introduce relations, i.e. take a quotient module, which reflect the fundamental relations that are imposed by the connected sum axiom and the classical and chimerical Reidemeister moves. The connected sum axiom defines a multiplication on the equivalence classes of the 4-valent planar graphs. In order that this multiplication be well-defined, associative, and commutative, one must impose the relations identifying all the possible connected sum forms involving the same irreducible graphs. The resulting quotient then defines an algebra over our field.

Because we are working with oriented chimerical graphs, the number of cases that must be considered is somewhat larger than in the unoriented case. In order to simplify the algebraic structure, defined above, we have identified the vertices which arise during the construction of the associated state with those which occur by virtue of the chimerical graph structure. As a consequence, the relations associated with the classical Reidemeister moves are sufficient to imply those required for the chimerical Reidemeister moves. Recall that the connected sum axiom implies that $[\![\Gamma_1 \# \Gamma_2]\!] = [\![\Gamma_1]\!][\![\Gamma_2]\!]$ and that, in addition, one has the

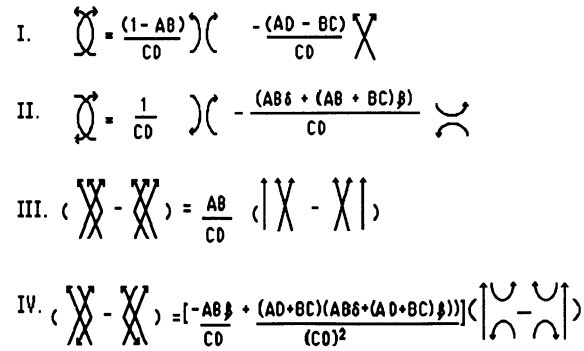

FIGURE 4.2. The oriented chimerical graph relations

relation $[\![\Gamma_1 \cup \Gamma_2]\!] = \delta[\![\Gamma_1]\!][\![\Gamma_2]\!]$. For the elementary chimerical graphs, $\mathbf{1}$, $\boldsymbol{\delta}$, $\boldsymbol{\beta}$, and γ, we shall employ the bold face notation for the equivalence class as well as the actual representation of the graph.

We consider the type I Reidemeister moves. The relation implied by these moves is the requirement that $\alpha \cdot \alpha^{-1} = 1$, i.e., one of the relations given earlier,

$$(A\boldsymbol{\delta} + C\boldsymbol{\beta})(B\boldsymbol{\delta} + D\boldsymbol{\beta}) = \mathbf{1}$$

where $\mathbf{1}$ denotes the equivalence class of the trivial knot serving as a multiplicative unit in the algebra.

For the type II Reidemeister move, there are two cases, depending upon the relative orientations of the strands. The identifying the state expansions associated to the local structures of the classical type II moves give the two expressions at the top of Figure 4.2. In order to illustrate the development of these formula we shall calculate one of the realtions required by one of the chimerical type II relations to show that it is implied by the first relation in 4.2.

In Figure 4.3 we show the expansion of the left-hand side of the chimerical type II Reidemeister move in terms of oriented planar 4-valent graphs. In Figure 4.4 we show the result of the application of the relation and the resulting algebraic simplification which demonstrates the invariance under the chimerical graph move.

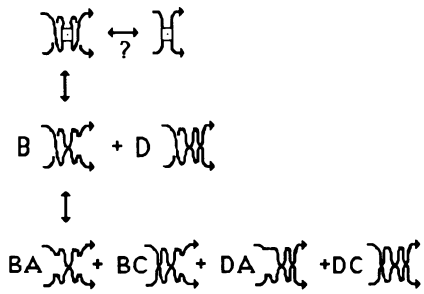

FIGURE 4.3.

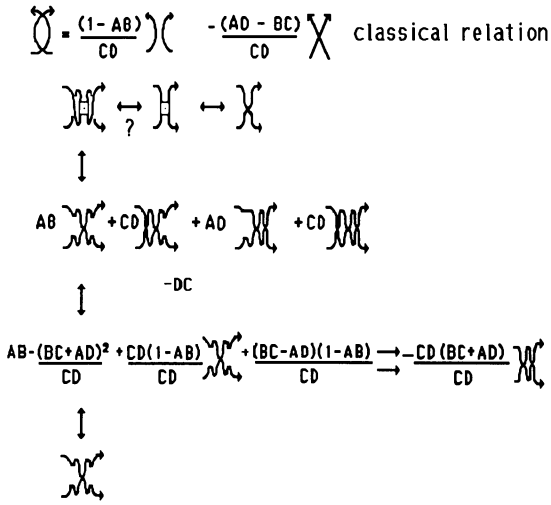

FIGURE 4.4

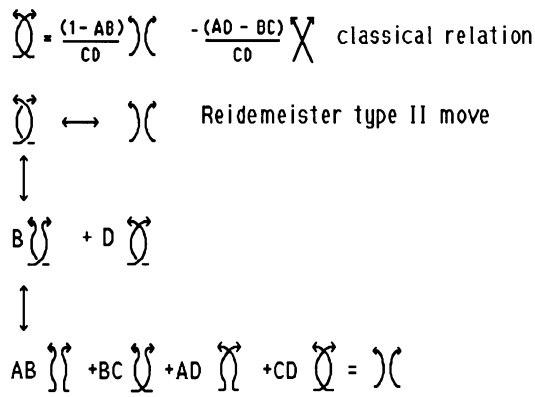

FIGURE 4.5

Invariance under the remaining chimerical moves is proved in corresponding ways from the relations (one needs to consider all possible orientation configurations). The graph relations are developed from the requirement of invariance of the classical Reidemeister moves by considering the state expansion an example of which, one of the type II Reidemeister moves, is shown in Figure 4.5. Kauffman and Vogel [18], have developed direct analogs of the relations in Figure 4.2 from the existence of the knot invariant polynomials. These relations are not independent but are given in this form because this is the manner in which they are encountered in the reduction of the planar graphs. Specifically, one can show that the fourth relation is a consequence of the second and third relations, that the first is a consequence of the second and the fourth and, as a consequence, the following proposition:

Proposition 4.1. *Relations* II *and* III *in Figure* 4.2 *generate the chimerical graph relations required for algebraic invariance under the chimerical and classical Reidemeister moves.*

Question. Do relations I and III in Figure 4.2 generate the chimerical graph relations required for the invariance under chimerical and classical Reidemeister moves?

That these relations suffice to reduce any oriented 4-valent graph in S^2 to an expression involving only the algebraic variables is the purpose of the next proposition. In order to achieve the definition of the chimerical graph polynomial by means of the state model one must be able to reduce the graphs and one must know that the reduction to an algebraic expression is unique, i.e. there are not further relations.

Theorem 4.2. *Every oriented* 4*-valent graph in* S^2 *can be reduced to a unique expression in terms of the algebraic variables.*

Sketch of the proof. Every oriented planar 4-valent graph is the projection of an oriented knot or link in standard ascending position, to use the language of [19], which is equivalent to a trivial presentation, i.e. without crossings, by a sequence of Reidemeister moves which never increases the number of crossings in the projection at any stage, cf. [19]. These Reidemeister moves have associated relations in the chimerical graph algebra which, paralleling the Reidemeister moves, never increase the number of vertices in the graph. This is accomplished by means of the selection of an "innermost generalized bigon" in the graph. An "innermost generalized bigon" is defined by the union of two portions of circuits meeting at precisely two vertices and which contain no other such bigons in one of its complementary regions. As a consequence, within the bigon, portions of circuits never cross more than once, otherwise an interior bigon would be created thereby. An example is shown in Figure 4.6. The utility of these generalized bigons was shown in [19] where a sequence of triangle transformations (these never increase the number of crossings) was described which created a bigon (with no vertices other than the two defining vertices) which could be used to reduce the number of crossings or, in the present case, reduce the number of vertices in the graph by means of a type II Reidemeister move. In our case we use type I or type II relations, according to the required orientation. We note that the analog of a type I Reidemeister move is accomplished by means of the connected sum axiom via multiplication by β or its inverse.

In order to show that any two such calculations, by means of a sequence of applications of the chimerical graph relations, gives the same final algebraic result one must consider the relationship between alternative calculation schemes. One can give a proof of this equality by means of a process analogous to that employed to show that the oriented polynomial is well defined, [19], as follows: Let Γ_n denote the collection of S^2 equivalence classes of oriented 4-valent graphs having no more than n vertices and assume, by induction, that each such graph has a representation in terms of the elementary variables that is

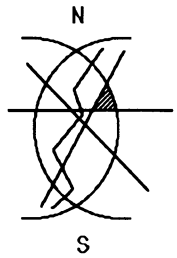

FIGURE 4.6. The generalized bigon

unchanged under any sequence of transformations, by means of the relations, remaining within Γ_n. The induction begins with Γ_0, in which there are no vertices and therefore every element has the expression $\delta^{c(\Gamma)-1}$, where $c(\Gamma)$ denotes the number of (chimerical) circuits in Γ.

The proof is then completed by showing that every $\Gamma \in \Gamma_{n+1}$ has a representation in terms of the elementary variables that is unchanged under any sequence of transformations remaining within Γ_{n+1}. These transformations are: the elimination of a monogon by multiplication by β, via the connected sum axiom; the elimination of a bigon, via relations I or II; and the application of relations III or IV. The first two immediately reduce the number of vertices while the latter preserve the number of crossings in one term and reduce the number in the remaining terms of the relation. By enumeration of the possible cases and direct calculation, one shows that if one is presented with two transformations from among the first two types applied to $\Gamma \in \Gamma_{n+1}$, then the resulting calculation gives the same algebraic expression. The basis of the argument is the observation that if the supports of the transformations, i.e. the regions within the circles which define the transformations and the associated relations, are disjoint, then the transformations commute and one can apply them in either order with the same resulting expression. One then continues the reduction algorithm to show that the final results are equal. The exceptional cases in which the supports are not disjoint are enumerated and calculated by means of the geometric relationships which arise in each of the cases to show that, there too, the transformations result in the same algebraic expression.

The same principle applies to the situation of the transformations of the latter types, i.e. the case of disjoint support allows one to apply the transformations in either order to achieve the same expression to which one may apply the reduction algorithm to achieve the same ultimate calculation. As above, one must enumerate all the cases in which the supports of the transformations are not disjoint and exploit the geometric relationships that are associated to each of the situations.

These enumerations and calculations are facilitated by the observation that if there is a generalized bigon whose support is disjoint from the union of the supports of the two transformations in question, the reduction algorithm

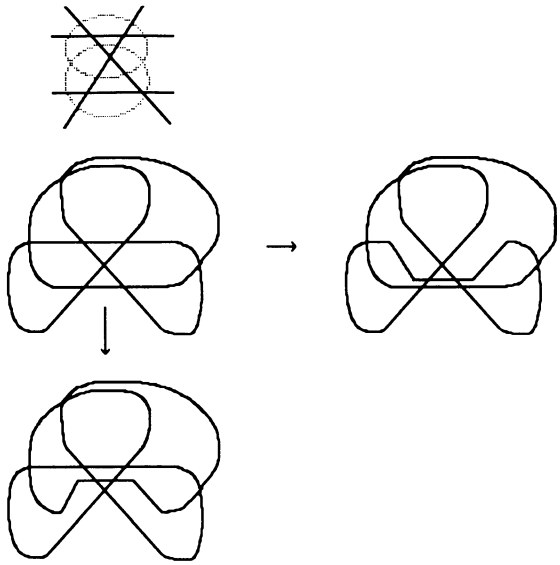

FIGURE 4.7

can be applied there to express the two calculations in question by means of equations within Γ_n where, by induction, the equality of the final expressions is already demonstrated. The impact of this observation is very significant as it implies an important reduction in the number of exceptional cases that must be considered because the complement of the supports of the transformations in question cannot contain any monogons or (generalized) bigons. In order to give some concrete indication of the structure of our method of argument, we outline one of the cases that arises in the argument.

Consider the case of two triangle moves whose supports, indicated by the dashed circles, intersect exactly as a neighborhood of one of the vertices, as shown in Figure 4.7. There are eight possible orientations that must be considered, up to S^2 equivalence. For each of these 8 orientations, there are a small number of possible connections such that there are no monogons nor (generalized) bigons in the complement. This condition implies that each connecting segment in the complement is embedded and intersects each of the other segments no more than once, thereby greatly restricting the number of possible cases. Nevertheless, there are 24 cases to be considered for each of the possible choices of orientations, one of which is shown in Figure 4.7. The actual number of cases can be reduced somewhat by the use of symmetries and global changes of orientation.

The result of the two triangle moves in question are also shown in Figure 4.7. Note that, in one case, there is a distant bigon move that can be employed to show that the result of applying that relation does not change the result of the calculation while in the other case, there is a distant trigon move (disjoint from the second trigon and the bigon as well) which can be applied to create a bigon

and therefore complete the calculation without changing the end result by virtue of the previous discussion concerning the relationship between applications of distant moves.

We complete the sketch of the proof of the theorem with the observation that any two calculation schemes involve only a finite number of graphs and, therefore, the entire sequences of transformations lie within some Γ_N, for some N, and therefore give the same final result.

This proof illustrates one of the difficulties associated with these graph polynomials, i.e. the complexity of the resulting algebraic expression and the associated problem of extracting combinatorially relevant information. The fundamental graphs, $\boldsymbol{\delta}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, as well as the additional elementary example, $\boldsymbol{\varepsilon}$, are shown in Figure 2.3.

By means of the graph relations one can calculate the representation of these graphs in terms of the algebraic and graphical variables. Recall that we have set $P_G(\boldsymbol{\delta}) = -(l^{-1} + l)m^{-1}$ and have defined $\beta \equiv P_G(\boldsymbol{\beta})$ and $\gamma \equiv P_G(\boldsymbol{\gamma})$. Application of the substitution of the variables A, B, C, and D according to the formulae given earlier gives

$$\begin{aligned} P_G(\boldsymbol{\varepsilon}) = & -m\beta(2m\beta^2 + 3\gamma(l^{-1} + l))(l^{-1} + l)^{-2} \\ & + im^{3/2}(l^{-1} - l)(m\beta^2 + \gamma(l^{-1} + l))^{3/2}(l^{-1} + l)^{-2} \\ & \times (m^2 - (l^{-1} + l)^2)^{-1/2}. \end{aligned}$$

Note that this expression in invariant under the change l goes to l^{-1} and complex conjugation. This is a reflection of the fact that $\boldsymbol{\varepsilon}$ is a planar graph.

5. A STATE MODEL FOR THE ALEXANDER-CONWAY, JONES, AND ORIENTED LINK POLYNOMIALS

In this section we shall discuss some of the algebraic implications of the reduction of the oriented chimerical graph state model developed in the previous section to the special cases of the Alexander-Conway, Jones, and oriented link polynomials. The results imply important reductions in structure which have corresponding implications for the combinatorial structure of the polynomial invariants of the classical knots and links.

Recall that if we set $l = i\alpha(D/C)^{1/2} = i$ and $m = i(BC - AD)/(CD)^{1/2} = -iz$, the result is the Conway polynomial and if, in addition, we set $z = (t^{-1/2} - t^{1/2})$, the result is the Alexander polynomial. We shall, therefore, first consider the state model associated to the Conway polynomial, by means of the first substitution, i.e. $l = i$. This is achieved by the following substitutions:

$$\begin{aligned} A &= -(\gamma\beta^{-2} - im)/(2\beta D) = (z - \gamma\beta^{-2})/(2\beta D), \\ B &= -(\gamma\beta^{-2} + im)\beta D/2 = (-z - \gamma\beta^{-2})\beta D/2, \\ C &= 1/(\beta^2 D) \end{aligned}$$

which imply that $\alpha = 1/(\beta D)$ and $\alpha^{-1} = \beta D$.

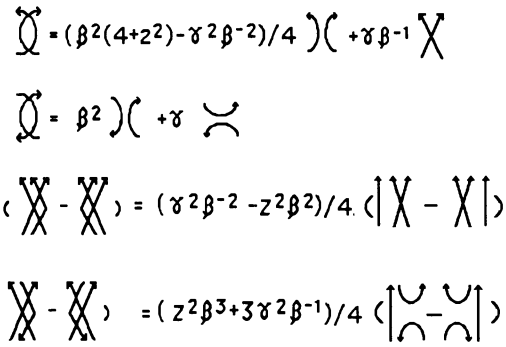

FIGURE 5.1. The general Alexander/Conway relations

The general relations are shown in Figure 5.1. If one is only interested in the applications of the state model to the case of classical oriented knots and links, one may make the following substitutions for the graphical variables: $\gamma = 0$ and $\beta = 2$. These choices are only one among the range of the possibilities and were chosen so as to reduce the complexity of the relations that arise in the graphical relations. The resulting relations are shown in Figure 5.2.

The expectation function or state formula for the Conway polynomial resulting from the state model expansion applied to an n crossing presentation of an oriented knot or link, L, is given by the following formula:

$$\alpha^{-\omega(L)} \sum_{\sigma \in S} A^{\varepsilon(\sigma,0,+)} B^{\varepsilon(\sigma,0,-)} C^{\varepsilon(\sigma,\chi,+)} D^{\varepsilon(\sigma,\chi,-)} [\![\sigma L]\!]$$

which can be expressed in the following form

$$2^{-n} \sum_{\sigma \in S} (im)^{\varepsilon(\sigma,0,+)} (-im)^{\varepsilon(\sigma,0,-)} [\![\sigma L]\!]$$

giving, upon substitution for m

$$2^{-n} \sum_{\sigma \in S} (z)^{\varepsilon(\sigma,0,+)} (-z)^{\varepsilon(\sigma,0,-)} [\![\sigma L]\!].$$

From this formulation it is easy to prove the following facts:

(0) The recursion: $\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$.

(i) The only terms contributing to the state expectation function are those having an odd number of (chimerical) circuits, i.e. those having an even number of circuits evaluate to 0. A proof can easily be given by induction on the number of chimerical circuits and the number of crossings in the graphical state. The case of 0 crossings follows from the definition of δ and requires that only the single component cases makes a contribution. All the other cases reduce to this case. In the case of induction, the elimination of monogons and bigons corresponds to multiplication by nonzero factors, 2, 4, or $4 + z^2$ and does not change the number of chimerical circuits in the graph. Thus one need only consider the effect of the application of the trigon relations. Consideration of

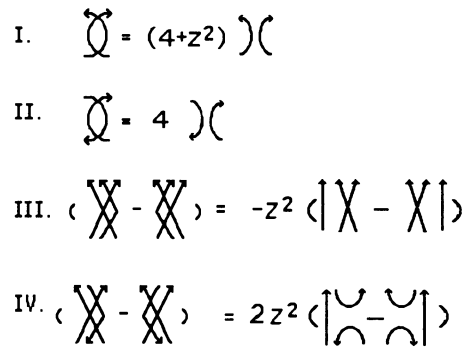

FIGURE 5.2. An evaluation of the Conway polynomial relations

all the possible cases shows that if the trigon involves an odd (even) number of circuits then the difference terms also involve an odd (even) number of circuits and, by induction, are thereby equal to 0 in the even case. Thus, the value is unchanged by the application of the trigon relations.

(ii) The constant term of the Conway polynomial of a knot is equal to one and all powers of the variable appearing in the polynomial are even, since the only way to insure that there is precisely one circuit in the graph resulting from the state associated to a knot is to have an even number of "0" vertex states in the expansion, cf. (i). In general, the power of z which appear are odd (even) if the number of components is even (odd).

(iii) The values given the specific oriented 4-valent planar graphs play a fundamental role in the state model. We note that $\nabla(\boldsymbol{\delta}) = 0$, $\nabla(\boldsymbol{\beta}) = 2$, $\nabla(\boldsymbol{\gamma}) = 0$, $\nabla(\boldsymbol{\varepsilon}) = 8 + 2z^2$, and $\nabla(\textbf{borromean}) = 4z^4$. The graphs associated to the standard projections the simplest knots have the following values:

$$\nabla(\mathbf{3}_1) = 8 + 2z^2, \qquad \nabla(\mathbf{4}_1) = 14, \qquad \nabla(\mathbf{5}_1) = 32 + 16z^2 + 2z^4,$$
$$\nabla(\mathbf{5}_2) = 32 + 8z^2, \qquad \nabla(\mathbf{6}_1) = 64, \qquad \nabla(\mathbf{6}_2) = 64 + 16z^2 \text{ and},$$
$$\nabla(\mathbf{6}_3) = 64 + 32z^2 + 4z^4.$$

(iv) Recall, that by the substitution, $z = (t^{-1/2} - t^{1/2})$, one a achieves a state model for the Alexander polynomial.

Another evaluation, rather different from the previous, gives an "interaction" model for the Alexander/Conway polynomial. Specifically, if we define $\gamma = z\beta^2$ we find the relations shown in Figure 5.3.

The expectation function or state formula for the Conway polynomial resulting from this "Interaction" state model expansion applied to an n crossing presentation of an oriented knot or link, L, is given by the following formula, in which we have taken $\beta = 1$:

$$\sum_{\sigma \in S} (0)^{\varepsilon(\sigma,0,+)} (-z)^{\varepsilon(\sigma,0,-)} [\![\sigma L]\!],$$

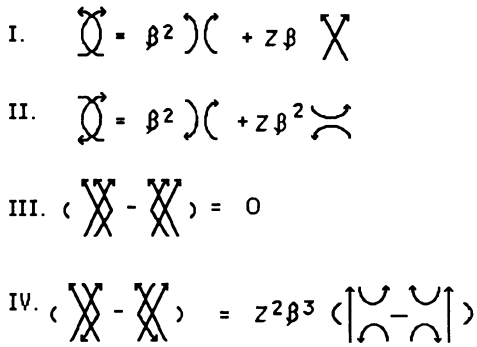

FIGURE 5.3. An Interaction model for the Alexander/Conway polynomial

whose nonzero terms in the summation arise only from the operations removing a negative crossing or the creation of vertices in the chimerical graph whose value is determined by the relations in Figure 5.3.

Rather than limiting the initial discussion of the Jones polynomial model to its specific evaluations we shall provide a development which gives an equivalent formulation of the oriented polynomial and which is well adapted to the Jones polynomial context. Specifically, we consider the change of variables $l = it^{-1}$ and $m = -iw(t^{-1} - t)$ and calculate as follows:

$$A = t^{-1}w^{1/2}(1-w^2)^{1/2}(\gamma - \beta^2 w)^{-1/2}D^{-1} - \beta(1-w^2)(\gamma - \beta^2 w)^{-1}D^{-1},$$
$$B = -\beta w D + t w^{3/2}(1-w^2)^{-1/2}(\gamma - \beta^2 w)^{1/2}D,$$

and

$$C = w^{-1}(1-w^2)(\gamma - \beta^2 w)^{-1}D^{-1}$$

implying that

$$\alpha = t^{-1}w^{-1/2}(1-w^2)^{1/2}(\gamma - \beta^2 w)^{-1/2}D^{-1}.$$

For example, we could invoke the additional relations (among other possibilities)

$$1 = (w^{-1} - w)^{1/2}(\gamma - \beta^2 w)^{-1/2}, \quad \text{and} \quad 1 = w^2(1 - \beta t^{-1})(1 - \beta t)$$

implying that

$$\delta = w^{-1},$$
$$\beta = [(t^{-1} + t) \pm w^{-1}(w^2(t^{-1} - t)^2 + 4)^{1/2}]/2, \quad \text{and}$$
$$\gamma = w^{-1} + (\beta^2 - 1)w.$$

Note that there is a choice of root to be made in the evaluation of β. Each choice has somewhat different implications for the final form of the state model for the associated version of the oriented polynomial. The state formula for

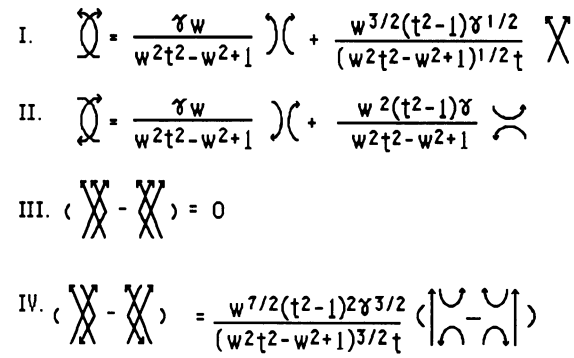

FIGURE 5.4. An Interaction model for the oriented polynomial

the oriented polynomial of an n crossing presentation of an oriented link, with respect to this model, is

$$w^n t^{\omega(L)} \sum_{\sigma \in S} (t^{-1} - \beta)^{\varepsilon(\sigma,0,+)} (t - \beta)^{\varepsilon(\sigma,0,-)} [\![\sigma L]\!].$$

Another evaluation gives a "Interaction" model for the oriented polynomial. Specifically, if we define $\beta = \gamma^{1/2} w^{1/2} t (1 - w^2 + w^2 t^2)^{-1/2}$ we find the relations shown in Figure 5.4.

The expectation function or state formula for the oriented polynomial resulting from this "Interaction" state model expansion applied to an n crossing presentation of an oriented knot or link, L, is given by the following formula, in which we have taken $\gamma = 1$:

$$\begin{aligned}&((w^{-1} - w + wt^2)^{1/2})^n t^{\omega(L)} \\ &\quad \times \sum [(t^{-1} - t) w^2 / (w^{-1} - w + wt^2)^{1/2}]^{\varepsilon(\sigma,0,+)} [0]^{\varepsilon(\sigma,0,-)} [\![\sigma L]\!]\end{aligned}$$

whose nonzero terms in the summation arise only from the operations removing a positive crossing or the creation of vertices in the chimerical graph whose value is determined by the relations in Figure 5.4.

Returning to the general model for the general of the Jones polynomial one has $w = -(t^{-1/2} + t^{1/2})^{-1}$ giving the evaluations:

$$\begin{aligned}\delta &= -(t^{-1/2} + t^{1/2}), \\ \beta &= [(t^{-1} + t) \pm ((t^{-1} - t)^2 + 4(t^{-1/2} + t^{1/2})^2)^{1/2}]/2 \\ &= [(t^{-1} + t) \pm ((t^{-1} + 2 + t)^2)^{1/2}]/2 \\ &= -1 \text{ or } t^{-1} + 1 + t,\end{aligned}$$

and, therefore,

$$\gamma = -(t^{-1/2} - t^{1/2}) \quad \text{or} \quad -(t^{-3/2} + 2t^{-1/2} + 2t^{1/2} + t^{3/2}).$$

The associated state formula expansion associated to this model for the Jones

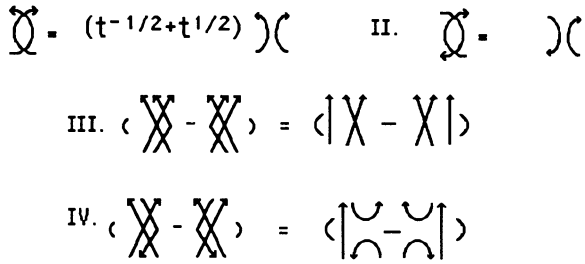

FIGURE 5.5. Jones polynomial graph relations

polynomial is

$$[-(t^{-1/2}+t^{1/2})^{-1}]^n t^{\omega(L)} \sum_{\sigma\in S} (t^{-1}+1)^{\varepsilon(\sigma,0,+)}(1+t)^{\varepsilon(\sigma,0,-)} [\![\sigma L]\!]$$

which can be rewritten in the form

$$[-1]^n t^{\omega(L)} \sum_{\sigma\in S} t^{-\varepsilon(\sigma,0,+)/2} t^{\varepsilon(\sigma,0,-)/2} (t^{-1/2}+t^{1/2})^{\varepsilon(\sigma,\chi)} [\![\sigma L]\!],$$

where $\varepsilon(\sigma,\chi)$ denotes the number of χ states of σ, both $+$ and $-$. These choices have direct implications for the chimerical graph algebra by means of the relations show in Figure 5.5. The relations and the choices above give the following values:

$$V(\boldsymbol{\delta}) = -(t^{-1/2}+t^{1/2}), \quad V(\boldsymbol{\beta}) = -1,$$
$$V(\boldsymbol{\gamma}) = (t^{-1/2}+t^{1/2}), \quad V(\boldsymbol{\varepsilon}) = (t^{-1/2}+t^{1/2})^2,$$

and

$$V(\textbf{borromean}) = 2(t^{-1/2}+t^{1/2})^2+1.$$

We conclude this section with the presentation of a "Interaction" model for the Jones polynomial by means of the substitution $w = -(t^{-1/2}+t^{1/2})^{-1}$ in the model for the oriented polynomial in which we employ the evaluation $\gamma = -t^{-1}-t$. The associated state formula expansion associated to this model for the Jones polynomial is

$$(-t^{1/4})^n t^{\omega(L)} \sum_{\sigma\in S} (t^{-3/4}-t^{1/4})^{\varepsilon(\sigma,0,+)}(0)^{\varepsilon(\sigma,0,-)} [\![\sigma L]\!].$$

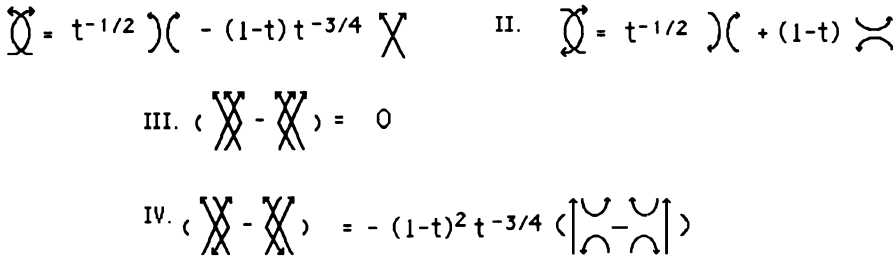

FIGURE 5.6. An Interaction model for the Jones polynomial

6. Semioriented algebraic invariants for chimerical graphs

In this section we shall present an extension of the semioriented polynomial invariant, defined for classical knots and links, to the case of oriented chimerical graphs via the averaging method which was employed for the oriented polynomials. A related approach is presented in Kauffman and Vogel, [18]. As in the previous case, the general averaging approach has the advantage of not requiring a proof, from first principles, that these classical invariants are well defined and is, therefore, the quickest route to the development of the semioriented chimerical graph invariants and their properties. We shall show that this new semioriented polynomial, for oriented chimerical 4-valent graphs, satisfies a recursion relation which is exactly the same as that which occurs for the semioriented polynomial associated to oriented knots and links. A second recursion relation allows one to express the invariant of a specific placement of an oriented chimerical 4-valent graph in terms of the invariants of oriented knots and links.

The recursion formulae for the semioriented invariants requires the development of some additional notation. Recall that the number of strands, or components, in a link L is denoted by $c(L)$. Here we extend this definition to the number of circuits in the chimerical graph. In addition, we define $\langle X, Y\rangle$, X and Y mutually disjoint oriented circuits (links, in the classical case), to be the algebraic linking numbers of X and Y. The orientation conventions that are to be employed in the recursive calculation, according as the crossing in L_+ involves the same or distinct strands of the link, are recalled in Figure 6.1.

Theorem 6.1 (Semioriented invariants for chimerical graphs). *There is a unique way of associating to each oriented chimerical graph in 3-dimensional Euclidean space, Γ, an algebraic function, $F_G(\Gamma)$, in the algebraic variables a and x, and elementary oriented chimerical graph variables β and γ, satisfying the connected sum axiom, such that spatially equivalent oriented chimerical graphs have the same associated algebraic function and such that*

(i) $F_G(U) = 1$, $F_G(\boldsymbol{\delta}) = -\delta \equiv (a^{-1} + a)x^{-1} - 1$, $F_G(\boldsymbol{\beta}) = \beta$, $F_G(\boldsymbol{\gamma}) = -\gamma$,

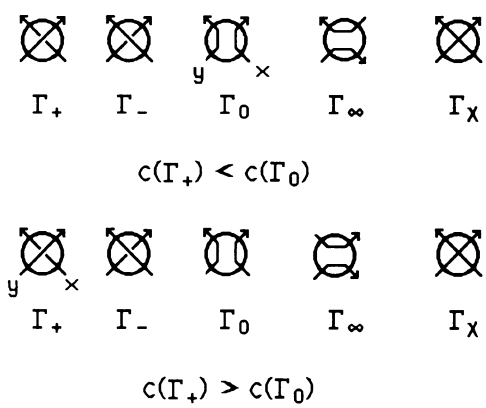

FIGURE 6.1.

and

(ii) *if* Γ_+, Γ_-, Γ_0, *and* Γ_∞ *are planar presentations of oriented chimerical graphs in each of which there are identified small circular regions containing either a single crossing or, in the last cases, no crossing at all, and such that outside these small circular regions shown in Figure* 6.1, *the planar presentations are exactly the same, then the semioriented polynomial satisfies one of the formulae, according as the crossing in* Γ_+ *involves the same or distinct components*:

(I) *If* $c(\Gamma_+) < c(\Gamma_0)$, *let* $\lambda = \langle X, \Gamma_0 - X\rangle$ *and* $aF_G(\Gamma_+) + a^{-1}F_G(\Gamma_-) = x[F_G(\Gamma_0) + (ia)^{-4\lambda}F_G(\Gamma_\infty)]$.

(II) *If* $c(\Gamma_+) > c(\Gamma_0)$, *let* $\mu = \langle X, \Gamma_+ - X\rangle$ *and* $aF_G(\Gamma_+) + a^{-1}F_G(\Gamma_-) = x[F_G(\Gamma_0) - (ia)^{-4\mu+2}F_G(\Gamma_\infty)]$.

(iii) *if* Γ_+, Γ_-, Γ_0 *and,* Γ_χ *are any four oriented chimerical graphs presentations that are identical except near one point where they are as in Figure* 6.1, *then*

$$F_G(\Gamma_\chi) = \rho(+)F_G(\Gamma_+) + \rho(-)F_G(\Gamma_-) + \rho(0)F_G(\Gamma_0),$$

where

$$\rho(+) \equiv [\alpha A/(A-B)C] = \frac{a^{-1}\beta}{x(1+\delta)} - \frac{ia^{-1}(x-a^{-1})(2\beta^2 - \gamma(\delta+1))^{1/2}}{x(1+\delta)(\delta-1)^{1/2}(x^2-\delta-2)^{1/2}},$$

$$\rho(-) \equiv [-\alpha^{-1}B/(A-B)C] = \frac{a\beta}{x(1+\delta)} + \frac{ia(x-a)(2\beta^2 - \gamma(\delta+1))^{1/2}}{x(1+\delta)(\delta-1)^{1/2}(x^2-\delta-2)^{1/2}},$$

$$\rho(0) \equiv [(A+B)/C] = \frac{-2\beta}{(1+\delta)} - \frac{i(a^{-1}-a)(2\beta^2 - \gamma(\delta+1))^{1/2}}{(1+\delta)(\delta-1)^{1/2}(x^2-\delta-2)^{1/2}}.$$

As in the development of the oriented polynomial for chimerical graphs, the first recursion relation derives from the goal of providing an extension of the semioriented polynomial invariant while the second recursion relation allows for the expression of the influence of the chimerical graph vertices in terms of the algebraic variables and the two graph variables, β and γ. There is an important difference between the form of the first recursion relation for classical knots and links and that given in the theorem for the extension to chimerical graphs. This is the occurrence of the factor $i = \sqrt{-1}$ in the "∞" term of the recursion. This factor derives from the fact that the linking number of two chimerical circuits is a multiple of $1/2$ rather than being an integer as is the case for the classical knots and links. Note that, as in the case of the knot and link invariants, δ can be expressed in terms of the algebraic variables. It is this later recursion relation which is the essential content of the theorem. The proof of the theorem is analogous to that given for the case of the oriented polynomial in §3.

The precise relationships determined by the requirement of invariance under spatial movements of the algebraic function to be associated to the chimerical graph follow the pattern established above. As a first step towards this result one

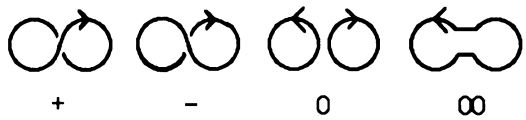

FIGURE 6.2. Determination of δ

may invoke only the first recursion relation in order to calculate the invariant in terms of elementary oriented chimerical graphs, i.e. one does not attempt to reduce the number of vertices of the graph but only simplify, as much as possible, the spatial placement of the graph by using crossing changes in the manner employed for classical knots and links.

In addition to using the theorem to give the recursive formula for calculation one also employs it to determine the value of δ, the invariant associated to the distant union of two unknotted circles shown as the "0" state in Figure 6.2.

The recursion identity gives the equation $a + a^{-1} = x[-\delta + 1]$ since the "$+$", "$-$", and ∞ cases are presentations of the trivial knot. Thus $\delta = -(a^{-1} + a)x^{-1} + 1$.

As with the oriented polynomial, the elementary consequences of the geometric theory of these invariants are simple applications of the recursion relations given in the theorem.

Proposition 6.2. *Suppose that Γ_1 and Γ_2 are two chimerical graphs in S^3, then*

(i) $F_G(\Gamma_1 \# \Gamma_2) = F_G(\Gamma_1)F_G(\Gamma_2)$, *where $\Gamma_1 \# \Gamma_2$ denotes any "connected sum" of Γ_1 and Γ_2.*

(ii) $F_G(\Gamma_1 \cup \Gamma_2) = \delta F_G(\Gamma_1)F_G(\Gamma_2)$, *where $\Gamma_1 \cup \Gamma_2$ is the "distant union" of Γ_1 and Γ.*

(iii) $\overline{F}_G(\Gamma_1)(a, x) = F_G(\overline{\Gamma}_1)(a, x)$, *where $\overline{\Gamma}_1$ is the mirror image of Γ_1, i.e. reverses all the crossings in a generic projection of Γ_1, and the conjugation in the algebra takes a to a^{-1}, i to i^{-1}, and leaves x unchanged.*

(iv) *If Γ^* denotes the chimerical graph Γ in which the orientation of the circuit X has been reversed, then $F_G(\Gamma^*) = (ia)^{4\langle X, \Gamma - X\rangle} F_G(\Gamma)$.*

The last property is used in the development of stereotopological indices for graphs since the change in the algebraic function of the variables often allows one to distinguish one placement of a chimerical graph in space from its mirror image in a way analogous to which the linking number and the oriented polynomial were employed earlier. The occurrence of the factor "i" is a further reflection of the fractional nature of the linking number for chimerical circuits. To illustrate the use of the recursion formula and these associated properties, we shall describe how the calculational methods would apply to the graphs, $\Gamma_{1/2}$ and $\Gamma_{-1/2}$, shown in Figure 2.6. First one can apply Theorem 6.1(ii) to show that

$$\begin{aligned} aF_G(\Gamma_{1/2}) + a^{-1}F_G(\Gamma_{-1/2}) &= x[F_G(\Gamma_0) - a^{-4(1/2)+2}F_G(\Gamma_\infty)] \\ &= x[F_G(\Gamma_0) - F_G(\Gamma_\infty)] \end{aligned}$$

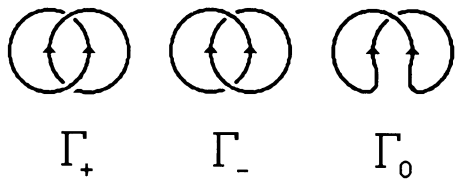

FIGURE 6.3

since $c(\Gamma_+) > c(\Gamma_0)$ and $\mu = \langle X, \Gamma_+ - X\rangle = 1/2$. Since Γ_0 and Γ_∞ are both β, this shows that

$$F_G(\Gamma_{-1/2}) = -a^2 F_G(\Gamma_{1/2})$$

as is implied by Theorem 6.2(iv). Thus, it is sufficient to calculate $F_G(\Gamma_{1/2})$, which can be accomplished by an application of Theorem 6.1(iii) as follows:

$$F_G(\Gamma_{1/2}) = \rho(+)F_G(\Gamma_+) + \rho(-)F_G(\Gamma_-) + \rho(0)F_G(\Gamma_0),$$

where Γ_+, Γ_-, and Γ_0 are shown in Figure 6.3. Since these are classical links, the polynomial invariant of the Γ_+ can be computed by recursion via part (ii) to find that $F_G(\Gamma_+) = -(a^{-3} + a^{-1})x^{-1} + a^{-2} + (a^{-3} + a^{-1})x$, the second is equivalent to the unlink of two components and gives $F_G(\delta) = (a^{-1}+a)x^{-1} - 1$, and the third is the trivial knot, contributing 1. Substitution in the formula gives

$$\begin{aligned} F_G(\Gamma_{1/2}) &= \rho(+)[-(a^{-3} + a^{-1})x^{-1} + a^{-2} + (a^{-3} + a^{-1})x] \\ &\quad + \rho(-)[(a^{-1} + a)x^{-1} - 1] + \rho(0) \\ &= [(a^{-2} - 1)\beta + ia^{-1}(1 - \delta)^{1/2}(x^2 - \delta - 2)^{1/2} \\ &\qquad \times(-2\beta^2 + \gamma(\delta + 1))^{-1/2}]/(1 + \delta) \end{aligned}$$

Since $\Gamma_{-1/2}$ is the mirror reflection of $\Gamma_{1/2}$, one may employ this to complete the calculation, or use the above formula in which the quantity $[-(a^{-3}+a^{-1})x^{-1}+a^{-2}+(a^{-3}+a^{-1})x]$ is replaced by $[(a^{-1}+a)x^{-1}-1]$ and $[(a^{-1}+a)x^{-1}-1]$ is replaced by $[-(a+a^3)x^{-1}+a^2+(a+a^3)x]$, or use Theorem 6.2(iv). By expressing as many of these quantities as possible by means of expressions which are invariant under mirror reflection/conjugation involution assists in the recognition of important relationships. Thus we employ δ, β and, γ as they represent planar graphs which are, therefore, unchanged under mirror reflection. From these calculations, one notes that the two graphs are achiral as their invariants are changed under complex conjugation and sending a to a^{-1} and that their invariants are interchanged as they are mirror imges of each other, as required by Proposition 6.2(iii).

7. A STATE MODEL FOR THE SEMIORIENTED CHIMERICAL GRAPH INVARIANT

The state model approach to the development of the semioriented polynomial invariant for chimerical graphs begins with the definition of the elementary states as in the case of the oriented polynomial. Thus, let Γ denote a generic

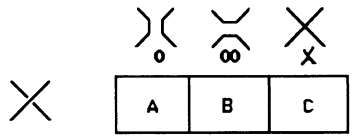

FIGURE 7.1. The fundamental semioriented state relations

presentation of an oriented chimerical graph. At each crossing of the presentation one makes the replacements according to the following table. The result is a formal algebraic sum of products of the formal variables A, B and, C times S^2 equivalence classes of oriented planar 4-valent graphs denoted $[\![\Gamma]\!]$, as above. Again, we introduce a normalization factor, a power of a variable denoted " α ", to compensate for the type I Reidemeister moves. We *define* the semioriented polynomial invariant, F_G, associated to the generic projection, Γ, of an oriented 4-valent graph to be

$$F_G(\Gamma) = (-1)^{c(\Gamma)-1}\alpha^{-\omega(\Gamma)}[\![\Gamma]\!], \tag{7.1}$$

as above. Recall that we extend the definition of $\omega(L)$ to $\omega(\Gamma)$ by assigning the crossing number 0 to each chimerical vertex. We shall see later that the natural home for this invariant is $Z[a^{\pm 1}, x^{\pm 1}, \beta, \gamma]$.

Let σ_+ and σ_- denote the $+1$ and -1 versions of the crossing, let 0 and ∞ denote the removed crossings and let χ denote the chimerical vertex form of the crossing. From the fundamental states formulae one derives the following formulae, with notation restricted to the fixed crossing under consideration:

$$\sigma_+ - \sigma_- = (A - B)[0 - \infty], \tag{7.2}$$

$$A\sigma_+ - B\sigma_- - (A^2 - B^2)0 = (A - B)C\chi. \tag{7.3}$$

The first formula gives rise to the recursion formula providing the connection of the polynomial to the semioriented polynomial while the second formula provides the averaging interpretation which can be exploited to prove the existence of the semioriented chimerical graph polynomial assuming the existence of the semioriented polynomial for classical knots and links, as above.

We shall, again, first discuss the development of the recursion formula. The initial step is the relation required in order to insure invariance under type I Reidemeister moves. The two cases are shown in the Figure 7.2. The connected sum axiom gives the following relations:

$$[\![\Gamma_+]\!] = (A\boldsymbol{\delta} + B + C\boldsymbol{\beta})[\![\Gamma]\!] = \alpha[\![\Gamma]\!],$$
$$[\![\Gamma_-]\!] = (B\boldsymbol{\delta} + A + C\boldsymbol{\beta})[\![\Gamma]\!] = \alpha^{-1}[\![\Gamma]\!],$$

where we define $\alpha = (A\boldsymbol{\delta} + B + C\boldsymbol{\beta})$ and $\alpha^{-1} = (B\boldsymbol{\delta} + A + C\boldsymbol{\beta})$. As before, we shall require the following relation:

$$(A\boldsymbol{\delta} + B + C\boldsymbol{\beta})(B\boldsymbol{\delta} + A + C\boldsymbol{\beta}) = \mathbf{1}. \tag{7.4}$$

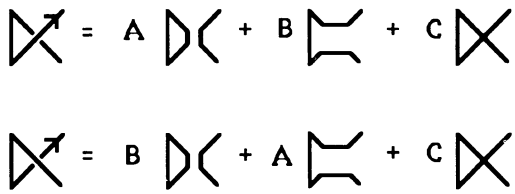

FIGURE 7.2. Type I Reidemeister move relations

The definition of α allows one to complete the determination of the defining formula for the semioriented polynomial. In order to do this we return to the first of the two relations arising out of the states formula which, after normalization, gives:

$$\begin{aligned} &i\alpha[(-1)^{c(\Gamma)-1}\alpha^{-1}[\![\sigma_+]\!]] - i\alpha^{-1}[(-1)^{c(\Gamma)-1}\alpha[\![\sigma_-]\!]] \\ &\quad = -i(A-B)[[(-1)^{c(\Gamma_0)-1}[\![0]\!]] + (i\alpha)^{-t}[(-1)^{c(\Gamma_\infty)-1}\alpha^{t}[\![\infty]\!]]] \end{aligned}$$

where the number of circuits in Γ, Γ_0, and Γ_∞, respectively, are related in such a way that the indicated sign relationship holds and "t", we recall, is determined by a somewhat more complicated recursive formulation involving two versions of the crossing relation depending upon the number of distinct strands involved in the crossings since this influences the choice of orientation and, thereby, the choice of normalization power in the recursion formula. The number of circuits in a chimerical graphs, Γ, is denoted by $c(\Gamma)$ and $\langle X, Y\rangle$ denotes the algebraic linking number of X and Y, two transverse oriented chimerical graphs, e.g. circuits. The orientation conventions that are to be employed in the recursive calculation, according as the crossing in Γ_+ involves the same or distinct strands of the link are shown in Figure 6.1. Thus, t is either $4\langle X, \Gamma_0 - X\rangle$, if $c(\Gamma_+) < c(\Gamma_0)$, or $4\langle X, \Gamma_+ - X\rangle - 2$, if $c(\Gamma_+) > c(\Gamma_0)$, as required. This formula gives, upon multiplication by $\alpha^{-\omega(\Gamma_0)}$, the recursion formulae for the semioriented polynomial, with $a = i\alpha$ and $x = -i(A-B)$, as stated in (ii) of Theorem 6.1.

Thus, modulo proving that the polynomial is well defined, one has a generalization of the semioriented polynomial by means of this state model applied to oriented chimerical graphs. We shall, as with the oriented polynomial above, reverse the argument to show that, assuming the existence of the semioriented polynomial for oriented classical knots and links, one may define the chimerical graph polynomial by an averaging process growing out of the second formula.

Suppose that the vertices of the oriented chimerical graph, Γ, are enumerated by $1, 2, 3, \ldots, k$ and σ_+ and σ_- denote operators on the vertices which replace them by the $+1$ or -1 crossing, respectively, σ_0, denote the operator which removes the crossing in the orientation respecting manner. Let S denote the set of all functions, ε, from $\{1, 2, \ldots, k\}$ into $\{-1, +1, 0\}$ each of which is called a link state since the result of applying the operator $\sigma_{\varepsilon(i)}$ to the ith crossing, for all the crossings results in an oriented knot or link, denoted

$\varepsilon\Gamma$. Let $\varepsilon+$, $\varepsilon-$ and ε_0 denote the number of $+1$ values, -1 values and 0 values, respectively, taken on by the link state ε. The second formula implies that

$$\begin{aligned}&\alpha[A/(A-B)C][(-1)^{s-1}\alpha^{-1}[\![\sigma_+]\!]]\\&\quad-\alpha^{-1}[B/(A-B)C][(-1)^{s-1}\alpha[\![\sigma_-]\!]]\\&\quad+[(A^2-B^2)/(A-B)C][(-1)^{s'-1}[\![\sigma_0]\!]]=[(-1)^{s-1}[\![\chi]\!]]\end{aligned}$$

which gives, upon multiplication by $\alpha^{-\omega(\Gamma_0)}$

$$\begin{aligned}&[\alpha A/(A-B)C][(-1)^{s-1}\alpha^{-\omega(\Gamma_+)}[\![\sigma_+]\!]]\\&\quad+[-\alpha^{-1}B/(A-B)C][(-1)^{s-1}\alpha^{-\omega(\Gamma_-)}[\![\sigma_-]\!]]\\&\quad+[(A+B)/C][(-1)^{s'-1}\alpha^{-\omega(\Gamma_0)}[\![\sigma_0]\!]]=[(-1)^{s-1}\alpha^{-\omega(\Gamma_\infty)}[\![\chi]\!]]\end{aligned}$$

where s, respectively s', denotes the number of circuits in the $+$, $-$, and χ, respectively 0, states.

As a consequence of this expansion of χ, we have the following weighted average formula for the semioriented chimerical graph invariant:

$$F_G(\Gamma)=\sum_{\varepsilon\in S}(\alpha A/C(A-B))^{\varepsilon+}(-\alpha^{-1}B/C(A-B))^{\varepsilon-}((A+B)/C)^{\varepsilon_0}F(\varepsilon\Gamma) \tag{7.4}$$

where $F(\varepsilon\Gamma)$ denotes the semioriented polynomial of the oriented knot or link, $\varepsilon\Gamma$. As with the oriented polynomial above, one may use this formula to define F_G by verifying that the quantity so defined is unchanged under the generalized Reidemeister moves associated to the graphs, thereby proving Theorem 6.1.

The result of this weighted averaging process is a Laurent polynomial in the variables A, B, C, and a and x, the latter two arising out of the semioriented polynomial. In order to achieve the final form of the promised result one must remove these former variables. This is accomplished by means of the following identifications:

$$\begin{aligned}A&=\frac{-\beta x^{1/2}(a^{-1}+a)^{1/2}(-a^{-1}-a+3x-x^3)^{1/2}}{(2\beta^2x-2\gamma x+\gamma(a^{-1}+a))^{1/2}(-a^{-1}-a+2x)}+\frac{ix(x-a^{-1})}{(-a^{-1}-a+2x)},\\B&=\frac{-\beta x^{1/2}(a^{-1}+a)^{1/2}(-a^{-1}-a+3x-x^3)^{1/2}}{(2\beta^2x-2\gamma x+\gamma(a^{-1}+a))^{1/2}(-a^{-1}-a+2x)}-\frac{ix(x-a)}{(-a^{-1}-a+2x)},\\C&=\frac{(a^{-1}+a)^{1/2}(-a^{-1}-a+3x-x^3)^{1/2}}{x^{1/2}(2\beta^2x-2\gamma x+\gamma(a^{-1}+a))^{1/2}}.\end{aligned}$$

The weighted average formula gives the following three factors, respectively:

$$\rho(+) \equiv [\alpha A/(A-B)C] = \frac{a^{-1}\beta}{x(1+\delta)} - \frac{ia^{-1}(x-a^{-1})(2\beta^2-\gamma(\delta+1))^{1/2}}{x(1+\delta)(\delta-1)^{1/2}(x^2-\delta-2)^{1/2}},$$

$$\rho(-) \equiv [-\alpha^{-1}B/(A-B)C] = \frac{\alpha\beta}{x(1+\delta)} + \frac{ia(x-a)(2\beta^2-\gamma(\delta+1))^{1/2}}{x(1+\delta)(\delta-1)^{1/2}(x^2-\delta-2)^{1/2}},$$

$$\rho(0) \equiv [(A+B)/C] = \frac{-2\beta}{(1+\delta)} - \frac{i(a^{-1}-a)(2\beta^2-\gamma(\delta+1))^{1/2}}{(1+\delta)(\delta-1)^{1/2}(x^2-\delta-2)^{1/2}}.$$

Substitution of these in the expansion formula above gives

$$F_G(\Gamma_\chi) = \rho(+)F_G(\Gamma_+) + \rho(-)F_G(\Gamma_-) + \rho(0)F_G(\Gamma_0)$$

proving the expansion formula in (iii) Theorem 6.1.

8. Algebraic aspects of the state model for the semioriented chimerical graph invariant

In addition to providing an extension of the semioriented polynomial from the category of classical knots and links to the category of chimerical graphs the state model provides a completely independent method of defining the oriented polynomial, even in the category of classical knots and links.

The development is accomplished by identifying the fundamental relations that must exist between the planar graphs that result from the invocation of the state model to transform a chimerical graph in space into an algebraic representative lying in the free module of oriented 4-valent planar graphs over the field of rational functions in A, B, and C. The result of the introduction of the necessary relations, the connected sum axiom and, a change of algebraic variables (to better represent the ultimate invariant) is the identification of two fundamental chimerical graph generators and a radical change of coefficient field.

Suppose that the vertices of the oriented chimerical graph, Γ, are enumerated by $1, 2, 3, \ldots, k$ and let σ_0 and σ_∞ denote operators on the crossings which replace them by the 0 or ∞ crossing removals, respectively, and let σ_χ, denote the operator which replaces the crossing in the diagram by a vertex in the oriented planar graph of the projection. Let S denote the set of all functions, ε, from $\{1, 2, \ldots, k\}$ into $\{0, \infty, \chi\}$. Each of these functions is called a link state since the result of applying the operator $\sigma_{\varepsilon(i)}$ to the ith crossing, for all the crossings, results in an oriented 4-valent planar graph, denoted $\varepsilon\Gamma$. Let $\varepsilon(0)$, $\varepsilon(\infty)$ and $\varepsilon(\chi)$ denote the number of 0 values, ∞ values and χ values, respectively, taken on by the link state ε. We define the state expansion of Γ as follows:

$$\alpha^{-\omega(\Gamma)} \sum_{\sigma\in S} A^{\varepsilon(0)} B^{\varepsilon(\infty)} C^{\varepsilon(\chi)} [\![\varepsilon\Gamma]\!], \tag{8.1}$$

recalling that α was defined to be $A\delta + B + C\beta$ and that α^{-1} was defined to be $B\delta + A + C\beta$, where δ and β denoted the variables associated to, re-

spectively, the distant union of two planar circles and the connected 4-valent planar graph with one vertex, $\boldsymbol{\beta}$. Thus, the state expansion of a chimerical graph can be considered as an element of the free module of S^2 equivalence classes of oriented 4-valent planar graphs, denoted $[\![\sigma\Gamma]\!]$, over the field of rational functions in variables A, B, C, δ, and β. In order to use this state expansion to determine chimerical isotopy invariants for chimerical graphs we shall again introduce relations, i.e. take a quotient module, which reflect the fundamental relations that are imposed by the connected sum axiom and the classical and chimerical Reidemeister moves. In addition, we shall identify graphs which differ only in their circuit orientations, i.e. we shall discard the orientation structure. This is done to reduce the complexity of the graph algebra and appears permissible because the resulting algebraic structure is still sufficiently rich to support the semioriented polynomial invariant of classical knots and links.

In order to simplify the algebraic structure, defined above, we have identified the vertices which arise during the construction of the associated state and those which occur by virtue of the chimerical graph structure. As a consequence, the relations associated with the classical Reidemeister moves are sufficient to imply those required for the chimerical Reidemeister moves.

We shall first consider the type I Reidemeister moves. The relation implied by these moves is the requirement that $\alpha \cdot \alpha^{-1} = 1$, i.e. one of the relations given earlier,

$$(A\boldsymbol{\delta} + B + C\boldsymbol{\beta})(B\boldsymbol{\delta} + A + C\boldsymbol{\beta}) = \mathbf{1}$$

where $\mathbf{1}$ denotes the equivalence class of the trivial knot serving as a multiplicative unit in the algebra. For the Reidemeister move of type II the identifying state expansion associated to the local structure of the move gives the expression at the top of Figure 8.1. Similarly, for the Reidemeister move of type III, the identifying state expansion associated to the local structure of the move gives the second relation shown in Figure 8.1. Kauffman and Vogel, [18], have developed direct analogs of the relations in Figure 8.1 from the existence of the knot invariant polynomials.

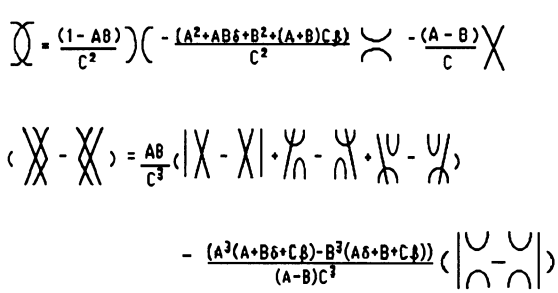

FIGURE 8.1. The semioriented chimerical graph relations

Invariance under the chimerical moves is proved from these relations in a manner analogous to that employed in §4, for the oriented invariant, giving the following theorem.

Proposition 8.1. *The relations shown in Figure* 8.1 *generate the chimerical graph relations required for algebraic invariance of the semioriented polynomial under the chimerical and classical Reidemeister moves.*

Theorem 8.2. *Every oriented* 4*-valent graph in* S^2 *can be reduced to a unique expression in terms of the algebraic variables by means of these relations.*

The proof of this theorem follows the model for the corresponding result for the oriented invariant, i.e. an inductive reduction to a large class of elementary archetypical cases each of which is verified by direct calculation. The argument is more complicated here to the extent that the lack of an orientation allows for a larger number of elementary configurations than was required in the oriented invariant case.

By means of the graph relations one can calculate the actual representation of the elementary graphs in terms of the algebraic and graphical variables. Recall that we have set $F_G(\boldsymbol{\delta}) = -\delta = (a^{-1} + a)x^{-1} - 1$ and have defined $\beta \equiv F_G(\boldsymbol{\beta})$ and $\gamma \equiv -F_G(\boldsymbol{\gamma})$. Application of the substitution of the variables A, B, and C according to the formulae given earlier gives

$$\begin{aligned} F_G(\boldsymbol{\varepsilon}) = {} & -(x^2(\delta-1)^2 - 4)^{1/2}(-2\beta^2 + \gamma(\delta+1))^2 \\ & + \beta(\beta^2(\delta-3) + 3\gamma(\delta+1))(\delta-1)^{1/2}(x^2-\delta-2)^{1/2}(-2\beta^2+\gamma(\delta+1))^{1/2} \end{aligned}$$

Note that this expression in invariant under the change a goes to a^{-1} and complex conjugation, indeed the representation contains no direct presence of a. This is again a reflection of the fact that $\boldsymbol{\varepsilon}$ is a planar graph and the rather nonessential role played by the orientation.

9. Another state model for the Jones polynomial?

In this section we shall discuss some of the algebraic implications of the reduction of the semioriented chimerical graph state model developed in the previous section to the special case of the Jones polynomial. The results imply important reductions in structure which have corresponding implications for the combinatorial structure of the polynomial invariants of the classical knots and links.

Recall that if we set $a = -t^{-3/4}$ and $x = t^{-1/4} + t^{1/4}$, in the semioriented polynomial the result is the Jones polynomial. The result of this substitution is

$$A = it^{1/4}, \quad B = -it^{-1/4}, \quad C = 0$$

which imply that $\alpha = it^{-3/4}$, $\alpha^{-1} = -it^{3/4}$. The result is precisely the Kauffman state model for the Jones polynomial, [17] in which the graphical term is not utilized. Since there are no vertex terms in the representation the result is an "interaction" model in the sense employed in §5.

The form of the semioriented chimerical graph relations, shown in Figure 8.1, shows that in order to achieve a semioriented interaction model one must set either $C = 0$, thereby eliminating the nontrivial graph terms in the model, or set either A or B to be 0. In the former case one has the equation, in terms of the variables a and x;

$$(ax^3 - 3ax + a^2 + 1)^{1/2}(a^2 + 1)^{1/2} = 0$$

implying that either $a = \pm i$, in which case $F_G(L) = (-1)^{c(L)-1}$, or

$$ax^3 - 3ax + a^2 + 1 = 0.$$

Setting $x = s^{-1} + s$ and substituting in the equation, one has

$$s^3a^2 + (1 + s^6)a + s^3 = 0$$

having solutions $a = s^{-3}$, s^3. This is the Jones polynomial. Alternatively, one may solve for a:

$$a = -x^3/2 + 3x/2 \pm (1 - x^2)(x^2 - 4)^{1/2}.$$

With this evaluation, one has

$$A = i((x^2 - 4)^{1/2} + x)/2, \quad B = i((x^2 - 4)^{1/2} - x)/2, \quad C = 0.$$

Note that $B = A^{-1}$, as is the case for the Kauffman state model of the Jones polynomial. Thus, by setting $C = 0$, one finds exactly the Jones polynomial or an essentially trivial invariant.

The other alternative, say $B = 0$, implies a relationship between the graphical variables, β and γ, and the algebraic variables, a and x. As a consequence one has

$$\gamma = -(2a^2s^5+3a^2s^3+2a^2s-2s^4a-2s^2a-a-s^6a-s^3)\beta^2(s^2+1)^{-1}(as-s^2-1)^{-1}a^{-1}$$

employing the a, s variables. Unlike the result in the oriented case, this is not sufficient to imply the existence of an interaction model as there remains a second term in the triangle relation in Figure 8.1. Requiring that this remaining term equal 0 leads to an equation without solution. Therefore, one has the following proposition:

Proposition 9.1. *The Jones polynomial and* $F_G(L) = (-1)^{c(L)-1}$ *are the only elementary semioriented interaction invariants which extend to the category of chimerical graphs.*

10. Some applications of the chimerical graph invariants

The purpose of this final section is to illustrate the use of these polynomial invariants by means of a couple of elementary examples associated to the trefoil knot and a comparison of their respective properties. The first is an embedding of $\boldsymbol{\beta}$ in which one segment is a trefoil and the second links this trefoil. The second is an embedding of Γ in which one circuit is a trefoil and the second

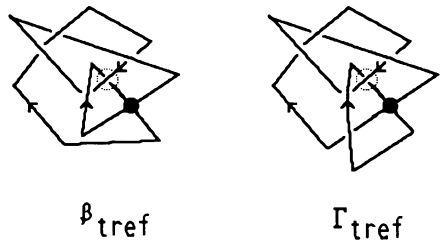

FIGURE 10.1

circuit links the trefoil in a positive fashion. The calculations of the associated invariants follows either the traditional pattern of reduction to simple cases by changing the crossing at an appropriate site, such as the ones identified in the figure, or by expansion via the weighted averaging formulae reducing the calculations to those associated to the oriented and semioriented invariants of classical knots and links. The full expansion of the semioriented graph invariant for either of these examples results in a very complicated algebraic expression, with respect to the current variables, and therefore one is led to seek evaluations of the variables which result in simpler expressions which are, nevertheless, sufficient to capture the desired properties. In the case of the first graph, β_{tref}, setting $\beta = 0$ and $x = 1$ gives the result;

$$\begin{aligned} &F_G(\beta_{\text{tref}})(a, 1, \beta = 0) \\ &\quad = \gamma^{1/2}(a^7 + 2a^6 + 3a^5 + 3a^4 + a^3 - a^2 - 2a - 2 - a^{-1})(a^{-1} + a)^{-1/2} \end{aligned}$$

implying the chimerical chirality of β_{tref}. For the case of the semioriented polynomial associated to Γ_{tref} one has the result

$$F_G(\Gamma_{\text{tref}})(a, 1, \beta = 0) = \gamma^{1/2}(a^{9/2} + a^{7/2} + 2a^{5/2} + a^{3/2} + a^{1/2})(a^{-1} + a)^{-1/2}$$

again implying the chimerical chirality of the graph. Such results are to be expected because of the chiral nature of the trefoil knot. If, however, one considers the analogous construction with the figure 8 knot, which is achiral, the chiral nature of the result is less evident. In the case of the embedding associated to Γ, the linking number between the two circuits is sufficient to imply the chirality. If, however, one considers the analagous embeddings of β, shown as the first two examples in Figure 10.2, the calculations give the following results:

$$\begin{aligned} &F_G(\boldsymbol{\beta}_3)(a, 1, \beta = 0) = \gamma^{1/2}(a^6 + a^5 + 2a^4 + a^3 - a - 1 - a^{-1})(a^{-1} + a^1)^{-1/2}, \\ &F_G(\boldsymbol{\beta}_7)(a, 1, \beta = 0) \\ &\quad = \gamma^{1/2}a^{-15/2}(a^8 + a^7 + a^6 - 2a^4 - 3a^3 - 2a - 1)(a^{-1} + a^1)^{-1/2}. \end{aligned}$$

For the third example, $\boldsymbol{\beta}_4$, one can show that, by a sequence of elementary moves, the mirror reflection is equivalent to the original position. Thus they have the same invariant, one which is unchanged under the involution of sending "a" to "a^{-1}" and taking the complex conjugate. This is a consequence

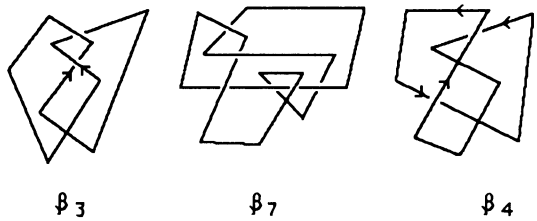

FIGURE 10.2

of the fact that $\rho(+)$ and $\rho(-)$ are interchanged and $\rho(0)$ is unchanged by this involution, and that the semioriented polynomials of the positive and negative Hopf links, h_+ and h_-, are interchanged and that of the figure 8 knot, 4_1, is invariant under the involution. Thus the involution leaves invariant the following expansion of the polynomial:

$$\begin{aligned} F_G(\boldsymbol{\beta}_4) = {} & \rho(+)^2 + \rho(-)^2 + \rho(0)^2 + \rho(+)\rho(-) + \rho(0)[\rho(+) + \rho(-)]F_G(\delta) \\ & + \rho(0)[\rho(+)F_G(h_+) + \rho(-)F_G(h_-)] + \rho(+)\rho(-)F_G(4_1)\,. \end{aligned}$$

Because there are examples of classical knots and links whose semioriented polynomials are invariant with respect to this involution and which are, nevertheless, distinct from their mirror images one encounters the same situation for the chimerical graphs.

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