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L^p BOUNDS FOR SPECTRAL MULTIPLIERS ON NILPOTENT GROUPS

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ABSTRACT. A criterion is given for the L^p boundedness of a class of spectral multiplier operators associated to left-invariant, homogeneous subelliptic second-order differential operators on nilpotent Lie groups, generalizing a theorem of Hörmander for radial Fourier multipliers on Euclidean space. The order of differentiability required is half the homogeneous dimension of the group, improving previous results in the same direction.

The Hörmander multiplier theorem on the group $\mathbb{R}^n$ gives a sufficient condition for a Fourier multiplier operator

$$\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi),$$

with $m \in L^\infty(\mathbb{R}^n)$, to extend to an operator bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$ and of weak type $(1, 1)$. Write $\mathbb{R}^+ = (0, \infty)$ and fix any auxiliary function $\eta \in C_0^\infty(\mathbb{R}^+)$, not identically zero. For $\beta \geq 0$ let L^2_β denote the L^2 Sobolev space of order β. Then the condition is that

$$\sup_t \|\eta(|\cdot|)m(t\cdot)\|_{L^2_\alpha(\mathbb{R}^n)} < \infty$$

for some $\alpha > n/2$ [H]. It is actually independent of the choice of η. Specialized to radial multipliers $m(|\xi|)$, the hypothesis becomes

$$\sup_t \|\eta(\cdot)m(t\cdot)\|_{L^2_\alpha(\mathbb{R})} < \infty, \tag{1}$$

still for some $\alpha > n/2$. This order of differentiability is essentially optimal, even in the radial case. For if $m(s) = s^{i\tau}$ for $s \in \mathbb{R}^+$, where $\tau \in \mathbb{R}$, then $\check{m}(x) = c_{n,\tau}|x|^{-n-i\tau}$ where $|c_{n,\tau}| \sim |\tau|^{n/2}$ as $|\tau| \to \infty$ [S2, pp. 51–52]. Thus the multiplier operator is readily seen to be of weak type $(1, 1)$ with a bound which grows at least as fast as $|\tau|^{n/2}$, which is comparable to the bound in (1) with $\alpha = n/2$.

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In the radial case we are dealing with spectral functions of the Laplace operator Δ. From this point of view it is more natural to write a multiplier operator as

$$\widehat{Tf}(\xi) = m(|\xi|^2)\hat{f}(\xi)$$

so that $T = m(\Delta)$. It is straightforward to check that m satisfies (1) if and only if $m(\cdot^2)$ does, so the formulation of the multiplier theorem is unaffected by the change of variables.

One can ask for results of this type for spectral multipliers of other operators. Consider the following class of subelliptic differential operators on nilpotent groups. Let $\mathbf{g}$ be a finite-dimensional nilpotent Lie algebra, and suppose that

$$\mathbf{g} = \bigoplus_{i=1}^{s} \mathbf{g}_i$$

as a vector space where $[\mathbf{g}_i, \mathbf{g}_j] \subset \mathbf{g}_{i+j}$ for all i, j, and that $\mathbf{g}_1$ generates $\mathbf{g}$ as a Lie algebra. Let G be the associated connected, simply connected Lie group. Consider any finite subset $\{X_k\}$ of $\mathbf{g}_1$ which spans $\mathbf{g}_1$. Each X_k may be identified with a unique left-invariant vector field on G, which we also denote by X_k. Then $\mathscr{L} = -\sum X_k^2$ is a left-invariant second-order differential operator. It is selfadjoint (its domain is $\{f \in L^2(G) : \mathscr{L}f \in L^2(G)\}$). $L^p(G)$ will be defined with respect to Haar measure, which is bi-invariant. $\mathscr{L}$ admits a spectral resolution $\mathscr{L} = \int_0^\infty \lambda \, dP_\lambda$. Concerning the spectral theorem we employ the notation and results in [RS]. Any bounded, Borel measurable function m on $[0, \infty)$ defines a bounded operator

$$m(\mathscr{L}) = \int_0^\infty m(\lambda)\, dP_\lambda \tag{2}$$

on $L^2(G)$. Associated to such a group is its so-called homogeneous dimension

$$D = \sum_j j \cdot \text{dimension}(\mathbf{g}_j).$$

Our multiplier theorem is then

Theorem 1. *Suppose m is a continuous function on $\mathbb{R}^+$ which satisfies* (1) *for some $\alpha > D/2$. Then $m(\mathscr{L})$ extends to an operator bounded on $L^p(G)$ for all $p \in (1, \infty)$ and of weak type* $(1, 1)$.

Corollary 2. *For each $p \in (1, \infty)$ and $\varepsilon > 0$ there exists $C < \infty$ such that for all $\tau \in \mathbb{R}$, $\mathscr{L}^{i\tau}$ is bounded on $L^p(G)$ with operator norm not exceeding*

$$C[1 + |\tau|]^{2|p^{-1}-2^{-1}|(D+\varepsilon)/2}.$$

Another special case is that of Riesz means: For $R > 0$ and $\beta \in \mathbb{C}$ let $\sigma_R(\lambda) = (1-(\lambda/R)^2)_+^\beta$. Then $\sigma_R(\mathscr{L})$ is bounded on L^p, $p \in (1, \infty)$, uniformly in R, provided the real part of β is larger than $(D-1)/2$. And $\sigma_R(\mathscr{L})f \to f$ in L^p norm for all $f \in L^p$, for the same range of β, as $R \to \infty$. See also

[Mü] for interesting sharper results involving mixed norms in the case of the Heisenberg group.

Hulanicki and Stein [FS, pp. 208–215] have shown that $\alpha > 3D/2+3$ suffices in Theorem 1. Hulanicki and Jenkins [HJ] proved that if m has compact support and belongs to C^k for some k larger than $D/2+1$ then $m(\mathscr{L})$ is given by convolution with an L^1 kernel. Earlier de Michele and Mauceri obtained a result on the Heisenberg group which concerns more general operators than spectral multipliers, but which assumes a higher order of differentiability for m. Müller [Mü] has also obtained related results on the Heisenberg group. A very general result of Stein [S1] applies when m is of Laplace transform type, a much more restrictive condition. We do not know whether the condition $\alpha > D/2$ is necessary, as in the Euclidean case.

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The main difficulty here is to translate comparatively abstract information on the spectral multiplier side into concrete control of convolution kernels, so that the Calderón-Zygmund theory of singular integrals might be invoked. Our proof is merely a refinement of the method of Hulanicki and Stein, who rely heavily on certain information on the fundamental solution for the heat equation

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \mathscr{L}\right) u(x, t) = 0 \quad \text{on } G \times [0, \infty), \\ u(x, 0) = f(x) \quad \text{on } G \end{cases}$$

or equivalently on the semigroup $e^{-t\mathscr{L}}$. Their basic estimate is that, roughly speaking, the fundamental solution decays exponentially as $|x| \to \infty$, for fixed t. Subsequently Jerison and Sánchez-Calle [JS] have shown[1] that the decay is $O(e^{-c|x|^2})$ (see below for the precise statement), which gives some immediate improvement on the result of Hulanicki and Stein. In order to obtain the putative optimal value $\alpha = D/2+\varepsilon$ we also need to exploit L^2 information derived from the spectral theorem in a sharper way.

To any L^∞ Borel measurable spectral multiplier m corresponds a unique tempered distribution, which we denote by $\check{m}$, on G such that

$$m(\mathscr{L})f = f * \check{m}$$

for all $f \in \mathscr{S}$, the Schwartz class on G. Let us hypothesize once and for all that all functions considered as spectral multipliers are to be Borel measurable. There is an analogue of Plancherel's theorem in our context:

[1]Sharper decay estimates have been proved by Davies [D], Kusuoka and Stroock [KS1, KS2], Melrose [M] and Varopoulos [V]. However in the present context of the sublaplacian on a stratified nilpotent group, the exponential squared decay is an immediate consequence of earlier work of Hulanicki and Jenkins, who prove that $\int h(x)e^{-\varepsilon|x|^2}\,dx < \infty$ for some $\varepsilon > 0$. See Lemma 1.13 of [HJ].

Proposition 3. *Let* $m \in L^\infty([0, \infty))$. *Then* $\check{m} \in L^2(G)$ *if and only if*

$$\int_0^\infty |m(\lambda)|^2 \lambda^{D/2} \frac{d\lambda}{\lambda} < \infty.$$

There exists a constant such that for all such m,

$$\|\check{m}\|_2^2 = c \int_0^\infty |m(\lambda)|^2 \lambda^{D/2} \frac{d\lambda}{\lambda}.$$

The proof is deferred until the end of the paper. We may assume henceforth that $D \geq 2$, whence $\alpha > 1$, for otherwise we are in $\mathbb{R}^1$ where all is known already.

For $r > 0$ define $\delta_r : \mathbf{g} \mapsto \mathbf{g}$ by $\delta_r X = r^j X$ for $X \in \mathbf{g}_j$, and extend by linearity. The exponential map identifies G with $\mathbf{g}$, so we may regard the δ_r instead as maps from G to itself. They are group automorphisms. Fix a homogeneous norm, that is, a function $|\cdot| : G \mapsto [0, \infty)$, C^∞ away from 0, with $|x| = 0 \Leftrightarrow x = 0$ where the latter 0 denotes the group identity element, and $|\delta_r x| = r|x|$ for all $r \in \mathbb{R}^+$, $x \in G$. Let $d\mu = t^{D/2}\, dt/t$ on $\mathbb{R}^+$.

Denote by $h_t(x)$ the heat kernel

$$e^{-t\mathscr{L}} f = f * h_t$$

for all $f \in L^2$. Then $h_t(x) = t^{-D/2} h(\delta_{t^{-1/2}} x)$, where $h \in \mathscr{S}$ (see for instance [FS]). Clearly $\|f * h_t\|_2 \to 0$ as $t \to \infty$ for all $f \in L^2$; in other words $\|e^{-t\mathscr{L}} f\|_2 \to 0$. Therefore $\| \int_0^\varepsilon dP_\lambda(f)\|_2 \to 0$ as $\varepsilon \to 0$. Consequently the point $\lambda = 0$ may be neglected in the spectral resolution, and we should regard our spectral multipliers as functions on $\mathbb{R}^+$ rather than on $[0, \infty)$. The fundamental estimate of [JS] is

$$|h(x)| \leq C e^{-c_0 |x|^2} \tag{3}$$

for some $c_0, C \in \mathbb{R}^+$.

Let m satisfy (1) for some $\alpha > D/2$, and for the present assume m is supported in $(\frac{1}{2}, 2)$. The strategy of the proof is to express m in terms of $\{e^{-t\mathscr{L}} : t \in \mathbb{R}^+\}$.[2] Following Hulanicki and Stein we make the substitution $\kappa = e^{-\lambda}$ and set

$$n(\kappa) = m(\lambda) = m(-\log \kappa).$$

n is supported on $(e^{-2}, e^{-1/2})$; extend it by 0 so as to regard it as a function on $[-\pi, \pi]$. Then $n \in L^2_\alpha$ since this space is diffeomorphism-invariant. We wish to decompose n as the sum of blocks $\sum_{2^j \leq |l| < 2^{j+1}} a_l e^{il\kappa}$, where j runs from 0 to ∞, but at the same time would like the blocks to inherit the property of vanishing identically in some fixed neighborhood of 0. The following elementary result is an adequate compromise. It is proved by taking the Fourier series for n and introducing an appropriately smooth partition of unity in the Fourier transform variable.

[2] Not every such expression helps; we were unable to obtain even Corollary 2 from the subordination identity $\mathscr{L}^{i\tau} = c_\tau \int_0^\infty e^{-t\mathscr{L}}\, t^{-i\tau} \frac{dt}{t}$.

Lemma 4. *n may be decomposed on $[-\pi, \pi]$ as $\sum_{j=0}^{\infty} n_j$ where n_j is periodic, so that*

(1) $\hat{n}_j(l) = 0$ *unless* $2^j \le |l| \le 2^{j+2}$ $(0 \le l \le 4$ *when* $j = 0)$,

(2) $\|n_j\|_{L^2(d\kappa)} \le C2^{-j\alpha}\|m\|$,

(3) $|n_j(\kappa)| + |\frac{d}{d\kappa} n_j(\kappa)| \le C_N 2^{-jN}\|m\|$ *for all* $|\kappa| \le e^{-4}$, *for all* $N < \infty$.

$\|m\|$ will denote $\|m\|_{L^2_\alpha(\mathbb{R}^+)}$. $\widehat{\ }$ denotes the Fourier transform on $[-\pi, \pi]$. Condition (2) just expresses the fact that $n \in L^2_\alpha$, while (3) results from the smoothness of the partition of unity in the variable l together with the vanishing of n near 0.

Since $n(0) = 0$, $n(\kappa) = \sum [n_j(\kappa) - n_j(0)]$. For $\lambda \in \mathbb{R}^+$ set

$$m_j(\lambda) = n_j(e^{-\lambda}) - n_j(0)$$

so that $m = \sum m_j$. Then

$$\begin{aligned} \|m_j\|^2_{L^2(d\mu)} &= \int_0^1 |n_j(\kappa) - n_j(0)|^2 (\log \kappa^{-1})^{D/2-1} \frac{d\kappa}{\kappa} \\ &\le C \int_{e^{-4}}^1 [|n_j(\kappa)|^2 + |n_j(0)|^2]\, d\kappa \\ &\quad + \int_0^{e^{-4}} |n_j(\kappa) - n_j(0)|^2 (\log \kappa^{-1})^{D/2-1} \frac{d\kappa}{\kappa} \\ &\le C2^{-2j\alpha}\|m\|^2 \end{aligned}$$

if N is chosen large enough in the lemma. Therefore by Proposition 3,

$$\|\check{m}_j\|_2 \le C2^{-j\alpha}\|m\|. \tag{4}$$

Each $\check{m}_j(x)$ decays quite rapidly as $|x| \to \infty$:

Lemma 5. *There exist $B, C < \infty$ and $\rho < 1$ such that for all $j \ge 0$ and all $|x| \ge B2^j$,*

$$|\check{m}_j(x)| \le C\rho^{|x|}\|m\|.$$

For the proof set $N = 2^j$ and assume $\|m\| = 1$. Write $n_j(\kappa) = \sum_l a_l e^{il\kappa}$. Then

$$m_j(\lambda) = \sum a_l e^{ile^{-\lambda}} - n_j(0),$$

whence

$$m_j(\mathscr{L}) = \sum a_l \sum_{k=0}^{\infty} \frac{(il)^k}{k!} e^{-k\mathscr{L}} - n_j(0)I$$

where I is the identity. In terms of kernels this means

$$\check{m}_j = \sum a_l \sum \frac{(il)^k}{k!} h_k - n_j(0)\delta_{x=0},$$

where $h_0 = \delta_{x=0}$, the Dirac mass at 0. All these Dirac masses are irrelevant for $x \neq 0$, so by (1) of Lemma 4,

$$|\check{m}_j(x)| \leq C2^j \max_l |a_l| \sum_{k=1}^{\infty} \frac{l^k}{k!} h_k(x).$$

Now by Stirling's formula, since h_k is bounded in sup norm uniformly in k,

$$\begin{aligned}\sum_{k \geq |x|} \frac{l^k}{k!} h_k(x) &\leq C \sum_{k \geq |x|} \frac{4^k N^k}{k!} \leq C \sum_{k \geq |x|} \left(\frac{4eN}{k}\right)^k \\ &\leq C \sum_{k \geq |x|} 2^{-k} \leq C2^{-|x|}\end{aligned}$$

if $|x| \geq BN$ and B is chosen large enough. On the other hand by (3),

$$\begin{aligned}\sum_{1 \leq k < |x|} \frac{l^k}{k!} h_k(x) &\leq C \sum_{1 \leq k < |x|} \frac{(4N)^k}{k!} k^{-D/2} e^{-c_0|x|^2/k} \\ &\leq C \sum_{1 \leq k} \frac{(4N)^k}{k!} e^{-c_0|x|} \leq Ce^{4N - c_0|x|} \leq Ce^{-(c_0/2)|x|}\end{aligned}$$

if B is chosen large enough relative to c_0. By Lemma 4, $\max_l |a_l| \leq C2^{-j\alpha}$, so summing over l

$$|\check{m}_j(x)| \leq C2^{j(1-\alpha)} e^{-\varepsilon|x|}$$

for some $\varepsilon > 0$. This is $\leq C\rho^{|x|}$ if ρ is sufficiently close to 1, since $|x| \geq B2^{j+2}$.

The next step is to sum over j. We already know that $\check{m} \in L^2(G)$. But more is true at infinity: letting $A_k = \{2^k \leq |x| \leq 2^{k+1}\}$,

$$\|\check{m}\|_{L^2(A_k)} \leq C2^{-k\alpha} \quad \text{for } k \geq 0. \tag{5}$$

For if c_1 is chosen large enough relative to B,

$$\sum_{j < k - c_1} \|\check{m}_j\|_{L^2(A_k)} \leq C|A_k|^{1/2} \sum_{j \leq k - c_1} \rho^{2^k} \leq C2^{kD/2} \rho^{2^k} k \leq C_N 2^{-kN}$$

for any N, while

$$\sum_{j \geq k - c_1} \|\check{m}_j\|_{L^2(A_k)} \leq C \sum_{j \geq k - c_1} 2^{-j\alpha} \leq C2^{-k\alpha}.$$

$|A_k| \sim 2^{Dk}$, which we have used above, so Hölder's inequality gives

$$\|\check{m}\|_{L^1(A_k)} \leq C2^{k(D/2-\alpha)}.$$

Thus $\check{m} \in L^1$, provided $\alpha > D/2$. In particular we have proved that if m is compactly supported in $\mathbb{R}^+$ and $m \in L^2_\alpha$, then $m(\mathscr{L})$ is bounded on all L^p.

Moreover $\lambda^n m(\lambda) \in L^2_\alpha$ for all $n \in \mathbf{Z}^+$, still assuming compact support of m. $[\lambda^n m(\lambda)]^\vee = \mathscr{L}^n \check{m}$, so we conclude that $\mathscr{L}^n \check{m} \in L^2(G)$ and satisfies (5), for all n.

In order to remove the restriction that m have compact support we exploit the dilation-invariance of $\mathscr{L}$. For any function f on G and any $t \in \mathbb{R}^+$ set $d_t f(x) = f(\delta_t x)$. Then $d_t^{-1} \mathscr{L} d_t = t^2 \mathscr{L}$, so $d_t^{-1} \sigma(\mathscr{L}) d_t = \sigma(t^2 \mathscr{L})$ for any $\sigma \in L^\infty(\mathbb{R}^+)$. Define $\sigma^t(\lambda) = \sigma(t\lambda)$. Then for any $f \in \mathscr{S}$, formally

$$\sigma^t(\mathscr{L}) f = d_{t^{-1/2}} \sigma(\mathscr{L}) d_{t^{1/2}} f = d_{t^{-1/2}}(d_{t^{1/2}} f * \check{\sigma}) = t^{-D/2} f * (d_{t^{-1/2}} \check{\sigma})$$

so that

$$\check{\sigma}^t(x) = t^{-D/2} \check{\sigma}(\delta_{t^{-1/2}} x). \tag{6}$$

Let m be a continuous function on $\mathbb{R}^+$ satisfying (1) for some $\alpha > D/2$, with no restriction on its support. Decompose $m = \sum_{j \in \mathbf{Z}} m^{(j)}$ where $m^{(j)}$ is supported on $(2^{j-1}, 2^{j+1})$, and so that if $\nu^{(j)}(\lambda) = m^{(j)}(2^j \lambda)$, then $\nu^{(j)} \in L^2_\alpha$ uniformly in j. Combining (5) and (6) then yields

$$\|\mathscr{L}^l (m^{(j)\vee})\|_2 \le C_l 2^{lj} 2^{jD/4} \tag{7}$$

and for $k \ge 0$

$$\|\mathscr{L}^l (m^{(j)\vee})\|_{L^2\{|x| \sim 2^k 2^{-j/2}\}} \le C_l 2^{jl} 2^{jD/4} 2^{-k\alpha}. \tag{8}$$

Now fix $\eta \in C_0^\infty(G)$, vanishing identically in a neighborhood of 0 but identically one in a neighborhood of $\{|x| = 1\}$. Set $\check{m}^t(x) = t^D \eta(x) \check{m}(\delta_t x)$. Our basic result on $\check{m}$ is

Lemma 6. *Suppose that $\alpha > D/2$ and that m satisfies* (1). *Then $\check{m}^t \in L^1$, uniformly in $t \in \mathbb{R}^+$. Moreover there exist finite $N(\alpha)$ and $C(\alpha)$ so that for any $\varepsilon \in (0, 1]$ and any $t \in \mathbb{R}^+$, $\check{m}^t$ may be decomposed as $g + h$ where $\|g\|_1 \le \varepsilon$ and $\|h\|_{C^1} \le C\varepsilon^{-N}$.*

This follows from (7) and (8), using the fact that there exists $M < \infty$ such that for any h supported on a compact set, $\mathscr{L}^M h \in L^2 \Rightarrow h \in C^1$. Since convolution with the distribution $\check{m}$ defines an operator bounded on L^2, the lemma implies Theorem 1 by standard Calderón-Zygmund analysis.

It remains only to prove Proposition 3. Consider any compact subinterval $[0, b] \subset [0, \infty)$ and let χ denote its characteristic function. Define $\phi \in L^2(G)$ by $\phi = \sigma(\mathscr{L}) h_1$ where $\sigma(\lambda) = \chi(\lambda) e^\lambda$. Observe that for any $\sigma, \tau \in L^\infty(0, b]$ with $\check{\tau} \in L^2$,

$$\sigma(\mathscr{L}) \check{\tau} = (\sigma\tau)^\vee. \tag{9}$$

For if $f \in L^1 \cap L^2$,

$$\begin{aligned} f * (\sigma\tau)^\vee &= (\sigma\tau)(\mathscr{L}) f = \sigma(\mathscr{L}) \tau(\mathscr{L}) f \\ &= \sigma(\mathscr{L})(f * \check{\tau}) = f * (\sigma(\mathscr{L}) \check{\tau}), \end{aligned}$$

the last step justified because $\check{\tau} \in L^2$, $\sigma(\mathscr{L})$ is bounded on L^2 and left-invariant, and $f \in L^1$.

Next observe, following Hulanicki and Stein, that $\sigma \in L^\infty(0, b] \Rightarrow \check{\sigma} \in L^2$. For $\sigma(\lambda) = \sigma(\lambda)e^{\lambda} \cdot e^{-\lambda}$, so by (9), $\check{\sigma} = (\sigma \cdot \exp)(\mathscr{L})h \in L^2$ since $h \in L^2$ and $\sigma(\lambda)e^{\lambda} \in L^\infty$.

Set $\Upsilon = \check{\chi} \in L^2$ and $\phi_t = e^{-t\mathscr{L}}\Upsilon$. There is the alternative expression $\Upsilon = (\chi e^{t\cdot})(\mathscr{L})h_t$ for all t, for $\chi(\lambda) = \chi(\lambda)e^{t\lambda} \cdot e^{-t\lambda}$ and (9) applies. Consider any $\sigma \in L^\infty(0, b]$, and look at $\langle \phi_t, |\sigma|^2(\mathscr{L})\phi_t \rangle$. On the one hand

$$\langle \phi_t, |\sigma|^2(\mathscr{L})\phi_t \rangle = \langle \Upsilon, e^{-2t\mathscr{L}}|\sigma|^2(\mathscr{L})\Upsilon \rangle = \int_0^\infty e^{-2t\lambda}|\sigma|^2(\lambda)\, d(\Upsilon, P_\lambda \Upsilon)$$

where $\{P_\lambda\}$ are the spectral projections and $d(\Upsilon, P_\lambda\Upsilon)$ is the positive, finite Borel measure which to each Borel set $B \subset \mathbb{R}^+$ assigns mass $\langle \Upsilon, \chi_B(\mathscr{L})\Upsilon \rangle$, where χ_B is the characteristic function of B. The last integral tends to

$$\int_0^\infty |\sigma|^2(\lambda)\, d(\Upsilon, P_\lambda\Upsilon)$$

as $t \to 0$. On the other hand $\phi_t = (\chi e^{-t\cdot})^\vee = \chi(\mathscr{L})h_t$, so

$$\begin{aligned}
\langle \phi_t, |\sigma|^2(\mathscr{L})\phi_t \rangle &= \langle \sigma(\mathscr{L})\phi_t, \sigma(\mathscr{L})\phi_t \rangle \\
&= \langle \sigma(\mathscr{L})\chi(\mathscr{L})h_t, \sigma(\mathscr{L})\chi(\mathscr{L})h_t \rangle \\
&= \langle \sigma(\mathscr{L})h_t, \sigma(\mathscr{L})h_t \rangle \\
&= \|h_t * \check{\sigma}\|_2^2.
\end{aligned}$$

Since $\check{\sigma} \in L^2$, this converges to $\|\check{\sigma}\|_2^2$ as $t \to 0$.

We conclude that

$$\|\check{\sigma}\|_2^2 = \int_0^b |\sigma|^2(\lambda)\, d(\Upsilon, P_\lambda\Upsilon) \tag{10}$$

for all $\sigma \in L^\infty(0, b]$. At most one Borel measure on $(0, b]$ can have this property, and b is arbitrary, so there exists a unique sigma-finite Borel measure on $\mathbb{R}^+$ satisfying

$$\|\check{\sigma}\|_2^2 = \int_0^\infty |\sigma|^2(\lambda)\, d\mu(\lambda)$$

for all σ with compact support in $[0, \infty)$. From the uniqueness and from (6) it follows that μ is homogeneous of degree $D/2 - 1$, which forces $d\mu = c\lambda^{D/2}d\lambda/\lambda$.

Since this paper was completed, our principal result has been reproved in a simple way, and extended, by Mauceri and Meda [MM]. We hope that our method nonetheless may be of some value.

REFERENCES

[CS] M. Christ and C. D. Sogge, *The weak type L^1 convergence of eigenfunction expansions for pseudodifferential operators*, Invent. Math. **94** (1988), 421–453.

[D] E. B. Davies, *Explicit constants for Gaussian upper bounds on heat kernels*, Amer. J. Math. **109** (1987), 319–333.

[FS] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Princeton Univ. Press, Princeton, N.J., 1982.

[H] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–139.

[HJ] A. Hulanicki and J. W. Jenkins, *Almost everywhere summability on nilmanifolds*, Trans. Amer. Math. Soc. **278** (1983), 703–715.

[JS] D. Jerison and A. Sánchez-Calle, *Estimates for the heat kernel for a sum of squares of vector fields*, Indiana Univ. Math. J. **35** (1986), 835–854.

[KS1] S. Kusuoka and D. Stroock, *Applications of the Malliavin calculus,* part III, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. **34** (1987), 391–442.

[KS2] ——, *Long time estimates for the heat kernel associated with a uniformly subelliptic symmetric second order operator*, Ann. of Math. **127** (1988), 165–189.

[MM] G. Mauceri and S. Meda, *Vector-valued multipliers on stratified groups*, Revista Mat. Iberoamericana **6** (1990), 141–154.

[M] R. Melrose, *Propagation for the wave group of a positive subelliptic second order differential operator*, Hyperbolic Equations and Related Topics, Academic Press, Boston, Mass., 1986.

[Mü] D. Müller, *On Riesz means of eigenfunction expansions for the Kohn-Laplacian*, preprint.

[RS] M. Reed and B. Simon, *Methods of modern mathematical physics.* I: *Functional analysis*, Academic Press, Orlando, 1980.

[S1] E. M. Stein, *Topics in harmonic analysis*, Princeton Univ. Press, Princeton, N.J., 1970.

[S2] ——, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.

[V] N. Th. Varopoulos, *Semi-groupes d'opérateurs sur les espaces L^p*, C.R. Acad. Sci. Paris **301** (1985), 865–868.

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