LINEAR TOPOLOGICAL CLASSIFICATIONS
OF CERTAIN FUNCTION SPACES

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Abstract. Some linear classification results for the spaces $C_p(X)$ and $C^*_p(X)$ are proved.

0. Introduction

If $X$ is a space then $C_p(X)$ denotes the set of all continuous real-valued functions on $X$ with the topology of pointwise convergence. We write $C^*_p(X)$ for the subspace of $C_p(X)$ consisting of all bounded functions. $R$ stands for the usual space of real numbers, $I$—for the unit segment $[0, 1]$ and $Q$ is the Hilbert cube $I^n$. If $n \geq 1$ then $\mu^n$ denotes the $n$-dimensional universal Menger compactum. Let $X$ be a separable metric space. A separable metric space $Y$ is called an $X$-manifold if $Y$ admits an open cover by sets homeomorphic to open subsets of $X$.

Results in [A1, A2 and Ps] show that the linear topological classification of the spaces $C_p(X)$ is very complicated. Below the linear topological classification results for the spaces $C_p(X)$ which I know are listed:

(1) Let $X$ and $Y$ be non-zero-dimensional compact polyhedra. Then $C_p(X) \sim C_p(Y)$ if and only if $\dim X = \dim Y$ [Pv]. Here the symbol "~" stands for linear homeomorphism.

(2) If $X$ is a locally compact subset of $R^n$ such that $\text{cl}((\text{Int}(X)) \cap (R^n - X)) \neq \emptyset$ then $C_p(X) \sim C_p(R^n)$ [Dr1].

(3) If $X$ is a 1-dimensional compact ANR with finite ramification points or a continuum $X$ is a one-to-one continuous image of $[0, \infty)$ then $C_p(X) \sim C_p(I)$ [KO].

For topological classification results of the spaces $C_p(X)$ see [BGM, BGMP, GH and M].

The aim of this paper is to prove the following results:

(4) $C_p(X) \sim C_p(Q)$ if and only if $X$ is a compact metric space containing a copy of $Q$.

(5) Let $X$ be a subset of $R^n$. Then $C_p(X) \sim C_p(I^n)$ iff $X$ is compact and $\dim X = n$.
(6) $C_p(X) \sim C_p(\mu^n)$ if and only if $X$ is an $n$-dimensional compact metric space containing a copy of $\mu^n$.

(7) $C_p(X) \sim C_p(l_2)$ provided $X$ is an $l_2$-manifold (by $l_2$ is denoted the separable Hilbert space).

(8) Let $X$ be one of the spaces $Q$, $t^n$ or $\mu^n$, and $Y$ be a locally compact subset of an $X$-manifold. Then $C_p(Y) \sim C_p(X)^\omega$ if and only if $Y$ contains a closed copy of the topological sum $\sum X_i$ of infinitely many copies of $X$.

Similar results are also proved for the spaces $C_p^*(X)$.

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1. Preliminaries

All spaces under discussion are Tychonoff and all mappings between topological spaces are continuous. By $L_p(X)$ is denoted the dual linear space of $C_p(X)$ with the weak (i.e. pointwise) topology. It is known that

$$L_p(X) = \left\{ \sum_{i=1}^{k} a_i \delta_{x_i} : a_i \in \mathbb{R} - (0) \text{ and } x_i \in X \text{ for each } i \leq k \right\}.$$  

Here $\delta_x$ is the Dirac measure at the point $x \in X$. We denote

$$P_\infty(X) = \left\{ \sum_{i=1}^{k} a_i \delta_{x_i} : a_i \in (0, 1) \text{ for each } i \text{ and } \sum_{i=1}^{k} a_i = 1 \right\}$$

and $\text{supp}(l) = (x_1, \ldots, x_k)$, where $l = \sum_{i=1}^{k} a_i \delta_{x_i} \in L_p(X)$.

Let $A$ be a closed subset of a space $X$. Consider the following conditions:

(i) There is a continuous linear extension operator $u: C_p(A) \rightarrow C_p(X)$ (recall that $u: C_p(A) \rightarrow C_p(X)$ is an extension operator if $u(f)|A = f$ for every $f \in C_p(A)$);

(ii) There is a continuous linear extension operator $u: C_p(A) \rightarrow C_p(X)$ and a positive constant $c$ such that $\|u(f)\| \leq c \cdot \|f\|$ for every $f \in C_p(A)$.

(iii) There is a regular extension operator $u: C_p(A) \rightarrow C_p(X)$ i.e. a continuous linear extension operator $u$ with $u(1_A) = 1_X$ and $u(f) \geq 0$ provided $f \geq 0$.

$A$ is said to be $l$-embedded (resp., $l^*$-embedded) in $X$ if the condition (i) (resp., the condition (ii)) holds. If (iii) is satisfied then $A$ is called strongly $l$-embedded in $X$. Dugundji [D] proved that every closed subset of a metric space $X$ is strongly $l$-embedded in $X$ (he did not state this explicitly in this form). It is known (see [AČ, Dr1]) that $A$ is $l$-embedded (resp., strongly $l$-embedded) in $X$ if and only if there is a mapping $r: X \rightarrow L_p(A)$ (resp., $r: X \rightarrow P_\infty(A)$) such that $r(x) = \delta_x$ for every $x \in A$. Such a mapping will be called an $L_p$-valued (resp., a $P_\infty$-valued) retraction. Every $L_p$-valued retraction $r: X \rightarrow L_p(A)$ defines a continuous linear extension operator $u_r: C_p(A) \rightarrow C_p(X)$ by setting
$u_r(f)(x) = r(x)(f)$. If the operator $u_r$ satisfies the condition (ii), $r$ is said to be a bounded $L_p$-valued retraction.

Let $u: C_p(A) \rightarrow C_p(X)$ be a continuous linear extension operator. Then the mapping $v(f, g) = u(f) + g$ is a linear homeomorphism from $C_p(A) \times C_p(A; A)$ onto $C_p(X)$, where

$$C_p(X; A) = \{g \in C_p(X): g|A = 0\}.$$

Analogously, if $A$ is $l^*$-embedded in $X$ then $C^*_p(A) \times C^*_p(X; A)$ is linearly homeomorphic to $C^*_p(X)$.

Let $\mathcal{H}$ be a family of bounded subsets of a space $X$ (i.e. $f|K$ is bounded for every $K \in \mathcal{H}$ and $f \in C_p(X)$) and $E$ be a linear topological subset of $C_p(X)$. Then we set

$$\left( \prod E \right)_{\mathcal{H}} = \{(f_1, \ldots, f_n, \ldots) \in E^\alpha: \lim_n \|f_n\|_K = 0 \text{ for every } K \in \mathcal{H}\}$$

and

$$\left( \prod E \right)^*_{\mathcal{H}} = \{(f_1, \ldots, f_n, \ldots) \in \left( \prod E \right)_{\mathcal{H}}: \sup_n \|f_n\| < \infty \}.$$

$(\prod E)_{\mathcal{H}}$ and $(\prod E)^*_{\mathcal{H}}$ are considered as topological linear subspaces of $C_p(X)^\alpha$. We write $(\prod E)_b$ and $(\prod E)_c$ (resp. $(\prod E)^*_b$ and $(\prod E)^*_c$) if $\mathcal{H}$ is the family of all bounded (resp., of all compact) subsets of $X$. In the above notations $\|f\|_K$ stands for the set $\sup\{|f(X)|: x \in K\}$. Let us note that if $X$ is pseudocompact and $E$ is a linear subset of $C_p(X)$, the space

$$(\prod E)^0_0 = \{(f_1, \ldots, f_n, \ldots) \in E^\alpha: \lim_n \|f_n\| = 0\}$$

is considered in [GH].

We need also the following notion: a space $X$ is said to be a $k_R$-space [N] if every function $f: X \rightarrow R$ is continuous provided that $f|K$ is continuous for each compact subset $K$ of $X$.

2. Linear topological classifications of $C_p(X)$

2.1 Lemma. Let $A$ be a strongly $l$-embedded (resp., $l$-embedded or $l^*$-embedded) subset of a space $X$. Then $A \times Y$ is strongly $l$-embedded (resp., $l$-embedded or $l^*$-embedded) in $X \times Y$ for every space $Y$.

Proof. Suppose $A$ is strongly $l$-embedded in $X$. So, there exists a $P_\infty$-valued retraction $r_1: X \rightarrow P_\infty(A)$. Define a mapping $r: X \times Y \rightarrow P_\infty(A \times Y)$ by setting

$$r(x, y) = \sum_{i=1}^{k} a_i \delta_{(x_i, y)}, \quad \text{where } r_1(x) = \sum_{i=1}^{k} a_i \delta_{x_i}. $$

It is easily shown that $r$ is a $P_\infty$-valued retraction. Thus, $A \times Y$ is strongly $l$-embedded in $X \times Y$. One can also prove that $r$ is a (bounded) $L_p$-valued retraction provided $r_1$ is a (bounded) $L_p$-valued retraction. Hence, if $A$ is $l$ (resp., $l^*$)-embedded in $X$ then $A \times Y$ is $l$ (resp., $l^*$)-embedded in $X \times Y$. 


Lemma. Let $A$ be an $l^*$-embedded subset of a space $X$. Then $(\prod C_p(X))_b$ is linearly homeomorphic to $(\prod C_p(A))_b \times (\prod C_p(X; A))_b$.

Proof. Let $u: C_p(A) \to C_p(X)$ be a continuous linear extension operator such that $\|u(f)\| \leq c \cdot \|f\|$ for every $f \in C^*_p(A)$, where $c > 0$. Since $\|f\| = \infty$ provided $f \in C_p(A) - C^*_p(A)$, the inequality $\|u(f)\| \leq c \cdot \|f\|$ holds for every $f \in C_p(A)$. Then the mapping $r: X \to L_p(A)$, defined by $r(x)(f) = u(f)(x)$, is an $L_p$-valued retraction. Consider the linear homeomorphism $v$ from $C_p(A) \times C_p(X; A)$ onto $C_p(X)$, $v(f, g) = u(f) + g$. Suppose $(f_1, \ldots, f_n, \ldots) \in C_p(A)_\omega$ and $(g_1, \ldots, g_n, \ldots) \in C_p(X; A)_\omega$. Put

$$H(K) = \text{cl}_A \left( \bigcup \{\text{supp}(r(x)) : x \in K\} \right),$$

where $K$ is a subset of $X$. Obviously, $\|u(f_n)\|_K \leq c \cdot \|f_n\|_H(K)$ for every $n \in \mathbb{N}$. By a result of Arhangel’skii [A2], $H(K)$ is a bounded subset of $A$ provided $K$ is a bounded subset of $X$. Hence, $(f_1, \ldots, f_n, \ldots) \in (\prod C_p(A))_b$ if and only if $(u(f_1), \ldots, u(f_n), \ldots)$ belongs to $(\prod C_p(X))_b$. Consequently, $(v(f_1, g_1), \ldots, v(f_n, g_n), \ldots)$ belongs to $(\prod C_p(X))_b$ if $(g_1, \ldots, g_n, \ldots) \in (\prod C_p(X; A))_b$ and $(f_1, \ldots, f_n, \ldots) \in (\prod C_p(A))_b$. Suppose

$$(v(f_1, g_1), \ldots, v(f_n, g_n), \ldots) \in (\prod C_p(X))_b.$$

Then $(f_1, \ldots, f_n, \ldots) \in (\prod C_p(A))_b$ because $v(f_n, g_n)|A = f_n$ for every $n$. Therefore $(u(f_1), \ldots, u(f_n), \ldots) \in (\prod C_p(X))_b$. So we have $(g_1, \ldots, g_n, \ldots) \in (\prod C_p(X; A))_b$. Thus, $(v(f_1, g_1), \ldots, v(f_n, g_n), \ldots)$ belongs to $(\prod C_p(X))_b$ if $(g_1, \ldots, g_n, \ldots) \in (\prod C_p(X; A))_b$ and $(f_1, \ldots, f_n, \ldots) \in (\prod C_p(A))_b$. Hence, the formula $v_0((f_1, \ldots, f_n, \ldots), (g_1, \ldots, g_n, \ldots)) = (v(f_1, g_1), \ldots, v(f_n, g_n), \ldots)$ defines a linear mapping from $(\prod C_p(A))_b \times (\prod C_p(X; A))_b$ onto $(\prod C_p(X))_b$ which is a homeomorphism.

Lemma. Let $A$ be an $l^*$-embedded subset of a space $X$. If every closed and bounded subset of $A$ is compact then $(\prod C_p(X \times Y))_c \sim (\prod C_p(A \times Y))_c \times (\prod C_p(X \times Y; A \times Y))_c$ for any space $Y$.

Proof. Let $u_1: C_p(A) \to C_p(X)$ be a continuous linear extension operator such that $\|u_1(f)\| \leq c \cdot \|f\|$ for every $f \in C^*_p(A)$, where $c > 0$, and $r_1: X \to L_p(A)$ be defined by $r_1(x)(f) = u_1(f)(x)$. Obviously, $r_1$ is an $L_p$-valued retraction. For a given space $Y$ the equality $r(x, y) = \sum_{i=1}^k a_i \delta_{r_i(y)}$, where $r_1(x) = \sum_{i=1}^k a_i \delta_{r_i(y)}$, defines an $L_p$-valued retraction from $X \times Y$ into $L_p(A \times Y)$. Next, set $u(f)(x, y) = r(x, y)(f)$ for every $(x, y) \in X \times Y$ and $f \in C_p(A \times Y)$. It is easily shown that $u: C_p(A \times Y) \to C_p(X \times Y)$ is a continuous linear extension operator.
Claim 1. $\|u(f)\| \leq c \cdot \|f\|$ for every $f \in C_p^*(A \times Y)$.

Fix a point $(x, y) \in X \times Y$ and an $f \in C_p^*(A \times Y)$. It follows from the definition of $u$ that

$$u(f)(x, y) = \sum_{i=1}^{k} a_i f(x_i, y), \quad \text{where } r_1(x) = \sum_{i=1}^{k} a_i \delta_{x_i}.$$ 

So, $|u(f)(x, y)| \leq \sum_{i=1}^{k} |a_i| \cdot \|f\|$. Take a function $g \in C_p^*(A)$ with $\|g\| = 1$ and $g(x_i) = \text{sgn}(a_i)$ for each $i = 1, \ldots, k$. Then $u_1(g)(x) = r_1(x)(g) = \sum_{i=1}^{k} |a_i|$. Since $\|u_1(g)\| \leq c \cdot \|g\|$, we have $\sum_{i=1}^{k} |a_i| \leq c$. Hence, $|u(f)(x, y)| \leq c \cdot \|f\|$. Claim 1 is proved.

Claim 2. For every compact subset $K$ of $X \times Y$ the set

$$H(K) = \{x \in A \times Y : \text{supp}(r(x, y)) \subseteq K\},$$

is also compact.

Let $n_X: X \times Y \to X$ and $n_Y: X \times Y \to Y$ be the natural projections. Then $n_X(K)$ and $n_Y(K)$ are compact subsets of $X$ and $Y$ respectively. By a result of Arhangel'skii [A2],

$$H_1(K) = \text{cl}_A \{x \in n_X(K) : x \in n_Y(K)\}$$

is a bounded subset of $A$. Thus, $H_1(K)$ is compact. So $H_1(K) \times n_Y(K)$ is a compact subset of $A \times Y$. Since $r(x, y)(K) = \text{supp}(r_1(x)) \times \{y\}$ for every point $(x, y) \in X \times Y$, we have $H(K) \subseteq H_1(K) \times n_Y(K)$. Hence, $H(K)$ is compact as a closed subset of $H_1(K) \times n_Y(K)$. Claim 2 is proved.

Now, the proof of Lemma 2.3 follows from the above two claims and the arguments used in the proof of Lemma 2.2.

2.4 Corollary. Let $X$ be a product of metric spaces and $A$ be an $l^*$-embedded subset of $X$. Then $(\prod C_p(X))_c \sim (\prod C_p(A))_c \times (\prod C_p(X; A))_c$.

Proof. Since $A$ is closed in $X$, every closed bounded subset of $A$ is compact. Thus, the proof follows from Lemma 2.3, where $Y$ is the one-point space.

2.5 Lemma. Suppose $X$ is a space such that both $X \times I$ and $X \times T$ are $k_R$-spaces, where $T = \{0, 1/n : n \in \mathbb{N}\}$. Then $C_p(X \times I)$ is linearly homeomorphic to $(\prod C_p(X \times I))_c$.

Proof. Since, by Lemma 2.1, $X \times T$ is strongly $l$-embedded in $X \times I$ we have

$$C_p(X \times I) \sim C_p(X \times T) \times C_p(X \times I; X \times T).$$

Let $I_n = [1/n + 1, 1/n]$ and $E_n = C_p(X \times I_n; X \times [1/n + 1, 1/n])$ for every $n \in \mathbb{N}$. Consider the set

$$(\prod E_n)_c = \left\{ (f_1, \ldots, f_n, \ldots) \in \prod E_n : \lim_{n} \|f_n\|_{X \times I_n} = 0 \right\}$$

for every compact subset $K$ of $X$, as a topological linear subset of $\prod \{E_n : n \in \mathbb{N}\}$. Since $X \times I$ is a $k_R$-space
we have $C_p(X \times I; X \times T) \sim (\prod E_n)_c$. Identifying each $E_n$ with the space $E = C_p(X \times I; X \times \{0, 1\})$ we get

(2) $C_p(X \times I; X \times T) \sim (\prod E)_c$.

Analogously, $C_p(X \times T) \sim C_p(X \times \{0\}) \times C_p(X \times T; X \times \{0\})$ and

$C_p(X \times T; X \times \{0\}) \sim (\prod C_p(X))_c$.

Thus,

(3) $C_p(X \times T) \sim C_p(X \times \{0\}) \times (\prod C_p(X))_c \sim (\prod C_p(X))_c$.

By Lemma 2.3, the following holds

(4) $(\prod C_p(X \times I))_c \sim (\prod C_p(X \times \{0, 1\}))_c \times (\prod E)_c$.

Obviously,

(5) $(\prod C_p(X \times \{0, 1\}))_c \sim (\prod C_p(X))_c \times (\prod C_p(X))_c \sim (\prod C_p(X))_c$.

So we have

$C_p(X \times I) \sim C_p(X \times T) \times C_p(X \times I; X \times T)$ by (1)

$\sim (\prod C_p(X))_c \times (\prod E)_c$ by (2) and (3)

$\sim (\prod C_p(X \times I))_c$ by (4) and (5).

2.6 Corollary. Let $X$ be as in Lemma 2.5. Then $C_p(X \times I)$ is homeomorphic to $C_p(X \times I)^\omega$.

Proof. S. Gul’ko and T. Hmyleva [GH] proved that $(\prod C_p(X))_0$ is homeomorphic to $C_p(X)^\omega \times (\prod C_p(X))_0$ for every pseudocompact space $X$. Using the same arguments one can see that $(\prod C_p(X))_c$ is homeomorphic to $C_p(X)^\omega \times (\prod C_p(X))_c$ for each $X$. Now, the proof of Corollary 2.6 follows from Lemma 2.5.

2.7 Lemma. Suppose a space $X$ contains an $l$-embedded copy $F_1$ of a space $Y$ and $Y$ contains an $l'$-embedded copy $F_2$ of $X$. Then $C_p(X) \sim C_p(Y)$ provided one of the following conditions is fulfilled:

(i) $C_p(Y) \sim (\prod C_p(Y))_b$;

(ii) $C_p(Y) \sim (\prod C_p(Y))_c \sim (\prod C_p(F_2))_c \times (\prod C_p(Y; F_2))_c$.

Proof. We have $C_p(X) \sim C_p(F_1) \times E_1$ and $C_p(Y) \sim C_p(F_2) \times E_2$, where $E_1 = C_p(X; F_1)$ and $E_2 = C_p(Y; F_2)$. Thus, $C_p(X) \sim C_p(Y) \times E_1$. Suppose $C_p(Y) \sim (\prod C_p(Y))_b$. By Lemma 2.2,

$$(\prod C_p(Y))_b \sim (\prod C_p(F_2))_b \times (\prod E_2)_b,$$

so

$$(\prod C_p(Y))_b \sim (\prod C_p(X))_b \times (\prod E_2)_b.$$
Therefore,

\[ C_p(Y) \sim \left( \prod C_p(Y) \right)_b \sim C_p(Y) \times \left( \prod C_p(Y) \right)_b \]

\[ \sim C_p(Y) \times \left( \prod C_p(X) \right)_b \times \left( \prod E_2 \right)_b. \]

Hence, \( C_p(X) \sim E_1 \times C_p(Y) \sim E_1 \times C_p(Y) \times (\prod C_p(X))_b \times (\prod E_2)_b \sim C_p(X) \times (\prod C_p(X))_b \times (\prod E_2)_b \sim (\prod C_p(X))_b \times (\prod E_2)_b \sim C_p(Y). \)

If condition (ii) is fulfilled we use the same arguments.

2.8 Theorem. (i) Let \( X \) be a subspace of \( R^n \). Then \( C_p(X) \sim C_p(I^n) \) if and only if \( X \) is compact and \( \dim X = n \);

(ii) \( C_p(X) \sim C_p(Q) \) if and only if \( X \) is a compact metric space containing a copy of \( Q \).

Proof. We prove only the first part of Theorem 2.8. The proof of (ii) is analogous to that of (i).

Suppose \( C_p(X) \sim C_p(I^n) \). Then by [A2 and A3] \( X \) is a compact metric space. Next, it follows from a result of Pavlovskii [Pv] that there is a nonempty open subset of \( I^n \) which can be embedded in \( X \). Thus, \( \dim X = n \).

Now, let \( X \) be a compact \( n \)-dimensional subset of \( R^n \). Then \( X \) contains a copy of \( I^n \). On the other hand \( X \) can be considered as a subset of \( I^n \).

Hence, by Corollary 2.4, \( (\prod C_p(I^n))_c \sim (\prod C_p(X))_c \times (\prod C_p(I^n) ; X))_c \). Since \( C_p(I^n) \sim (\prod C_p(I^n))_c \) (see Lemma 2.5), we derive from Lemma 2.7(ii) that \( C_p(X) \sim C_p(I^n) \).

2.9 Theorem. Let \( \mu^n \) be the \( n \)-dimensional universal Menger compactum. Then \( C_p(X) \sim C_p(\mu^n) \) if and only if \( X \) is an \( n \)-dimensional compact metric space containing a copy of \( \mu^n \).

Proof. Let \( C_p(X) \sim C_p(\mu^n) \). Then, by results of Arhangel’skii [A2, A3] and Pestov [Ps], \( X \) is an \( n \)-dimensional compact metric space. It follows from [Pv] that there exists an open subset of \( \mu^n \) which can be embedded in \( X \). But each open subset of \( \mu^n \) contains a copy of \( \mu^n \) [Bt]. Thus, \( X \) contains a copy of \( \mu^n \).

Suppose \( X \) is an \( n \)-dimensional compact metric space containing a copy of \( \mu^n \). Since \( X \) can be embedded in \( \mu^n \), by Lemma 2.7(ii) and Corollary 2.4 it is enough to show that \( C_p(\mu^n) \sim (\prod C_p(\mu^n))_c \). For proving this fact we need the following result of Dranishnikov [Dr2]: There is a mapping \( f_n \) from \( \mu^n \) onto \( Q \) such that \( f_n^{-1}(P) \) is homeomorphic to \( \mu^n \) for every LC\(^{n-1}\)C\(^{n-1}\)-compact subspace \( P \) of \( Q \). Now, consider \( Q \) as a product \( Q_1 \times I \), where \( Q_1 \) is a copy of \( Q \). Let \( T = \{ 0, 1/k ; k \in N \} \) and \( T^* = f_n^{-1}(Q_1 \times T) \). Then

\[ C_p(\mu^n) \sim C_p(T^*) \times C_p(\mu^n ; T^*) \]

and

\[ C_p(T^*) \sim C_p(f_n^{-1}(Q_1 \times \{ 0 \})) \times C_p(T^* ; f_n^{-1}(Q_1 \times \{ 0 \})). \]
Since each of the sets $f_n^{-1}(Q_1 \times \{1/k\}), \ k \in \mathbb{N}$, and $f_n^{-1}(Q_1 \times \{0\})$ is homeomorphic to $\mu^n$, we have

$$C_p(T^*; f_n^{-1}(Q_1 \times \{0\})) \sim \left( \prod C_p(\mu^n) \right)_c$$

and

$$C_p(f_n^{-1}(Q_1 \times \{0\})) \sim C_p(\mu^n).$$

Thus,

$$C_p(T^*) \sim C_p(\mu^n) \times \left( \prod C_p(\mu^n) \right)_c \sim \left( \prod C_p(\mu^n) \right)_c \times C_p(T^*). \quad (7)$$

Finally,

$$C_p(\mu^n) \sim C_p(T^*) \times C_p(\mu^n; T^*) \quad \text{by (6)}$$

$$\sim \left( \prod C_p(\mu^n) \right)_c \times C_p(T^*) \times C_p(\mu^n; T^*) \quad \text{by (7)}$$

$$\sim \left( \prod C_p(\mu^n) \right)_c \times C_p(\mu^n) \sim \left( \prod C_p(\mu^n) \right)_c.$$

2.10 **Theorem.** Let $X$ be a metric space and $\tau$ be an infinite cardinal. Suppose $Y$ is an $I^*$-embedded subspace of the product $X^\tau$ and $Y$ contains an $I^*$-embedded copy of $X^\tau$. Then $C_p(Y) \sim C_p(X^\tau)$.

**Proof.** By Corollary 2.4 and Lemma 2.7(ii), it is enough to show that $C_p(X^\tau) \sim \left( \prod C_p(X^\tau) \right)_c$. Since $\tau$ is infinite we have $X^\tau = (X^\omega)^\tau$. So we can suppose that $X$ is not discrete. Thus, there exists a nontrivial converging sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ with $\lim x_n = x_0$. Let $T = \{x_0, x_n; n \in \mathbb{N}\}$. By Lemma 2.1, $X^\tau \times T$ is $I^*$-embedded in $X^\tau \times X$. Therefore,

$$C_p(X^\tau) \sim C_p(X^\tau \times T) \times C_p(X^\tau \times X; X^\tau \times X).$$

But $C_p(X^\tau \times T) \sim C_p(X^\tau \times \{x_0\}) \times C_p(X^\tau \times T; X^\tau \times \{x_0\})$ because $X^\tau \times \{x_0\}$ is also $I^*$-embedded in $X^\tau \times T$. Since $X^\tau \times T$ is a $k_R$-space [N] we have $C_p(X^\tau \times T; X^\tau \times \{x_0\}) \sim \left( \prod C_p(X^\tau) \right)_c$. Hence,

$$C_p(X^\tau \times T) \sim C_p(X^\tau \times \{x_0\}) \times \left( \prod C_p(X^\tau) \right)_c \sim \left( \prod C_p(X^\tau) \right)_c \times C_p(X^\tau \times T \times \{x_0\}).$$

Then

$$C_p(X^\tau) \sim C_p(X^\tau \times T) \times C_p(X^\tau \times X; X^\tau \times X)$$

$$\sim \left( \prod C_p(X^\tau) \right)_c \times C_p(X^\tau \times T) \times C_p(X^\tau \times X; X^\tau \times T)$$

$$\sim \left( \prod C_p(X^\tau) \right)_c \times C_p(X^\tau) \sim \left( \prod C_p(X^\tau) \right)_c.$$
2.11 Corollary. Let \( X \) be a separable metric space and \( \tau > \omega \). Then \( C_p(X^\tau) \sim C_p(Y) \) for every closed \( G_\delta \)-subset \( Y \) of \( X^\tau \).

Proof. Suppose \( Y \) is a closed \( G_\delta \)-subset of \( X^\tau \). It is well known (see for example [PP]) that modulo a permutation of the coordinates, \( Y = Z \times X^{\tau - \omega} \), where \( Z \) is a closed subset of \( X^{\omega} \). Thus, by Lemma 2.1, \( Y \) is \( l^\tau \)-embedded in \( X^\tau \). On the other hand \( \{z\} \times X^{\tau - \omega} \) is an \( l^\tau \)-embedded copy of \( X^\tau \) in \( Y \) for each \( z \in Z \). Now, Theorem 2.10 completes the proof.

2.12 Corollary. Let \( U \) be a functionally open subset of \( R^\tau \), \( \tau \geq \omega \). Then \( C_p(U) \sim C_p(R^\tau) \).

Proof. Modulo a permutation of the coordinates, \( U = V \times R^{\tau - \omega} \), where \( V \) is open in \( R^{\omega} \). Obviously, \( U \) contains an \( l^\tau \)-embedded copy of \( R^\tau \). Since there is an embedding of \( V \) in \( R^{\omega} \) as a closed subset, by Lemma 2.1, \( U \) can be \( l^\tau \)-embedded in \( R^\tau \). Thus, by Theorem 2.10, \( C_p(U) \sim C_p(R^\tau) \).

Let \( f \) be a mapping from a space \( X \) onto a space \( Y \). Recall that a continuous linear operator \( u: C_p(X) \rightarrow C_p(Y) \) is said to be an averaging operator for \( f \) if \( u(h \circ f) = h \) for every \( h \in C_p(Y) \). If \( f \) admits a regular averaging operator \( u: C_p(X) \rightarrow C_p(Y) \) we can define a mapping \( r: Y \rightarrow P_{\omega}(X) \) by the formula \( r(y)(g) = u(g)(y) \). The mapping \( r \) has the following property [Dr1]: \( \text{supp}(r(y)) \) is contained in \( f^{-1}(y) \) for each \( y \in Y \). Conversely, if there is a mapping \( r: Y \rightarrow P_{\omega}(X) \) such that \( \text{supp}(r(y)) \subset f^{-1}(y) \) for every \( y \in Y \), then the formula \( u(g)(y) = r(y)(g) \) defines a regular averaging operator \( u \) for \( f \).

It is easily seen that if \( u \) is a regular averaging operator for \( f \) the mapping \( v(g) = (u(g), g - u(g) \circ f) \) is a linear homeomorphism from \( C_p(X) \) onto \( C_p(Y) \times E \), where \( E = \{g - u(g) \circ f; g \in C_p(X)\} \). Dranishnikov proved [Dr1, Theorem 9] that \( C_p(R^n) \sim C_p(U) \) for every open subset \( U \) of \( R^n \). The same arguments are used in the proof of Proposition 2.13 below.

2.13 Proposition. Let \( \{U_i; i \in N\} \) be an infinite locally finite functionally open cover of a space \( X \). Suppose there is a space \( Y \) with \( C_p(\text{cl}_X(U_i)) \sim C_p(Y) \) for each \( i \in N \). Then \( C_p(X) \sim C_p(Y)^\omega \) provided \( X \) contains an \( l \)-embedded copy of a topological sum \( \bigoplus_{i=1}^\infty F_i \) such that \( C_p(F_i) \sim C_p(Y) \) for every \( i \in N \).

Proof. For every \( i \in N \) take an \( f_i \in C_p(X) \) such that \( f^{-1}_i(0) = X - U_i \) and \( f_i \geq 0 \). Without loss of generality we can suppose that \( \sum_{i=1}^\infty f_i = 1 \). Let \( f \in C_p(\sum \text{cl}_X(U_i)) \) such that \( f|\text{cl}_X(U_i) = f_i|\text{cl}_X(U_i) \). Consider the natural mapping \( p: \sum \text{cl}_X(U_i) \rightarrow X \) with all preimages finite. Let \( r: X \rightarrow P_{\omega}(\sum \text{cl}_X(U_i)) \) be defined by \( r(x) = \sum \{f(y) \cdot \delta_y; y \in p^{-1}(x)\} \). It is easily seen that \( r \) is continuous and \( \text{supp}(r(x)) \subset p^{-1}(x) \) for every \( x \in X \). Thus, there is a regular averaging operator \( u: C_p(\sum \text{cl}_X(U_i)) \rightarrow C_p(X) \) for \( p \). Hence, \( C_p(\sum \text{cl}_X(U_i)) \) is linearly homeomorphic to \( C_p(X) \times E \), where \( E \) is a linear subspace of \( C_p(\sum \text{cl}_X(U_i)) \). Since \( \sum F_i \) is \( l \)-embedded in \( X \) we have \( C_p(X) \sim C_p(\sum F_i) \times C_p(\sum F_i) \). Observe that
\[
C_p \left( \sum \text{cl}_X(U_i) \right) \sim \prod_{i=1}^\infty C_p(\text{cl}_X(U_i)) \sim C_p(Y)^\omega \sim C_p \left( \sum F_i \right).
\]
Now, using the technique of Pelczynski [P] and Bessaga [B] we have
\[ C_p(X) \sim C_p \left( \sum F_i \right) \times C_p \left( X \setminus \sum F_i \right) \sim C_p(Y) \omega \times C_p \left( X \setminus \sum F_i \right) \]
\[ \sim (C_p(Y) \omega \times \ldots \times C_p(Y) \omega \times \ldots) \times C_p(Y) \omega \times C_p \left( X \setminus \sum F_i \right) \]
\[ \sim (C_p(X) \times E \times \ldots \times C_p(X) \times E \times \ldots) \times C_p(X) \]
\[ \sim C_p(X) \omega \times E \omega \sim (C_p(X) \times E) \omega \sim C_p \left( \sum \text{cl}_X(U_i) \right) \omega \sim C_p(Y) \omega. \]

2.14 Theorem. Let \( Y \) be a noncompact separable metric space and \( X \) be one of the spaces \( Q, I^n, \mu^n, l_2 \). Then \( C_p(Y) \sim C_p(X) \omega \) provided \( Y \) is an \( X \)-manifold.

Proof. Let \( \{ U_i : i \in \mathbb{N} \} \) be an infinite locally finite open cover of \( Y \) such that each \( \text{cl}_Y(U_i) \) is regularly closed subset of \( X \). It is clear that a topological sum \( \sum F_i \) of infinitely many regularly closed subsets \( F_i \) of \( X \) is contained in \( Y \) as a closed subset. Since each of the sets \( \text{cl}_Y(U_i) \) and \( F_i, i \in \mathbb{N} \), contains a closed copy of \( X \), it follows from Theorem 2.8, Theorem 2.9 and Theorem 2.10 that \( C_p(\text{cl}_Y(U_i)) \sim C_p(F_i) \sim C_p(X) \) for every \( i \in \mathbb{N} \). Hence, by Proposition 2.13, \( C_p(Y) \sim C_p(X) \omega \).

2.15 Theorem. Let \( U \) be a functionally open subset of \( I^\tau \) and \( \tau \) be an uncountable cardinal. Then \( C_p(U) \sim C_p(I^\tau) \omega \).

Proof. There exists a projection \( p \) from \( I^\tau \) onto a countable face of \( I^\tau \) such that \( p^{-1}(p(U)) = U \) (see [PP]). Take a locally finite open cover \( \{ U_i : i \in \mathbb{N} \} \) of \( p(U) \) such that \( \text{cl}_{I^\tau}(p^{-1}(U_i)) \subset U \) for every \( i \in \mathbb{N} \). Since each \( \text{cl}_{I^\tau}(p^{-1}(U_i)) \) is a closed \( G_\delta \)-subset of \( I^\tau \), by Corollary 2.11, \( C_p(\text{cl}_{I^\tau}(p^{-1}(U_i))) \sim C_p(I^\tau) \).

Now, let \( \{ x_i : i \in \mathbb{N} \} \) be a closed discrete infinite subset of \( p(U) \). So, the topological sum \( \sum p^{-1}(x_i) \) is \( l \)-embedded in \( U \) (by Lemma 2.1) and obviously, each \( p^{-1}(x_i) \) is homeomorphic to \( I^\tau \). Thus, by Proposition 2.13, \( C_p(U) \sim C_p(I^\tau) \omega \).

2.16 Theorem. Let \( X \) be one of the spaces \( Q, I^n, \mu^n, \) and \( Y \) be a locally compact subset of an \( X \)-manifold. Then \( C_p(Y) \sim C_p(X) \omega \) if and only if \( Y \) contains a closed copy of the topological sum \( \sum X \) of infinitely many copies of \( X \).

Proof. The proof of the part "if" is based on a Dranishnikov's idea from [Dr1, Theorem 9'], where it is shown that \( C_p(P) \sim C_p(R^n) \) for every locally compact subset \( P \) of \( R^n \) with \( \text{cl}_{R^n}(\text{Int}(P)) \cap (R^n - P) \neq \emptyset \).

Suppose \( Y \) is a locally compact subspace of an \( X \)-manifold \( Z \) and contains a closed copy of the topological sum \( \sum X \). Then \( C_p(Y) \sim C_p(\sum X) \times C_p(Y \setminus \sum X) \). Next, take a locally finite open cover \( \{ V_i : i \in \mathbb{N} \} \) of \( Y \) such that each \( \text{cl}_Y(V_i) \) is compact. For every \( i \in \mathbb{N} \) there exists an open subset \( U_i \).
of $Z$ such that $V_i = U_i \cap Y = U_i \cap \text{cl}_Y(V_i)$. Since every set $V_i$ is closed in $U_i$, $\sum V_i$ is closed in $\sum U_i$. Thus, $C_p(\sum U_i) \sim C_p(\sum V_i) \times C_p(\sum U_i ; \sum V_i)$. Let $\{f_i : i \in N\}$ be a partition of unity subordinated to the cover $\{V_i : i \in N\}$. Define a continuous mapping $r: Y \to P_\infty(\sum V_i)$ as in the proof of Proposition 2.13 and by the same arguments we get that $C_p(\sum V_i)$ is linearly homeomorphic to $C_p(Y) \times E$, where $E$ is a linear subspace of $C_p(\sum V_i)$. It follows from Theorem 2.14 that $C_p(U_i) \sim C_p(X)\omega$ for every $i \in N$. Hence

$$C_p(X)\omega \sim C_p\left(\sum U_i\right) \sim C_p\left(\sum V_i\right) \times C_p\left(\sum U_i ; \sum V_i\right)$$

$$\sim C_p(Y) \times E \times C_p\left(\sum U_i ; \sum V_i\right).$$

Now, using the scheme of Pelczynski and Bessaga we get $C_p(Y) \sim C_p(X)\omega$.

Suppose there is a linear homeomorphism $\theta$ from $C_p(\sum X) = C_p(X)\omega$ onto $C_p(Y)$. Let $K$ be the set $\{y \in Y; \text{ every neighborhood of } y \text{ in } Y \text{ contains a copy of } X\}$. We use the following property of $X$ (for $Q$ and $I^n$ this is obvious, and for $\mu^n$ see [Bt]):

(*) Every open subset of $X$ contains a copy of $X$.

Now we show that $K$ is nonempty. Indeed, by [Pv], $Y$ contains an open subset of $\sum X$. So, by (*), $Y$ contains a copy $F$ of $X$ and $F \subset K$. Obviously $K$ is closed in $Y$ and it follows also from (*) that $Y - K$ does not contain a copy of $X$. Next, assume $K$ is compact. Consider the set

$$L = \text{cl}\left(\bigcup\{\text{supp}(\theta^*(\delta_y)): y \in K\}\right),$$

where $\theta^*: L_p(Y) \to L_p(\sum X)$ is the dual homeomorphism of $\theta$. By a result of Arhangel'skii [A2], $L$ is a compact subset of $\sum X$. Therefore, there is a $k \in N$ such that $L \subset \sum_{i=1}^k X_i$. Let $P = \sum_{i=1}^k X_i$, $f \in C_p(\sum X; P)$ and $y \in K$. We have $\theta^*(\delta_y)(f) = \delta_y(\theta(f)) = \theta(f)(y)$. But $\theta^*(\delta_y)(f) = 0$ because $\text{supp}(\theta^*(\delta_y)) \subset P$. Thus, $\theta(f)$ belongs to $C_p(Y; K)$ for every $f \in C_p(\sum X; P)$. Let $p$ be the linear projection from $C_p(\sum X) = C_p(P) \times C_p(\sum X; P)$ onto $C_p(\sum X; P)$. Then $\theta \circ p \circ \theta^{-1}: C_p(Y; K) \to \theta(C_p(\sum X; P))$ is a continuous linear retraction. This means that there is a closed linear subspace $E$ of $C_p(Y; K)$ such that $C_p(Y; K)$ is linearly homeomorphic to $C_p(\sum X; P) \times E$. Clearly, $C_p(Y; K) \sim C_p(Y/K; (K))$, where $(K)$ is the identification point of $K$ in the quotient space $Y/K$. Analogously, $C_p(\sum X; P) \sim C_p((\sum X)/P; (P))$. Since $C_p(Y/K) \sim R \times C_p(Y/K; (K))$ and

$$C_p\left(\left(\sum X\right)/P; (P)\right) \times R \sim C_p\left(\left(\sum X\right)/P\right),$$

we get that $C_p(Y/K) \sim C_p((\sum X)/P) \times E$. Now, we need the following result of Dranishnikov [Dr1, Theorem 6]: Let $X_1$ and $X_2$ be compact metric spaces and $C_p(X_1)$ be linearly homeomorphic to a product $C_p(X_2) \times E_1$. Then $\dim X_2 \leq \dim X_1$. Actually, it is proved that $X_2$ is a union of countably many compact subsets which are embeddable in $X_1$. It follows from Dranishnikov's arguments that the last statement remains valid if $X_1$ and $X_2$ are separable locally compact.
metric spaces. Hence, there is a countable family \( \{F_i : i \in N\} \) of compact subsets of \( (\sum X)/P \) such that \( (\sum X)/P = \bigcup \{F_i : i \in N\} \) and each \( F_i \) can be embedded in \( Y/K \). Since \( (\sum X)/P \) has the Baire property, there exists an \( i_0 \in N \) with \( \text{Int}(F_{i_0}) \neq \emptyset \). Then the set \( \text{Int}(F_{i_0}) - \{(P)\} \) is both open in \( \sum X \) and embeddable in \( Y/K \). Thus, by (\( \ast \)), \( Y/K \) contains a copy of \( X \). So \( Y - K \) contains also a copy of \( X \). But we have already seen that this is not possible. Therefore \( K \) is not compact.

Take a countable infinite discrete family \( \{W_i : i \in N\} \) in \( K \) consisting of open subsets of \( K \). Let \( W_i^* \) be an open subspace of \( Y \) with \( W_i^* \cap K = W_i \) for each \( i \in N \). For every \( i \in N \) there is a copy \( X_i \) of \( X \) such that \( X_i \subset W_i^* \). It follows from (\( \ast \)) that \( X_i \subset K \) because \( Y - K \) does not contain a copy of \( X \). Hence, \( X_i \subset W_i \) for every \( i \in N \). So \( \{X_i : i \in N\} \) is a discrete family in \( K \). Thus, \( \sum X_i \) is a closed subset of \( Y \).

2.17 Corollary. Let \( X \) be a locally compact (n-dimensional) separable metric space. Then \( C_p(X) \sim C_p(Q)^{\alpha} \) (resp., \( C_p(X) \sim C_p(\mu^\alpha) \)) if and only if \( X \) contains a closed copy of the topological sum \( \sum Q \) (resp., \( \sum \mu^\alpha \)).

Proof. Since \( X \) can be embedded in \( Q \) (resp., in \( \mu^\alpha \)), the proof follows from Theorem 2.16.

3. Linear topological classifications of \( C_p^*(X) \)

The proofs of the Lemmas 3.1–3.4 below are similar to the proofs of the corresponding lemmas from §2.

3.1 Lemma. Let \( A \) be an \( l^* \)-embedded subset of a space \( X \). Then \( (\prod C_p^*(X))^*_{\tau} \sim (\prod C_p^*(A))^*_{\tau} \times (\prod C_p^*(X;A))^*_{\tau} \).

3.2 Lemma. Let \( A \) be an \( l^* \)-embedded subset of a space \( X \). If every closed bounded subset of \( A \) is compact then \( (\prod C_p^*(X \times Y))^*_{\tau} \sim (\prod C_p^*(A \times Y))^*_{\tau} \times (\prod C_p^*(X \times Y))^*_{\tau} \) for any space \( Y \).

3.3 Corollary. Let \( A \) be an \( l^* \)-embedded subset of a product \( X \) of metric spaces. Then
\[
(\prod C_p^*(X))^*_{\tau} \sim (\prod C_p^*(A))^*_{\tau} \times (\prod C_p^*(X;A))^*_{\tau}.
\]

3.4 Lemma. Suppose \( X \) is a space such that both \( X \times T \) and \( X \times I \) are \( k_R \)-spaces, where \( T = \{0, 1/n : n \in N\} \). Then we have \( C_p^*(X \times I) \sim (\prod C_p^*(X \times I))^*_{\tau} \).

3.5 Corollary. Let \( X = \sum I^\tau \) be a topological sum of infinitely many copies of \( I^\tau \), \( \tau \geq 1 \). Then \( C_p^*(X) \sim (\prod C_p^*(X))^*_{\tau} \).

3.6 Lemma. Suppose a space \( X \) contains an \( l^* \)-embedded copy \( F_1 \) of a space \( Y \) and \( Y \) contains an \( l^* \)-embedded copy \( F_2 \) of \( X \). Then:
(i) \( C_p^*(X) \sim (\prod C_p^*(X))^*_{\tau} \sim C_p^*(Y) \) if \( C_p^*(Y) \sim (\prod C_p^*(Y))^*_{\tau} ; \)
(ii) \( C_p^*(X) \sim (\prod C_p^*(X))^*_{\tau} \sim C_p^*(Y) \) if \( C_p^*(Y) \sim (\prod C_p^*(Y))^*_{\tau} \times (\prod C_p^*(F_2))^*_{\tau} \).
Proof. Let $C^*_p(Y) \sim (\prod C^*_p(Y))_b$. Using the same arguments as in the proof of Lemma 2.7(i), one can show that $C^*_p(X) \sim C^*_p(Y)$. Next, by Lemma 3.1, we have

$$
(\prod C^*_p(X))_b \sim (\prod C^*_p(F_1))_b \times (\prod C^*_p(X; F_1))_b
$$

and

$$
(\prod C^*_p(Y))_b \sim (\prod C^*_p(F_2))_b \times (\prod C^*_p(Y; F_2))_b
$$

Thus,

$$
(\prod C^*_p(X))_b \sim (\prod C^*_p(F_1))_b \times (\prod C^*_p(X; F_1))_b
$$

Using the same arguments we can prove that $(\prod C^*_p(X))_b \sim C^*_p(X)$ if $C^*_p(Y) \sim (\prod C^*_p(F_2))_b \times (\prod C^*_p(Y; F_2))_b$. 

3.7 Corollary. Let $\{X_i: i \in \mathbb{N}\}$ be an infinite family of spaces such that each $X_i$ is strongly $l$-embedded in a space $Y$ and contains a strongly $l$-embedded copy $Y_i$ of $Y$. Then $C^*_p(\sum Y_i) \sim (\prod C^*_p(\sum Y_i))_b \sim C^*_p(\sum X_i)$ if $C^*_p(\sum Y_i) \sim (\prod C^*_p(\sum Y_i))_b$.

Proof. Let for each $i$ $u_i: C_p(X_i) \rightarrow C_p(Y)$ be a regular extension operator. Then the mapping $u: C_p(\sum X_i) \rightarrow C_p(\sum Y_i)$, defined by $u(f) = \sum u_i(f|X_i)$ is also a regular extension operator. Thus, $\sum X_i$ is $l^*$-embedded in $\sum Y_i$. Analogously, $\sum Y_i$ is $l^*$-embedded in $\sum X_i$. Now the proof follows from Lemma 3.6(i).

3.8 Theorem. Let $X$ be a metric space and $\tau$ be an infinite cardinal. Suppose $Y$ is an $l^*$-embedded subspace of the product $X^\tau$ and $Y$ contains an $l^*$-embedded copy of $X^\tau$. Then $C^*_p(Y) \sim C^*_p(X^\tau)$.

Proof. By Corollary 3.3 and Lemma 3.6(ii), it is enough to show that $C^*_p(X^\tau) \sim (\prod C^*_p(X^\tau))_c$. The last can be proved using the same arguments as in the proof of Theorem 2.10.

3.9 Corollary. Let $X$ be a separable metric space and $\tau > \omega$. Then $C^*_p(X^\tau) \sim C^*_p(Y)$ for every closed $G_\delta$-subset $Y$ of $X^\tau$. 

3.10 Corollary. Let \( U \) be a functionally open subset of \( R^\tau, \tau \geq \omega \). Then \( C^*_p (R^\tau) \sim C^*_p (U) \).

The proofs of Corollaries 3.9 and 3.10 are similar respectively to the proofs of Corollaries 2.11 and 2.12.

3.11 Proposition. Let \( \sum \mu^n_i \) be a topological sum of infinitely many copies of the \( n \)-dimensional Menger compactum. Then \( C^*_p (\sum \mu^n_i) \sim (\prod C^*_p (\sum \mu^n_i))^c \).

Proof. For each \( i \in N \) take a mapping \( f^n_i \) from \( \mu^n_i \) onto a copy \( Q_i \) of the Hilbert cube \( Q \) such that \( (f^n_i)^{-1}(P) \) is homeomorphic to \( \mu^n \) for every \( LC^{n-1} \& C^{n-1} \)-compact subspace \( P \) of \( Q_i \) (see [Dr2]). Define \( f_n: \sum \mu^n_i \rightarrow \sum Q_i \) by \( f_n|\mu^n_i = f^n_i \). Consider \( Q_i \) as a product \( Q^i_1 \times I \), where \( Q^i_1 \) is a copy of \( Q \).

Let \( T_i = Q_i^1 \times \{0, 1/k : k \in N\} \) and \( T = f_n^{-1}(\sum T_i) \). Then we have

\[
C^*_p \left( \sum \mu^n_i \right) \sim C^*_p (T) \times C^*_p \left( \sum \mu^n_i; T \right)
\]

and

\[
C^*_p (T) \sim C^*_p \left( \prod \left( (Q^i_1 \times \{0\}) \right) \right) \times C^*_p \left( T; f_n^{-1} \left( \left( \sum (Q^i_1 \times \{0\}) \right) \right) \right).
\]

Since each of the sets \( f_n^{-1} \left( \sum (Q^i_1 \times \{0\}) \right) \) and \( f_n^{-1} \left( \sum (Q^i_1 \times \{1/k\}) \right) \) for \( k \in N \) is homeomorphic to \( \sum \mu^n_i \), the following holds

\[
C^*_p \left( f_n^{-1} \left( \sum (Q^i_1 \times \{0\}) \right) \right) \sim C^*_p \left( \sum \mu^n_i \right)
\]

and

\[
C^*_p \left( T; f_n^{-1} \left( \sum (Q^i_1 \times \{0\}) \right) \right) \sim \left( \prod C^*_p \left( \sum \mu^n_i \right) \right)^c.
\]

Thus,

\[
C^*_p (T) \sim \left( \prod C^*_p \left( \sum \mu^n_i \right) \right)^c \times \left( \prod C^*_p \left( \sum \mu^n_i \right) \right)^c
\]

\[
= \left( \prod C^*_p \left( \sum \mu^n_i \right) \right)^c \times \left( \prod C^*_p \left( \sum \mu^n_i \right) \right)^c.
\]

Finally we get

\[
C^*_p \left( \sum \mu^n_i \right) \sim C^*_p (T) \times C^*_p \left( \sum \mu^n_i; T \right)
\]

\[
\sim \left( \prod C^*_p \left( \sum \mu^n_i \right) \right)^c \times C^*_p (T) \times C^*_p \left( \sum \mu^n_i; T \right)
\]

\[
\sim \left( \prod C^*_p \left( \sum \mu^n_i \right) \right)^c \times C^*_p \left( \sum \mu^n_i \right) \sim \left( \prod C^*_p \left( \sum \mu^n_i \right) \right)^c.
\]

3.12 Lemma. Suppose \( p \) is a mapping from a space \( X \) onto a space \( Y \) such that for every compact subset \( K \) of \( Y \) the preimage \( p^{-1}(K) \) is also compact.
Let \( p \) admit a regular averaging operator \( u: C_p(X) \to C_p(Y) \). Then \( C_p^*(X) \sim C_p^*(Y) \times E_1 \) and \( \left( \prod C_p^*(Y) \right)_c \sim \left( \prod C_p^*(Y) \right)_c^* \times \left( \prod E_1 \right)_c^* \), where \( E_1 = \{ g - u(g) \circ p : g \in C_p^*(X) \} \).

**Proof.** Consider the mapping \( r: Y \to P_\infty(X) \) defined by \( r(y)(g) = u(g)(y) \) for all \( g \in C_p(X) \). We have \( \text{supp}(r(y)) \subset p^{-1}(y) \) for each \( y \in Y \). The last implies that \( \|u(g)\|_K \leq \|g\|_{p^{-1}(K)} \) for every \( g \in C_p^*(X) \) and \( K \subset Y \). Hence, \( u(C_p^*(X)) = C_p^*(Y) \) and the mapping \( v(g) = (u(g), g - u(g) \circ p) \) is a linear homeomorphism from \( C_p^*(X) \) onto \( C_p^*(Y) \times E_1 \). Next, let \((g_1, \ldots, g_n, \ldots) \in \left( \prod C_p^*(X) \right)_c \) and \( K \) be a compact subset of \( Y \). Since, \( \|u(g_n)\|_K \leq \|g_n\|_{p^{-1}(K)} \) and \( p^{-1}(K) \) is compact, we have \((u(g_1), \ldots, u(g_n), \ldots) \in \left( \prod C_p^*(Y) \right)_c \) and \((g_1 - u(g_1) \circ p, \ldots, g_n - u(g_n) \circ p, \ldots) \in \left( \prod E_1 \right)_c^* \). Obviously, \((g_1, \ldots, g_n, \ldots) \in \left( \prod C_p^*(X) \right)_c \) if \((u(g_1), \ldots, u(g_n), \ldots) \in \left( \prod C_p^*(Y) \right)_c \) and \((g_1 - u(g_1) \circ p, \ldots, g_n - u(g_n) \circ p, \ldots) \in \left( \prod E_1 \right)_c^* \). Thus, the mapping

\[
v_0(g_1, \ldots, g_n, \ldots) = ((u(g_1), \ldots, u(g_n), \ldots), (g_1 - u(g_1) \circ p, \ldots, g_n - u(g_n) \circ p, \ldots))
\]

is a linear homeomorphism from \( \left( \prod C_p^*(X) \right)_c \) onto \( \left( \prod C_p^*(Y) \right)_c \times \left( \prod E_1 \right)_c^* \).

3.13 **Proposition.** Let \( \{U_i : i \in \mathbb{N}\} \) be an infinite locally finite functionally open cover of a space \( X \). Suppose there is a space \( Y \) such that \( C_p^*(Y) \sim C_p^*(\sum \text{cl}_X(U_i)) \sim \left( \prod C_p^*(\sum \text{cl}_X(U_i)) \right)_c^* \). Then \( C_p^*(X) \sim C_p^*(Y) \) if \( X \) contains an \( l^* \)-embedded copy of \( Y \).

**Proof.** There exists a natural mapping \( p \) from \( \sum \text{cl}_X(U_i) \) onto \( X \) such that \( p^{-1}(K) \) is compact for every compact subset \( K \) of \( X \). As in the proof of Proposition 2.13 we conclude that \( p \) admits a regular averaging operator \( u: C_p \left( \sum \text{cl}_X(U_i) \right) \to C_p(X) \).

By Lemma 3.12, \( \left( \prod C_p^*(\sum \text{cl}_X(U_i)) \right)_c^* \sim \left( \prod C_p^*(\sum \text{cl}_X(U_i)) \right)_c^* \times \left( \prod E_1 \right)_c^* \), where \( E_1 = \{ g - u(g) \circ p : g \in C_p^*(\sum \text{cl}_X(U_i)) \} \). Since \( Y \) is \( l^* \)-embedded in \( X \), \( C_p^*(X) \sim C_p^*(Y) \times C_p^*(X; Y) \). Then we have

\[
C_p^*(X) \sim C_p^*(Y) \times C_p^*(X; Y) \sim \left( \prod C_p^* \left( \sum \text{cl}_X(U_i) \right) \right)_c^* \times C_p^*(X; Y)
\]

\[
\sim \left( \prod C_p \left( \sum \text{cl}_X(U_i) \right) \right)_c^* \times C_p^* \left( \sum \text{cl}_X(U_i) \right) \times C_p^*(X; Y)
\]

\[
\sim \left( \prod C_p \left( \sum \text{cl}_X(U_i) \right) \right)_c^* \times C_p^*(Y) \times C_p^*(X; Y)
\]

\[
\sim \left( \prod C_p \left( \sum \text{cl}_X(U_i) \right) \right)_c^* \times C_p^*(X)
\]

\[
\sim \left( \prod C_p^*(X) \right)_c^* \times \left( \prod E_1 \right)_c^* \times C_p^*(X) \sim \left( \prod C_p^*(X) \right)_c^* \times \left( \prod E_1 \right)_c^*
\]

\[
\sim \left( \prod C_p \left( \sum \text{cl}_X(U_i) \right) \right)_c^* \sim C_p^*(Y).
\]
3.14 **Theorem.** Suppose $X$ is a noncompact $Y$-manifold, where $Y$ is one of the spaces $Q$, $I^n$, $\mu^n$, $l_2$. Then $C_p^*(X) \sim C_p^*(\sum Y)$.

*Proof.* Let $\{U_i; i \in N\}$ be an infinite locally finite open cover of $X$ such that each $cl_X(U_i)$ is regularly closed subset of $Y$. By Corollary 3.5, Proposition 3.11 and Theorem 3.8 we have $C_p^*(\sum Y) \sim (\prod C_p^*(\sum Y))^*$. Since each set $cl_X(U_i)$ is closed in $Y$ and contains a closed copy of $Y$, it follows from Corollary 3.7 that $(\prod C_p^*(\sum cl_X(U_i)))^* \sim C_p^*(\sum Y)$. Obviously $X$ contains a closed copy of $\sum Y$. Thus, by Proposition 3.13, $C_p^*(X) \sim C_p^*(\sum Y)$.

3.15 **Theorem.** Let $U$ be a functionally open subset of $I^\omega$ and $\tau$ be an uncountable cardinal. Then $C_p^*(U) \sim C_p^*(\sum I^\omega)$.

*Proof.* Take a projection $p$ from $I^\omega$ onto a countable face $I^{\omega_0}$ of $I^\omega$ such that $p^{-1}(p(U)) = U$ (for the existence of such projection see [PP]). Now, let $\{U_i; i \in N\}$ be a locally finite open cover of $p(U)$ such that $cl_{p(U)}(U_i) \subset p(U)$ for each $i \in N$. Then $\{p^{-1}(U_i); i \in N\}$ is an infinite locally finite functionally open cover of $U$ with $cl_{p(U)}(p^{-1}(U_i)) \subset U$ for every $i \in N$. Since $p$ is an open mapping we have $cl_{p(U)}(p^{-1}(U_i)) = p^{-1}(cl_{p(U)}(U_i))$. Thus, by Lemma 2.1, each set $cl_{p(U)}(p^{-1}(U_i))$ is strongly $l$-embedded in $I^\omega$ and contains a strongly $l$-embedded copy of $I^\omega$. Hence, it follows from Corollary 3.5 and Corollary 3.7 that $C_p^*(\sum cl_{p(U)}(p^{-1}(U_i)))) \sim C_p^*(\sum I^\omega)$. On the other hand $U$ contains an $l^*$-embedded copy of $\sum I^\omega$ (see the proof of Theorem 2.15). Therefore, by Proposition 3.13, $C_p^*(U) \sim C_p^*(\sum I^\omega)$.

3.16 **Theorem.** Let $Y$ be one of the spaces $Q$, $I^n$, $\mu^n$ and $X$ be a locally compact subset of a $Y$-manifold. Then $C_p^*(X) \sim C_p^*(\sum Y)$ if $X$ contains a closed copy of $\sum Y$.

*Proof.* Let $X$ be a locally compact subspace of a $Y$-manifold $Z$ and let $X$ contain a closed copy of $\sum Y$. Then $C_p^*(X) \sim C_p^*(\sum Y) \times C_p^*(X; \sum Y)$. Take an infinite locally finite open cover $\{V_i; i \in N\}$ of $X$ such that each set $cl_X(V_i)$ is compact and $cl_X(V_i) \subset U_i$, where $U_i$ is an open subset of $Y$. Thus, each $cl_X(V_i)$ is contained in a copy $Y_i$ of $Y$. Let $u: C_p^*(\sum cl_X(V_i)) \rightarrow C_p(X)$ be a regular averaging operator for the natural mapping $p: \sum cl_X(V_i) \rightarrow X$. As in the proof of Proposition 3.13, we get $(\prod C_p^*(\sum cl_X(V_i)))^* \sim (\prod C_p^*(X))^* \times (\prod E)^*$, where $E$ is a linear subspace of $C_p^*(\sum cl_X(V_i))$. Since $\sum cl_X(V_i)$ is a closed subset of $\sum Y_i$, by Corollary 3.3 we have $(\prod C_p^*(\sum Y_i))^* \sim (\prod C_p^*(\sum cl_X(V_i)))^* \times (\prod G)^*$, where $G = C_p^*(\sum Y_i; \sum cl_X(V_i))$. Thus,

\[
(\prod C_p^*(\sum Y_i))^* \sim (\prod C_p^*(X))^* \times (\prod E)^* \times (\prod G)^*.
\]

Then

\[
C_p^*(X) \sim C_p^*(\sum Y) \times C_p^*(X; \sum Y) \\
\sim (\prod C_p^*(\sum Y))^* \times C_p^*(X; \sum Y)
\]
because \( C_p^*(\sum Y) \sim (\prod C_p^*(\sum Y))_c^* \) (see Corollary 3.5 and Proposition 3.11).

Hence
\[
C_p^*(X) \sim \left( \prod C_p^*(\sum Y) \right)_c^* \times C_p^*(X; \sum Y) \\
\sim \left( \prod C_p^*(\sum Y) \right)_c^* \times C_p^*(\sum Y) \times C_p^*(X; \sum Y) \\
\sim \left( \prod C_p^*(\sum Y) \right)_c^* \times C_p^*(X) \\
\sim C_p^*(X) \times \left( \prod C_p^*(X) \right)_c^* \times \left( \prod E \right)_c^* \times \left( \prod G \right)_c^* \\
\sim \left( \prod C_p^*(\sum Y) \right)_c^* \sim C_p^*(\sum Y).
\]

**Added in proof.** After this paper was submitted for publication Arhangel'skii [A4] introduced the notion of an \( S \)-stable space. A space \( X \) is \( S \)-stable if \( C_p(X) \sim C_p(X \times S) \), where \( S = \{0, 1/n, n \in \mathbb{N}\} \). Obviously, if \( X \times S \) is a \( k_R \)-space, then \( X \) is \( S \)-stable iff \( (\prod C_p(X))_c^* \sim C_p(X) \).

An elementary proof of the \( S \)-stability of \( \mu^n \) (without using Dranishnikov's results, see the proof of this fact in our Theorem 2.9) is given in [A4]. Arhangel'skii [A4] generalized our Theorem 2.8(ii) by proving that if a compact metric space \( X \) contains a subspace \( Y \) with \( C_p(Y) \sim C_p(Q) \) then \( C_p(X) \sim C_p(Q) \).

**References**


