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ITERATION OF A COMPOSITION OF EXPONENTIAL FUNCTIONS

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ABSTRACT. We show that for certain complex parameters $\lambda_1, \ldots, \lambda_{n-1}$ and λ_n the Julia set of the function

$$e^{\lambda_1 e^{\cdot^{\cdot^{\lambda_{n-1}e^{\lambda_n z}}}}}$$

is the whole plane $\mathbb{C}$. We denote by Λ the set of n-tuples $(\lambda_1, \ldots, \lambda_n)$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ for which the equation

$$e^{\lambda_1 e^{\cdot^{\cdot^{\lambda_{n-1}e^{\lambda_n z}}}}} - z = 0$$

has exact two real solutions. In fact, one of them is an attracting fixed point of

$$e^{\lambda_1 e^{\cdot^{\cdot^{\lambda_{n-1}e^{\lambda_n z}}}}},$$

which is denoted by q. We also show that when $(\lambda_1, \ldots, \lambda_n) \in \Lambda$, the Julia set of

$$e^{\lambda_1 e^{\cdot^{\cdot^{\lambda_{n-1}e^{\lambda_n z}}}}}$$

is the complement of the basin of attraction of q. The ideas used in this note may also be applicable to more general functions.

A continued composition of n exponential functions

$$f_i(z) = e^{\lambda_i z}, \qquad \lambda_i \in \mathbb{C},\ i = 1, 2, \ldots, n,$$

shall be denoted by the symbol $E_{\lambda_1, \ldots, \lambda_n}$ which is an abbreviation for

$$e^{\lambda_1 e^{\cdot^{\cdot^{\lambda_{n-1}e^{\lambda_n z}}}}}$$

that is, each e^{λ_i} is used as the exponent of the preceding.

It had until 1981 been an open problem, proposed by Fatou [9], whether $\mathscr{T}(e^z) = \mathbb{C}$. In 1981 Misiurewicz [11] proved this conjecture answering this sixty year old question of Fatou. In 1984, Baker and Rippon [4] studied the

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sequence of iterations of $E_a(z) = e^{az}$ where a is a nonzero complex parameter (similar results were obtained by Devaney [5]). They analysed the way in which $\mathscr{N}(E_a)$ divides the plane and the various possible limit functions for convergent subsequences of $\{E_a^n\}_{n=0}^{\infty}$ in components of $\mathscr{N}(E_a)$. They proved that there are no limit functions (and so $\mathscr{T}(E_a) = \mathbb{C}$) provided $\lim_{n\to\infty} E_a^n(0) = \infty$, in particular this is the case for all real $a > \frac{1}{e}$. This extends the result of Misiurewicz for $a = 1$. In this paper we study the Julia set of the function

$$E_{\lambda_1,\dots,\lambda_n}(z) = e^{\lambda_1 e^{\cdot^{\cdot^{\cdot^{\lambda_{n-1}e^{\lambda_n z}}}}}}$$

where $\lambda_1, \dots, \lambda_n$ are complex parameters, and extend Baker and Rippon's result for the case $n = 1$. Some aspects of the convergence of the sequence of the natural iterates of this function were also studied by Thron [13] in 1957 and Shell [12] in 1959.

The Julia set of an entire function f can be defined as either the set of points at which the family $\{f^n\}_{n=0}^{\infty}$ of iterations of f is not normal or as the closure of the set of repelling periodic points [1].

Throughout this paper we denote the set of complex numbers by $\mathbb{C}$, the set of real numbers by $\mathbb{R}$, the Julia set of $E_{\lambda_1,\dots,\lambda_n}$ by $\mathscr{T}(E)$ and the normal set of $E_{\lambda_1,\dots,\lambda_n}$ by $\mathscr{N}(E)$.

We need the following known results:

Theorem A. *Unless $f(z)$ is a rational function of order 0 or 1 the set $\mathscr{T}(f)$ has the following properties (proved for rational functions in [7, 8] and for entire functions in [9]):*

(1) *$\mathscr{T}(f)$ is a nonempty perfect set.*
(2) *$\mathscr{T}(f^n) = \mathscr{T}(f)$ for any integer $n \geq 1$.*
(3) *$\mathscr{T}(f)$ is completely invariant under the mapping $z \to f(z)$, i.e., if α belongs to $\mathscr{T}(f)$ then so do $f(\alpha)$ and every solution β of $f(\beta) = \alpha$.*

Theorem B. *Let D be a domain of the complex plane with at least three boundary points and let f be analytic in $\overline{D}$, except that if D is unbounded f need not be analytic at ∞. Let f map D into itself and suppose that no subsequence of $\{f^n\}_{n=0}^{\infty}$ has the identity map as a limit in D (in particular this is so if f is not a univalent map of D onto D). Then the whole sequence $\{f^n\}_{n=0}^{\infty}$ converges in D to a constant limit $\alpha \in \overline{D}$* [3].

We denote by $\mathscr{S}$ the set of all finite singularities of $f^{-1}(z)$ and $\mathscr{E}$ the set of points of the form $f^n(s)$, $s \in \mathscr{S}$, $n = 0, 1, 2, \dots$. Then a point belongs to $\mathscr{E}$ precisely when it is a finite singularity of some inverse function $f^{-n}(z)$ of an iterate of $f(z)$ [2].

To study the stable behavior of a transcendental entire function f we need to discuss the possible limit functions of subsequences of $\{f^n\}_{n=0}^{\infty}$ in the domains concerned. The following results are developed by Baker [2].

Theorem C. *Let $\mathscr{E}'$ denote the derived set of $\mathscr{E}$, then any constant limit of a sequence $\{f^{n_k}(z)\}_{k=0}^{\infty}$ in a component of the normal set $\mathscr{N}(f)$ belongs to*

$$\mathscr{L} = \mathscr{E} \cup \mathscr{E}' \cup \{\infty\} = \overline{\mathscr{E}} \cup \{\infty\}.$$

Theorem D. *If the set $\mathscr{L}$ defined in Theorem C has an empty interior and a connected complement, then no sequence $\{f^{n_k}\}_{k=0}^{\infty}$ has a nonconstant limit function in any component of $\mathscr{N}(f)$.*

A point $\omega \in \widehat{\mathbb{C}}$ is an asymptotic value for a map f if there is a path $\alpha: [0, 1) \to \mathbb{C}$ such that $\lim_{t\to 1}\alpha(t) = \infty$ and $\lim_{t\to 1} f\circ\alpha(t) = \omega$.

Let U be a connected component (domain) of $\mathscr{N}(f)$. U is preperiodic if there exist integers p and q such that $f^{p+q}(U) = f^{p}(U)$; it is periodic if $p = 0$. A component of $\mathscr{N}(f)$ which is neither periodic nor preperiodic is wandering.

A map f is called of critically finite type (or simply finite type) if f has finitely many critical values and asymptotic values. The following theorem is due to L. Goldberg and L. Keen, and describes the stable behavior of finite type entire functions [10].

Theorem E. *If $f: \mathbb{C} \to \mathbb{C}$ is of finite type, then f has no wandering domain.*

We remark that the application of the last result mentioned above is essential in the proof of the following theorem:

Theorem 1. *Let $\mathscr{S}$ be the set of the finite singularities of the inverse function of $E_{\lambda_1,\ldots,\lambda_n}$. If each forward orbit of $s \in \mathscr{S}$ tends to ∞, then $\mathscr{T}(E) = \mathbb{C}$.*

In particular, if $\lambda_i > 0$ for $i = 1, 2, \ldots, n$ and the forward orbit of 0 tends to ∞, then $\mathscr{T}(E) = \mathbb{C}$.

In order to prove the theorem we need the following lemmas.

Lemma 2. *Finite type maps are closed under composition.*

Proof. Assume that two maps f and g are of finite type.

If α is a critical value of $f(g(z))$, then there exists β such that

$$f'(g(\beta))g'(\beta) = 0 \quad \text{and} \quad f(g(\beta)) = \alpha.$$

If $f'(g(\beta)) = 0$, then α is a critical value of f as well; if $g'(\beta) = 0$, then there is at least one critical value of g corresponding to α under f. It follows that the number of critical values of $f(g(z))$ is less than or equal to the sum of the critical values of f and g.

If α is an asymptotic value of $f(g(z))$, then there exists a critical path $\Gamma: [0, 1) \to \mathbb{C}$ such that

$$\lim_{t\to 1}\Gamma(t) = \infty \quad \text{and} \quad \lim_{t\to 1} f(g(\Gamma(t))) = \alpha.$$

We claim that there exists γ which is either a finite number or ∞ such that

$$\lim_{t\to 1} g(\Gamma(t)) = \gamma.$$

Otherwise, let us denote by M the set of the limiting points of $g(\Gamma(t))$ as $t \to 1$. Then $f(z) = \alpha$ for each $z \in M$. Since f is a nonconstant entire map, M does not contain any limit points, that is M is a discrete set. Suppose that M contains more than one point. Arbitrarily pick two different points $\gamma_1, \gamma_2 \in M$, then they can be separated by two disjoint closed discs D_1 and D_2 with $\gamma_i \in D_i$ where $i = 1, 2$ and

$$(D_1 \cup D_2 \backslash \{\gamma_1, \gamma_2\}) \cap M = \varnothing.$$

But the fact that each γ_i is a limiting point implies that the curve $g(\Gamma(t))$ where $t \in [0, 1)$ must frequently enter each disc D_i in fact infinitely many times. Thus the curve $g(\gamma(t))$ where $t \in [0, 1)$ intersects the circles C_i which are the boundaries of D_i infinitely many times as $t \to 1$. Since C_1, say, is compact, there must be a limiting point γ_0 on it, and this limit point does not belong to M. Contradiction! Therefore M consists of a single point γ which is an asymptotic value of g. Since g is continuous, the image of Γ is also a path. Thus the number of the asymptotic values of $f(g(z))$ is less than or equal to the number of the asymptotic values of $g(z)$. Q.E.D.

As an immediate consequence, $E_{\lambda_1, \dots, \lambda_n}$ is of finite type.

Lemma 3. *The only finite singularities of* $E^{-1}_{\lambda_1, \dots, \lambda_n}$ *are*

$$0, 1, e^{\lambda_1}, e^{\lambda_1 e^{\lambda_2}}, \dots, e^{\lambda_1 e^{\cdot^{\cdot^{\cdot^{\lambda_{n-1}e^{\lambda_n}}}}}}.$$

Proof. This follows from the simple facts that $E_{\lambda_1, \dots, \lambda_n}$ does not have any algebraic singularity and the inverse function

$$E^{-1}_{\lambda_1, \dots, \lambda_n}(z) = \frac{1}{\lambda_n} \ln \left(\frac{1}{\lambda_{n-1}} \ln \left(\cdots \left(\frac{1}{\lambda_1} \ln z \right) \cdots \right) \right)$$

is well defined if and only if $z \neq 0, 1, e^{\lambda_1}, e^{\lambda_1 e^{\lambda_2}}, \dots, e^{\lambda_1 e^{\cdot^{\cdot^{\cdot^{\lambda_{n-1}e^{\lambda_n}}}}}}$. Q.E.D.

Therefore, with Baker's notation

$$\mathcal{S} = \left\{ 0, 1, e^{\lambda_1}, e^{\lambda_1 e^{\lambda_2}}, \dots, e^{\lambda_1 e^{\cdot^{\cdot^{\cdot^{\lambda_{n-1}e^{\lambda_n}}}}}} \right\}$$

and

$$\mathcal{L} = \text{Closure}\{E^k_{\lambda_1, \dots, \lambda_n}(s), s \in \mathcal{S}, k = 1, 2, \dots\}.$$

We are going to show the following lemma.

Lemma 4. *Under the assumptions of Theorem* 1, *the complement of* $\mathcal{L}$ *is connected.*

Proof. It suffices to show that if P is a countable subset of $\mathbb{C}$, then the complement $\mathbb{C} \backslash P$ of P is connected. Towards this end, pick arbitrarily two points

$q_1, q_2 \in \mathbb{C}\backslash P$. Through each q_i, $i = 1, 2$, there exists a family L_i of uncountably many straight lines. There must be $l_i \in L_i$, $i = 1, 2$, such that l_1 and l_2 have an intersection and contain no points of P. It follows from the fact that $\mathbb{C}\backslash P$ is path-connected that $\mathbb{C}\backslash P$ is connected. Q.E.D.

Now we are ready to prove Theorem 1:

Proof of the Theorem 1. The set $\mathscr{L}$ has empty interior and connected complement. According to a theorem of I. N. Baker [2], each limit function of the family $\{E^k_{\lambda_1, \dots, \lambda_n}\}_{k=0}^{\infty}$ must be either a constant belonging to $\mathscr{L}$ or ∞.

It remains to show that the family $\{E^k_{\lambda_1, \dots, \lambda_n}\}_{k=0}^{\infty}$ is normal nowhere.

Suppose that $\mathscr{N}(E)$ is not empty. Let U be a component of $\mathscr{N}(E)$. Since $E_{\lambda_1, \dots, \lambda_n}$ is critically finite, $E_{\lambda_1, \dots, \lambda_n}$ does not possess a wandering domain [6]. Thus there exist nonnegative integers l and m such that $G = E^m_{\lambda_1, \dots, \lambda_n}(U)$ is invariant under $g = E^l_{\lambda_1, \dots, \lambda_n}$.

It follows from Theorem B that the whole sequence $\{g^k\}_{k=0}^{\infty}$ converges in G to a constant limit which belongs to $\overline{G}$. Denote this limit by α. Since G is invariant under g, if α is finite, we have $g(\alpha) = \alpha$. This is to say that α is periodic.

By the hypothesis of the theorem it is clear that for each $s \in \mathscr{S}$,

$$\lim_{k\to\infty} E^k_{\lambda_1, \dots, \lambda_n}(s) = \infty.$$

Thus $s \in \mathscr{S}$ is not eventually periodic, and

$$\begin{aligned}\mathscr{L} &= \mathrm{Closure}\{E^k_{\lambda_1, \dots, \lambda_n}(s), s \in \mathscr{S}, k = 1, 2, \dots\} \\ &= \{E^k_{\lambda_1, \dots, \lambda_n}(s), s \in \mathscr{S}, k = 1, 2, \dots\} \cup \{\infty\}.\end{aligned}$$

Applying Theorem C, $\alpha \in \mathscr{L}\backslash\{\infty\}$ cannot be periodic. Contradiction! Therefore, α must be ∞.

It follows from

$$\lim_{k\to\infty} E^{kl}_{\lambda_1, \dots, \lambda_n}(z) = \lim_{k\to\infty} g^k(z) = \infty$$

uniformly on G that

$$\lim_{k\to\infty} E^{kl-1}_{\lambda_1, \dots, \lambda_n}(z) = \infty$$

uniformly on G.

Consequently, for each $j \geq 0$,

$$\lim_{k\to\infty} E^{kl-j}_{\lambda_1, \dots, \lambda_n}(z) = \infty$$

uniformly on G.

For each sequence

$$\{E^{k_m}_{\lambda_1, \dots, \lambda_n}\}_{m=0}^{\infty} \subset \{E^k_{\lambda_1, \dots, \lambda_n}\}_{k=0}^{\infty},$$

there exists an integer $j \geq 0$ such that there is a subsequence

$$\{E^{k_{m_t}}_{\lambda_1,\dots,\lambda_n}\}_{t=0}^{\infty} \subset \{E^{k_m}_{\lambda_1,\dots,\lambda_n}\}_{m=0}^{\infty},$$

which is a subsequence of $\{E^{kl-j}_{\lambda_1,\dots,\lambda_n}\}_{k=0}^{\infty}$. Thus we have

$$\lim_{t\to\infty} E^{k_{m_t}}_{\lambda_1,\dots,\lambda_n}(z) = \infty$$

in G. Furthermore we conclude that the whole sequence $\{E^{k}_{\lambda_1,\dots,\lambda_n}\}_{k=0}^{\infty}$ has limit ∞.

Now we claim that the sequence $\{(E^{k}_{\lambda_1,\dots,\lambda_n})'\}_{k=0}^{\infty}$ of derivatives of $E^{k}_{\lambda_1,\dots,\lambda_n}$, $k = 0, 1, \dots,$ also tends to ∞ on G. In fact,

$$(E_{\lambda_1,\dots,\lambda_n})'(z) = \prod_{i=1}^{n} \lambda_i E_{\lambda_i,\dots,\lambda_n}(z),$$

and so, according to the chain rule

$$\begin{aligned}(E^{k}_{\lambda_1,\dots,\lambda_n})'(z) &= \prod_{j=1}^{k} (E_{\lambda_1,\dots,\lambda_n})'(E^{j-1}_{\lambda_1,\dots,\lambda_n}(z)) \\ &= \prod_{i=1}^{n}\prod_{j=0}^{k-1} \lambda_i^k E_{\lambda_i,\dots,\lambda_n}(E^{j}_{\lambda_1,\dots,\lambda_n}(z)).\end{aligned}$$

Therefore,

$$\ln|(E^{k}_{\lambda_1,\dots,\lambda_n})'(z)| = k\sum_{i=1}^{n}\ln|\lambda_i| + \sum_{j=0}^{k-1}\sum_{i=1}^{n}\ln|E_{\lambda_i,\dots,\lambda_n}(E^{j}_{\lambda_1,\dots,\lambda_n}(z))|.$$

Since

$$\begin{aligned}\lim_{k\to\infty}|E^{k+1}_{\lambda_1,\dots,\lambda_n}(z)| &= \lim_{k\to\infty}|E_{\lambda_1,\dots,\lambda_n}(E^{k}_{\lambda_1,\dots,\lambda_n}(z))| \\ &= \lim_{k\to\infty}|e^{\lambda_1 E_{\lambda_2,\dots,\lambda_n}(E^{k}_{\lambda_1,\dots,\lambda_n}(z))}| = \infty,\end{aligned}$$

for $z \in G$, it follows that

$$\lim_{k\to\infty}|E_{\lambda_2,\dots,\lambda_n}(E^{k}_{\lambda_1,\dots,\lambda_n}(z))| = \infty.$$

With a similar argument, consequently we have

$$\lim_{k\to\infty}|E_{\lambda_i,\dots,\lambda_n}(E^{k}_{\lambda_1,\dots,\lambda_n}(z))| = \infty$$

for $i = 1, \dots, n$, which implies that

$$\lim_{k\to\infty}\ln|(E^{k}_{\lambda_1,\dots,\lambda_n})'(z)| = \infty,$$

and our assertion follows immediately. It follows from the Bloch-Landau Theorem that if D is a disc contained in G, then $E^{k}_{\lambda_1,\dots,\lambda_n}(D)$ contains a disk of arbitrarily large radius. Since $\mathscr{T}(E) \neq \emptyset$, there exists an integer $k_0 > 0$ such that

$$E^{k_0}_{\lambda_1, \dots, \lambda_n}(D) \cap \mathscr{T}(E) \neq \emptyset$$

which is impossible, since $D \subset G \subset \mathscr{N}(E)$.

In the case where all $\lambda_i > 0$, the argument is much simpler since the fact that

$$\lim_{k \to \infty} E^k_{\lambda_1, \dots, \lambda_n}(0) = \infty$$

implies that each forward orbit of a real number tends to ∞. Particularly, each forward orbit of $s \in \mathscr{S} \subset \mathbb{R}$ tends to ∞, and the result follows immediately. Q.E.D.

Now we focus our attention on the case when $\lambda_i > 0$, $i = 1, 2, \dots, n$, and $E_{\lambda_1, \dots, \lambda_n}$ has exactly two distinct positive fixed points. From the convexity of the graph of $E_{\lambda_1, \dots, \lambda_n}$, for z real it follows that of these one is attracting and the other is repelling.

We denote the attracting one by q and the repelling one by p as shown in Figure 1 (where the dotted line signifies the horizontal asymptote of the function). Also, from the convexity of $E_{\lambda_1, \dots, \lambda_n}$, clearly $q > p$.

Noting that there exists $\varepsilon > 0$ such that

$$|E'_{\lambda_1, \dots, \lambda_n}(p)| > 1 + \varepsilon.$$

We have the following theorem:

Theorem 6. *Let* $\lambda_i > 0$, $i = 1, 2, \dots, n$. *If* $E_{\lambda_1, \dots, \lambda_n}$ *has an attracting fixed point* q, *then* $\mathscr{T}(E)$ *is the complement of the basin of attraction of* q.

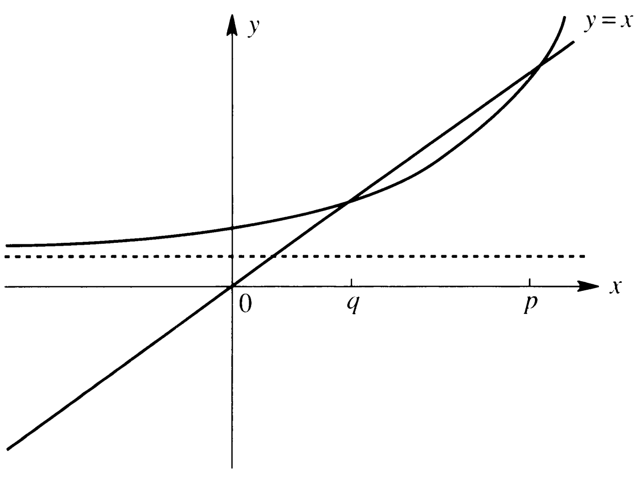

FIGURE 1

Proof. Write $z = x + iy$ with $x, y \in \mathbb{R}$. Let $H = \{z \in \mathbb{C} | x < p\}$. Since, for $z \in H$,

$$\begin{aligned}|E_{\lambda_1, \dots, \lambda_n}(z)| &= |E_{\lambda_1, \dots, \lambda_{n-1}}(e^{\lambda_n x + i\lambda_n y})| \\ &\leq E_{\lambda_1, \dots, \lambda_{n-1}}(e^{\lambda_n x}) < E_{\lambda_1, \dots, \lambda_n}(p) = p,\end{aligned}$$

the images of H under $\{E^k_{\lambda_1, \dots, \lambda_n}\}_{k=0}^{\infty}$ are bounded by the disc centered at 0 with radius p. Clearly the interval $[0, q] \subset H$ is contained in the basin of attraction of q. It follows from Vitali's Convergence Theorem that H is included in the basin of attraction of q. Hence $\mathscr{T}(E)$ lies to the right of the vertical line $x = p$.[1] Furthermore we can show that $\mathscr{T}(E)$ is the complement of this basin.

We denote by $\mathscr{A}$ the set of points $z \in \mathbb{C} \backslash \overline{H}$ such that $E_{\lambda_1, \dots, \lambda_n}(z) \in \mathbb{C} \backslash \overline{H}$. In the next stage, we examine the magnitude of the derivative $E'_{\lambda_1, \dots, \lambda_n}$ of $E_{\lambda_1, \dots, \lambda_n}$ on $\mathscr{A}$. In fact we are going to prove that

$$|E'_{\lambda_1, \dots, \lambda_n}(z)| \geq 1 + \varepsilon \tag{7}$$

for each $z \in \mathscr{A}$ by a contradiction. Suppose that $|E'_{\lambda_1, \dots, \lambda_n}(z)| < 1 + \varepsilon$ for some $z \in \mathscr{A}$. Let us write

$$|E_{\lambda_i, \dots, \lambda_n}(z)| = e^{A_i} \quad \text{and} \quad E_{\lambda_i, \dots, \lambda_n}(p) = e^{P_i}$$

for $i = 1, 2, \dots, n$. Then

$$|E_{\lambda_i, \dots, \lambda_n}(z)| \geq E_{\lambda_i, \dots, \lambda_n}(p)$$

if and only if $A_i \geq P_i$ for each $i = 1, 2, \dots, n$. For confirmation, we argue as follows:

We note the following two cases:

(1) If $j = 1$, then since $z \in \mathscr{A}$, and thus $E_{\lambda_1, \dots, \lambda_n}(z) \in \mathscr{A} \subset \mathbb{C} \backslash \overline{H}$,

$$e^{A_1} = |E_{\lambda_1, \dots, \lambda_n}(z)| > p = E_{\lambda_1, \dots, \lambda_n}(p) = e^{P_1},$$

and so $A_1 > P_1$.

(2) If $j > 1$, suppose that there exists j such that $A_j < P_j$. We obtain the following consequence:

$$e^{A_{j-1}} = e^{\lambda_{j-1} e^{A_j} \cos B_j} \leq e^{\lambda_{j-1} e^{A_j}} < e^{\lambda_{j-1} e^{P_j}} = e^{P_{j-1}}$$

where B_j is some real value. Hence $A_j < P_j$ implies $A_{j-1} < P_{j-1}$, and so in particular $A_1 < P_1$ which contradicts case (1).

[1] Notice that $p \in \mathscr{T}(E)$, so $\mathscr{T}(E)$ is not strictly to the right of the line $x = p$. The idea of this argument is due to Devaney [6].

Now we claim that (7) holds, for otherwise

$$\begin{aligned}\operatorname{Re} E_{\lambda_1,\dots,\lambda_n}(z) \le |E_{\lambda_1,\dots,\lambda_n}(z)| &= \frac{|E'_{\lambda_1,\dots,\lambda_n}(z)|}{\prod_{i=1}^{n}\lambda_i \prod_{i=2}^{n}|E_{\lambda_i,\dots,\lambda_n}(z)|} \\ &\le \frac{|E'_{\lambda_1,\dots,\lambda_n}(z)|}{\prod_{i=1}^{n}\lambda_i \prod_{i=2}^{n}E_{\lambda_i,\dots,\lambda_n}(p)} \\ &= \frac{p|E'_{\lambda_1,\dots,\lambda_n}(z)|}{E'_{\lambda_1,\dots,\lambda_n}(p)} < \frac{p(1+\varepsilon)}{1+\varepsilon} = p\end{aligned}$$

which contradicts $E_{\lambda_1,\dots,\lambda_n}(z) \in \mathbb{C}\backslash\overline{H}$ and thus shows our assertion.

Let D be a closed disc with radius δ and $E^k_{\lambda_1,\dots,\lambda_n}(D) \subset \mathbb{C}\backslash\overline{H}$ for all $k \ge 0$. Then

$$|E'_{\lambda_1,\dots,\lambda_n}(E^k_{\lambda_1,\dots,\lambda_n}(z))| \ge 1+\varepsilon$$

for $z \in D$ and all k. Hence

$$|(E^k_{\lambda_1,\dots,\lambda_n})'(z)| = \prod_{i=1}^{k} |E'_{\lambda_1,\dots,\lambda_n}(E^{i-1}_{\lambda_1,\dots,\lambda_n}(z))| \ge (1+\varepsilon)^k,$$

which tends to ∞ as $k \to \infty$.

It follows from the Bloch-Landau Theorem that $E^k_{\lambda_1,\dots,\lambda_n}(D)$ contains a disc with arbitrary large radius for k sufficiently large. Thus there exists an integer $k_0 > 0$ such that $E^{k_0}_{\lambda_1,\dots,\lambda_n}(D) \cap H \ne \varnothing$. But this is absurd. The contradiction shows the impossibility of $E^k_{\lambda_1,\dots,\lambda_n}(D)$ staying in $\mathbb{C}\backslash\overline{H}$ for all k. Thus the complement of the basin of attraction of q is nowhere dense in $\mathbb{C}$. As an immediate consequence, $\{E^k_{\lambda_1,\dots,\lambda_n}\}_{k=0}^{\infty}$ is normal nowhere in the complement of the basin. This is equivalent to saying that $\mathcal{T}(E)$ is the complement of the basin of the attraction of q. Q.E.D.

Remark. In order to study the dynamical behavior of a given function f, we often need to investigate the possible limit functions of subsequences of $\{f^n\}_{n=0}^{\infty}$ in the set of normality. The ideas used in this work may also be applicable to more general functions in the following sense:

If we assume that f is of critically finite type, the finiteness theorem combining Baker's results provides a useful tool for finding the relevant limit functions. In this case, the set $\mathcal{S}$ only consists of finitely many points. This implies that $\mathcal{E}$ is a countable set. If furthermore we assume the complement of $\mathcal{L}$ is connected and the interior of $\mathcal{L}$ is empty (this occurs, for instance, when $\mathcal{L}$ happens to be a countable set), then Baker's results enable us to give a better estimate of the possible limit functions. To precisely determine the limit functions of subsequences of $\{f^n\}_{n=0}^{\infty}$, we need to examine the order of the growth of $\{f^n\}_{n=0}^{\infty}$ in the domain concerned. One way to do this examination is to consider the sequence of derivatives $\{(f^{n_k})'\}_{k=0}^{\infty}$ of $\{f^{n_k}\}_{k=0}^{\infty}$. If some disc in the domain concerned is expanded under the iterations of f, with the Bloch-Landau Theorem we are able to establish a contradiction and conclude $\mathscr{T}(f) = \mathbb{C}$. To see an application to the family of the composition of sine functions with n parameters, let

$$S_{\lambda_1}(z) = \lambda_1 \sin z$$

and

$$S_{\lambda_1, \dots, \lambda_{k+1}}(z) = S_{\lambda_1, \dots, \lambda_k}(\lambda_{k+1} \sin z)$$

for $k = 1, \dots, n$. Since finite type maps are closed under composition by Lemma 2, $S_{\lambda_1, \dots, \lambda_n}$ is of critically finite type. It is easy to check that the finite singularities of $S^{-1}_{\lambda_1, \dots, \lambda_n}$ are

$$\pm\lambda_1, \lambda_1 \sin(\pm\lambda_2), \dots, S_{\lambda_1, \dots, \lambda_{n-1}}(\pm\lambda_n).$$

With the same method as we used in the proof of Theorem 1, one can show that if each forward orbit of finite singularities of $S^{-1}_{\lambda_1, \dots, \lambda_n}$ tends to ∞, then the Julia set of $S_{\lambda_1, \dots, \lambda_n}$ is the whole plane.

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