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CONTINUITY OF TRANSLATION IN THE DUAL OF $L^{\infty}(G)$ AND RELATED SPACES

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ABSTRACT. Let X be a Banach space and G a locally compact Hausdorff group that acts as a group of isometric linear operators on X. The operation of $x \in G$ on X will be denoted by L_x. We study the set X_c of elements $\mu \in X$ such that $x \mapsto L_x\mu$ is continuous with respect to the topology on G and the norm-topology on X. The spaces X studied include $M(G)^*$, $\mathrm{LUC}(G)^*$, $L^{\infty}(G)^*$, $\mathrm{VN}(G)$, and $\mathrm{VN}(G)^*$. In most cases, characterizations of X_c do not appear to be possible, and we give constructions that illustrate this. We relate properties of X_c to properties of G. For example, if X_c is sufficiently small, then G is compact, or even finite, depending on the case. We give related results and open problems.

0. INTRODUCTION

Let X be a Banach space and G a locally compact Hausdorff group that acts as a group of isometric linear operators on X. The operation of $x \in G$ on X will be denoted by L_x. We denote by X_c or $(X)_c$ (depending on how complicated the name for X is) the set of elements $\mu \in X$ such that $x \mapsto L_x\mu$ is continuous with respect to the topology on G and the norm-topology on X. We study X_c when X is one of the spaces $M(G)^*$, $\mathrm{LUC}(G)^*$, $L^{\infty}(G)^*$, $\mathrm{VN}(G)$, and $\mathrm{VN}(G)^*$, and (with less emphasis) some other spaces. We also consider elements of $(X^*)_c$ in relation to translation-invariant means on X.

When $X = X_c$, $\mu * f$ is well defined for every $\mu \in M(G)$ and every $f \in X$. We can then define by repeated applications of duality $\mu * \tau$ for all $\mu \in M(G)$ and all $\tau \in X^*$ or $X^{*\cdots*}$. If $\tau \in (X^{*\cdots*})_c$, we get a second definition of "$\mu * \tau$," by using the continuity of $x \mapsto L_x\tau$. Those definitions of convolution do not always coincide. See [Ru2] for the proof that there are translation-invariant

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means τ on $L^\infty(G)$ for which the two convolutions disagree for absolutely continuous μ's.

In addition to discussing continuity properties of translation, we will discuss measurability and semicontinuity properties. By *weak measurable* we mean that $x \mapsto \langle L_x\mu, f\rangle$ is measurable for all f in the space in question and all μ in its dual space. We call the group action *lower semicontinuous* if for each $x \in X$ and each $\varepsilon > 0$, the set $\{g \in G : \|L_g x - x\| > \varepsilon\}$ is open in G.

Our motivation comes from the well-known fact that $M(G)_c = L^1(G)$ and from some of the ideas in [GLL].

There are three general patterns that arise. (i) $X_c = X$ if and only if G is discrete (this is usually easily seen). (ii) If X_c is "small," then G is "small" (usually compact). (iii) If X_c is very small (especially if it has the "wrong" norm), then G is finite. The specific meaning of the terms in quotes will vary with the space X. One also hopes to find that X_c is a "natural" space; but, that rarely seems to occur, and in fact "natural" generally seems to coincide with "small" or "very small."

In all cases, the operation of G on X will be translation, or the (multiple) dual of translation. Translation can be thought of as convolution with a point mass, and we shall often use that point of view. Sometimes left translation will appear as L_x^* when a dual of it is being considered.

We begin with a brief study of $(M(G)^*)_c$ in §1, and then turn to $(\mathrm{LUC}(G)^*)_c$ in §2. Note that our $\mathrm{LUC}(G)$ is the space of "right uniformly continuous functions" as defined in [HR, vol. 1, p. 275].

In §3, we discuss $(L^\infty(G)^*)_c$. Since $L^\infty(G)_c = \mathrm{LUC}(G)$, the left uniformly continuous functions on G, there is nothing more that we can say about $L^\infty(G)_c$. In §4 we deal with $\mathrm{VN}(G)_c$. In §5 we consider $(\mathrm{VN}(G)^*)_c$. For the reader not familiar with $\mathrm{VN}(G)$, we point out that for a locally compact abelian group G with dual group Γ, $\mathrm{VN}(G) = L^\infty(\Gamma)$. Of course, translation in $\mathrm{VN}(G)$ corresponds to multiplication by a character in $L^\infty(\Gamma)$, so behavior at infinity is what is important for continuity. In particular, $(L^\infty(\Gamma))_c$ contains $C_0(\Gamma)$ (but is not equal to it, in general).

§6 contains miscellaneous results and open problems. In the Appendix we give proofs of (well-known) results that we cannot find in the literature in the form we need.

Δ will denote the spectrum of the C^*-algebra being discussed (which algebra will be clear from the context).

1. Continuity of translation in $M(G)$ and $M(G)^*$

Let $M(G)$ denote the space of regular Borel measures on the locally compact group G and $M(G)^*$ the dual space of $M(G)$. See [GM] for information about $M(G)$.

We denote the set of bounded Borel functions on G by $B_\infty(G)$.

Our first theorem consists of a summary of well-known and elementary facts.

Proposition 1.1. *Let G be a locally compact group. Then the following hold.*

(i) $M(G)_c = L^1(G)$.

(ii) $M(G)_c = \mathrm{LUC}(G)$ *if and only if G is finite.*

(iii) $(M(G)^*)_c \subseteq B_\infty(G)$ *if and only if G is discrete.*

(iv) $(M(G)^*)_c \supseteq B_\infty(G)$ *if and only if G is discrete.*

(v) $(M(G)^*)_c = M(G)^*$ *if and only if G is discrete.*

Proof. (i) is well known. See [GM, 8.2.1] for another proof and history. Here is a simple proof. For $\nu \in M(G)_c$, and $f \in C_{00}(G)$, the vector integral $\int f(y)L_y\, d\nu$ exists and coincides with the ordinary convolution $f * \nu$, which is in $L^1(G)$. Hence, ν can be approximated in norm by elements of $L^1(G)$, and hence $\nu \in L^1(G)$.

(ii) Let $M(G)_c = \mathrm{LUC}(G)$, so $L^1(G) = \mathrm{LUC}(G)$ by (i). Thus, every constant function is in $L^1(G)$, so G is compact. If G were not discrete, then $L^1(G)$ would contain functions that are unbounded, and that would contradict the assumption $M(G)_c = \mathrm{LUC}(G)$.

(iii) Let χ be the linear functional such that $\langle \chi, \mu\rangle = \int d\mu_d$ where μ_d denotes the discrete part of μ. If G is not discrete, then $\chi \in M(G)^*$, and $\chi \notin B_\infty(G)$, so one direction of (iii) follows. On the other hand, if G is discrete, then $(M(G)^*)_c = M(G)^*$, $M(G) = L^1(G)$, and $M(G)^* = L^\infty(G)$. This proves (iii).

(iv) If G is discrete, then $B_\infty(G) = l^\infty(G_d)$, as we have just seen, and the assertion follows trivially. If G is not discrete, then there exist $f \in B_\infty(G)$ that are not left uniformly continuous. Such f cannot be elements of $(M(G)^*)_c$.

(v) follows from (iii)–(iv), since $B_\infty(G) \subseteq M(G)^*$ always holds. □

A subspace Y of $M(G)$ is an *L-subspace* if it is closed and if $\mu \in Y$ and $\nu \ll \mu$ imply $\nu \in Y$.

Proposition 1.2. *Let $M(G) = \sum_\alpha X_\alpha$ be a decomposition of $M(G)$ into mutually singular L-subspaces which are translation-invariant. Let $\{f_\alpha\}$ be a set of left uniformly continuous, uniformly bounded, functions on G. Define χ by $\langle \chi, \mu\rangle = \sum_\alpha \int f_\alpha\, d\Pi_\alpha \mu$, where Π_α is the projection of $M(G)$ onto X_α. Then $\chi \in (M(G)^*)_c$.*

Proof. This is immediate.

Proposition 1.3. *Let G be a locally compact group. Then the following hold.*

(i) *If $\chi \in (M(G)^*)_c$, then the restriction of χ to $L^1(G)$ agrees with an element of* $\mathrm{LUC}(G)$.

(ii) *If G is abelian and $\chi \in \Delta M(G)$ is such that $x \mapsto \langle \chi, \delta(x) * \mu\rangle$ is continuous for some μ with $\langle \chi, \mu\rangle \neq 0$, then the restriction of χ to $M_d(G)$ agrees with a continuous character on G.*

Proof. We leave the proof of (i) to the reader and only sketch the proof of (ii).

Note that

$$\langle \chi, \delta_x * \mu - \mu \rangle = \langle \chi, \delta_x \rangle \langle \chi, \mu \rangle - \langle \chi, \mu \rangle .$$

And, of course, $\langle \chi, \delta_x \rangle$ is the value of the character χ at $x \in G$. □

Proposition 1.4. *Let G be a nondiscrete locally compact group. Then the following hold.*

(i) *There exists $\mu \in M(G)$ such that $x \mapsto L_x \mu$ is not weak measurable from G to $M(G)$.*

(ii) *The action of G on $M(G)^*$ is neither lower semicontinuous nor weak* measurable from G to $M(G)^*$.*

Remark. The action of G on $M(G)$ is always lower semicontinuous, since $M(G)$ is the dual space of $C_0(G)$, where the action is continuous. See [GLL] for details.

Proof. We first prove the nonlower semicontinuity. Let $g \in G$, $g \neq e$, and define $\phi \in M(G)^*$ by $\langle \phi, \mu \rangle = \mu(\{e\}) - \mu(\{g\})$. Then

$$\|L_x \phi - \phi\| = \begin{cases} 2 & \text{for } x = g, g^{-1}; \\ 1 & \text{for } x \neq e, g, g^{-1}; \\ 0 & \text{for } x = e. \end{cases}$$

Therefore $\{x : \|L_x \phi - \phi\| > 3/2\}$ is not open and the action of G is not lower semicontinuous.

We now prove the nonmeasurability. We assume the axiom of choice, of course, so that there is a nonmeasurable subset E of G (see Lemma A.2 for a formal statement and a sketch of a proof). Let $\chi \in M(G)^*$ be such that $\langle \chi, \mu \rangle = 0$ for all continuous $\mu \in M(G)$ and $\langle \chi, \mu \rangle = \mu(E)$ for all discrete μ. Let $\mu = \delta(e)$ where e is the identity of G. Then $x \mapsto \langle L_x \chi, \mu \rangle = \langle \chi, L_x \mu \rangle$ is the characteristic function of E, so $x \mapsto L_x \mu$ is not weak measurable and $x \mapsto L_x \chi$ is not weak* measurable. □

Remark. An alternative proof of the nonlower semicontinuity can be obtained by adapting the nonmeasurability proof, as follows. If, instead of being nonmeasurable, E is a dense subgroup without interior (see Lemma A.1), then $\|L_x \chi - \chi\| = 0$ if $x \in E$ and $\|L_x \chi - \chi\| = 1$ if $x \notin E$. The nonlower semicontinuity follows at once.

2. Continuity of translation in LUC*

Let LUC = LUC(G) denote the space of left uniformly continuous functions on the locally compact group G. Then LUC is a commutative C^*-subalgebra of $L^\infty(G)$, and therefore LUC has a maximal ideal space (spectrum) $\Delta = \Delta\,\mathrm{LUC}(G)$. The dual space $\mathrm{LUC}^* = \mathrm{LUC}(G)^*$ is the space $M(\Delta)$ of regular Borel measures on Δ. The group action on $\mathrm{LUC}(G)^*$ is weak* continuous (that is obvious) and hence lower semicontinuous [GLL].

Proposition 2.1. *Let G be a locally compact group. Then*

$$(\mathrm{LUC}(G)^*)_c = \mathrm{LUC}(G)^*$$

if and only if G is discrete.

Proof. Point evaluation at $g \in G$ is an element of $\mathrm{LUC}(G)^*$. Of course, if $x \in G$ and $x \neq e$, then there exists $f \in \mathrm{LUC}(G)$ such that $\|f\| = 1$, $f(xg) = 1$, and $f(g) = -1$. Therefore $\|L_x^* \delta_g - \delta_g\| = 2$. Proposition 2.1 now follows.

Proposition 2.2. *Let G be a locally compact group. Then $(\mathrm{LUC}(G)^*)_c$ is an L-subspace of* $\mathrm{LUC}(G)^*$.

Proof. Note that $\mathrm{LUC}(G)$ is a commutative C^*-algebra, so

$$\mathrm{LUC}(G) = C(\Delta\,\mathrm{LUC}(G)) \quad \text{and} \quad \mathrm{LUC}(G)^* = M(\Delta\,\mathrm{LUC}(G)),$$

where $\Delta\,\mathrm{LUC}(G)$ is the maximal ideal space of $\mathrm{LUC}(G)$. It follows that $\mathrm{LUC}(G)$ is dense in $L^1(\mu)$ for every $\mu \in \mathrm{LUC}(G)^*$. Thus, it will suffice to show that if $\mu \in (\mathrm{LUC}(G)^*)_c$ and $f \in \mathrm{LUC}(G)$, then $f\mu \in (\mathrm{LUC}(G)^*)_c$, which is a $2 - \varepsilon$ argument and is left to the reader. □

We will use the following lemma in the proof of Proposition 2.4.

Lemma 2.3. *Let W be a translation-invariant closed subspace of* $\mathrm{LUC}(G)$. *Let $\mu \in (W^*)_c$. If X is a translation-invariant subspace of $L^\infty(G)$ containing W, then there exists $\nu \in X_c^*$ such that the restriction of ν to W agrees with μ.*

Proof. Indeed, $(W^*)_c$ is a Banach module under convolution by $L^1(G)$. The continuity of translation means that the approximate units of $L^1(G)$ are approximate units for the module action. The Cohen Factorization Theorem [HR, vol. II, pp. 268–270] applies, and we can write $\mu = f * \mu'$, where $f \in L^1(G)$ and $\mu' \in (W^*)_c$. Let ϕ be an extension of μ' to an element of $\mathrm{LUC}(G)^*$, and define $\nu \in (W^*)_c$ by $\langle \nu, g\rangle = \langle \phi, f * g\rangle$ for $g \in W$. Since, for $f \in L^1(G)$ and $g \in W$, the vector integral $\int f(y) L_y g\, dy$ is norm-convergent, and since ϕ commutes with this integral, we have $\nu = \mu$ on W. □

Proposition 2.4. *Let G be a locally compact group. Then $(\mathrm{LUC}(G)^*)_c = L^1(G)$ if and only if G is compact.*

Proof. If G is compact, then $\mathrm{LUC}(G) = C(G)$, $\mathrm{LUC}(G)^* = M(G)$ and therefore $(\mathrm{LUC}(G)^*)_c = L^1(G)$ by (the proof of) [GM, 8.3.1] or Proposition 1.1(i).

If G is not compact, then $C_0(G)$ does not contain nonzero constants. Define the linear functional μ on $\mathbf{C} \oplus C_0(G)$ by $\langle \mu, a + f\rangle = a$, for all $a \in \mathbf{C}$ and $f \in C_0(G)$. Use Lemma 2.3 to extend μ to an element (call it μ also) of $(\mathrm{LUC}(G)^*)_c$. Since μ annihilates $C_0(G)$, $\mu \notin L^1(G)$. □

Remark. We may extend Proposition 2.4 by the following: G is compact if and only if every element of $(\mathrm{LUC}(G)^*)_c$ is either in $L^1(G)$ or absolutely continuous with respect to a translation-invariant mean on $\mathrm{LUC}(G)$. As we shall see in Theorem 3.10 (with $W = \mathrm{LUC}(G)$), all noncompact groups G are such that there exist $\mu \in (\mathrm{LUC}(G)^*)_c$ which are singular with respect to all translation-invariant means.

Lemma 2.5. *Let G be a nondiscrete locally compact group. Let $x \in \Delta\mathrm{LUC}(G)$ and U be a neighborhood of the identity e of G. Then there exist an infinite set of $g \in U$ such that $L_g x \neq x$ and the $L_g x$ are distinct.*

Proof. See [GnL] for an argument that proves that there exists one $g \in U$ such that $L_g x \neq x$. We will use that assertion in what follows. We may suppose that U is a compact neighborhood of e. Now, suppose to the contrary that $\{L_g x : g \in U\} = \{x_1, \dots, x_n\}$ for some integer $n \geq 1$. Fix $1 \leq j \leq n$ and consider the set $X_j = \{g : L_g x = x_j\}$. Then $X_j = \bigcap_{f \in \mathrm{LUC}(G)} \{g : f(L_g x) = f(x_j)\}$. Each of the sets in the intersection is closed, since we are considering $\mathrm{LUC}(G)$. Hence, X_j is closed. Hence, the set U is a finite union of the closed sets $X_1, \dots, X_n$. One of those sets, say X_1, must contain a neighborhood W of some element g_0. Let V be a neighborhood of e such that $VW \subseteq X_1$. Then $L_g x = x_1$ for all $g \in W$. But there exists $h \in V$ such that $L_h x_1 \neq x_1$. But then $L_h L_g x \neq x_1$, a contradiction. □

Theorem 2.6. *Let G be a nondiscrete locally compact group and let μ be a measure in $\mathrm{LUC}(G)^*$ with a nonzero discrete part. Then μ is not an element of $(\mathrm{LUC}(G)^*)_c$.*

Proof. Let $\mu = \sum_1^\infty \alpha_j x_j + \omega$, where the $x_j \in \Delta$ and ω is a continuous measure. We may assume $\alpha_1 = \max|\alpha_j| > 0$. Let $0 < \varepsilon < \alpha_1/5$. Choose an integer $k > 0$ such that $\sum_{j>k} |\alpha_j| < \varepsilon$. Choose disjoint compact neighborhoods $V_1, \dots, V_k$ of $x_1, \dots, x_k$. Let U be a compact neighborhood of e such that $Ux_j \subseteq V_j$ for $1 \leq j \leq k$. By Lemma 2.5, there exists $g \in U$ such that $gx_1 \neq x_1$. Since $\mathrm{LUC}(G) = C(\Delta)$, there exists $f \in \mathrm{LUC}(G)$ such that $f = 0$ outside of V_1, $\int |f|\, d\omega + \int |f|\, dL_g\omega < \varepsilon$, $\|f\|_\infty = 1$, $f(x_1) = 1$, and $f(gx_1) = 0$.

Then

$$\|g\mu - \mu\| \geq |\langle g\mu - \mu, f\rangle| \geq |\alpha_1| - 2\varepsilon - 2\varepsilon \geq |\alpha_1|/5. \quad \square$$

3. Continuity of translation in $L^{\infty *}$

We begin the study of $L^{\infty *}$ with a summary of some old results about continuity and measurability. First, though, we remind the reader that L^∞ may be identified with the set of continuous functions on a compact space, so $L^{\infty *}$ may be identified with the space of all regular Borel measures on that same space. Hence, the usual notions of absolute continuity and singularity apply to elements of $L^{\infty *}$.

Proposition 3.1. *Let G be a locally compact group. The following are equivalent:*

(i) *G is discrete.*

(ii) $L^{\infty}(G)_c = L^{\infty}(G)$.
(iii) $(L^{\infty}(G)^*)_c = L^{\infty}(G)^*$.

Proof. The equivalence of (i) and (ii) amounts to the (well-known) assertion that not every bounded measurable function on a locally compact group is continuous, except when the group is discrete. For the equivalence of (i) and (iii), let μ be any element of $L^{\infty}(G)^*$ whose restriction to $C_0(G)$ is evaluation at the identity e of G. Then $\|L_x\mu - \mu\| = 2$ (just take the supremum against elements of $C_0(G)$ whenever $x \neq e$). □

Proposition 3.2. *Let G be a locally compact group. If G is not discrete, and G is amenable as a discrete group, then there exist $\mu, \nu \in L^{\infty}(G)^*$ such that $x \mapsto L_x\mu$ is not weak* measurable as a function from G to $L^{\infty *}$, and $x \mapsto L_x\nu$ is not lower semicontinuous.*

Proof. The assertion about nonmeasurability is a restatement of a result of Rudin [Ru3]. Here are the details. We may assume that G is σ-compact. Let N be a compact normal subgroup of G such that G/N is metrizable and nondiscrete. (For a proof of the existence of N, see Lemma A.4.) By [Ru3], there exists $f \in L^{\infty}(G/N)$ such that for every $\phi : G/N \to [0, 1]$, there exists $\mu \in \Delta L^{\infty}(G/N)$ for which $\phi(x) = \langle f, L_x\mu\rangle$ for all $x \in G/N$. The standard duality argument (using the canonical inclusion $L^{\infty}(G/N) \subseteq L^{\infty}(G)$) shows we may identify f with an element of $L^{\infty}(G)$ and μ with an element of $\Delta L^{\infty}(G)$. [Indeed, the Šilov boundary of $L^{\infty}(G/N)$ is the entire maximal ideal space and $L^{\infty}(G/N)$ can be identified with a closed subalgebra of $L^{\infty}(G)$. Thus, point evaluations lift from the maximal ideal space of $L^{\infty}(G/N)$ to $L^{\infty}(G)$.] We let ϕ be the characteristic function of a nonmeasurable subset of G/N, so that the inverse image E' of E in G is nonmeasurable. This establishes the assertion of nonweak* measurability.

For the nonlower semicontinuity, we let H_0 be a countable dense subgroup of G/N and let H be the pre-image of H_0 in G. [Such H_0 exists because G/N is σ-compact and metrizable.]

Let $f \in L^{\infty}(G/N) \subseteq L^{\infty}(G)$ and $\tau \in \Delta L^{\infty}(G)$ be such that

$$\langle L_x\tau, f\rangle = \begin{cases} 0 & \text{if } x \in H_0N; \\ 1 & \text{if } x \notin H_0N. \end{cases}$$

Let $\{m_\alpha\}$ be a net of discrete measures on H such that for every $x \in H$, $\|L_x m_\alpha - m_\alpha\| \to 0$. (Such a net exists by the amenability of G_d.) Let ν be a weak* accumulation point of $\{m_\alpha * \tau\}$. Then $L_x\nu = \nu$ for all $x \in H$. Of course,

$$\langle L_x\nu, f\rangle = \lim\langle L_x m_\alpha * \tau, f\rangle = \begin{cases} 0 & \text{if } x \in H; \\ 1 & \text{if } x \notin H. \end{cases}$$

This implies that

$$\{x : \|L_x\nu - \nu\| > 1/2\} \supseteq \{x : |\langle L_x\nu - \nu, f\rangle| > 1/2\} = G \setminus H,$$

and

$$\{x : \|L_x\nu - \nu\| > 1/2\} \subseteq G \setminus H.$$

Since H is dense and without interior, the nonlower semicontinuity follows. □

Remarks. (i) A result of Talagrand [T] can be used to give a different proof of the assertion about nonmeasurability in Proposition 3.2(ii). The result is this: if Martin's axiom is assumed, then a function $f \in L^\infty(G)$ is Riemann integrable if and only if for every $\mu \in L^\infty(G)^*$, $x \mapsto \langle L_x\mu, f\rangle$ is measurable.

(ii) For $\mathbf{R}$ and $\mathbf{T}$, there are more constructive proofs of the assertion about nonlower semicontinuity in Proposition 3.2; see Theorem 3.4(ii)–(iii).

(iii) In all cases, $L^1(G) \subseteq (L^\infty(G)^*)_c$. We have not been able to characterize $(L^\infty(G)^*)_c$.

(iv) Since translation is continuous for the predual of $L^\infty(G)$, translation is lower semicontinuous on $L^\infty(G)$. But the failure of weak* measurability in $L^\infty(G)^*$ implies the failure of weak measurability of translation in $L^\infty(G)$. Hence, lower semicontinuity does not imply weak measurability.

We now show that the study of continuity under translation of measures μ on $\Delta L^\infty(G)$ for a locally compact group G can be reduced to the case of nonnegative measures.

Proposition 3.3. *Let G be a locally compact group and let $\mu \in L^\infty(G)^*$. Let $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ denote the decomposition of μ where the μ_j are all nonnegative, $\mu_1 \perp \mu_2$, and $\mu_3 \perp \mu_4$.*

(i) *If $x \mapsto L_x\mu$ is continuous, then $x \mapsto L_x\mu_j$ is continuous for $j = 1, \ldots, 4$.*
(ii) *If $x \mapsto L_x\mu$ is continuous, then $x \mapsto L_x|\mu|$ is continuous.*
(iii) *The converse of* (ii) *is false.*

Proof. (i) Assume that $x \mapsto L_x\mu$ is continuous. Fix $x \in G$. It is obvious that the real and complex parts of μ translate continuously. We may therefore assume that μ is real. We give the proof for $j = 1$.

Let A denote the set where μ and $\delta_x * \mu$ are both positive. Let B denote the set where μ is positive and $\delta_x * \mu$ is negative. And let C denote the set where μ is negative and $\delta_x * \mu$ is positive. Then

$$\begin{aligned}\|\mu - \delta_x * \mu\| &\geq \|(\mu - \delta_x * \mu)_{|A}\| + \|(\mu - \delta_x * \mu)_{|B}\| + \|(\mu - \delta_x * \mu)_{|C}\| \\ &\geq \|(\mu - \delta_x * \mu)_{|A}\| + \|(\mu)_{|B}\| + \|(-\delta_x * \mu)_{|C}\| \\ &= \|\mu_1 - \delta_x * \mu_1\|.\end{aligned}$$

This proves the continuity of $x \mapsto L_x\mu_1$.

(ii) Assume that $x \mapsto L_x\mu$ is continuous. Fix $\varepsilon > 0$. We shall show that for each fixed x,

$$\| L_x|\mu| - |\mu| \,\| \leq \|L_x\mu - \mu\| + \varepsilon. \tag{3.1}$$

This will suffice to complete the proof.

Recall that for each $\nu \in L^\infty(G)^*$, $\|\nu\| = \sup\sum_{j=1}^k |\nu(E_j)|$, where the supremum is taken over all finite Borel partitions $\{E_j\}$ of $\Delta L^\infty(G)$.

We consider $\|L_x|\mu| - |\mu|\,\|$. Since $L_x|\mu| = |L_x\mu|$, we have

$$\sum(L_x|\mu|(E_j) - |\mu|(E_j)|) = \sum ||L_x\mu|(E_j) - |\mu|(E_j)|,$$

for any Borel partition $\{E_j\}_{j=1}^J$, and for suitable finite Borel partitions $\{E_{j,k}\}$ of E_j, $1 \le j \le J$, we have

$$\begin{aligned}\sum(|L_x|\mu|(E_j) - |\mu|(E_j)|) &\le \sum_j \left|\sum_k |L_x\mu(E_{j,k})| - \sum_k |\mu(E_{j,k})|\right| + \varepsilon \\ &\le \sum_j \left|\sum_k |L_x\mu(E_{j,k}) - \mu(E_{j,k})|\right| + \varepsilon \\ &= \sum_{j,k} |(L_x\mu - \mu)(E_{j,k})| + \varepsilon \\ &\le \|L_x\mu - \mu\| + \varepsilon.\end{aligned}$$

Since the preceding holds for all finite Borel partitions $\{E_j\}$ of $\Delta L^\infty(G)$, (3.1) follows.

(iii) We give an example. Let $G = \mathbf{R}$. Let ω be a translation-invariant mean on $L^\infty(\mathbf{R})$ and let f be an idempotent function in $L^\infty(\mathbf{R})$ such that $x \mapsto L_x f\omega$ is not continuous. (See Theorem 3.4(i) for an example of such an ω and f.) Let $\mu = (1 - 2f)\omega$. (Here, "1" obviously denotes the constant function.) Then $\omega = |\mu|$, so $x \mapsto L_x|\mu|$ is continuous, while $x \mapsto L_x\mu$ cannot be continuous. □

Remarks. (i) The conclusion and proof of Proposition 3.3(i) hold for order-preserving group actions. For example, they hold for $\mathrm{LUC}(G)^*$ in place of $L^\infty(G)^*$. The proof does not hold for translation on $\mathrm{VN}(G)$, although a similar result does hold; see Proposition 4.1.

(ii) We do not know whether Proposition 3.3(iii) holds for all nondiscrete groups.

We now show that $(L^\infty(\mathbf{R})^*)_c$ is not an L-space. The method gives an alternative proof of the nonlower semicontinuity of the action of $\mathbf{R}$ on $L^\infty(\mathbf{R})^*$, as well as that of the action on $L^\infty(\mathbf{T})^*$: the relevant assertions are included in the statement of the next result.

Theorem 3.4. (i) *There exists a translation-invariant mean* $\mu \in L^\infty(\mathbf{R})^*$ *and an idempotent function* $f \in L^\infty(\mathbf{R})$ *such that* $f\mu \notin (L^\infty(\mathbf{R})^*)_c$. *In particular,* $(L^\infty(\mathbf{R})^*)_c$ *is not an* L*-space.*

(ii) *The action of* $\mathbf{R}$ *on* $L^\infty(\mathbf{R})^*$ *is not lower semicontinuous.*

(iii) *The action of* $\mathbf{T}$ *on* $L^\infty(\mathbf{T})^*$ *is not lower semicontinuous.*

Proof. (i) We let μ be any weak* accumulation point of $\{n^{-1}\chi_{[n,2n]}m_{\mathbf{R}}\}$. Then μ is a translation-invariant mean. We define f as the characteristic function of the union $E = \bigcup E_n$, where

$$E_n = 2n + \bigcup_{j=0}^{\frac{1}{2}10^n - 1} (2j/10^n, (2j+1)/10^n).$$

Then $\langle \mu, f \rangle = \frac{1}{4}$, because E contains half of each interval $(2k, 2k+1)$ and is disjoint from each interval $(2k-1, 2k)$.

Define $x_j = \sum_{r=j}^{\infty} 10^{-r}$ for all $j \geq 1$, $f = \chi_E$, and $\nu = f\mu$. For $1 \leq j \leq k < \infty$, we have

$$|(E + x_j) \cap E \cap [2k, 2k+1]| = \left(\frac{1}{2} - \frac{[10^k x_j] + 1}{2 \cdot 10^k} \right) \frac{1}{9},$$

which converges to $\frac{1}{18}(1 - x_j)$ as $k \to \infty$.

Thus, $\mu((E+x_j) \cap E) = \frac{1}{18}(1-x_j) \to \frac{1}{18}$, and $\|L_{x_j} \nu - \nu\| = \frac{1}{4} + \frac{1}{4} - \frac{2}{18}(1-x_j) \to \frac{1}{2} - \frac{1}{9} > 0$. It follows that $\nu \notin (L^\infty(\mathbf{R})^*)_c$.

(ii) We use the notation of (i). For $1 \leq j \leq n$, let $x_{j,n} = \sum_{r=j}^{n} 10^{-r}$. If $k > n$ and E, μ, ν are as in (i), then

$$|(E + x_{j,n}) \cap E \cap (2k, 2k+1)| = \frac{1}{2} - \frac{x_{j,n}}{2},$$

so $\mu((E + x_{j,n}) \cap E) = \frac{1}{4}(1 - x_{j,n})$ and

$$\|L_{x_{j,n}} \nu - \nu\| = \frac{1}{4} + \frac{1}{4} - \frac{1}{2}(2 - x_{j,n}) = \frac{x_{j,n}}{2},$$

which converges to $x_j/2$ as $n \to \infty$. Thus, for $j = 1$ (in fact, for every $j \geq 1$), $\|L_{x_j} \nu - \nu\| > \frac{1}{4}$ (see the end of the proof of (i) above). Since $x_j \leq \frac{1}{9}$, we have $x_{j,n}/2 \leq x_j/2 < \frac{1}{4}$. The set $\{x \in \mathbf{R} : \|L_x \nu - \nu\| > \frac{1}{4}\}$ is not open, since it contains $x_1 = \frac{1}{9}$, but not a neighborhood of x_1, as it does not contain any $x_{1,n}$ for $j > 1$. This concludes the proof of (ii).

(iii) We identify $\mathbf{T}$ with the set $[0, 1)$. We let ν be a weak* accumulation point in $L^\infty(\mathbf{T})^*$ of the sequence of L^1-functions $f_n = n\chi_{[1-1/n, 1]}$. The restriction of ν to $C(\mathbf{T})$ is the point mass at the identity. For $k \geq 1$, let $m_k = 10^{2^k}$ and $\mu_k = \frac{1}{m_k} \sum_{j=1}^{m_k} L_{j/m_k} \nu$. Let μ be a weak* accumulation point of the μ_k. Then μ is a mean on $L^\infty(\mathbf{T})$ that is invariant with respect to translation by finite sums of the form $\sum_{j,k} j/m_k$.

Let $C_1 = [0, 1) \setminus \bigcup_{j=0}^{9} (\frac{j}{10}, \frac{j}{10} + \frac{1}{100})$. For $k > 1$, let

$$C_k = C_{k-1} \Big\backslash \bigcup_{j=0}^{m_{k-1}-1} \left(\frac{j}{m_{k-1}}, \frac{j}{m_{k-1}} + \frac{1}{m_k} \right).$$

Let $C = \bigcap_{m=1}^{\infty} C_m$. Then C is a compact set with no interior and with nonzero Lebesgue measure. (By starting with a large value of m, the Lebesgue measure of C could be made to be as close to 1 as desired.) For $a \in \mathbf{T}$,

$$\langle L_a\mu, \chi_C\rangle = \lim_k \lim_m \langle L_a\mu_k, \chi_{C_m}\rangle,$$

by an obvious abuse of notation.

We claim that $\langle \mu, \chi_C\rangle = |C|$ (the Lebesgue measure of C). Indeed, for each fixed k and all $m \geq k$, $\langle \mu_k, \chi_{C_m}\rangle = |C_k|$, and the claim follows. We also claim that, if the sequence $\{\varepsilon_j\}$ consists mostly of 0's, with an occasional 1, then $a = \sum \varepsilon_j 10^{-j}$ is such that $\langle L_a\mu, \chi_C\rangle = 0$. Indeed, suppose that for $j \geq 1$, $a_j = \sum_{r=j}^{\infty} 10^{-2^{2^r}}$. Then $L_{a_j - a_l}\mu = \mu$ for $j > l$, so $L_{a_j}\mu = L_{a_l}\mu$ for all $1 \leq j, l < \infty$. For fixed j and $k = 2j$, we have

$$\langle L_{a_j}\mu_k, C_{k+1}\rangle = \langle L_{a_{j+1}}\mu_k, C_{k+1}\rangle = 0,$$

because $0 < a_{j+1} < 10^{-2^{2j+1}} = 10^{-2^{k+1}}$ and because all the intervals of the form $(l \cdot 10^{-2^k}, l \cdot 10^{-2^k} + 10^{-2^{k+1}})$ are missing from C_{k+1}. Since C_m decreases for increasing m, we have

$$\langle L_{a_j}\mu, C\rangle = \lim_k \lim_m \langle L_{a_j}\mu_k, C_m\rangle = \lim_k \lim_m \langle L_{a_m}\mu_k, C_{n(m)}\rangle = 0.$$

Thus, $\|L_b\mu - \mu\| = 0$ for all finite sums $b = \sum_{j,k} j \cdot 10^{-2^k}$, and $\|L_{a_j}\mu - \mu\| = |C|$ for all j. It follows that $\{x : \|L_x\mu - \mu\| > \frac{|C|}{2}\}$ is not open. □

Remarks. (i) If μ is a translation-invariant mean constructed as in (i) above and $\varepsilon > 0$ is given, then there exists a compact open subset $E \subseteq \Delta L^{\infty}(\mathbf{R})$ with measure $\mu(E) > \frac{1}{2} - \varepsilon$ and a sequence $\{g_n\}$ tending to zero such that for all sufficiently large n, we have

(a) $\mu(L_{g_n}E \cap E) = 0$, and

(b) Each g_n is the limit of a sequence $\{g_{n,k}\}$ with $\mu(L_{g_{n,k}}E \cap E) > \frac{1}{2} - \varepsilon$.

So the set E, which is almost half of the spectrum, behaves in a very strange way. To find E, we let $\eta = 10^{-m} < \varepsilon$ and let F be the union of the sets

$$F_k = k + \bigcup_{r=0}^{\frac{1}{2}10^{2^k}-1} \left(\frac{2r}{10^{2^k}}, \frac{2r+1}{10^{2^k}}\right) \cap (k, k+1-\eta).$$

Then χ_E is the Gel′fand transform of χ_F. We set $g_n = \sum_{r=n}^{\infty} 10^{-2^r}$ and $g_{n,k} = \sum_{r=n}^{k} 10^{-2^r}$. The indices $n > m$ are "sufficiently large" in the preceding.

(ii) The proof of Theorem 3.4(iii) above actually shows more. Namely, that there is a probability measure $\mu \in L^{\infty}(\mathbf{T})^*$ and a compact subset E of $\Delta L^{\infty}(G)$ such that χ_E is the Gel′fand transform of χ_C and $\mu(L_g E) = \mu(E)$ for all g in the (dense) subgroup H of elements of $\mathbf{T}$ with finite decimal expansion, and there is a sequence $g_n \to 0$ such that $\mu(L_{g_n}E) = 0$ for all n. We know that $L_{g_n}\mu$ is carried by E^c (the complement of E) and that $\mu(E^c)$ can be made arbitrarily small by changing C, and that after such changing of E the very same g_n still works so we also can obtain (by an increasing limit in E, so to speak)

$$\|L_g\mu - \mu\| = \begin{cases} 0 & \text{for all } g \in H; \\ 2 & \text{for suitable } g \text{ arbitrarily close to } 0. \end{cases}$$

Let ω be a translation-invariant mean on G. For a nonzero element $\mu \in (L^\infty(G)^*)_c$, we have $|\mu| \in (L^\infty(G)^*)_c$ by Proposition 3.3(ii), so

$$\nu(f) = \int \langle L_x|\mu|, f\rangle \, d\omega$$

exists for every $f \in L^\infty(G)$ and $\mu_\omega = \nu/\|\nu\|$ defines a translation-invariant mean.

Proposition 3.5. *Let G be compact, let $\mu \in (L^\infty(G)^*)_c$ and let ω be a translation-invariant mean on $L^\infty(G)$. Then $\mu \ll \mu_\omega$.*

Proof. Clearly, we may assume that $\mu \geq 0$, $\|\mu\| = 1$, and in consequence $\nu = \mu_\omega$.

Let $\varepsilon > 0$ be given. To show that $\mu \ll \mu_\omega$, we must find $\delta > 0$ such that whenever $f \in L^\infty(G)$ with $0 \leq f \leq 1$ and $\langle \mu_\omega, f\rangle < \delta$, we then have $\langle \mu, f\rangle < \varepsilon$. Let U be a compact neighborhood of e such that $\|L_y\mu - \mu\| < \frac{\varepsilon}{2}$ for $y \in U$. Because ω is a translation-invariant mean, $\omega(U) \neq 0$, so the number $\delta = \frac{\varepsilon}{2}\omega(U) > 0$. Let $f \in L^\infty(G)$ with $0 \leq f \leq 1$ and $\langle \mu_\omega, f\rangle < \delta$. Then

$$\langle \mu_\omega, f\rangle = \int_G \langle L_y\mu, f\rangle \, d\omega \geq \int_U \langle L_y\mu, f\rangle \, d\omega \geq \omega(U)\left(\langle \mu, f\rangle - \frac{\varepsilon}{2}.\right)$$

Hence, $\omega(U)\langle \mu, f\rangle < \delta + \frac{\varepsilon}{2}\omega(U) = \varepsilon\omega(U)$. □

Remark. Proposition 3.5 suggests the possibility that $(L^\infty(G)^*)_c$ is an L-subspace (band). We have shown (see Theorem 3.4) that is not the case for $G = \mathbf{R}$, and we suspect it is false even for (some?) compact groups. Of course, it is exactly in the compact case that establishing this possibility would give a complete characterization of $(L^\infty(G)^*)_c$ as the L-space generated by the translation-invariant means on $L^\infty(G)$. In the noncompact case, we show (Theorem 3.10) that the situation is even worse: for many G there are elements of $(L^\infty(G)^*)_c$ that are not absolutely continuous with respect to a translation-invariant mean (nor, in many cases, with respect to Haar measure as well—see also Theorem 3.7). We explore in the remainder of this section variations on those two themes: how close is $(L^\infty(G)^*)_c$ to being a band? How close is $(L^\infty(G)^*)_c$ to containing only elements absolutely continuous with respect to a translation-invariant mean? See also the Remarks following Theorem 3.4 for some calculations related to that subspace question.

Lemma 3.6. *Let G be a locally compact group. Let $x \in \Delta L^\infty(G)$.*

(i) *If G is nondiscrete, then $\delta_x \notin (L^\infty(G)^*)_c$.*

(ii) *If $\delta_x \not\perp \mu$ for some translation-invariant mean μ, then G is finite.*

Proof. (i) By Lemma 2.5, every neighborhood U of e contains an element g such that $gx \neq x$. Of course, then $\|\delta_{gx} - \delta_x\| = 2$, so (i) follows.

(ii) Suppose first that G is not discrete and that $\delta_x \not\perp \mu$ for some translation-invariant mean μ. Then μ contains a nonzero point mass at x. Let U be any neighborhood of e. By Lemma 2.5 (or [GnL]), there exists an infinite set g_j of points in U such that $L_{g_j}x$ are distinct. Hence, μ contains an infinite set of equal point masses. That contradicts the finiteness of μ and proves (ii) for nondiscrete G.

If G is discrete, then for any $x \in \Delta L^\infty(G)$, $L_g x \neq x$ for all $g \in G$ [Rup, Corollary 4.8]. We now argue as in the nondiscrete case. □

Theorem 3.7. *Let G be a unimodular locally compact group with an infinite closed discrete subgroup H. Then there exists an element $\mu \in (L^\infty(G)^*)_c$ that is singular with respect to every translation-invariant mean on G and with respect to $L^1(G)$.*

Proof. If G is discrete (so $H = G$ will do), then the theorem asserts the existence of $\mu \in L^{\infty*}$ such that $\mu \perp \nu$ for all translation-invariant means ν; and $\mu \perp L^1(G)$. Choose $\mu = \delta(x)$ in Lemma 3.6(ii), where $x \in \Delta L^\infty(G)$ is at infinity, that is, annihilates $C_0(G)$.

We thus may assume that G is not discrete. Let U be a compact neighborhood of the identity e of G such that $UU^{-1} \cap H = \{e\}$. Let ϕ be an element of $L^\infty(H)^*$ that is singular to all invariant means and to all elements of $L^1(H)$ (for example, $\phi = \delta_x$ as in Lemma 3.6). For $f \in L^\infty(G)$ and $h \in H$, we define $Tf(h) = \int_{Uh} f\,du$. Then $Tf \in L^\infty(H)$. We set $\langle \mu, f\rangle = \langle \phi, Tf\rangle$. Then clearly $\mu \in L^\infty(G)^*$. Furthermore, we claim that $\lim_{x\to e} T_x f = Tf$ with convergence in norm *uniformly* for f in the unit ball of $L^\infty(G)$. It will follow that $\mu \in (L^\infty(G)^*)_c$.

We establish that claim. Note that for $f \in L^\infty(G)$, $x \in G$, and $h \in H$,

$$T_x f(h) = \int_{Uh} (L_x f)(t)\,dt = \int_{x^{-1}Uh} f(t)\,dt.$$

It follows that

$$\begin{aligned} |T_x f(h) - T(h)| &\le \left| \int_{x^{-1}Uh} f(t)\,dt - \int_{Uh} f(t)\,dt \right| \\ &\le \|f\|_\infty \int |\chi_{x^{-1}Uh} - \chi_{Uh}|\,dt = \|f\|_\infty \|L_x \chi_{Uh} - \chi_{Uh}\|_1 . \end{aligned}$$

Since G is unimodular, the last line is independent of h. Since U is compact, $\chi_U \in L^1(G)$. The claim now follows.

The linear functional μ has one property we use: if f is the characteristic function of UH, then $f\mu = \mu$.

Now consider the subspace Y of $L^\infty(G)$ consisting of functions that are constant on each set of the form Uh for $h \in H$, and zero outside UH. Then Y is isomorphic to $L^\infty(H)$. Let ν be a positive translation-invariant mean on $L^\infty(G)$. Then the restriction of ν to Y gives a translation-invariant linear functional on $L^\infty(H)$, though it may be zero.

We argue by contradiction, and suppose that $\mu \not\perp \nu$. Then ν cannot be zero on Y. Indeed let f be the characteristic function of UH. Then $f\mu = \mu$, as observed above. Let $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous with respect to ν and μ_s is singular with respect to ν. If $f\nu = 0$, then $\mu_a = f\mu_a \ll f\nu = 0$. Hence $f\nu \neq 0$. Hence ν is not zero on Y. Hence ν restricts to a translation-invariant functional (of norm possibly smaller than one) on Y. Since the restriction of μ to Y is exactly ϕ, the restriction of ν is singular with respect to ϕ. Therefore there exists a sequence $\{f_n\}$ of functions in the unit ball of Y (which we identify with $L^\infty(H)$) such that $f_n\mu \to \mu$ and $f_n\nu \to 0$ (both in norm). Hence $\nu \perp \mu$.

Furthermore, if $\mu \not\perp L^1(G)$, then the restriction of μ to Y is not singular with respect to $L^1(H)$, another contradiction of the choice of ϕ. □

For abelian groups, we have another version of Theorem 3.7; this is a special case of Theorem 3.10; we include it because its proof is different from those of Theorem 3.7 and Theorem 3.10.

Proposition 3.8. *Let G be a noncompact abelian group and let W be a C^*-translation-invariant subalgebra of $L^\infty(G)$ that contains $AP(G)$. Then there exists $\mu \in (W^*)_c$, $\mu \geq 0$, such that every nonnegative extension of μ to $L^\infty(G)$ is singular with respect to every translation-invariant mean on $L^\infty(G)$.*

Proof. We first construct μ. Let ν be a probability measure on the Bohr compactification bG of G such that ν is singular with respect to m_{bG} and such that the support of $\hat{\nu}$ in the dual group Γ of G is compact (here Γ has its regular—nondiscrete—topology). Such a ν can be found by taking a Riesz product on bG based on a relatively compact sequence in Γ. Let ω be an extension of ν to $L^\infty(G)$. Fix $f \in L^1(G)$ such that $\|f\|_1 \leq 2$, $\operatorname{Supp}\hat{f}$ is compact, and $\hat{f} = 1$ on $\operatorname{Supp}\hat{\nu}$. Define a linear functional μ on W by

$$g \mapsto \langle \omega, f * g \rangle = \langle \mu, g \rangle.$$

We claim that μ has the required properties. First, because of the convolution with f, $\mu \in (L^\infty(G)^*)_c$, so the restriction of μ to W is in $(W^*)_c$. Second, μ extends ν. Finally, if μ were not singular with respect to a translation-invariant mean τ, then the restriction of μ to $AP(G)$ would not be singular with respect to the restriction of τ. (Here we use the nonnegativity of μ.) Since ν is singular, this cannot happen. □

For some nonabelian groups and with $W = AP(G)$, the conclusion of Proposition 3.8 is false, as we now show. Note that $SL(2, \mathbf{R})$ satisfies the hypotheses of the next result.

Proposition 3.9. *Let G be a locally compact group such that $AP(G)$ is finite dimensional. Then every element of $AP(G)^*$ is absolutely continuous with respect to the translation-invariant mean on $AP(G)$.*

Proof. Let $\overline{G}^{ap}$ denote the AP-compactification of G. Then $\overline{G}^{ap}$ is a finite group of cardinality n, and $AP(G) = C(\overline{G}^{ap})$. Define for each $f \in AP(G)$, $m(f) = n^{-1}\sum\{\overline{f}(x) : x \in \overline{G}^{ap}\}$, where $\overline{f}(x) = \langle x, f\rangle$ for each $x \in \overline{G}^{ap} \subseteq AP(G)^*$. Then m is the unique translation-invariant mean on $AP(G)$. Of course, if $\mu \in AP(G)^*$, then μ is a measure on the finite discrete space $\overline{G}^{ap}$. □

Let $CB(G)$ denote the continuous bounded functions on G.

Theorem 3.10. *Let G be a noncompact locally compact group. Let W be a translation-invariant C^*-subalgebra of $L^\infty(G)$ such that $W' = W \cap CB(G)$ separates points of G. Then there exists a nonnegative $\mu \in (W^*)_c$ such that the restriction of μ to W' is singular with respect to every translation-invariant mean on W'. Thus, every extension of μ to a nonnegative element of $L^\infty(G)^*$ is singular to every translation-invariant mean on $L^\infty(G)$.*

Proof. We let Δ be the maximal ideal space of W'. For each $x \in G$, let δ_x denote the point evaluation at $x : \langle\delta_x, h\rangle = h(x)$ for all $h \in W'$. If $x \in G$, then $x \mapsto \delta_x$ is (weak*) continuous from $G \to W'^*$ and one-to-one. Let $f \neq 0$ be a continuous function on G with compact support K. Define μ by

$$\langle\mu, h\rangle = \int h(t)f(t)\,dt \quad \text{for } f \in W.$$

Then $\mu_{|W'}$ (considered as a measure on Δ) is nonzero and has support contained in $\widetilde{K} = \{\delta_x : x \in K\}$. We claim that $m(\widetilde{K}) = 0$ for every translation-invariant mean m on W'. Indeed, since G is not compact, there exists a sequence $\{x_i\}$ of elements of G such that

$$x_iK \cap x_jK = \varnothing \quad \text{for } i \neq j.$$

Hence,

$$(x_iK)^\sim \cap (x_jK)^\sim = \varnothing \quad \text{for } i \neq j.$$

If $m((x_iK)^\sim) \neq 0$, with m translation invariant, then $m(\bigcup_i (x_iK)^\sim) = \infty$, a contradiction. We claim that $x \mapsto \delta_x * \mu$ is continuous. To see this, fix $x \in G$ and $h \in W$. Then

$$\langle\delta_x * \mu, h\rangle = \langle\mu, L_xh\rangle = \int h(xt)f(t)\,dt = \int h(t)f(x^{-1}t)\,dt.$$

Hence, if $x_\alpha \to x$, then

$$\|\delta_{x_\alpha} * \mu - \delta_x * \mu\| \le \|L_{x_\alpha}f - L_xf\|_1 \to 0.$$

Since translation-invariant means are nonnegative, restriction of a translation-invariant mean to W' is a (scalar multiple of a) translation-invariant mean on W' (if W' does not contain the constants, by a "translation-invariant mean" on W' we mean a positive translation-invariant functional on W' of norm one), and the restriction to W' of any nonnegative extension of μ will agree with μ. □

Let G be a locally compact group, and let $L^1(G)^0$ denote the annihilator of $L^1(G)$ in $L^\infty(G)^{**}$. Since $L^\infty(G)$ is a commutative Banach algebra, $L^\infty(G)^{**}$ is also a commutative Banach algebra, $L^\infty(G)^*$ is an L-space of bounded Borel measures on the maximal ideal space of $L^\infty(G)$, and $L^1(G)^0$ is a weak* closed ideal in $L^\infty(G)^{**}$. Because of the L-space structure, there exists an idempotent $z \in L^\infty(G)^{**}$ such that $L^1(G)^0 = zL^\infty(G)^{**}$. We define P to be the projection on $L^\infty(G)^*$ given by

$$\langle P\mu, f\rangle = \langle \mu, (1-z)f\rangle \quad \text{for all } f \in L^\infty(G).$$

Lemma 3.11. *In the preceding circumstances, the following hold*:

(i) *For each* $x \in G$,

$$L_x(hk) = L_x(h)L_x(k) \quad \textit{for all } h, k \in L^\infty(G)^{**}.$$

(ii) *For each* $x \in G$, $L_xL^1(G)^0 = L^1(G)^0$, $L_x^*z = z$, *and* $L_xP = P$.

(iii) *For each* $x \in G$, $L_x^*P = PL_x^*$.

(iv) *If* m *is a left translation-invariant mean on* $L^\infty(G)$, *then* $Pm \in L^1(G)$ *is a positive left invariant functional. If* G *is not compact, then* $Pm = 0$ *and* $(1-P)m = m$.

Proof. (i) This follows from the fact that $L^\infty(G)$ is weak* dense in $L^\infty(G)^{**}$, and that multiplication in $L^\infty(G)^{**}$ is separately continuous in the weak* topology.

(ii) Let $f \in L^1(G)^0$, $\phi \in L^1(G)$, and $x \in G$. Then $\langle L_x^{**}f, \phi\rangle = \langle f, L_x^*\phi\rangle$. Hence, $L_xL^1(G)^0 = L^1(G)^0$. Furthermore, $L_x^{**}z$ is also an identity on $L^1(G)^0$. To see this, let $f \in L^1(G)^0$. Then

$$\begin{aligned}(L_x^{**}z)(f) &= (L_x^{**}z)(L_x^{**}(L_{x^{-1}}^{**}f)) \\ &= L_x^{**}(z(L_{x^{-1}}^{**}f)) \quad \text{by (i)} \\ &= L_x^{**}(L_{x^{-1}}^{**}f) = f.\end{aligned}$$

It follows that $L_x^{**}z = z$.

(iii) We apply (ii) at the second to third equality below. Let $x \in G$, $\phi \in L^\infty(G)^*$, and $f \in L^\infty(G)^{**}$. Then

$$\begin{aligned}\langle L_x^*(1-P)\phi, f\rangle &= \langle (1-P)\phi, L_x^{**}f\rangle = \langle \phi, zL_x^{**}f\rangle \\ &= \langle \phi, L_x^{**}zf\rangle = \langle (1-P)(L_x^*)\phi, f\rangle.\end{aligned}$$

(iv) Since P is a projection from an idempotent element of $L^\infty(G)^{**}$, Pm is a positive measure (functional). By (iii), Pm is also invariant. Of course, if G is noncompact, then the only invariant element of $L^1(G)$ is the zero measure, and (iv) follows. □

Proposition 3.12. *Let* G *be a locally compact group, and let* W *be a left translation-invariant* W^**-subalgebra of* $L^\infty(G)$. *If* W *does not admit a left translation-invariant mean of the form* μ_f, *where*

$$\langle \mu_f, h\rangle = \int f(t)h(t)\,dt \quad \textit{for all } h \in W$$

and where $f \in L^1(G)$, *then every* μ_f *is singular with respect to each left translation-invariant mean on* W *when regarded as measures on* $\Delta(W^{**})$.

Proof. In this case, W_*, the unique predual of W, is exactly $\{\mu_f : f \in L^1(G)\}$. The theorem now follows from Lemma 3.11, which is valid with $L^\infty(G)$ replaced by W.

Corollary 3.13. *Let* G *be a noncompact locally compact group. Then every nonzero element of* $L^1(G)$ *is singular with respect to every left translation-invariant mean on* $L^\infty(G)$ *when regarded as measures on* $\Delta(L^\infty(G)^{**})$.

Theorem 3.14. *Let* G *be a locally compact group. Then the following are equivalent.*

(i) $(L^\infty(G)^*)_c = L^1(G)$.
(ii) $L^\infty(G)$ *has a unique left translation-invariant mean.*

Proof. If (i) holds, then $\mathrm{LUC}(G)^*_c = L^1(G)$ also (by Lemma 2.3), so G is compact by Proposition 2.4. If (ii) holds, then [Ch] shows that G is compact. We therefore may assume that G is compact, for both directions of the equivalence.

(i) $\Rightarrow$ (ii) First, remember that (since G is compact), m_G is a translation-invariant mean. If ω is a translation-invariant mean, then (by the hypothesis of (i)), $\omega \in L^1(G)$, so $\omega = m_G$, so there is at most one translation-invariant mean on G, and (ii) follows.

(ii) $\Rightarrow$ (i) By the first paragraph, we may assume that G is compact. Let $\mu \in (L^\infty(G)^*)_c$. Since G is compact, Haar measure m_G is a translation-invariant mean, so Proposition 3.5 may be applied, with the conclusion that μ is absolutely continuous with respect to the translation-invariant mean μ_{m_G}. By (ii), $\mu_{m_G} = m_G$. □

Remark. If G is amenable as a discrete group (and therefore amenable in its original topology as well), then $L^\infty(G)$ has more than one left translation-invariant mean; see [Gn1, Ru2] for a proof. However, for $n \geq 3$ and $G = SO(n, \mathbf{R})$ the situation is different: $L^\infty(G)$ has a unique left translation-invariant mean (see [M, Dr]).

4. Continuity of translation in VN(G)

Let $\mathrm{VN} = \mathrm{VN}(G)$ denote the von Neumann algebra of the locally compact group G; that is, $\mathrm{VN}(G)$ is the dual space of $A(G)$. Then VN is a C^*-subalgebra of the bounded operators on $L^2(G)$.

We define a number of norms and spaces as follows. For $f \in L^1(G)$, define $\|f\|_{C^*_\lambda} = \|\lambda(f)\|$, where λ is the regular representation of G. For $f \in L^1(G)$, we define

$$\|f\|_{C^*} = \sup\{\|\pi(f)\| : \pi \text{ is a continuous unitary representation of } G\}.$$

We define $C^*_\lambda(G)$ to be the completion of $L^1(G)$ in the norm $\|\cdot\|_{C^*_\lambda}$ and $C^*(G)$ to be the completion of $L^1(G)$ in the norm $\|\cdot\|_{C^*}$. We set $B_\lambda(G) = C^*_\lambda(G)^*$

and $B(G) = C^*(G)^*$. It is always the case that $B_\lambda(G)$ is closed in $B(G)$. If G is amenable, then $B(G) = B_\lambda(G)$. We let $C_\delta^*(G)$ be the closure in $\mathrm{VN}(G)$ of $L^1(G_d)$. Finally, $C_\delta^*(G_d)$ is the closure in $\mathrm{VN}(G_d)$ of $L^1(G_d)$. Here G_d is G with the discrete topology. If G_d is amenable, then $C_\delta^*(G) = C_\delta^*(G_d)$ [DR1, Proposition 3.4].

Proposition 4.1. *Let G be a locally compact group. If f and f^* are both in $\mathrm{VN}(G)_c$, then $f_+ \in \mathrm{VN}(G)_c$, where f_+ is the nonnegative part of $\mathrm{Re}(f) = \frac{1}{2}(f + f^*)$. If G is compact or f normal, the hypothesis on f^* is superfluous.*

Proof. For the first part, we may suppose that $f = f^*$. Let P be a spectral projection of f such that $fP = f_+$. Then

$$\begin{aligned}\|L_x f - f\| &= \sup_{\|\eta\|\,\|\xi\|\leq 1} |\langle (L_x f - f)\eta, \xi\rangle| \geq \sup_{\|\eta\|\,\|\xi\|\leq 1} |\langle (L_x f - f)P\eta, \xi\rangle| \\ &= \sup_{\|\eta\|\,\|\xi\|\leq 1} |\langle (L_x f_+ - f_+)\eta, \xi\rangle| = \|L_x f_+ - f_+\|.\end{aligned}$$

This proves the first part. We always have $f \in \mathrm{VN}(G)_c$ if and only if $|f^*| = |ff^*|^{1/2} \in \mathrm{VN}(G)_c$. [Indeed, $f \in \mathrm{VN}(G)_c$ implies $|ff^*|^{1/2} \in \mathrm{VN}(G)_c$ (as it is a limit of polynomials in ff^*). Conversely, suppose that $|f^*| \in \mathrm{VN}(G)_c$, and let $f = u|f|$ be the polar decomposition of f. Then $|f^*|u \in \mathrm{VN}(G)_c$. But $|f^*|u = uu^*|f^*|u = u|f| = f$. Hence $f \in \mathrm{VN}(G)_c$.] If f is normal, then $|f^*| = |f|$, so f is in $\mathrm{VN}(G)_c$ if and only if f^* is. The same assertion holds for compact G and any $f \in \mathrm{VN}(G)$ by Theorem 4.4(ii) below. □

Proposition 4.2. *Let G be a locally compact group. Then $\mathrm{VN}(G)_c = \mathrm{VN}(G)$ if and only if G is discrete.*

Proof. Point evaluation at $g \in G$ is an element of $\mathrm{VN}(G)$. Of course, if $x \in G$, then there exists $f \in A(G)$ such that $\|f\| = 1$, $f(xg) = 1$, and $f(g) = 0$. Therefore $\|L_x^*\delta_g - \delta_g\| \geq 1$. □

Let G be a locally compact group. Then $\mathrm{VN}(G)$ and $L^2(G)$ are both subspaces of the algebraic dual of $A(G) \cap C_c(G)$. Hence, the intersection $\mathrm{VN}(G) \cap L^2(G)$ is well defined. In fact, we can illustrate this more precisely, as follows. Let $f \in L^2(G) \cap L^1(G)$. Then $g \mapsto \int fg\,dx$ defines a linear functional on $A(G)$, thus giving rise to an element of $\mathrm{VN}(G) = A(G)^*$. On the other hand, f operates on $L^2(G)$ by convolution, thus giving rise to an element of $\mathrm{VN}(G)$. These two elements of $\mathrm{VN}(G)$ are the same. Thus, such an f can be thought of as an element of $L^2(G) \cap \mathrm{VN}(G)$, and the intersection is not empty.

Proposition 4.3. *Let G be a unimodular locally compact group. Then the following hold.*

(i) $\mathrm{VN}(G)_c \cap L^2(G)$ *is norm dense in* $\mathrm{VN}(G)_c$.

(ii) *If* $\mathrm{VN}(G) \cap L^2(G)$ *is norm dense in* $\mathrm{VN}(G)$, *then* G *is discrete.*

Proof. (i) Obviously,

$$L^1(G) \cap L^2(G) * \mathrm{VN}(G)_c \subseteq \mathrm{VN}(G)_c, \tag{4.1}$$

and the set on the left side of (4.1) is norm-dense in $\mathrm{VN}(G)_c$, since $L^1(G) \cap L^2(G)$ is norm-dense in $L^1(G)$. On the other hand

$$L^1(G) \cap L^2(G) * \mathrm{VN}(G)_c \subseteq L^2(G), \tag{4.2}$$

since if $g \in A(G) \cap L^2(G)$, $f \in L^1(G) \cap L^2(G)$, and $\mu \in \mathrm{VN}(G)_c$, then

$$|\langle f * \mu, g\rangle| = |\langle \mu, \check{f} * g\rangle| \leq \|\mu\|_{\mathrm{VN}} \|f\|_2 \|g\|_2.$$

In the above, the function $\check{f}$ is defined by $\check{f}(x) = f(x^{-1})$. Therefore $\|f*\mu\|_2 \leq \|\mu\|_{\mathrm{VN}} \|f\|_2$. Now (i) follows.

(ii) Suppose that G is not discrete. Let $f \in \mathrm{VN}(G) \cap L^2(G)$. We claim that $\|f - \mathrm{Id}\|_{\mathrm{VN}} \geq 1$, where Id denotes the identity in $\mathrm{VN}(G)$; that is, Id is evaluation of $f \in A(G)$ at the identity e of G. Let $g \in A(G)$ be such that $g(e) = 1$, $\|g\|_A \leq 1$, and the support of g is concentrated in a small neighborhood U of e. If U is sufficiently small, then $\langle f, g\rangle$ is near 0, while $\langle \mathrm{Id}, g\rangle = 1$. Hence $\|f - \mathrm{Id}\|_{\mathrm{VN}} \geq 1$. □

Theorem 4.4. *Let* G *be a locally compact group. Then the following hold.*

(i) $C^*_\lambda(G) \subseteq \mathrm{VN}(G)_c$.

(ii) $\mathrm{VN}(G)_c = C^*_\lambda(G)$ *if and only if* G *is compact.*

(iii) $\mathrm{VN}(G)_c = L^1(G)$ *if and only if* G *is finite.*

Proof. (i) is obvious from the definition. We prove (ii) (for which the assertion of (i) provides the motivation). It is obvious that for all groups, $C^*_\lambda(G) \subseteq \mathrm{VN}(G)_c$. If G is compact, then $L^2(G) \subseteq L^1(G)$, and so by Proposition 4.3(i), $L^1(G)$ is norm-dense in $\mathrm{VN}(G)_c$, that is, $\mathrm{VN}(G)_c = C^*_\lambda(G)$.

Now suppose that $\mathrm{VN}(G)_c = C^*_\lambda(G)$, and that G is *not* compact. Then by [Gn3], there exists $T \in \mathrm{VN}(G)$ and $S \in C^*_\lambda(G)$ such that either $ST \notin C^*_\lambda(G)$, or $TS \notin C^*_\lambda(G)$. In the first case, the operator $ST \in \mathrm{VN}(G)_c$. In the second case, the operator $S^*T^* \in \mathrm{VN}(G)_c$ and $T^*S^* = (ST)^* \notin C^*_\lambda(G)$. We have a contradiction.

(iii) If $\mathrm{VN}(G)_c = L^1(G)$, then $C^*_\lambda(G) = L^1(G)$, so $L^1(G)$ can be renormed as a C^*-algebra. Hence G must be finite; see [Ga]. □

Corollary 4.5. *Let* G *be an infinite compact group. Suppose that* G_d *(and hence* G*) is amenable. Then* $\mathrm{VN}(G)_c \cap C^*_\delta(G) = \{0\}$.

Proof. Indeed, by Theorem 4.4, $\mathrm{VN}(G)_c = C^*_\lambda(G)$. Dunkl and Ramirez [DR1] show that if G_d is amenable, then $C^*_\lambda(G) \cap C^*_\delta(G) = \{0\}$. □

Remark. There exist compact infinite groups G such that

$$C_\lambda^*(G) \subseteq C_\delta^*(G),$$

so that, in particular,

$$\mathrm{VN}(G)_c \subseteq C_\delta^*(G).$$

For example, $SO(n, \mathbf{R})$ for $n \geq 3$ is such a group; see [CLR] (for $n \geq 5$) and [Dr] (for $n = 3, 4$). On the other hand, $M_d(G) \cap C_\lambda^*(G) = \{0\}$ for all nondiscrete G as is shown by a $2 - \varepsilon$ argument. [$M_d(G)$ is the set of discrete measures.]

5. Continuity of translation in $\mathrm{VN}(G)^*$

As in §4, we let $\mathrm{VN} = \mathrm{VN}(G)$ denote the von Neumann algebra of the locally compact group G; that is, $\mathrm{VN}(G)$ is the dual space of $A(G)$; see [E]. In this section, however, we study $\mathrm{VN}(G)^*$ and $(\mathrm{VN}(G)^*)_c$.

We will sometimes use the three spaces $C_\lambda^*(G)$, $C_\delta^*(G)$, and $C_\delta^*(G_d)$ as defined in §4.

Theorem 5.1. *Let G be a locally compact group. Then the following hold.*

(i) $(\mathrm{VN}(G)^*)_c = \mathrm{VN}(G)^*$ *if and only if G is discrete.*

(ii) *If G is abelian and nondiscrete, then $x \mapsto L_x\mu$ is neither lower semicontinuous from G to $\mathrm{VN}(G)^*$ nor weak* measurable.*

Proof. (i) If G is not discrete, δ_e is not in the VN-norm closure of the span of $\{\delta_x : x \in G, x \neq e\}$. (To see this, suppose that $\theta = \sum_1^n \alpha_j \delta_{x_j}$ is a finite sum of point masses none of which is the identity. Let $f \in A(G)$ such that $f(e) = 1$, $\|f\|_{A(G)} = 1$, and $f(x_j) = 0$ for $1 \leq j \leq n$. Then $\|\delta_e - \theta\|_{\mathrm{VN}} \geq 1$. Hence δ_e is not in that closure.) The Hahn-Banach Theorem shows that there exists $\mu \in \mathrm{VN}(G)^*$ such that $\langle \mu, \delta_e \rangle = 1$ and $\langle \mu, \delta_x \rangle = 0$ for all $x \neq e$. Then

$$\|\delta_x * \mu - \mu\|_{\mathrm{VN}(G)^*} \geq |\langle \delta_x * \mu - \mu, \delta_e \rangle| = 1 \quad \text{for all } x \neq e.$$

(ii) Let H be a nonmeasurable subgroup of G. (Such exist by Lemma A.3.) We let W be the norm-closed subalgebra of $\mathrm{VN}(G)$ generated by $\{\delta(x) : x \in H\}$. As the characteristic function 1_H of H is positive-definite, there is an idempotent measure τ on the Bohr compactification $b\Gamma$ of the dual group Γ of G such that the Fourier-Stieltjes transform of τ is the characteristic function of H. Convolution against τ gives a projection from $AP(\Gamma)$ onto W, so W is a direct summand of $AP(\Gamma)$. Therefore there exists $\chi \in AP(\Gamma)^*$ such that $\langle \chi, f \rangle = \hat{f}(0)$ if $f \in W$ and $\langle \chi, f \rangle = 0$ for all f in the closure of $M_d(G \backslash H)$. Extend χ to an element of VN^*. Let $\{m_\alpha\}$ be a net of discrete probability measures on H such that for every $x \in H$, $\|\delta(x) * m_\alpha - m_\alpha\| \to 0$. Let ϕ be an accumulation point of $\{m_\alpha * \chi\}$. Then $L_x\phi = \phi$ for all $x \in H$. For all $f \in AP(\Gamma)$, $\langle L_x\phi - \phi, f \rangle = \langle L_x\chi - \chi, f \rangle$. Also,

$$\|l_x\chi - \chi\| \geq |\langle L_x\chi - \chi, \delta(e)\rangle| = \begin{cases} 0 & \text{if } x \in H; \\ 1 & \text{otherwise}. \end{cases}$$

Hence $G \setminus H = \{x : \|L_x\phi - \phi\| > \frac{1}{2}\}$. This set is not open, since otherwise H would be closed (and therefore measurable). This gives the nonlower semicontinuity. The nonmeasurability is obvious, since $\langle L_x\chi, \delta(0)\rangle$ is the characteristic function of H. □

Let G be a locally compact group. Then $M_d(G)$ is a subspace of $\mathrm{VN}(G)$, and restriction gives a natural mapping of $\mathrm{VN}(G)^*$ onto $C_\delta^*(G)^*$. In particular, evaluation at points of G gives a mapping Π of $\mathrm{VN}(G)^*$ into the bounded complex-valued functions on G. We can say more, however.

Proposition 5.2. *Let G be a locally compact group. Then the following hold.*

(i) Π *maps* $\mathrm{VN}(G)^*$ *into* $B(G_d)$.
(ii) Π *maps* $(\mathrm{VN}(G)^*)_c$ *into* $B(G)$.
(iii) *If G is amenable, then* Π *maps* $(\mathrm{VN}(G)^*)_c$ *onto* $B(G)$.
(iv) *If G_d is amenable, then* Π *maps* $\mathrm{VN}(G)^*$ *onto* $B(G_d)$.

Proof. (i) Let $\phi \in \mathrm{VN}(G)^*$, and let $f \in M_d(G)$. Then $\langle \Pi\phi, f\rangle = \langle \phi, f\rangle$. Thus,

$$|\langle \Pi\phi, f\rangle| \leq \|\phi\|_{\mathrm{VN}(G)^*}\|f\|_{C_\delta^*(G)} \leq \|\phi\|_{\mathrm{VN}(G)^*}\|f\|_{C^*(G_d)}.$$

Therefore $\|\Pi\phi\|_{B(G_d)} \leq \|\phi\|_{\mathrm{VN}(G)^*}$.

(ii) If $\phi \in (\mathrm{VN}(G)^*)_c$, then $\Pi\phi \in C(G)$. In particular, $\Pi\phi \in C(G) \cap B(G_d)$. Therefore $\Pi\phi \in B(G)$ by [E].

(iii) If G is amenable, then there exists a bounded approximate identity $\{f_\alpha\}$ in $A(G)$. Furthermore, this bounded approximate identity may be chosen so that for each $x \in G$, $\lim_\alpha \|L_x f_\alpha - f_\alpha\| = 0$. See the Appendix, Lemma A.5, where this assertion is stated formally and a proof is given. Fix $f \in B(G)$. Then $\|f_\alpha f\| \leq C\|f\|$ and $f_\alpha f \to f$ uniformly on compact sets. Let ϕ be any accumulation point of $\{f_\alpha f\}$ in $\mathrm{VN}(G)^*$ (such exists since $A(G) \subseteq A(G)^{**} = \mathrm{VN}(G)^*$). Obviously, $\Pi\phi = f$. It remains to show that $\phi \in (\mathrm{VN}(G)^*)_c$.

If $\varepsilon > 0$ is given, then there exists a neighborhood U of e such that $\|L_x f - f\| < \varepsilon$ for all $x \in U$. Then

$$\begin{aligned}\|L_x\phi - \phi\| &\leq \limsup_\alpha \|L_x f_\alpha f - f_\alpha f\| \\ &\leq \limsup_\alpha [\|(L_x f_\alpha)(L_x f) - (L_x f_\alpha)f\| + \|(L_x f_\alpha)f - f_\alpha f\|] \\ &\leq (\sup \|f_\alpha\|)\varepsilon + \|f\| \limsup_\alpha \|L_x f_\alpha - f_\alpha\| = C\varepsilon.\end{aligned}$$

Hence, $\phi \in (\mathrm{VN}(G)^*)_c$.

(iv) If G_d is amenable, then $C_\delta^*(G) = C_\delta^*(G_d)$ by [DR1, Proposition 3.4]. Hence, $C_\delta^*(G_d)$ is a closed subspace of $\mathrm{VN}(G)$. By duality, $B(G_d)$ is a quotient of $\mathrm{VN}(G)^*$. That quotient mapping is, obviously, the mapping Π. □

Remarks. (i) The mapping Π is one-to-one if and only if G is finite. This amounts to the assertion that $C_\delta^*(G)$ equals $\mathrm{VN}(G)$, which can only happen when G is finite, by Proposition 5.5.

(ii) If G is discrete, $\Pi\,\mathrm{VN}(G)^* = B_\lambda(G)$, by duality.

When G is a compact abelian group, the identification of $(\mathrm{VN}(G)^*)_c$ is particularly easy. We give the background, and then the formal assertion and its proof.

Suppose that G is a locally compact *abelian* group. Then $\mathrm{VN}(G) = L^\infty(\Gamma)$, where Γ is the dual group of G. Translation in $\mathrm{VN}(G)$ corresponds to multiplication of elements of $L^\infty(\Gamma)$ by characters on Γ. The mapping Π corresponds to a projection of $\mathrm{VN}(G)^*$ onto $M(b\Gamma)$, given by restriction to the almost periodic functions on Γ. Furthermore, because translation in $\mathrm{VN}(G)$ corresponds to multiplication by characters, if $\mu \in (\mathrm{VN}(G)^*)_c$ is identified with the corresponding measure on $\Delta\,\mathrm{VN}(G)$, then the positive part $\mu_+ \in (\mathrm{VN}(G)^*)_c$ also.

Proposition 5.3. *Let G be a compact abelian group. Let $\mu \in \mathrm{VN}(G)^*$. Then $\mu \in (\mathrm{VN}(G)^*)_c$ if and only if $|\mu| = \sum_{j=1}^\infty \beta_j$, where $\beta_j \geq 0$ and $\Pi\beta_j$ is a multiple of a character on G (distinct for different j), and the convergence in norm is absolute.*

Proof. It suffices to prove the assertions for $\mu \geq 0$.

(i) Suppose that $\mu \in (\mathrm{VN}(G)^*)_c$. Then $\Pi\mu \in B(G) = A(G)$ by Theorem 5.1(ii), so $\Pi\mu = \sum \lambda_i \gamma_i$ with $\lambda_i \geq 0$ and $\gamma_i \in \Gamma$.

The mapping Π is induced by the projection P of $\Delta\,\mathrm{VN}(G) \to b\Gamma$ given by restriction to the almost periodic functions on Γ. Let $\beta_j = \chi_{P^{-1}\{\gamma_j\}}\mu$. Since $P^{-1}\{\gamma_j\} \cap P^{-1}\{\gamma_l\} = \varnothing$ whenever $j \neq l$, $\|\mu\| = \sum \|\beta_j\|$.

(ii) Now suppose that $\mu = \sum \beta_j$, each $\beta_j \geq 0$, $\Pi\beta_j$ is a nonnegative point mass at the character γ_j, and $\sum \|\beta_j\| = C < \infty$. Let $\varepsilon > 0$ and choose $n > 0$ such that $\sum_{j>n} \|\beta_j\| < \varepsilon$. Let U be a neighborhood of $0 \in G$ such that $|\langle x, \gamma_j\rangle - 1| < \varepsilon/C$ for $1 \leq j \leq n$. Then

$$\|L_x\mu - \mu\| \leq \sum_{j\leq n} \|(\langle x, \gamma_j\rangle - 1)\beta_j\| + 2\varepsilon < 3\varepsilon.$$

It follows that $\mu \in (\mathrm{VN}(G)^*)_c$. $\square$

Since $A(G)$ is a closed subset of its second dual, which is $\mathrm{VN}(G)^*$, Π maps $A(G)$ one-to-one into $B(G)$. In all cases, $A(G) \subseteq (\mathrm{VN}(G)^*)_c$. This raises the obvious question: When does $A(G) = (\mathrm{VN}(G)^*)_c$?

Theorem 5.4. *Let G be a locally compact group. Suppose that $(\mathrm{VN}(G)^*)_c = A(G)$. Then the following hold.*

(i) *If* G *is amenable, then* G *is compact.*
(ii) *If* G_d *is amenable, then* G *is finite.*

Proof. (i) Indeed, since the mapping Π maps $(\mathrm{VN}(G)^*)_c$ onto $B(G)$ by Proposition 5.2(iii), $A(G) = B(G)$, so G is compact.

(ii) By part (i), we may assume that G is compact. (If G_d is amenable, so is G, as a simple argument shows.)

Suppose that G is compact and infinite, so not discrete. Let W be the closed subalgebra of $\mathrm{VN}(G)$ generated by the union of $C^*_\lambda(G)$ and $C^*_\delta(G)$. Then, in fact, W is the direct sum of $C^*_\lambda(G)$ and $C^*_\Delta(G)$, by [DR1]. This uses the amenability of G_d and the nondiscreteness of G. Let $\phi \in W^*$ be such that $\phi = 0$ on $C^*_\lambda(G)$ and $\langle \phi, \mu\rangle = \langle 1, \mu\rangle$ for all $\mu \in C^*_\delta(G)$, where $1 \in A(G)$ is the function constantly one. Let ϕ' be an extension of ϕ to an element of $\mathrm{VN}(G)^*$. Then $\Pi\phi' = 1$ but $\phi' \neq 1$, since $\phi' = 0$ on $C^*_\lambda(G)$. If $\phi' \in (\mathrm{VN}(G)^*)_c$, we would be done. We now modify ϕ'.

By the amenability of G_d, there exists a net of discrete probability measures $\{m_\alpha\}$ such that for all $x \in G$, $\|R_x m_\alpha - m_\alpha\| \to 0$, where R_x is right translation. Let $\chi \in \mathrm{VN}(G)^*$ be any weak* accumulation point of $\{m_\alpha * \phi'\}$. It is easy to see that $L_x\chi = \chi$ for all $x \in G$. Hence, $\chi \in (\mathrm{VN}(G)^*)_c$. Of course, since χ is a weak* accumulation point of the $m_\alpha * \phi'$, χ is zero on $C^*_\lambda(G)$, while $\langle \chi, f\rangle = \langle 1, f\rangle$ for all $f \in C^*_\delta(G)$. Hence $\Pi\chi = 1$ and $\chi \neq 1$. □

Remark. If G is amenable and $\Pi(\mathrm{VN}(G)^*)_c = A(G)$, then G is compact. The proof is the same as for Theorem 5.4(i).

Proposition 5.5. *Let* G *be a locally compact group. Then the following are equivalent.*

(i) Π *maps* $\mathrm{VN}(G)^*$ *one-to-one onto its image.*
(ii) $C^*_\delta(G) = \mathrm{VN}(G)$.
(iii) G *is finite.*

Proof. (i) ⇒ (ii) follows by an application of the Hahn-Banach Theorem, since $C^*_\delta(G)$ is a closed subspace of $\mathrm{VN}(G)_c$.

(ii) ⇒ (iii) This is a consequence of [Gn2, Theorem 4]; we give a direct proof. If (ii) holds, then

$$L^1(G) * C^*_\delta(G) = L^1(G) * \mathrm{VN}(G) = \mathrm{VN}(G)_c.$$

But it is always the case that $C^*_\lambda(G) \supseteq L^1(G) * C^*_\delta(G)$, and that $\mathrm{VN}(G)_c \supseteq C^*_\lambda(G)$. Therefore $\mathrm{VN}(G)^* = C^*_\lambda(G)$, so G is compact by Theorem 4.4

It remains to show that G is discrete. Since $\mathrm{VN}(G) = C^*_\delta(G)$, $\mathrm{VN}(G)$ has a unique topological invariant mean. This follows from [DR2, Theorems 2.8 and 2.11]. By [Ren], G must be discrete.

That (iii) ⇒ (i) is obvious. □

Remark. If the equivalent conditions of Proposition 5.5 are satisfied, then, of course, the image of Π is $A(G) = B(G) = B(G_d)$, since G is finite.

6. Miscellaneous results and open questions

We consider in this section results related to those in the preceding sections and questions we believe to be open.

When G acts continuously on the Banach space X, then G acts lower semicontinuously on the dual space X^* of X [GLL]. If the action of G on X is discontinuous, the action on X^* may still be lower semicontinuous. On the other hand, lower semicontinuous action on X does not necessarily imply any continuity or measurability (in the sense of Bourbaki) property for the actin on X^*. We see this from the following example.

Example 6.1. Let $G = \mathbf{T}$ and let X be the set of functions on $\mathbf{T}_d$ that vanish at infinity: $X = C_0(\mathbf{T}_d)$. Then the following hold.

(i) Translation from $\mathbf{T}$ to X is not lower semicontinuous, though it is weakly measurable.

(ii) Translation from $\mathbf{T}$ to X^* is lower semicontinuous (and weak* Borel).

(iii) Translation from $\mathbf{T}$ to X^{**} is not weak* measurable and not lower semicontinuous.

Proof. (i) The nonlower semicontinuity is easy. We think of $\mathbf{T}$ as the interval $[0, 2\pi)$. Let h be the function that is 1 at 0, -1 at π, and zero everywhere else. Then for $x \neq 0$, $\|\delta_x * h - h\| = 2$ if $x = \pi$ and $\|\delta_x * h - h\| = 1$ if $x \neq \pi$. Hence $\{x : \|\delta_x * h - h\| > 1\}$ is not open, and G does not act lower semicontinuously. The map $x \mapsto \langle L_x\mu, f\rangle$ has countable support for all countably supported μ and countably supported f, and there are no other such μ's or f's. The weak measurability now follows.

(ii) Let $a\delta_x$ and $b\delta_y$ be point masses in $X^* = L^1(\mathbf{T}_d)$. Then

$$\|a\delta_x - b\delta_y\| = \begin{cases} |a| + |b| & \text{if } x \neq y; \\ |a - b| & \text{if } x = y. \end{cases}$$

Thus, moving point masses apart does not decrease norms. Hence, for all μ with finite support, and all $\varepsilon > 0$, $A = \{x : \|\delta_x * \mu - \mu\| > \varepsilon\}$ is open. Indeed, if $y \in A$ and $\lambda > 0$ is less than the distances between all points in the support of $|\mu| + |\delta_y * \mu|$, then $\{z : |y - z| < \lambda\} \subseteq A$. Of course, then by a standard 2ε argument, this holds for all $\mu \in L^1(\mathbf{T}_d)$. Hence, $\mathbf{T}$ acts lower semicontinuously on X^*. The assertion about weak* measurability follows exactly as for (i).

(iii) Let χ be the characteristic function of a nonmeasurable subset $N \subseteq \mathbf{T}$. Then $\langle \delta_x \check{\chi}, \delta_0\rangle = \chi(x)$, so translation of $\check{\chi}$ is not even weak* measurable. We obtain the nonlower semicontinuity of the action of X^{**} by taking χ to be the characteristic function of the rationals (elements of finite order) in $\mathbf{T}$. □

Here are some questions that appear to be open, with references to results related to them. Some of our questions are quite specific; others are rather open-ended.

1. Does lower semicontinuity of the action of G on X^* always imply weak* measurability? Lower semicontinuity does not imply weak measurability of the group action; see Proposition 1.4 and the Remark after it.

2. If $x \mapsto L_x\mu$ is weakly measurable from G to $M(G)$, is $\mu \in L^1(G)$? (Yes, if μ is a Riesz product and $G = \mathbf{T}$.) In the case of abelian G, such a μ must be in $\operatorname{Rad} L^1(G) = \{\nu \in M(G) : \hat{\nu}(\chi) = 0 \text{ for all } \chi \in \Delta M(G)\backslash\widehat{G}\}$; see [GM, 8.3.4].

3. If G is compact, and $\mu \in (L^\infty(G)^*)_c$ and $f \in L^\infty(G)$, is $f\mu \in (L^\infty(G)^*)_c$? This question asks if Proposition 3.3 extends to $L^\infty(G)^*$. An affirmative answer would give a complete characterization of $(L^\infty(G)^*)_c$ for compact G, by Proposition 3.5.

4. Does Proposition 3.3(iii) hold for all nondiscrete groups?

5. Is the action of G on $\mathrm{VN}(G)^*$ never semicontinuous? never Borel?

6. Let Rm denote the space of bounded Riemann-integrable functions on the compact group G; that is, Rm is the set of all bounded Borel functions f on G such that there exists a Borel function g equal to f except on a null set and g is continuous off of a null set. Then Rm is a commutative C^*-subalgebra of $L^\infty(G)$. It is not hard to show [S] that $(Rm^*)_c = L^1(G)$. Of course, Rm is translation-invariant. Furthermore, a straightforward argument shows that there exists a largest closed translation-invariant subalgebra X of $L^\infty(G)$ with $(X^*)_c = L^1(G)$. What is X? Is X larger then Rm?

7. Do there exist translation-invariant C^*-subalgebras $X \neq Rm$ of $L^\infty(G)$ for which $(X^*)_c$ is an L-space? An L-space not equal to $L^1(G)$?

8. Let G be a locally compact abelian group, and S the structure semigroup of $M(G)$ (see [GM, Chapter 5]). Is the mapping from $\Delta M(G) \times S \to \mathbf{C}$ given by evaluation at the element of S (semi)continuous in each variable separately? (This question is motivated by a result of B. E. Johnson [J], which states that a function separately continuous on a product space is measurable with respect to each product of Borel measures.)

9. Does the result of [J] also apply to translation-invariant means on L^∞?

10. Does $(\mathrm{VN}(G)^*)_c = A(G)$ imply G is finite for all locally compact groups? See Theorem 5.4.

11. $\mathrm{VN}(G)^* = L^\infty(G)$ occurs if and only if G is finite (just dualize and apply Theorem 4.4(iii)). Can $L^\infty(G)$ be dense in $\mathrm{VN}(G)^*$?

12. In the preceding questions, replace "translates continuously" by "translates measurably" (norm, weak, or weak*).

A. Appendix

We give here three results which seem to be in the folklore, but for which we can give no adequate reference, a fourth, for which our proof seems to be simpler than most, and a fifth result, which may be new, but which is proved by old methods.

Lemma A.1. *Let G be a nondiscrete locally compact group. Then G has an open subgroup that has a dense subgroup H of empty interior.*

Proof. Let L be a σ-compact open subgroup of G. Let K be a compact, normal subgroup of L such that L/K is metrizable and nondiscrete (see Lemma A.4 for a proof of the existence of such a K). Then L/K has a dense subsequence. Let E be the subgroup of L/K generated by such a sequence. Then E is countable, so it cannot have interior. Let H be the pre-image of E in L. Then H is a dense subgroup of L that has no interior. □

We use a more complicated version of the same idea for the proof of the next assertion.

Lemma A.2. *Let G be a nondiscrete locally compact group. Then G has a subset that is not Haar measurable.*

Proof. Let L be a σ-compact open subgroup of G. Let K be a compact, normal subgroup of L such that L/K is metrizable. Then the Haar measure on, and the measurable subsets of, L/K form a measure and σ-algebra that are isomorphic to Lebesgue measure on either $\mathbf{T}$ or $\mathbf{R}$ (depending on whether L/K is compact or not). The pull-back of a nonmeasurable subset of $[0, 1]$ to L/K and then to L (and hence to a subset of G), yields the required nonmeasurable set. We omit the remaining details and verifications. □

Hewitt and Ross [HR, vol. I, 16.13(d)] give a proof of the fact that every compact abelian group has a nonmeasurable subgroup. The next lemma gives the extension to nondiscrete, not necessarily compact abelian groups. The proof here is essentially that of [HR].

Lemma A.3. *Let G be a nondiscrete locally compact abelian group. Then G has a subgroup H such that*

(i) *H is not Haar measurable.*

(ii) *The subgroup H is such that for any open subgroup K of G of the form $K = \mathbf{R}^n \times C$ where C is compact, $K/(K \cap H)$ is countably infinite.*

(iii) *Any subgroup H' having the property of* (ii) *is necessarily nonmeasurable.*

Proof. (i) is immediate from (ii)–(iii).

(ii) We consider various cases. We consider (in effect) G to be compactly generated, and the cases to follow come from the structure theorem for locally compact abelian groups: G has an open subgroup of the form $\mathbf{R}^n \times C$, where C is compact and $n \geq 0$.

Case I. $G = \mathbf{R}$. Let $\mathbf{Q}$ denote the rational numbers in $\mathbf{R}$, and let $E_{\mathbf{R}}$ be a Hamel basis for $\mathbf{R}$ over $\mathbf{Q}$. We may assume that $1 \in E_{\mathbf{R}}$. Let $H_{\mathbf{R}} = \mathbf{Q}(E_{\mathbf{R}} \setminus \{1\})$, the $\mathbf{Q}$-linear span of $E_{\mathbf{R}} \setminus \{1\}$. Then $\mathbf{R} = H_{\mathbf{R}} \oplus \mathbf{Q}$, and the assertion of (ii) follows.

Case II. $G = \mathbf{T}$. We define $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, where $\mathbf{Z} \subseteq \mathbf{Q}$ is the set of integers. Then $\mathbf{R} \to \mathbf{T}$ maps $H_{\mathbf{R}}$ onto a subgroup $H_{\mathbf{T}}$ of elements of $\mathbf{T}$ and $\mathbf{Q}$ maps onto the subgroup $\mathbf{Q}'$ of elements of finite order. Of course, $\mathbf{T} = \mathbf{Q}' \oplus H_{\mathbf{T}}$, and the assertion of (ii) follows.

Case III. G has an open subgroup of the form $K = \mathbf{R}^n \times C$, where C is compact and $n > 1$. Let $H_{\mathbf{R}}$ be as in Case I, and let $H = H_{\mathbf{R}} \times \mathbf{R}^{n-1} \times C$. Then H is dense in the open subgroup K, and K/H is countable.

To see that H has the required property, we must show that whenever $K_1 = \mathbf{R}^{n_1} \times C_1$ is the product of Euclidean space with a compact abelian group C_1, then $K_1/(K_1 \cap H)$ is countable.

First note that since K_1 is open, $K_1 \cap K$ is open, and $K_1/(K_1 \cap K)$ is discrete. But since $K_1 \cap K$ is open in $K_1 = \mathbf{R}^{n_1} \times C_1$, $K_1 \cap K = \mathbf{R}^{n_1} \times C_2$, where C_2 is also compact. It follows that $K_1/(K_1 \cap K)$ is compact as well as discrete; that is, that $K_1 \cap K$ has finite index in K_1. Then

$$\text{(A.1)} \qquad \begin{aligned} \#K_1/(K_1 \cap H) &\leq (\#K_1/(K_1 \cap K))(\#(K_1 \cap K)/(H \cap K_1)) \\ &\leq \#K_1/(K_1 \cap H)\#K/H, \end{aligned}$$

which is countably infinite. Thus, H has the required property.

Case IV. G has an open compact subgroup K and the torsion subgroup T of K is such that K/T is countably infinite. Let $H = T$. Suppose that K_1 is a compact open subgroup of G. Then $K_1/(K_1 \cap K)$ is compact and discrete, so the countability of $K_1/(K_1 \cap H)$ follows just as in (A.1).

Case V. G has an open compact subgroup K and the torsion subgroup T of K is such that $N = K/T$ is uncountable. Then by [F, vol. I, Theorem 1.1], N is torsion-free. Let E be a maximal independent subset of N. For $x \in N$, $x \neq 0$ and $m \in \mathbf{Z}$, $m \neq 0$, let $\frac{1}{m}x$ denote the unique (because N is torsion-free) element $y \in N$ such that $my = x$ whenever it exists, and e when it does not exist. Then clearly $\bigcup_{m=1}^{\infty} \frac{1}{m} Gp(E) = N$, where $Gp(E)$ is the group generated by E. Let F be a one element subset of E, let Π denote the quotient mapping of $K \to N$, and let $H = \Pi^{-1}(\bigcup_{m=1}^{\infty} \frac{1}{m} Gp(E \backslash F))$. To see that H has the required property, we argue as follows. Since $T \subseteq H$, it is enough to show that $N/(\bigcup_{m=1}^{\infty} \frac{1}{m} Gp(E\backslash F))$ is countable, that is, we may assume that $H = (\bigcup_{m=1}^{\infty} \frac{1}{m} Gp(E\backslash F))$ and that $T = \{0\}$. Then every element $x \in N$ has the form $x = \frac{1}{m}(\sum_{j=1}^{k} \pm x_j) + \frac{r}{s} f$, where the $x_j \in E\backslash F$, and m, r, s are integers with $m, s > 0$. Since there are only a countable number of possibilities for the choices for r, s and since E is independent and N torsion-free, $N/(\bigcup_{m=1}^{\infty} \frac{1}{m} Gp(E\backslash F))$ is countable.

If K_1 is any open subgroup of G, then $K_1/(K_1 \cap K)$ is discrete and compact, and the calculation used in (A.1) completes the proof.

Case VI. G has a compact-open subgroup K with torsion subgroup T such that K/T is finite, so the torsion subgroup T is open. (This is the final case.) We may take $K = T$. Since K is compact and abelian, the (nonzero) elements of K must have a finite upper bound p for their orders, because of the Baire category theorem. This bound is called the *exponent* of K. For groups with finite exponent, Theorem 17.2 of [F] applies: $K = \bigoplus_{i \in I} K_i$ is a direct sum of finite cyclic groups K_i. Let $F \subset I$ be countably infinite with $F \neq I$, so $H = \bigoplus_{i \in I\backslash F} K_i$ has K/H countably infinite.

If K_1 is a compact and open subgroup of G, then K_1 must also be a torsion subgroup. As before, $K_1 \cap K$ has finite index in K_1, so the calculation of (A.1) shows that $K_1/(K_1 \cap H)$ is countable. This ends the proof of (ii).

(iii) If K is compact and open, then K is the countable union of the cosets of $K \cap H'$, so each of these cosets must be nonmeasurable, and H' is nonmeasurable. We may therefore assume that K has the form $\mathbf{R}^n \times C$ where $n > 1$ and C is a compact (possibly trivial) group. Suppose that $H' \cap K$ were measurable. Since $K = \bigcup_{n=1}^{\infty}(K \cap H' + x_n)$, where $x_n \in K$, $m(H' \cap K) > 0$. By Steinhaus's Theorem [GM, 8.3.4], $(K \cap H') + (K \cap H')$ contains a neighborhood of the identity. Since $K \cap H'$ is a subgroup, it is thus an open subgroup of $K = \mathbf{R}^n \times C$. In particular, $K \cap H'$ contains a set of the form $U \times \{0\}$, where U is open in $\mathbf{R}^n$. It follows at once that $K \cap H' = \mathbf{R}^n \times C'$, where C' is an open subgroup of C. But $K/(K \cap H') = C/C'$ is countable, so C' cannot be both open and of infinite index in the compact group C. This contradiction shows that $K \cap H'$ is not measurable. □

Remark. Our original proof of (iii) above used the continuity of translation in $L^1(G)$, which is more in keeping with the spirit of this paper (but the argument was much longer).

The next result is known in many forms; we believe our proof is simpler than most. See [HR, I, pp. 71 and 83] for the more common version and for some of the history. The result seems to be due originally to Kakutani and Kodaira [KK, Satz 6], whose proof involves representations.

Lemma A.4. *Let G be a σ-compact, nondiscrete locally compact group. Then G has a closed normal subgroup N such that G/N is metrizable, N has measure zero and G/N is nondiscrete. Furthermore, if $\{f_j\}$ is a countable sequence of Baire functions on G, then N may be chosen so that the f_j are constant on cosets of N.*

Proof. Let V_j be a sequence of compact subsets of G such that $V_j \subseteq V_{j+1}$ for all $j \geq 1$ and $G = \bigcup_{j \geq 1} V_j$. Let $\{U_j\}$ be a sequence of symmetric compact neighborhoods of e whose measures decrease to 0 such that $U_{j+1}U_{j+1} \subseteq U_j$ and $xU_{j+1}x^{-1} \subseteq U_j$ for all $x \in V_j$ and all $j \geq 1$. Let $N = \bigcap_{j=1}^{\infty} U_j$. Then N and G/N have the required properties. Indeed, $m_G(N) = 0$, since $m_G(N) \leq m_G(U_j)$ for all j, while G/N is metrizable, since the sets U_jN form a basis for G/N at eN. Since N has zero Haar measure, G/N is not discrete.

The second assertion follows easily from the first. We may assume that the f_j are continuous and compactly supported (that is where we use "Baire" in the hypothesis). Choose the U_k as above, but subject to the further restriction that $\sup |f_j(xy) - f_j(x)| + |f_j(yx) - f_j(x)| < 2^{-k}$ where the supremum is taken over all $y \in U_k$, $x \in G$, and $j \leq k$. The required conclusion follows easily. □

Lemma A.5. *If G is amenable, then $A(G)$ contains a bounded approximate identity $\{f_\alpha\}$ such that $g_\alpha(x) = \|L_x f_\alpha - f_\alpha\|$ converges to 0 uniformly on compact subsets of G.*

Proof. We use a slight modification of the standard construction [L], [P, pp. 70, 72, 96]. Our index set $\{\alpha\}$ will be the set of pairs (ε, K) where $\varepsilon > 0$, and K ranges over compact subsets of G containing the identity e. Of course, $(\varepsilon, K) > (\varepsilon_1, K_1)$ if and only if $\varepsilon < \varepsilon_1$ and $K \supseteq K_1$. For each pair (ε, K), set $K' = K^{-1}K$. Since G is amenable, there exists a compact subset U of G such that $0 < |U|$ and $|K'U| < (1+\varepsilon)|U|$.

The function $u_{\varepsilon, K} = \frac{1}{|U|} 1_{KU} * \check{1}_U$ is 1 on K and has norm bounded by

$$|KU|^{1/2}|U|^{-1/2} \leq \sqrt{1+\varepsilon}.$$

Hence $\{u_{\varepsilon, K}\}$ yields a bounded approximate identity for $A(G)$. For $x \in K$, we have

$$\begin{aligned} \|L_x u_{\varepsilon, K} - u_{\varepsilon, K}\|A &= \frac{1}{|U|}\|L_x * 1_{KU} - 1_{KU}\|_2 \|1_U\|_2 \\ &\leq \frac{|x^{-1}KU \Delta KU|^{1/2}}{|U|^{1/2}}. \end{aligned}$$

Because $x^{-1}K$ and K are subsets of K', and both contain the identity, e,

$$\begin{aligned} |x^{-1}KU\Delta KU| &\leq |x^{-1}KU\Delta U| + |KU\Delta U| \\ &= |x^{-1}KU \backslash U| + |KU\backslash U| < 2\varepsilon |U|, \end{aligned}$$

we have $\|L_x u_{\varepsilon, K} - u_{\varepsilon, K}\| < \sqrt{2\varepsilon}$ for all $x \in K$. □

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