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CHANGE OF VARIABLE RESULTS FOR A_p- AND REVERSE HÖLDER RH_r-CLASSES

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ABSTRACT. We study conditions under which the map

$$T_{h,\gamma}w(x) = w(h(x))|h'(x)|^\gamma$$

maps the Muckenhoupt class A_p into A_q and the reverse Hölder class RH_{r_1} into RH_{r_2}.

INTRODUCTION

In this paper we will study change of variable transformations $T_{h,\gamma}$ defined, for $\gamma \in \mathbf{R}$ and $h\colon \mathbf{R} \to R$ a homeomorphism, by $T_{h,\gamma}w(x) = w \circ h(x) \cdot |h'(x)|^\gamma$, acting on A_p and the reverse Hölder classes, RH_r. The A_p classes were introduced by Muckenhoupt [9] and consist precisely of those weights w for which, e.g., the Hardy-Littlewood maximal operator $Mf(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f|$ is bounded on L^p_w. If T is an operator such that $\|Tf\|_{p,w} \le c\|f\|_{p,w}$, $w \in A_p$, and if, say $w \circ h \cdot |h'| \in A_p$ for $w \in A_p$, then as a simple change of variables shows, the operator $Sg(t) = T(g \circ h) \circ h^{-1}(t)$ is also bounded on L^p_w. We will give necessary and sufficient conditions on h' so that $T_{h,\gamma}: A_p \to A_q$, and we will see that in such a case $q \ge p$. An indispensable tool is Rubio de Francia's extrapolation theorem [3, p. 448] which easily leads to a necessary condition on h'. This condition is also sufficient and the proof of this requires some detailed properties of A_p and RH_r which are collected in §1. We show that these results agree for $1 - p_0 \le \gamma \le 1$ and $p_0 = q_0$ with those found earlier in [6]. For $\gamma > 1$ and $\gamma < 1 - p_0$, the condition that characterizes when $T_{h,0}$ is onto is necessary and sufficient for $T_{h,\gamma}$ to map $A_{p_0} \to A_{p_0}$. We also show that the integral of an A_p weight (suitably interpreted) is an A_{p+1} weight.

We will then study how RH_r acts under $T_{h,\gamma}$. Since $\bigcup_{r<\infty} RH_r = \bigcup_{p>1} A_p = A_\infty$, one expects that analogous results hold. However, the statements and the proofs are different from the A_p case. In particular, since $RH_r = \bigcup_{p>1} A^r_p$ [6], where $A^r_p = \{w : w^r \in A_p\}$, an extension of the above mentioned extrapolation theorem is needed which deals with operators bounded on L^p_w, $w \in A^r_p$. This

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will be done in §4, and the remaining sections are then devoted to considering when $T_{h,\gamma} : RH_{r_1} \to RH_{r_2}$. In particular, we will show that $r_2 \le r_1$. An interesting by-product is that $T_{h,\gamma} : RH_r \to RH_r$ is onto iff $|h'| \in A_\infty$, when $\gamma = \frac{1}{r}$, and $\log|h'| \in \mathrm{clos}_{\mathrm{BMO}}(L^\infty)$ when $\gamma \neq \frac{1}{r}$.

A description (without proofs) of the results obtained at an early stage of our work can be found in [5].

1. Background and preliminary results

This section is devoted to notation and background material which will be needed in our later development. We will provide references for all the results for which it is easy to refer to a source, and we will provide proofs for these which we were unable to find in precisely the form stated. Throughout the paper we will refer to the statements as (Pn), $n = 1, 2, \ldots$.

(P1) For $1 < p < \infty$ we say that $w \in A_p$ iff

$$\frac{1}{|I|}\int_I w\left(\frac{1}{|I|}\int_I w^{1-p'}\right)^{p-1} \le c < \infty, \qquad 1/p + 1/p' = 1,$$

and c independent of I [9]. We denote by $A_p(w)$ the infimum of all such c. We have that $A_q(w) \le A_p(w)$, $q \ge p$, and so we can define $A_\infty(w) = \lim_{p\to\infty} A_p(w)$. The other classes with which we are concerned are $A_\infty = \bigcup_{p>1} A_p$, $A_1 = \{w : Mw \le cw\}$, where $Mw(x) = \sup_{x\in I}\frac{1}{|I|}\int_I w$, is the Hardy-Littlewood maximal function,

$$RH_r = \left\{w : \left(\frac{1}{|I|}\int_I w^r\right)^{1/r} \le c\frac{1}{|I|}\int_I w\right\};$$

we set

$$RH_r(w) = \inf\left\{c : \left(\frac{1}{|I|}\int_I w^r\right)^{1/r} \le c\frac{1}{|I|}\int_I w \text{ for all } I\right\}, \qquad r > 1.$$

If $s > r$, then $RH_r(w) \le RH_s(w)$. We also have $A_\infty = \bigcup_{r<\infty} RH_r$ [3].

(P2) $w \in A_p$ iff w and $w^{1-p'}$ are in A_∞ [3, p. 408]; in fact,

$$[6,\ 3.2] \qquad \max\{A_\infty(w), A_\infty(w^{1-p'})^{p-1}\} \le A_p(w) \le A_\infty(w)A_\infty(w^{1-p'})^{p-1}.$$

In particular, $w \in A_p$ iff $w^{1-p'} \in A_{p'}$.

(P3) $w \in RH_r$ iff $w^r \in A_\infty$ [13]; in fact,

$$[6,\ 3.1] \qquad \frac{A_\infty(w^r)^{1/r}}{A_\infty(w)} \le RH_r(w) \le A_\infty(w^r)^{1/r}.$$

(P4) Let $0 < \sigma < \infty$. Then

$$w^\sigma \in A_p \to \{w \in A_{1+(p-1)/\sigma} \text{ iff } w \in A_\infty\}.$$

Proof. The necessity is clear since $A_{1+(p-1)/\sigma} \subset A_\infty$. The sufficiency is known if $\sigma \geq 1$ [3, p. 394]. If $0 < \sigma < 1$, since $w \in A_\infty$, by (P3), $w^\sigma \in RH_{1/\sigma}$. Thus, if $1 < p < \infty$,

$$A_p(w^\sigma) \geq \frac{1}{|I|}\int_I w^\sigma \left(\frac{1}{|I|}\int_I w^{\sigma(1-p')}\right)^{p-1}$$
$$\geq \frac{1}{RH_{1/\sigma}(w^\sigma)}\left(\frac{1}{|I|}\int_I w\right)^\sigma \cdot \left(\frac{1}{|I|}\int_I w^{1-[1+\sigma/(p-1)]}\right)^{[(p-1)/\sigma]\sigma}.$$

Hence

$$A_{1+(p-1)/\sigma}(w)^\sigma \leq A_p(w^\sigma) RH_{1/\sigma}(w^\sigma).$$

The argument also works for $p = 1$.

(P5) For $1 \leq \sigma < \infty$, we let $A_p^\sigma = \{w : w^\sigma \in A_p\}$, and we denote by $A_p^\sigma(w) = A_p(w^\sigma)$, $1 \leq p \leq \infty$. With this notation we have $RH_r = A_\infty^r$ by (P3). We note that $A_p^\sigma \subset A_p$.

(P6) $A_p^\sigma = A_{1+(p-1)/\sigma} \cap RH_\sigma$.

Proof. If $w \in A_p^\sigma$, then $w \in A_{1+(p-1)/\sigma}$ and $w^\sigma \in A_\infty$ or $w \in RH_\sigma$. Conversely, if $w \in A_{1+(p-1)/\sigma} \cap RH_\sigma$, then $w^\sigma \in A_\infty$ and thus by (P4), since $w = (w^\sigma)^{1/\sigma} \in A_{1+(p-1)/\sigma}$, $w^\sigma \in A_p$.

(P7) We shall use the notation of [6] and say that $P \prec Q$ iff for any $T \geq Q$, P is majorized by an expression depending only on T.

With this notation we have for $1 < p < \infty$,

$$(1) \qquad w \in A_p \quad \text{iff} \quad \|Mf\|_{p,w} \leq c\|f\|_{p,w}, \qquad c \prec A_p(w).$$

(P8) If $w \in A_p$, then there is $\tau > 1$ such that $w^\tau \in A_p$ and matters can be arranged so that $\tau \prec A_p(w)$, $A_p(w^\tau) \prec A_p(w)$ [3, p. 397]. An analogous statement holds for RH_r [11].

(P9) $w \in \bigcap_{p>1} A_p$ iff $1/w \in \bigcap_{r<\infty} RH_r$ and $w \in A_\infty$.

Proof. If $w \in \bigcap_{p>1} A_p$, then $w \in A_\infty$ and for $1 < p < 2$,

$$\left(\frac{1}{|I|}\int_I w^{1-p'}\right)^{p-1} \leq \frac{c}{\frac{1}{|I|}\int_I w} \leq c\frac{1}{|I|}\int_I \frac{1}{w}.$$

Hence $1/w \in RH_{p'-1}$. Conversely, $1/w \in RH_{p'-1}$ for $1 < p < 2$, and thus $w^{1-p'} \in A_\infty$. This and $w \in A_\infty$ give $w \in A_p$ (P2).

(P10) Let $1 \leq p_1 < p_2 < \infty$ and $w_j \in A_{p_j}$, $j = 1, 2$. If $0 \leq \theta \leq 1$, then $A_{p_\theta}(w_1^\theta w_2^{1-\theta}) \leq A_{p_1}(w_1)^\theta A_{p_2}(w_2)^{1-\theta}$, where $p_\theta = \theta p_1 + (1-\theta)p_2$.

Proof. For the proof, let

$$L = \frac{1}{|I|}\int_I w_1^\theta w_2^{1-\theta}\left(\frac{1}{|I|}\int_I w_1^{\theta(1-p_\theta')} w_2^{(1-\theta)(1-p_\theta')}\right)^{p_\theta - 1} = K \cdot H.$$

Note that

$$K \le \left(\frac{1}{|I|}\int_I w_1\right)^{\theta}\left(\frac{1}{|I|}\int_I w_2\right)^{1-\theta},$$

and in H we apply Hölder's inequality with exponents

$$r = \frac{1-p_1'}{1-p_\theta'}\frac{1}{\theta} = \frac{p_\theta - 1}{p_1 - 1}\frac{1}{\theta} \quad \text{and} \quad r' = \frac{p_\theta - 1}{(1-\theta)(p_2 - 1)},$$

which gives us our desired inequality.

(P11) Let $\gamma \in \mathbf{R}$. We are throughout concerned with mappings $T_{h,\gamma}$ defined by $T_{h,\gamma}w = w \circ h \cdot |h'|^{\gamma}$. We need the following result. If for some $1 \le p < \infty$, $T_{h,\gamma} : A_p \to A_q$, then $h' \in A_\infty$.

Proof. This will follow if we can show that $u \in \mathrm{BMO}$ implies $u \circ h \in \mathrm{BMO}$ [7]. If $u \in \mathrm{BMO}$, then there is $\lambda > 0$ such that $e^{\lambda u} \in A_2$ [3, p. 409], and hence $e^{\lambda u} = w_1/w_2$ for some $w_1, w_2 \in A_1$ [3, p. 436]. From this we get

$$e^{\lambda u \circ h} = \frac{w_1 \circ h \cdot |h'|^{\gamma}}{w_2 \circ h \cdot |h'|^{\gamma}} \quad \text{and} \quad \lambda u \circ h = \log w_1 \circ h \cdot |h'|^{\gamma} - \log w_2 \circ h \cdot |h'|^{\gamma}.$$

Since the log's are in BMO [3, p. 409], the proof is complete.

(P12) Throughout the paper we will be concerned with homeomorphisms $h: \mathbf{R} \to \mathbf{R}$ for which h and h^{-1} are locally absolutely continuous. By (P11), we will know that $h' \in A_\infty$ and since A_∞ weights are not integrable, h will be onto. We may assume that $h' \ge 0$. For such h we have that $(h^{-1})' \in RH_q$ iff $h' \in A_{q'}$, and $RH_q((h^{-1})') = A_{q'}(h')^{1/q'}$ [6, 2.5]. In particular, $h' \in \bigcap_{p>1} A_p$ iff $(h^{-1})' \in \bigcap_{r<\infty} RH_r$, and $h' \in A_\infty$ iff $(h^{-1})' \in A_\infty$.

(P13) If $T_{h,\gamma} : RH_{r_1} \to RH_{r_2}$, then $h' \in A_\infty$.

For the proof, simply observe that if $u \in \mathrm{BMO}$, then $e^{\lambda u} \in RH_{r_1}$ for some $\lambda > 0$ and argue as in (P11).

In general we recommend the references [2 and 3] for a more detailed study of A_p and RH_r.

2. Mappings from A_p to A_q

For $\gamma \in \mathbf{R}$ and h a homeomorphism as in §1, we denote by $T_{h,\gamma}$ the mapping $T_{h,\gamma}w = w \circ h \cdot h'^{\gamma}$. The following theorem shows that $T_{h,\gamma}$ cannot improve A_p.

Theorem 2.1. *If* $1 < p_0 < \infty$ *and* $A_{q_0}(T_{h,\gamma}w) \prec A_{p_0}(w)$, $w \in A_{p_0}$, *then* $q_0 \ge p_0$.

Proof. Assume $1 < q_0 < p_0$. The hypothesis implies that by (P7)

$$\int Mf^{q_0} w \circ h \cdot h'^{\gamma} \le c \int f^{q_0} w \circ h \cdot h'^{\gamma}$$

with $c \prec A_{p_0}(w)$. With the change of variables $t = h(x)$ we obtain

$$\int Tg^{p_0} w \le c \int g^{p_0} w, \qquad c \prec A_{p_0}(w),$$

where

$$g = (f \circ h^{-1})^{q_0/p_0} \frac{1}{(h' \circ h^{-1})^{(1-\gamma)/p_0}} \quad \text{or} \quad f = h'^{(1-\gamma)/q_0} (g \circ h)^{p_0/q_0}$$

and

$$Tg = \frac{1}{(h' \circ h^{-1})^{(1-\gamma)/p_0}} [M\{h'^{(1-\gamma)/q_0} (g \circ h)^{p_0/q_0}\} \circ h^{-1}]^{q_0/p_0}.$$

We can apply now Rubio de Francia's extrapolation theorem [3, p. 448] and get for $1 < p < \infty$,

$$\int Tg^p w \le c \int g^p w, \qquad w \in A_p.$$

We undo the change of variables with $t = h(x)$ and get in terms of f with $w \equiv 1$,

$$(*) \qquad \int (Mf)^{pq_0/p_0} h'^{1-(1-\gamma)p/p_0} \le c \int f^{pq_0/p_0} h'^{1-(1-\gamma)p/p_0}.$$

Hence, if $p = p_0/q_0 > 1$, then

$$\int Mf \cdot h'^{1-(1-\gamma)/q_0} \le c \int f h'^{1-(1-\gamma)/q_0}.$$

If $\gamma = 1 - q_0$, this norm inequality is impossible, and if $\gamma \ne 1 - q_0$, it will lead to a contradiction as follows. If there is $w \in L^1_{\text{loc}}(\mathbf{R})$ such that

$$\text{(i)} \qquad \int Mfw \le c \int fw,$$

then $w = 0$ a.e. To prove this, we may assume that w is continuous. For, if (i) holds, then in view of the translation invariance it also holds with w replaced by $k * w$, $k \in C_c(\mathbf{R})$, $k \ge 0$. If $w \not\equiv 0$, then $w > 0$ a.e., otherwise (i) cannot hold for f with $\operatorname{supp} f \subset \{x : w(x) = 0\}$. Again by translation invariance, we may suppose that $w(0) > 0$. Then w is bounded away from 0 on $[0, \eta]$, $0 < \eta < \frac{1}{2}$. If

$$f(x) = \frac{1}{x \log^2 x}, \qquad 0 < x < \eta,$$

then

$$Mf(x) = \frac{1}{x \log 1/x}$$

and (i) gives

$$\int_0^\eta \frac{w(x)}{x \log 1/x} \le c \int_0^\eta \frac{w(x)}{x \log^2 x}.$$

Hence,

$$\int_0^\eta \frac{w(x)}{x \log^2 x} = \infty,$$

and thus the continuity of w shows that $w(0) = \infty$, contradicting $w(0) < \infty$. This completes the proof.

A slightly different proof that there are no regular measures for which

$$\int Mf(x)\,d\mu(x) \le C \int f(x)\,d\mu(x)$$

can be found in [3, p. 468].

We will now give a characterization of those homeomorphisms h for which $T_{h,\gamma}$ maps A_{p_0} into A_{q_0}. From the previous result we may suppose that $q_0 \ge p_0$.

Theorem 2.2. *For* $1 < p_0 \le q_0 < \infty$ *and* $\gamma \in \mathbf{R}$, *the following statements are equivalent*:

(1) $A_{q_0}(T_{h,\gamma}w) \prec A_{p_0}(w)$, $w \in A_{p_0}$,

(2) $h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0}$, $1 < p < \infty$, *and* $h' \in A_\infty$.

Before we prove Theorem 2.2 we remark that in (2) the condition $h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0}$ implies that $h' \in A_\infty$ except if $1 - q_0 < \gamma < 1$. To see this, we may assume that $\gamma \le 1 - q_0$ by (P5) and thus $(1-\gamma)p/p_0 - 1 > (1-\gamma)1/p_0 - 1 \ge 0$. By (P2), $h'^{[(1-\gamma)p/p_0-1](r'-1)} \in A_{r'}$, $r = pq_0/p_0$ or $h'^{((1-\gamma)p-p_0)/(pq_0-p_0)} \in A_{r'}$. Since $(1-\gamma)p - p_0 \ge pq_0 - p_0$, $h' \in A_\infty$ by (P5).

Proof of Theorem 2.2. $(1) \to (2)$. This is a repetition of the proof of Theorem 2.1 up to the inequality $(*)$. By (P7), $h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0}$ and $h' \in A_\infty$ by (P11).

$(2) \to (1)$. Let $w \in A_{p_0}$. Then there is $\tau > 1$ so that $w^\tau \in A_{p_0}$ and $A_{p_0}(w^\tau) \prec A_{p_0}(w)$ (P8). Let

$$L = \frac{1}{|I|}\int_I w \circ h \cdot h'^\gamma \left(\frac{1}{|I|}\int_I w \circ h h'^{\gamma(1-q_0')}\right)^{q_0-1} \equiv K \cdot H.$$

If $t = h(x)$, $h(I) = J$, then

$$K \le \frac{|J|}{|I|}\left(\frac{1}{|J|}\int_J w^\tau\right)^{1/\tau}\left(\frac{1}{|J|}\int_J (h^{-1})'^{(1-\gamma)\tau'}\right)^{1/\tau'}.$$

Thus for $\gamma > 1$ and $\rho = (\tau\gamma - 1)/\tau(\gamma - 1)$,

$$K \le A_\rho(h^{-1\prime})^{\gamma-1}\left(\frac{|J|}{|I|}\right)^\gamma \left(\frac{1}{|J|}\int_J w^\tau\right)^{1/\tau}$$

while for $1/\tau \le \gamma \le 1$ (in which case $0 \le (1-\gamma)\tau' \le 1$),

$$K \le \left(\frac{|J|}{|I|}\right)^\gamma \left(\frac{1}{|J|}\int_J w^\tau\right)^{1/\tau},$$

and for $\gamma < 1/\tau$,

$$K \le RH_{(1-\gamma)\tau'}(h^{-1\prime})^{1-\gamma}\left(\frac{|J|}{|I|}\right)^\gamma \left(\frac{1}{|J|}\int_J w^\tau\right)^{1/\tau}.$$

In an analogous way we estimate H and obtain

$$H \leq \left(\frac{1}{|J|}\int_J w^{\tau(1-p_0')}\right)^{1/\tau(p_0'-1)} \left(\frac{|J|}{|I|}\right)^{q_0-1} \cdot \left(\frac{1}{|J|}\int_J (h^{-1\prime})^{\sigma'[1+\gamma(q_0'-1)]}\right)^{(q_0-1)/\sigma'},$$

where $\sigma = \tau(1-p_0')/(1-q_0') = \tau(q_0-1)/(p_0-1)$. We proceed now in three cases:

Case 1. $\gamma \geq 1$. From the above inequalities of H and K we get

$$L \leq A_{p_0}(w^\tau)^{1/\tau} A_p(h^{-1\prime})^{\gamma-1} \left(\frac{|J|}{|I|}\right)^{q_0-1+\gamma} \left(\frac{1}{|J|}\int_J (h^{-1\prime})^{\sigma'[1+\gamma(q_0'-1)]}\right)^{(q_0-1)/\sigma'}.$$

We claim now that $h' \in \bigcap_{p>\tau_0} A_p$, $\tau_0 = (q_0+\gamma-1)/(p_0+\gamma-1)$, and $(h^{-1})' \in \bigcap_{p>1} A_p$ if $\gamma > 1$. Since $h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0}$, $1 < p < \infty$, the claim is clear for $\gamma = 1$, while if $\gamma > 1$, we get that $h' \in \bigcap_{r<\infty} RH_r$ (P3) and hence $(h^{-1})' \in \bigcap_{p>1} A_p$ (P12). We also observe that by (P4),

$$h' \in A_{1+(pq_0-p_0)/(p_0+p(\gamma-1))}.$$

Since $(pq_0-p_0)/(p_0+p(\gamma-1)) > (q_0-p_0)/(p_0+\gamma-1)$, it follows that $h' \in \bigcap_{p>\tau_0} A_p$, with $\tau_0 = 1+(q_0-p_0)/(p_0+\gamma-1) = (q_0+\gamma-1)/(p_0+\gamma-1)$. By (P12), $(h^{-1})' \in \bigcap_{r<\tau_0'} RH_r$, $\tau_0' = (q_0+\gamma-1)/(q_0-p_0)$.

Next we claim that $\alpha \equiv \sigma'[1+\gamma(q_0'-1)] < \tau_0'$. Since

$$\sigma' = \tau(q_0-1)/(\tau(q_0-1)-(p_0-1))$$

we see that $\alpha = \tau(q_0-1+\gamma)/(\tau(q_0-1)-(p_0-1)) > 1$. The inequality $\alpha < \tau_0'$ is equivalent with $\tau(q_0-p_0) < \tau(q_0-1)-p_0+1$ or $\tau-1 < (\tau-1)p_0$.

Consequently the last two factors of L become

$$\left\{\frac{\left(\frac{1}{|J|}\int_J (h^{-1})'^{\sigma'[1+\gamma(q_0'-1)]}\right)^{\frac{1}{\sigma'[1+\gamma(q_0'-1)]}}}{|I|/|J|}\right\}^{q_0-1+\gamma} \leq RH_{\sigma'[1+\gamma(q_0'-1)]}(h^{-1\prime})^{q_0-1+\gamma}.$$

Case 2. $1-q_0 \leq \gamma < 1$. Then $h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0}$, $1 < p < \infty$, implies that $1/h' \in \bigcap_{r<\infty} RH_r$ (P3), and since $h' \in A_\infty$, $h' \in \bigcap_{p>1} A_p$ (P9), and thus $(h^{-1})' \in \bigcap_{r<\infty} RH_r$ (P12). From the previous inequalities of K and H we obtain

$$L \leq A_{p_0}(w^\tau)^{1/\tau} \left\{\frac{\left(\frac{1}{|J|}\int_J (h^{-1})'^{(1-\gamma)\tau'}\right)^{\frac{1}{(1-\gamma)\tau'}}}{|I|/|J|}\right\}^{1-\gamma}$$

$$\cdot \left\{\frac{\left(\frac{1}{|J|}\int_J (h^{-1})'^{\sigma'[1+\gamma(q_0'-1)]}\right)^{\frac{1}{\sigma'[1+\gamma(q_0'-1)]}}}{|I|/|J|}\right\}^{q_0+\gamma-1},$$

where the last two factors are absent in case $(1-\gamma)\tau' \le 1$, or $\sigma'[1+\gamma(q_0'-1)] \le 1$, by Hölder's inequality.

Case 3. $\gamma < 1 - q_0$. As in Case 2, we get that $(h^{-1})' \in \bigcap_{r<\infty} RH_r$. We claim now that

$$(h^{-1})' \in \bigcap_{p>\delta_0} A_p, \qquad \delta_0 = \frac{1-\gamma-p_0}{1-\gamma-q_0}.$$

Since $1/h' \in A_\infty$ by (P4),

$$\frac{1}{h'} \in A_{1+(pq_0-p)/((1-\gamma)p-p_0)}.$$

Since $(pq_0-p_0)/((1-\gamma)p-p_0) \ge (q_0-p_0)/((1-\gamma)-p_0)$, we get that $1/h' \in \bigcap_{p>\sigma_0} A_p$, $\sigma_0 = 1+(q_0-p_0)/(1-\gamma-p_0)$. From this we see that

$$h' \in \bigcap_{r<\delta_0'} RH_r$$

and hence (P12), $(h^{-1})' \in \bigcap_{p>\delta_0} A_p$.

We also note that, if $\sigma'[1+\gamma(q_0'-1)] \equiv 1-\rho'$, then $p > \delta_0$. Using our previous inequalities on K, H we see that

$$L \le A_{p_0}(w^\tau)^{1/\tau} \left\{ \frac{\left(\frac{1}{|J|}\int_J (h^{-1})'^{(1-\gamma)\tau'}\right)^{1/(1-\gamma)\tau'}}{|I|/|J|} \right\}^{1-\gamma}$$

$$\cdot \left\{ \frac{|I|}{|J|} \left(\frac{1}{|J|}\int_J (h^{-1})'^{1-\rho'} \right)^{\rho-1} \right\}^{1-\gamma-q_0}$$

$$\le A_{p_0}(w^\tau)^{1/\tau} RH_{(1-\gamma)\tau'}(h^{-1\prime})^{1-\gamma} A_p(h^{-1\prime})^{1-\gamma-q_0}.$$

The proof of Theorem 2.2 is now complete.

Remark. For some applications we need to have conditions on h so that $w\circ h \in A_p$ whenever $w \in A_p^\tau$. We state this as

Corollary 2.3. *Let* $1 \le \tau_0 < \infty$ *and assume that* $h' \in \bigcap_{p>\tau_0} A_p$. *Then, if* $w \in A_{p_0}^{\tau_0}$ *for some* $1 < p_0 < \infty$, $w\circ h \in A_{p_0}$.

Proof. This is essentially the case $\gamma = 0$ of Theorem 2.2. Since $w^{\tau_0} \in A_{p_0}$, there is $\tau > \tau_0$ so that $w^\tau \in A_{p_0}$, and thus in the estimate for L in Case 2 of Theorem 2.2, τ can be taken bigger than τ_0, $\gamma = 0$, $p_0 = q_0$, and $\sigma = \tau$. Since $h' \in \bigcap_{p>\tau_0} A_p$, by (P12), $(h^{-1})' \in \bigcap_{r<\tau_0'} RH_r$ and thus

$$L \le A_{p_0}(w^\tau)^{1/\tau} RH_{\tau'}(h^{-1\prime})^{p_0}.$$

3. Corollaries of Theorem 2.2

In this section we will present some consequences of Theorem 2.2 and indicate in what sense Theorem 2.2 generalizes known results.

Corollary 3.1. *If* $1 < p_0 \le q_0 < \infty$, *then* $A_{q_0}(w \circ h \cdot h') \prec A_{p_0}(w)$, $w \in A_{p_0}$, *if and only if* $h' \in \bigcap_{p>q_0/p_0} A_p$.

Proof. This is $\gamma = 1$ of Theorem 2.2.

Corollary 3.2. *If* $1 < p_0 \le q_0 < \infty$ *and* $\gamma > 1$, *then* $A_{q_0}(T_{h,\gamma}w) \prec A_{p_0}(w)$, $w \in A_{p_0}$, *if and only if* $h' \in \bigcap_{r<\infty} RH_r \cap \bigcap_{p>\tau_0} A_p$, *where* $\tau_0 = (q_0+\gamma-1)/(p_0+\gamma-1)$.

Proof. If $A_{q_0}(t_{h,\gamma}w) \prec A_{p_0}(w)$, then $h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0}$, $1 < p < \infty$. Hence by (P3), $h' \in \bigcap_{r<\infty} RH_r$. Since $h' \in A_\infty$, we use (P4) to get

$$h' \in A_{1+(pq_0-p_0)/(p_0+(\gamma-1)p)}.$$

Since $(pq_0 - p_0)/(p_0 + p(\gamma - 1)) > (q_0 - p_0)/(p_0 + \gamma - 1)$ we see that

$$h' \in \bigcap_{p>\tau_0} A_p.$$

For the sufficiency first observe that $h' \in A_{\overline{p}} \cap RH_r$, for every $1 < r < \infty$ and $\overline{p} > \tau_0$. Since $A_{\overline{p}} \cap RH_r = A_q^r$, $q = 1 + r(\overline{p} - 1)$ of (P6) we see that $h'^r \in A_{1+r(\overline{p}-1)}$. We fix now $1 < p < \infty$, and let $r = 1 + (\gamma - 1)p/p_0$.

We claim that, if $\overline{p}$ is chosen so that $1 + r(\overline{p} - 1) = pq_0/p_0$, then $\overline{p} > \tau_0$, and this would complete the proof. Since $r = 1 + (\gamma - 1)p/p_0$, we get that

$$\overline{p}[1 + (\gamma - 1)p/p_0] = pq_0/p_0 + (\gamma - 1)p/p_0 = p/p_0(q_0 + \gamma - 1)$$

or

$$\overline{p} = \frac{p(q_0 + \gamma - 1)}{p_0 + (\gamma - 1)p} > \frac{q_0 + \gamma - 1}{p_0 + \gamma - 1}.$$

The special case $p_0 = q_0$ of Corollary 3.2 is of independent interest.

Corollary 3.3. *If* $p_0 = q_0$ *and* $\gamma > 1$, *then* $A_{p_0}(T_{h,\gamma}w) \prec A_{p_0}(w)$, $w \in A_{p_0}$, *if and only if* $\log h' \in \mathrm{clos}_{\mathrm{BMO}}(L^\infty)$.

Proof. For the necessity, since $h'^{1+(\gamma-1)p/p_0} \in A_p$, $1 < p < \infty$, we get $h' \in \bigcap_{r<\infty} RH_r$. Since also $h' \in A_\infty$, by (P4) $h' \in A_{1+(p-1)/(1+(\gamma-1)p/p_0)}$ or $h' \in \bigcap_{p>1} A_p$. From this by (P9), $1/h' \in \bigcap_{r<\infty} RH_r$. All of this implies that $\log h' \in \mathrm{clos}_{\mathrm{BMO}}(L^\infty)$ by [3, p. 474].

For the sufficiency we need to show that $h' \in \bigcap_{r<\infty} RH_r \cap \bigcap_{p>1} A_p$ by Corollary 3.2. Since $\log h' \in \mathrm{clos}_{\mathrm{BMO}}(L^\infty)$ is equivalent with $h' \in \bigcap_{r<\infty} RH_r$ and $1/h' \in \bigcap_{r<\infty} RH_r$ [3, p. 474], we only need to apply (P9) to see that $h' \in \bigcap_{p>1} A_p$ (note that $h' \in A_\infty$).

Remark. If $\gamma > 1$ and $A_{p_0}(T_{h,\gamma}w) \prec A_{p_0}(w)$, $w \in A_{p_0}$, then the mapping $T_{h,\gamma} : A_{p_0} \to A_{p_0}$ is onto. For the proof, let $w \in A_{p_0}$. We claim that $(h^{-1})' \in$

$\bigcap_{r<\infty} RH_r \cap \bigcap_{p>1} A_p$. Since by Corollary 3.2, $h' \in \bigcap_{r<\infty} RH_r$ we get from (P12) that $(h^{-1})' \in \bigcap_{p>1} A_p$. Similarly, $h' \in \bigcap_{p>1} A_p$ and again $(h^{-1})' \in \bigcap_{r<\infty} RH_r$. Hence Corollary 3.2 applied to h^{-1} gives that $u = w \circ h^{-1} \cdot (h^{-1})'^{\gamma} \in A_{p_0}$, and $u \circ h \cdot h'^{\gamma} = w$.

Corollary 3.4. *If* $1 < p_0 \le q_0 < \infty$ *and* $1 - q_0 \le \gamma < 1$ *then* $A_{q_0}(T_{h,\gamma} w) \prec A_{p_0}(w)$, $w \in A_{p_0}$, *if and only if* $h' \in \bigcap_{p>1} A_p$.

Proof. The necessity is easy, since $h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0} \subseteq A_\infty$, $1 < p < \infty$. Then $1/h' \in \bigcap_{r<\infty} RH_r$ (P3), and since also $h' \in A_\infty$, $h' \in \bigcap_{p>1} A_p$ (P9). This holds for all $\gamma < 1$.

For the sufficiency we need to consider two cases.

Case 1. $1 - p_0 < \gamma < 1$. Then, for $1 < p < 1 + \delta \equiv p_0/(1-\gamma)$, $1 > 1 + (\gamma - 1)p/p_0 > 0$. Since $h' \in A_r$, $r > 1$, we get from (P4)

$$h'^{1+(\gamma-1)p/p_0} \in A_{1+[1+(\gamma-1)p/p_0](r-1)},$$

$1 < p < 1 + \delta$. We want the index $1 + [1 + (\gamma - 1)p/p_0](r - 1) = pq_0/p_0$ for some $r > 1$. We solve for r and obtain

$$r = \frac{p}{p_0} \frac{q_0 + \gamma - 1}{1 + (\gamma - 1)p/p_0} = p \frac{q_0 + \gamma - 1}{p_0 + (\gamma - 1)p} > 1.$$

Hence for $1 < p < p_0/(1-\gamma)$, $h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0}$.

If $p \ge p_0/(1-\gamma)$, then $1 + (\gamma - 1)p/p_0 \le 0$. Since $h' \in A_{r'}$, $r' > 1$, we get that $h'^{1-r} \in A_r$. If $1 - r = 1 + (\gamma - 1)p/p_0$, then $r = (1-\gamma)p/p_0 \le pq_0/p_0$. Hence $h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0}$ for $p \ge p_0/(1-\gamma)$.

Case 2. $1 - q_0 \le \gamma \le 1 - p_0$. Since $(\gamma - 1)p \le -p_0 p < -p_0$, we see that $1 + (\gamma - 1)p/p_0 < 0$, $p > 1$. The idea of the proof is to establish the result for the endpoints $\gamma = 1 - p_0$, $1 - q_0$, and then use the convexity property (P10).

If $\gamma = 1 - p_0$, then $h'^{1+(\gamma-1)p/p_0} = h'^{1-p}$. Since $h' \in \bigcap_{r>1} A_r$, we see that $h'^{1-p} \in A_p \subset A_{pq_0/p_0}$. If $\gamma = 1 - q_0$, then $h'^{1+(\gamma-1)p/p_0} = h'^{1-pq_0/p_0} \in A_{pq_0/p_0}$ as before. If $1 - q_0 < \gamma < 1 - p_0$, then for some $0 < \theta < 1$, $\gamma = \theta(1 - q_0) + (1-\theta)(1-p_0) = 1 - (\theta q_0 + (1-\theta)p_0)$. We use now (P10) and obtain

$$h'^{\theta(1-pq_0/p_0)+(1-\theta)(1-p)} = h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0}.$$

Remark. Corollary 3.4 generalizes Theorem 2.7 of [6] where the case $p_0 = q_0$, $1 - p_0 \le \gamma \le 1$ was considered. It also shows that if $1 < p_0 < q_0 < \infty$, $1 - q_0 \le \gamma < 1$, and $A_{q_0}(T_{h,\gamma} w) \prec A_{p_0}(w)$, $w \in A_{p_0}$, then $T_{h,\gamma}$ can be extended to $A_{q_0} \supsetneqq A_{p_0}$.

Corollary 3.5. *If* $1 < p_0 \le q_0 < \infty$ *and* $\gamma < 1 - q_0$, *then* $A_{q_0}(T_{h,\gamma}w) \prec A_{p_0}(w)$, $w \in A_{p_0}$, *if and only if* $1/h' \in \bigcap_{r<\infty} RH_r \cap \bigcap_{r>\sigma_0} A_r$, *where* $\sigma_0 = 1 + (q_0 - p_0)/(1 - \gamma - p_0)$.

Proof. We first prove the necessity. By Theorem 2.2, $h' \in A_\infty$ and for $p > 1$,

$$h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0}.$$

By (P3), $1/h' \in \bigcap_{r<\infty} RH_r$. Since $(1 - \gamma)p/p_0 - 1 > pq_0/p_0 - 1 > 0$ and $1/h'^{((1-\gamma)p/p_0-1)} \in A_{pq_0/p_0}$, we get from (P4), since $1/h' \in A_\infty$,

$$1/h' \in A_{1+(pq_0-p_0)/((1-\gamma)p-p_0)}, \qquad p > 1,$$

or $1/h' \in \bigcap_{r>\sigma_0} A_r$, since $(pq_0 - p_0)/((1 - \gamma)p - p_0) > (q_0 - p_0)/((1 - \gamma) - p_0)$.

For the sufficiency we first note that $1-\gamma-p_0 > q_0-p_0$ and hence $1 \le \sigma_0 < 2$. Thus $1/h' \in A_2$, and hence $h' \in A_2 \subset A_\infty$.

Next, we claim that

$$h'^{1+(\gamma-1)p/p_0} \in A_{pq_0/p_0}, \qquad 1 < p < \infty.$$

By (P6), $1/h' \in A_{\overline{p}} \cap RH_r = A_q^r$, $1 < r < \infty$ and $\overline{p} > \sigma_0$, where $q = 1+r(\overline{p}-1)$. Hence

$$1/h'^r \in A_{1+r(\overline{p}-1)}.$$

We fix now $p > 1$, and let $r = (1 - \gamma)p/p_0 - 1$. If we choose $\overline{p}$ so that $1 + r(\overline{p} - 1) = pq_0/p_0$, then $\overline{p} = 1 + (pq_0 - p_0)/(p(1 - \gamma) - p_0) > \sigma_0$. The proof is now complete.

As before in Corollary 3.3 the case $q_0 = p_0$ is of special interest.

Corollary 3.6. *If* $p_0 = q_0$ *and* $\gamma < 1-p_0$, *then* $A_{p_0}(T_{h,\gamma}w) \prec A_{p_0}(w)$, $w \in A_{p_0}$, *if and only if* $\log h' \in \mathrm{clos}_{\mathrm{BMO}}(L^\infty)$, *and the mapping is onto.*

Proof. From [3, p. 474] and (P9), $1/h' \in \bigcap_{r<\infty} RH_r \cap \bigcap_{r>1} A_r$ if and only if $\log h' \in \mathrm{clos}_{\mathrm{BMO}}(L^\infty)$. The "onto" part is established as in the Remark to Corollary 3.3.

Remark 1. Another proof of the corollaries was proposed by the referee. It consists of using the results of §5 (see the remark at the end of that section) and then of proving the sufficiency as follows. Write $T_{h\gamma}w = v \circ hh'$ and $(T_{h\gamma}w)^{1-q_0'} = u \circ hh'$ and show that $u, v \in A_\infty$.

Remark 2. The results of this section will be needed later when we investigate the same problems for RH_r. However, as an application at the present stage we will prove that the integral of an A_p-weight is an A_{p+1}-weight, a property which is well known for weights of the form $|x|^\alpha$, i.e., $|x|^\alpha \in A_p$ if and only if $-1 < \alpha < p - 1$, so that $|x|^{\alpha+1} \in A_{p+1}$.

For $w \in A_p$ let $W(x) = |\int_0^x w|$.

Theorem 3.7. (i) *If* $1 < p < \infty$ *and* $w \in A_p$, *then* $W \in \bigcap_{r<\infty} \cap A_{p+1}$.

(ii) *If* $w \in A_1$, *then* $W \in \bigcap_{r<\infty} RH_r \cap \bigcap_{p>2} A_p$.

Proof. (i) Let $\sigma = \inf\{p\colon w \in A_p\}$. Then $w \in \bigcap_{p>\sigma} A_p$. We let $h(x) = \int_0^x w$ and observe that h is a homeomorphism satisfying our overall hypothesis and $h' = w$. For $n = 1, 2, \dots$, $|x|^n \in A_q^\sigma$, $q > n\sigma + 1$, and thus by Corollary 2.3, $W^n \in A_q \subset A_\infty$. This gives us $W \in \bigcap_{r<\infty} RH_r$ by (P3), and the case $n = 1$ implies that $W \in A_{p+1}$.

(ii) is now easy to prove. If $w \in A_1$, then $w \in A_p$, $p > 1$, and so $W \in \bigcap_{r<\infty} RH_r \cap A_{p+1}$, for every $p > 1$.

We note that $w \equiv 1$ gives $W(x) = |x|$ and thus (ii) cannot be strengthened to $W \in A_2$.

The question now arises under what conditions derivatives are again weights.

Theorem 3.8. *Let* w *be locally absolutely continuous with* $w' \geq 0$ *and let* $1 < p < \infty$. *Then* $w' \in A_p$ *if and only if* $A_p(w - w_\sigma) \leq c < \infty$, $\sigma > 0$, *where* $w_\sigma(x) = w(x - \sigma)$, c *independent of* σ.

Proof. If $w' \in A_p$, then $w(x) - w(x - \sigma) = \int_{x-\sigma}^x w' = k_\sigma * w'(x)$, where $k_\sigma = \chi_{[0,\sigma]}$. By [3, p. 467], $A_p(w - w_\sigma) \leq A_p(w') = c$.

Conversely, assume that $A_p(w - w_\sigma) \leq c < \infty$, $\sigma > 0$. Since $A_p(\lambda u) = A_p(u)$, $\lambda > 0$, we see that $A_p((w - w_\sigma)/\sigma) \leq c$, $\sigma > 0$. Thus

$$\frac{1}{|I|}\int_I \frac{w - w_\sigma}{\sigma}\left(\frac{1}{|I|}\int_I\left(\frac{w - w_\sigma}{\sigma}\right)^{1-p'}\right)^{p-1} \leq c < \infty,$$

and hence

$$\liminf_{\sigma\downarrow 0}\frac{1}{|I|}\int_I \frac{w - w_\sigma}{\sigma}\left(\liminf_{\sigma\downarrow 0}\frac{1}{|I|}\int_I\left(\frac{w - w_\sigma}{\sigma}\right)^{1-p'}\right)^{p-1} \leq c.$$

By Fatou's lemma

$$\frac{1}{|I|}\int_I w'\left(\frac{1}{|I|}\int_I w'^{1-p'}\right)^{p-1} \leq c$$

and $w' \in A_p$.

4. Extrapolation theorems

The program that we set ourselves in the next two sections is to study the transformations $T_{h,\gamma}$ on RH_r and to obtain characterizations so that $T_{h,\gamma}\colon RH_{r_1} \to RH_{r_2}$. As we have seen, in the A_p-case Rubio de Francia's extrapolation theorem played an indispensable role (Theorem 2.1 and the implication (1) → (2) of Theorem 2.2). Since $RH_{\sigma_0} = A_\infty^{\sigma_0} = \bigcup_{p<\infty} A_p^{\sigma_0}$, it is not surprising that we need an extrapolation theorem that applies to operators T bounded on $L_w^{p_0}$ for $w \in A_{p_0}^{\sigma_0}$. Since $A_{p_0}^{\sigma_0} \subsetneq A_{p_0}$, one expects that extrapolation to all $p > 1$ is unlikely. In fact, we will see that extrapolation is possible to all $p_0/\sigma_0' + 1/\sigma_0 < p \leq p_0$.

We fix now $1 < p_0$, $\sigma_0 < \infty$ and we fix

(1) $$\frac{p_0}{\sigma_0'} + \frac{1}{\sigma_0} < p < p_0 .$$

We let

(2) $$r = \frac{p_0 - 1}{p - 1} > 1$$

and

(3) $$t = r\frac{p_0 - p}{p_0 - 1} = \frac{p_0 - p}{p - 1} .$$

We observe that

$$\frac{\sigma_0 t}{r} = \sigma_0 \frac{p_0 - p}{p_0 - 1} < \frac{\sigma_0}{p_0 - 1}(\sigma_0 p_0 - \sigma_0 p_0 + p_0 - 1)\frac{1}{\sigma_0} = 1 .$$

Hence we can choose $0 < \delta < 1$ such that

(4) $$\frac{\sigma_0 t}{\delta r} < 1 .$$

Finally, let

(5) $$k = k(p) = \frac{\delta r \sigma_0}{\delta r - \sigma_0 t} .$$

It is clear that $k > \sigma_0$.

The proof of the extrapolation theorem will follow the lines of the weak type extrapolation theorem in [10].

Lemma 4.1. *Let $w \in A_p^k$ and let $g \geq 0$ be in $L^{p/(p_0-p)}(w)$. Then there exists $G \geq g^{1/r}$ such that*

(i) $w(G > y) \leq c/y^{rp/(p-p_0)} \int g^{p/(p_0-p)} w$, *with $c \prec A_p(w^k)$, and*

(ii) $(G^{-r}w)^{\sigma_0} \in A_{p_0}$ *with $A_{\sigma_0}\{(G^{-r}w)^{\sigma_0}\} \prec A_p(w^k)$.*

Proof. We note that $p_0' < p'$ and $w^{1-p'}$, $w^{k(1-p')} \in A_{p'}$. Define $h \geq 0$ by

$$g^{p/(p_0-p)} w = h^{p(p_0-1)/(p_0-p)} w^{1-rp'}$$

and let

$$H = \{M_r(h^{1/t} w^{1-p'}) w^{p'-1}\}^t ,$$

where $M_r \varphi = \{M(\varphi^r)\}^{1/r}$. Since $w \in A_p$ we have

$$\begin{aligned} w(M_r(h^{1/t} w^{1-p'}) > y^{1/t}) &= w(M(h^{r/t} w^{r(1-p')}) > y^{r/t}) \\ &\leq \frac{C}{y^{pr/t}} \int h^{pr/t} w^{1-rp'}, \qquad C \prec A_p(w),\ A_p(w) \leq A_p(w^k)^{1/k} . \end{aligned}$$

Hence

(6) $$w(Hw^{t(1-p')} > y) \leq \frac{C}{y^{pr/t}} \int h^{pr/t} w^{1-rp'}, \qquad C \prec A_p(w^k) .$$

We define now the function G by

$$G = H^{(p_0-1)/r} w^{-(p_0-p)/(p-1)}. \tag{7}$$

Our first claim is $G \geq g^{1/r}$. To see this simply observe that $M_r\varphi \geq \varphi$ so that $H \geq h$. Since $p_0 - 1 = r(p-1)$ and

$$h = g^{1/(p_0-1)} w^{(p_0-p)/p(p_0-1)} w^{(rp'-1)(p_0-p)/p(p_0-1)},$$

it follows that $G \geq g^{1/r}$.

We prove now (i). Since $t = (p_0 - p)/(p-1)$,

$$\begin{aligned} w(G > y) &= w(H^{(p_0-1)/r} w^{-(p_0-p)/(p-1)} > y) \\ &= w(H^{(p_0-1)/r} w^{t(1-p')(p_0-1)/r} > y) \\ &= w(H w^{t(1-p')} > y^{r/(p_0-1)}) \\ &\leq \frac{C}{y^{pr/(p_0-p)}} \int h^{p(p_0-1)/(p_0-p)} w^{1-rp'} \\ &\leq \frac{C}{y^{pr/(p_0-p)}} \int g^{p/(p_0-p)} w, \qquad C \prec A_p(w^k), \end{aligned}$$

by (6) and the definition of g.

In order to prove (ii), it suffices to show that $(G^{-r}w)^{\sigma_0(1-p_0')} \in A_{p_0'}$. We use the definition of H and (7) and get $(G^{-r}w)^{\sigma_0(1-p_0')} = M(h^{r/t} w^{r(1-p')})^{\sigma_0 t/r} \cdot w^{\gamma}$, where $\gamma = (1-p_0')k(1-\sigma_0 t/\delta r)$. Hence this equals

$$M(h^{r/t} w^{r(1-p')})^{\delta\sigma_0 t/\delta r} w^{(1-p_0')k(1-\sigma_0 t/\delta r)}.$$

Since $M(\ )^{\delta} \in A_1 \subset A_{p_0'}$ [3], $w^r \in A_{p_0}$, and $0 < \sigma_0 t/\delta r < 1$, the above expression is in $A_{p_0'}$. Moreover, [8, p. 19], $A_1\{M_r(\ )\}$ depends only on r.

We are now ready to prove a weak type extrapolation theorem.

Theorem 4.2. *Let $1 < p_0$, $\sigma_0 < \infty$, and let T be an operator satisfying: $w \in A_{p_0}^{\sigma_0}$ implies*

$$w\{|Tf| > y\} \leq \frac{C}{y^{p_0}} \int |f|^{p_0} w, \qquad C \prec A_{p_0}(w^{\sigma_0}).$$

Then for $p_0/\sigma_0' + 1/\sigma_0 < p < p_0$ there is $k = k(p) > \sigma_0$ such that for $w \in A_p^k$,

$$w\{|Tf| > y\} \leq \frac{C}{y^p} \int |f|^p w, \qquad C \prec A_p(w^k).$$

Proof. The proof uses properties of Lorentz spaces $L(p, q; \mu)$ which are defined as follows. Let $f_\mu^*(t) = \inf\{y\colon \mu\{|f(x)| > y\} \leq t\}$, be the rearrangement of f with respect to the measure μ. Then we say that $f \in L(p, q; \mu)$ iff

$$\|f\|_{p,q,\mu} = \left\{\int_0^\infty [t^{1/p} f_\mu^*(t)]^q \frac{dt}{t}\right\}^{1/q} < \infty.$$

For the properties of $L(p, q; \mu)$ spaces see [4, 12].

Let $w \in A_p^k$ where k is given by (5). We note that

$$\|f\|_{p,w}^{p_0} = \||f|^{p_0}\|_{p/p_0, w} = \int |f|^{p_0} g^{-1} w,$$

for some $\int g^{p/(p_0-p)} w = 1$. Let $\tau = p_0/p > 1$ and $r = (p_0 - 1)/(p-1)$ as in (2). For $s > 0$ let $E_s = \{|Tf| > s\}$. Then with G as in Lemma 4.1

$$\begin{aligned} s^{p_0} w(E_s)^{p_0/p} &= s^{p_0} \left(\int \chi_{E_s} w \right)^\tau = s^{p_0} \left(\int \chi_{E_s} G^r G^{-r} w \right)^\tau \\ &\le s^{p_0} \|\chi_{E_s}\|_{\tau, 1; G^{-r}w}^\tau \|G^r\|_{\tau', \infty; G^{-r}w}^\tau . \end{aligned}$$

We will first estimate $\|G^r\|_{\tau', \infty; G^{-r}w}^\tau$. By Lemma 4.1

$$G^{-r} w(G^r > y) \le \frac{1}{y} \int_{(G^r > y)} w \le \frac{C}{y^{p/(p_0-p)+1}} \int g^{p/(p_0-p)} w \le \frac{C}{y^{\tau'}} .$$

Hence $(G^r)^*_{G^{-r}w}(t) \le C/t^{1/\tau'}$ and so $\|G^r\|_{\tau', \infty; G^{-r}w} \le C$, $C \prec A_p(w^k)$.

For the estimation of $\|\chi_{E_s}\|_{\tau, 1; G^{-r}w}$ we note that by (ii) of Lemma 4.1,

$$\begin{aligned} G^{-r} w(\chi_{E_s} > y) = G^{-r} w(E_s) \chi_{[0,1)}(y) &\le \frac{C}{s^{p_0}} \int |f|^{p_0} G^{-r} w \cdot \chi_{[0,1)}(y) \\ &\le \frac{C}{s^{p_0}} \int |f|^{p_0} g^{-1} w \chi_{[0,1)}(y) \quad (\text{since } G \ge g^{1/r}) \\ &= \frac{C}{s^{p_0}} \|f\|_{p,w}^{p_0} \chi_{[0,1)}(y) . \end{aligned}$$

From this we see that

$$(\chi_{E_s})^*_{G^{-r}w}(t) \le \chi_{[0,R]}(t), \qquad R = \frac{C}{s^{p_0}} \|f\|_{p,w}^{p_0},$$

and thus

$$\|\chi_{E_s}\|_{\tau, 1; G^{-r}w} \le \int_0^R t^{1/\tau - 1}\, dt = cR^{1/\tau} = c\left(\frac{1}{s^{p_0}} \|f\|_{p,w}^{p_0} \right)^{1/\tau} .$$

We combine the two estimates and obtain

$$s^{p_0} w(E_s)^{p_0/p} \le c s^{p_0} \frac{1}{s^{p_0}} \|f\|_{p,w}^{p_0} .$$

Remark. If $p_0/\sigma_0' + 1/\sigma_0 < p < p_0$ and T is sublinear, then for $w \in A_p^k$,

$$\int |Tf|^p w \le C \int |f|^p w, \qquad C \prec A_p(w^k).$$

This follows from the Marcinkiewicz interpolation theorem. Note that $k > \sigma_0$.

In the study of how $T_{h,\gamma}$ acts on A_1, i.e., the $p_0 = 1$ case of Theorem 2.2, we need an extrapolation theorem for an operator T which is bounded on $L^{p_0}(w)$ for $w \in A_{r_0}$ where $1 \le r_0 < \infty$.

Lemma 4.3. *Let* $1 \le r_0 < \infty$, $1 < p_0 < \infty$, $\max(r_0, p_0) < p < \infty$ *and* $w \in A_{r_0}$. *If* $g \ge 0$ *is in* $L^{(p/p_0)'}(w)$, *then there is* $G \ge g$ *such that*

(i) $\|G\|_{(p/p_0)',w} \le c\|g\|_{(p/p_0)',w}$, $c \prec A_{r_0}(w)$;

(ii) $Gw \in A_{r_0}$ *and* $A_{r_0}(Gw) \prec A_{r_0}(w)$.

Proof. Let $t = (p - p_0)/(p-1)$. Then $0 < t < 1$ and $p'/t = (p/p_0)'$. Since $w \in A_{r_0} \subset A_p$ we have $w^{1-p'} \in A_{p'}$, and thus we can choose by (P8) $1 < r < p'$ such that $A_{p'/r}(w^{1-p'}) \prec A_{p'}(w^{1-p'}) \prec A_p(w) \le A_{r_0}(w)$. Define

$$G = \{M_r(g^{1/t}w)w^{-1}\}^t,$$

where $M_r f = M(f^r)^{1/r}$. Then $G \ge g$ and

$$\int G^{p'/t}w = \int M(g^{r/t}w^r)^{p'/r}w^{1-p'} \le c\int g^{p'/t}w$$

with $c \prec A_{r_0}(w)$ by (P7). This proves (i) and for (ii) simply observe that

$$Gw = M_r(g^{1/t}w)^t w^{1-t}.$$

Since $M_r(g^{p'/t}w) \in A_1$, with $A_1\{M_r(\ )\}$ depending only on r [8, p. 19] and $w \in A_{r_0}$, $0 < t < 1$, we see that by (P10), $Gw \in A_{r_0}$ and $A_{r_0}(Gw) \prec A_{r_0}(w)$.

Theorem 4.4. *Assume that* T *is an operator satisfying*: $\|Tf\|_{p_0,w} \le c\|f\|_{p_0,w}$ *for some* $1 < p_0 < \infty$ *and* $w \in A_{r_0}$, $c \prec A_{r_0}(w)$, *and* $1 \le r_0 < \infty$. *Then for* $\max(r_0, p_0) < p < \infty$ *and* $w \in A_{r_0}$, $\|Tf\|_{p,w} \le c\|f\|_{p,w}$.

Proof. We note that $\|Tf\|_{p,w}^{p_0} = \||Tf|^{p_0}\|_{p/p_0,w} = \int |Tf|^{p_0}gw$, for some $g \ge 0$ in $L^{(p/p_0)'}(w)$ with $\|g\|_{(p/p_0)',w} = 1$. We choose now $G \ge g$ as in Lemma 4.3 and obtain

$$\begin{aligned}\|Tf\|_{p,w}^{p_0} &= \int |Tf|^{p_0}gw \le c\int |f|^{p_0}Gw \\ &\le c\|f^{p_0}\|_{(p/p_0),w}\|G\|_{(p/p_0)',w} \le c\|f\|_{p,w}^{p_0}.\end{aligned}$$

Remarks. (1) If $r_0 > p_0$ and T is sublinear, then by the Marcinkiewicz interpolation theorem, $\|Tf\|_{p,w} \le c\|f\|_{p,w}$, $w \in A_{r_0}$ and $p_0 \le p < \infty$.

(2) It is in general not possible to extrapolate to $1 < p < p_0$. As an example, let $1 < r_0 < p_0 < \infty$ and consider $Tf = M(f^{p_1})^{1/p_1}$, where $p_0 = r_0 p_1$. Then $\|Tf\|_{p_0,w} \le c\|f\|_{p_0,w}$ iff $w \in A_{r_0}$, $c \prec A_{r_0}(w)$. If $\|Tf\|_{p,w} \le c\|f\|_{p,w}$ for some $p_1 < p < p_0$, then $w \in A_{p/p_1} \subsetneqq A_{r_0}$.

5. Multipliers from RH_{r_1} to RH_{r_2}

In this section we will discuss some results needed later, and we will characterize those positive functions φ for which $\varphi \cdot RH_{r_1} \subset RH_{r_2}$. We will use repeatedly (P3) which say that $RH_r = A_\infty^r$ and that $A_\infty^r(w)$ controls $RH_r(w)$.

Lemma 5.1. *If* $w_j \in RH_{r_j}$, $j = 1, 2$, *and* $0 \le \theta \le 1$, *then* $w_1^{\sigma} \cdot w_2^{1-\sigma} \in RH_r$, *where* $r = r_1\theta + r_2(1-\theta)$, $\sigma = r_1\theta/r$, *and* $A_\infty^r(w_1^\sigma w_2^{1-\sigma}) \le A_\infty^{r_1}(w_1)^\theta A_\infty^{r_2}(w_2)^{1-\theta}$.

Proof. By (P3), $w_j^{r_j} \in A_p$, $p \ge p_0$, $j = 1, 2$. Hence using (P10),

$$A_p(w_1^{r_1\theta} w_2^{r_2(1-\theta)}) \le A_p(w_1^{r_1})^\theta A_p(w_2^{r_2})^{1-\theta}.$$

Since $r_1\theta = r\sigma$, $r_2(1-\theta) = r(1-\sigma)$, we only need to let $p \to \infty$ to complete the proof.

Lemma 5.2. *The following statements are equivalent for* $\varphi : \mathbf{R} \to \mathbf{R}_+$:

(i) $\varphi \in \bigcap_{r<\infty} RH_r$,

(ii) $A_\infty(w^r\varphi^r) \prec A_\infty(w^r)$, $1 < r < \infty$.

Proof. This is Theorem 3.5 of [6] except for the explicit "$\prec$" statement of (i) $\to$ (ii). Fix $1 < r < \infty$ and $w \in A_\infty^r$. By (P1) there is $1 < p_0 < \infty$ such that $A_{p_0}(w^r) \le 2A_\infty(w^r)$. We choose now $\tau > 1$ so that $A_{p_0}(w^{\tau r}) \prec A_{p_0}(w^r)$ by (P8). Since by (i), $\varphi^{\tau' r} \in A_\infty$ we have $1 < p_1 < \infty$ so that $\varphi^{t' r} \in A_{p_1}$. We fix $p \ge \max(p_0, p_1)$ and observe that by Hölder's inequality

$$A_p(w^r\varphi^r) \le A_p(w^{\tau r})^{1/\tau} A_p(\varphi^{\tau' r})^{1/\tau'}.$$

Hence $A_\infty(w^r\varphi^r) \le A_{p_0}(w^{\tau r})^{1/\tau} A_{p_1}(\varphi^{\tau' r})^{1/\tau'} \prec A_{p_0}(w^r) \le 2A_\infty(w^r)$.

Theorem 5.3. *The following statements are equivalent for* $1 < r < \infty$:

(i) $h' \in A_\infty$;

(ii) *if* $w \in RH_r$, *then* $w \circ h \cdot h'^{1/r} \in RH_r$, *and* $A_\infty^r(w \circ h \cdot h'^{1/r}) \prec A_\infty^r(w)$.

Proof. (ii) $\to$ (i) follows by taking $w \equiv 1$. Conversely, if $h' \in A_\infty$, then $h' \in \bigcap_{p>\sigma} A_p$ for some $1 < \sigma < \infty$. Let $w \in RH_r = A_\infty^r$. Thus $w \in A_p^r$ for some $1 < p < \infty$, and consequently $(w \circ h)^r \cdot h' \in A_{\sigma p}$ by Corollary 3.1. Hence $w \circ h \cdot h'^{1/r} \in RH_r$. The "norm" relation can be obtained as in Lemma 5.2.

Theorem 5.4. *The following statements are equivalent for* $\varphi : \mathbf{R} \to \mathbf{R}_+$ *and* $1 < r_2 \le r_1$:

(i) $\varphi \in \bigcap_{r<r_0} RH_r$, $r_0 = r_1 r_2/(r_1 - r_2)$;

(ii) *if* $w \in RH_{r_1}$, *then* $w \cdot \varphi \in RH_{r_2}$, *and* $A_\infty^{r_2}(w \circ \varphi) \prec A_\infty^{r_1}(w)$.

Proof. The case $r_1 = r_2$ is in [6], so we assume $1 < r_2 < r_1$.

(ii) $\to$ (i). If $w \equiv 1$, then $\varphi \in RH_{r_2}$ and hence $\varphi^{r_2/r_1} \in RH_{r_1}$, and thus $\varphi^{1+r_2/r_1} \in RH_{r_2}$. We continue this process and obtain

$$\varphi^\tau \in RH_{r_2} \quad \text{for } \tau < \frac{r_1}{r_1 - r_2} = \sum_{n\ge 0} \left(\frac{r_2}{r_1}\right)^n.$$

Consequently, $\varphi \in \bigcap_{r<r_0} RH_r$.

(i) $\to$ (ii). Let $w \in RH_{r_1}$. Then $w \in RH_{\tau r_1}$ for some $\tau > 1$ by (P8). Since $\lambda = \tau r_1 r_2/(\tau r_1 - r_2) < r_1 r_2/(r_1 - r_2)$, (i) implies that $\varphi \in RH_\lambda$. We use now Lemma 5.1 with $\sigma = 1/2$ to obtain that $(w \cdot \varphi)^{1/2} \in RH_r$, where $r = \tau r_1\theta + \lambda(1-\theta)$ and $\frac{1}{2} = \tau r_1\theta/r$. Thus $\theta = \lambda/(\lambda + \tau r_1)$ and an easy calculation gives $r = 2r_2$. Hence $(w \cdot \varphi)^{2r_2/2} = (w\varphi)^{r_2} \in A_\infty$ or $w\varphi \in RH_{r_2}$. The "$\prec$" also follows from Lemma 5.1.

Remark. We will prove later (Corollary 6.3) that whenever (ii) holds for some $r_1, r_2 > 1$, then $r_2 \le r_1$.

Corollary 5.5. *If $w \in RH_{r_1}$, $\varphi \in RH_{r_2}$, and $r_1 r_2 > r_1 + r_2$, then $A^r_\infty(w\varphi) \prec A^{r_1}_\infty(w)$, $w \in A^{r_1}_\infty$, where $r = r_1 r_2/(r_1 + r_2)$.*

Proof. We have $r_2 = r_1 r/(r_1 - r)$ and the result follows since $\varphi \in RH_{r_2} \subset RH_r$, $r < r_2$.

Remark. It is possible to prove Theorem 5.3 without reference to Corollary 3.1. In fact, using the geometric characterization of $w \in A_\infty$ [3, p. 404], it follows easily that $w \in A_\infty \to w \circ hh' \in A_\infty$ iff $h' \in A_\infty$. This can be used to give an alternate proof of Corollary 3.1 (see Remark 1 after Corollary 3.6).

6. Mappings from RH_{r_1} to RH_{r_2}

In this section we will show that whenever $w \circ h \cdot h'^{\gamma} \in RH_{r_2}$ for every $w \in RH_{r_1}$, then $r_2 \le r_1$. We will see that matters can be reduced to the case $\gamma = 0$. Here the extrapolation theorem of §4 comes into play as well as part of Theorem 2.1. We will state the results in terms of $A^r_\infty = RH_r$.

Theorem 6.1. *If $A^{r_2}_\infty(w \circ h) \prec A^{r_1}_\infty(w)$, $w \in A^{r_1}_\infty$, then $r_2 \le r_1$.*

Proof. Assume that $r_2 > r_1$. Choose $p_0 > 1$ so that $r_2 > 1 + p_0(r_1 - 1)$. We first claim that $A_{q_0}(w \circ h) \prec A_{p_0}(w^{r_1})$, where $q_0 = 1 + (p_0 - 1)/r_2$. Since by (P2),

$$A_{q_0}(w \circ h) \le A_\infty(w \circ h) \cdot A_\infty(w \circ h^{1-q_0'})^{q_0 - 1},$$

and

$$A_\infty(w \circ h) \le A_\infty(w \circ h^{r_2})^{1/r_2} \prec A_\infty(w^{r_1}),$$

while $A_\infty(w \circ h^{1-q_0'})^{q_0-1} = A_\infty(w \circ h^{r_2(1-p_0')})^{(p_0-1)/r_2} \prec A_\infty(w^{r_1(1-p_0')})^{p_0-1}$, we see that $A_{q_0}(w \circ h) \prec A_\infty(w^{r_1}) \cdot A_\infty(w^{r_1(1-p_0')})^{p_0-1} \le A_{p_0}(w^{r_1})^2$ by (P2).

The claim gives us by (P7)

$$\int Mf^{q_0} w \circ h \le c \int f^{q_0} w \circ h, \qquad w \in A^{r_1}_{p_0},$$

$c \prec A_{p_0}(w^{r_1})$. As in the proof of Theorem 2.1 we change variables and reduce matters to

$$\int Tg^{p_0} w \le c \int g^{p_0} w, \qquad c \prec A_{p_0}(w^{r_1}),$$

where

$$g = (f \circ h^{-1})^{q_0/p_0} \frac{1}{(h' \circ h^{-1})^{1/p_0}},$$

and

$$Tg = \frac{1}{(h' \circ h^{-1})^{1/p_0}} [M\{h'^{1/q_0}(g \circ h)^{p_0/q_0}\} \circ h^{-1}]^{q_0/p_0}.$$

We apply now the extrapolation theorem of §4 and conclude that for $p_0/r_1' + 1/r_1 < p \le p_0$, $\int Tg^p w \le c \int g^p w$, $w \in A_p^k$. As before we let $w \equiv 1$ and undo our change of variables to get

$$(*) \qquad \int M f^{pq_0/p_0} h'^{1-p/p_0} \le c \int f^{pq_0/p_0} h'^{1-p/p_0}.$$

We claim now that $p_0/q_0 > p_0/r_1' + 1/r_1$. Since $q_0 = 1 + (p_0 - 1)/r_2$, an easy calculation shows that this is equivalent with $r_2 > 1 + p_0(r_1 - 1)$, valid by our choice in the beginning. Hence in $(*)$ we can let $p = p_0/q_0$ and obtain

$$\int Mf \cdot h'^{1-1/q_0} \le c \int f h'^{1-1/q_0}.$$

We have seen in the proof of Theorem 2.1 that this implies $h' \equiv 0$ which is impossible.

The next theorem shows that we can reduce matters to $\gamma = 0$.

Theorem 6.2. *Assume that* $1 < r_1$, $r_2 < \infty$ *and* $\gamma \in \mathbf{R}$. *If* $A_\infty^{r_2}(T_{h,\gamma}w) \prec A_\infty^{r_1}(w)$, $w \in A_\infty^{r_1}$, *then* $r_2 \le r_1$, *where* $T_{h,\gamma}w = w \circ h \cdot h'^\gamma$.

Proof. We assume that $r_2 > r_1$ and we will consider the five cases: $0 \le \gamma < 1/r_1$; $-\infty < \gamma < 0$; $1/r_1 < \gamma \le 1/r_1 + 1/r_2$; $\gamma > 1/r_1 + 1/r_2$; and $\gamma = 1/r_1$.

Since by (P13, P11), $(h^{-1})' \in A_\infty$, we have from Theorem 5.3, $T_{h^{-1},1/r_1}$: $RH_{r_1} \to RH_{r_1}$. Thus $w \cdot h'^{\gamma - 1/r_1} \in RH_{r_2} \subset RH_{r_1}$ for $w \in RH_{r_1}$, and $A_\infty^{r_2}(w \cdot h'^{\gamma-1/r_1}) \prec A_\infty^{r_1}(w)$. By Theorem 5.4, $h'^{\gamma-1/r_1} \in \bigcap_{r<\infty} RH_r$.

(i) $0 \le \gamma < 1/r_1$.

We may assume that $\gamma > 0$ by Theorem 6.1. By the above

$$\frac{1}{h'^\alpha} \in \bigcap_{r<\infty} RH_r, \qquad \alpha > 0,$$

so that in particular, $1/h'^\gamma$ is a pointwise multiplier from RH_{r_2} to RH_{r_2}. Thus, by Lemma 5.2, $A_\infty^{r_2}(w \circ h) = A_\infty^{r_2}(w \circ h \cdot h'^\gamma 1/h'^\gamma) \prec A_\infty^{r_2}(T_{h,\gamma}w) \prec A_\infty^{r_1}(w)$, and the case $\gamma = 0$ applies.

(ii) $-\infty < \gamma < 0$.

Since $h' \in A_\infty$, by Theorem 5.3 and the hypothesis $A_\infty^{r_1}(w \circ h \cdot h'^{1/r_1}) \prec A_\infty^{r_1}(w)$, $A_\infty^{r_2}(w \circ h \cdot h'^\gamma) \prec A_\infty^{r_1}(w)$. We use now Lemma 5.1 and get for $0 \le \theta \le 1$, $\sigma = r_1\theta/r$, $r = r_1\theta + r_2(1-\theta)$,

$$A_\infty^r(w \circ h \cdot h'^{\sigma\gamma + (1-\sigma)/r_1}) \prec A_\infty^{r_1}(w \circ h \cdot h'^{1/r_1})^\theta \cdot A_\infty^{r_2}(w \circ h \cdot h'^\gamma)^{1-\theta} \prec A_\infty^{r_1}(w).$$

We note that

$$\sigma\gamma + \frac{1-\sigma}{r_1} = \frac{1}{r}\left[r_1\theta\gamma + r_2(1-\theta)\frac{1}{r_1}\right].$$

This expression is positive for all θ satisfying $\theta(r_1\gamma - r_2/r_1) > -r_2/r_1$ or $0 < \theta < r_2/(r_2 - r_1^2\gamma)$. We fix now such a θ and observe that

$$A_\infty^r(w \circ h \cdot h'^\alpha) \prec A_\infty^{r_1}(w), \qquad \alpha = \frac{r_1\theta\gamma + r_2/r_1(1-\theta)}{r}.$$

Since $\gamma < 0$, $r_1\theta\gamma + r_2/r_1(1-\theta) < \theta + r_2/r_1(1-\theta)$, and thus $0 < \alpha < 1/r_1$. We are thus in the case (i).

(iii) $1/r_1 < \gamma \le 1/r_1 + 1/r_2$.

In the beginning of the proof we observed that $A_\infty^{r_2}(w \cdot h'^{\gamma - 1/r_1}) \prec A_\infty^{r_1}(w)$. Since $(h^{-1})' \in A_\infty$, by Theorem 5.3, for $w \in RH_{r_1}$, $w \circ h^{-1} \cdot (h' \circ h^{-1})^{\gamma - 1/r_1} \cdot (h^{-1})'^{1/r_2} = w \circ h^{-1} \cdot (h^{-1})'^{1/r_2 + 1/r_1 - \gamma}$ is in RH_{r_2} and

$$A_\infty^{r_2}(w \circ h^{-1} \cdot (h^{-1})'^{1/r_2+1/r_1-\gamma}) \prec A_\infty^{r_1}(w).$$

Since $r_2 > r_1$ we see that

$$0 \le \frac{1}{r_2} + \frac{1}{r_1} - \gamma < \frac{1}{r_1}$$

and again (i) applies.

(iv) $\gamma > 1/r_1 + 1/r_2$.

Again by Theorem 5.3 and the hypothesis we have $A_\infty^{r_1}(w \circ h \cdot h'^{1/r_1}) \prec A_\infty^{r_1}(w)$, and $A_\infty^{r_2}(w \circ h'^\gamma) \prec A_\infty^{r_1}(w)$. As in the case (ii), by Lemma 5.1

$$A_\infty^r(w \circ h \cdot h'^{\sigma/r_1 + (1-\sigma)\gamma}) \prec A_\infty^{r_1}(w),$$

where $\sigma = r_1\theta/r$, $r = r_1\theta + r_2(1-\theta)$, and $0 \le \theta \le 1$.

We wish to reduce this case to (iii), and hence we have to show that for some $0 < \sigma < 1$, $1/r_1 < \sigma/r_1 + (1-\sigma)\gamma \le 1/r_1 + 1/r$. Since $\gamma > 1/r_1$ we get $1/r_1 < \sigma/r_1 + (1-\sigma)\gamma$ for any $0 < \sigma < 1$. When $\sigma \uparrow 1$, $\sigma/r_1 + (1-\sigma)\gamma \to 1/r_1$ and $1/r_1 + 1/r \to 2/r_1$.

(v) $\gamma = 1/r_1$.

For $w = 1$, $h'^{1/r_1} \in RH_{r_2}$ and hence $h'^{r_2/r_1} \in A_\infty$. For $0 < \theta < 1$, $h'^{\theta r_2/\theta r_1} \in A_\infty$ or $h'^\theta \in RH_{r_2/r_1\theta}$. Since $r_2 > r_1$, we can choose θ so that $r_2/(1 + r_1\theta) > r_1$. We apply now Corollary 5.5 and obtain with $r = r_2/(1 + r_1\theta)$,

$$A_\infty^r(w \circ h \cdot h'^{1/r_1} \cdot h'^\theta) \prec A_\infty^{r_1}(w).$$

The previous two cases now apply and the proof of Theorem 6.2 is complete.

Corollary 6.3. *Let* $1 < r_1, r_2 < \infty$ *and* $\varphi : \mathbf{R} \to \mathbf{R}_+$. *If* $A_\infty^{r_2}(w\varphi) \prec A_\infty^{r_1}(w)$, *then* $r_2 \le r_1$.

Proof. Since $\varphi \in A_\infty$, $h(x) = \int_0^x \varphi$ is a homeomorphism satisfying our overall hypothesis and $h' = \varphi$. Hence by Theorem 5.3 and the hypothesis

$$A_\infty^{r_2}(w \circ h \cdot h'^{1+1/r_1}) \prec A_\infty^{r_1}(w \circ h \cdot h'^{1/r_1}) \prec A_\infty^{r_1}(w).$$

Theorem 6.2 now gives $r_2 \le r_1$.

7. Characterizations of $T_{h,\gamma} : RH_{r_1} \to RH_{r_2}$

We are now ready to characterize those homeomorphisms h for which $A_\infty^{r_2}(T_{h,\gamma}w) \prec A_\infty^{r_1}(w)$, where, as before, $T_{h,\gamma}w = w \circ h \cdot h'^{\gamma}$. By Theorem 6.2 we may assume that $r_2 \le r_1$.

Theorem 7.1. *For* $1 < r_2 \le r_1$ *and* $\gamma > 1/r_2$ *the following statements are equivalent*:

(i) $h' \in \bigcap_{r<\tau} RH_r$, $\tau = (\gamma - 1/r_1)r_1 r_2/(r_1 - r_2)$;

(ii) $A_\infty^{r_2}(T_{h,\gamma}w) \prec A_\infty^{r_1}(w)$, $w \in A_\infty^{r_1}$.

Proof. (i) → (ii). Since $h' \in A_\infty$, by Theorem 5.3, $A_\infty^{r_1}(w \circ h \cdot h'^{1/r_1}) < A_\infty^{r_1}(w)$. Since $h'^{\gamma-1/r_1} \in \bigcap_{r<r_0} RH_r$, $r_0 = r_1r_2/(r_1 - r_2)$, we can apply Theorem 5.4 to get

$$A_\infty^{r_2}(w \circ h \cdot h'^{\gamma}) = A_\infty^{r_2}(w \circ h \cdot h'^{1/r_1} \cdot h'^{\gamma-1/r_1}) \prec A_\infty^{r_1}(w \circ h \cdot h'^{1/r_1}) \prec A_\infty^{r_1}(w),$$

which is the desired result.

(ii) → (i). Since

$$\begin{aligned} A_\infty^{r_2}(w \cdot h'^{\gamma-1/r_1}) &= A_\infty^{r_2}(w \cdot (h^{-1})' \circ h^{1/r_1} \cdot h'^{\gamma}) \\ &\prec A_\infty^{r_1}(w \circ h^{-1} \cdot (h^{-1})'^{1/r_1}) \prec A_\infty^{r_1}(w), \end{aligned}$$

and since $(h^{-1})' \in A_\infty$, we see that $h'^{\gamma-1/r_1}$ is a multiplier from RH_{r_1} to RH_{r_2}, $r_2 \le r_1$, and Theorem 5.4 completes the proof.

Theorem 7.2. *For* $1 < r_2 \le r_1$ *and* $1/r_1 \le \gamma \le 1/r_2$, *the following are equivalent*:

(i) $h' \in A_\infty$,

(ii) $A_\infty^{r_2}(T_{h,\gamma}w) \prec A_\infty^{r_1}(w)$, $w \in A_\infty^{r_1}$.

Proof. (ii) → (i) follows from (P13).

(i) → (ii). Since $A_\infty^{r_1} \subset A_\infty^{r_2}$, we have by Theorem 5.3,

$$A_\infty^{r_2}(w \circ h \cdot h'^{1/r_1}) \prec A_\infty^{r_1}(w)$$

and

$$A_\infty^{r_2}(w \circ h \cdot h'^{1/r_2}) \prec A_\infty^{r_1}(w).$$

Hence by Lemma 5.1, for $0 \le \theta \le 1$,

$$A_\infty^{r_2}(w \circ h \cdot h'^{\theta/r_1 + (1-\theta)/r_2}) \prec A_\infty^{r_1}(w).$$

Theorem 7.3. *For* $1 < r_2 \le r_1$ *and* $\gamma < 1/r_1$ *the following are equivalent*:

(i) $h' \in \bigcap_{p>\sigma_0} A_p$, $\sigma_0 = (r_1 - r_1r_2\gamma)/(r_2 - r_1r_2\gamma)$.

(ii) $A_\infty^{r_2}(T_{h,\gamma}w) \prec A_\infty^{r_1}(w)$.

Proof. (i) → (ii). By (P12), $(h^{-1})' \in \bigcap_{r>\sigma_0'} RH_r$, and

$$\sigma_0' = \frac{r_1 r_2}{r_1 - r_2}\left(\frac{1}{r_2} - \gamma\right).$$

By Theorem 5.4, $(h^{-1})'^{1/r_2-\gamma}$ is a multiplier from RH_{r_1} to RH_{r_2}. Hence $A^{r_2}_\infty(w\circ h\cdot h'^\gamma) = A^{r_2}_\infty(w\circ h\cdot (h^{-1})'\circ h^{1/r_2-\gamma}\cdot h'^{1/r_2}) \prec A^{r_2}_\infty(w\cdot (h^{-1})'^{1/r_2-\gamma}) \prec A^{r_1}_\infty(w)$ by Theorem 5.3 since $h' \in A_\infty$.

(ii) → (i). Since $(h^{-1})' \in A_\infty$, we have

$$A^{r_2}_\infty(w\cdot (h'\circ h^{-1})^\gamma (h^{-1})'^{1/r_2}) \prec A^{r_2}_\infty(w\circ h\cdot h'^\gamma) \prec A^{r_1}_\infty(w).$$

Hence $(h^{-1})'^{1/r_2-\gamma}$ is a multiplier from RH_{r_1} to RH_{r_2} and thus by Theorem 5.4, $(h^{-1})'^{1/r_2-\gamma} \in \bigcap_{r<r_0} RH_r$, where $r_0 = r_1 r_2/(r_1 - r_2)$. This gives $(h^{-1})' \in \bigcap_{r<\sigma_0'} RH_r$, and this gives (i) by (P12).

The special case $r_2 = r_1$ of Theorems 7.1 and 7.3 may be of interest. In particular the following statements are equivalent:

(i) $h' \in \bigcap_{r<\infty} RH_r$;

(ii) for some $\gamma > 1/r_1$, $A^{r_1}_\infty(T_{h,\gamma}w) \prec A^{r_1}_\infty(w)$;

(iii) for every $\gamma > 1/r_1$, $A^{r_1}_\infty(T_{h,\gamma}w) \prec A^{r_1}_\infty(w)$.

An analogous statement holds for $\gamma < 1/r_1$.

In the limiting case $r_1 = r_2 = 1$ we have the following result.

Theorem 7.4. (1) *If* $\gamma > 1$, *then* $A_\infty(w\circ h\cdot h'^\gamma) \prec A_\infty(w)$ *iff* $h' \in \bigcap_{r<\infty} RH_r$.

(2) *If* $\gamma = 1$, *then* $A_\infty(w\circ h\cdot h') \prec A_\infty(w)$ *iff* $h' \in A_\infty$.

(3) *If* $\gamma < 1$, *then* $A_\infty(w\circ h\cdot h'^\gamma) \prec A_\infty(w)$ *iff* $h' \in \bigcap_{p>1} A_p$.

Proof. (1) For the necessity, first observe that $h'^\gamma \in A_\infty$ and so $h' \in A_\infty$ since $\gamma > 1$. By (P12), $(h^{-1})' \in A_\infty$. Since for $w \in A_\infty$, $w \in A^r_\infty$ for some $r > 1$, by Theorem 7.2, $w\circ h^{-1}\cdot (h^{-1})'^{1/r} \in A_\infty$. Therefore, $w\cdot h'^{\gamma-1/r} \in A_\infty$ and $h'^{\gamma-1/r}$ is a multiplier from A_∞ to A_∞. The result now follows from [6, 3.5].

Conversely, let $w \in A_\infty$. Then for $r > 1$, close to 1, $A_\infty(w^r) \prec A_\infty(w)$. Since $h' \in \bigcap_{r<\infty} RH_r$, by Theorem 7.1,

$$A_\infty(w\circ h\cdot h'^\gamma) \prec A^r_\infty(w\circ h\cdot h'^\gamma) \prec A^r_\infty(w) \prec A_\infty(w).$$

(2) The necessity is clear, and for the sufficiency simply observe that

$$A_\infty(w\circ h\cdot h') = A^r_\infty(w\circ h^{1/r}\cdot h'^{1/r}) \prec A^r_\infty(w^{1/r}) = A_\infty(w)$$

by Theorem 7.2.

(3) For the sufficiency, let $w \in A_\infty$. Choose $p < \infty$ so large that $1 - p \le \gamma < 1$ and $A_p(w) \le 2A_\infty(w)$. By Corollary 3.4,

$$A_\infty(w\circ h\cdot h'^\gamma) \le A_p(w\circ h\cdot h'^\gamma) \prec A_p(w) \le 2A_\infty(w).$$

Conversely, $h' \in A_\infty$ and so $(h^{-1})' \in A_\infty$ by (P12). Hence

$$A_\infty(w\cdot h'^{\gamma-1}) \prec A_\infty(w\circ h^{-1}\cdot (h^{-1})') \prec A_\infty(w),$$

and $h'^{\gamma-1}$ is a pointwise multiplier from A_∞ to A_∞. Since $\gamma < 1$, $1/h' \in \bigcap_{r<\infty} RH_r$ by [6, 3.5], and, since $h' \in A_\infty$, we get $h' \in \bigcap_{p>1} A_p$ by (P9).

We will now present a theorem giving conditions under which the mapping $T_{h,\gamma}w = w \circ h \cdot h'^{\gamma}$ is onto.

Theorem 7.5. *Let* $1 < r < \infty$.

(1) *If* $\gamma = 1/r$, *then* $T_{h,\gamma} : RH_r \to RH_r$ *is onto iff* $h' \in A_\infty$.

(2) *If* $\gamma \neq 1/r$, *then* $T_{h,\gamma} : RH_r \to RH_r$ *is onto iff* $\log h' \in \mathrm{clos}_{\mathrm{BMO}}(L^\infty)$.

Proof. (1) If $T_{h,\gamma} : RH_r \to RH_r$, then by taking $w \equiv 1$ we see that $h'^{1/r} \in RH_r$ or $h' \in A_\infty$ (P3). Conversely, if $h' \in A_\infty$, then $(h^{-1})' \in A_\infty$. If $w \in RH_r$, by Theorem 5.3, $u \equiv w \circ h^{-1} \cdot (h^{-1})'^{1/r} \in RH_r$ and so $u \circ h \cdot h'^{1/r} = w$.

(2) First consider $\gamma < 1/r$. If $T_{h,\gamma} : RH_r \to RH_r$ is onto, then for $u \in RH_r$ there is $w \in RH_r$ so that $w \circ h \cdot h'^{\gamma} = u$. Then $w = u \circ h^{-1} \cdot (h^{-1})'^{\gamma}$ and thus by Theorem 7.3, $(h^{-1})' \in \bigcap_{p>1} A_p$. Also by the same theorem, $h' \in \bigcap_{p>1} A_p$, and this gives by Corollary 3.4 that $1/h' = h^{-1\prime} \circ h \in \bigcap_{p>1} A_p$. By [3, p. 474], $\log h' \in \mathrm{clos}_{\mathrm{BMO}}(L^\infty)$.

If, conversely, $\log h' \in \mathrm{clos}_{\mathrm{BMO}}(L^\infty)$, then h' and $1/h'$ are in $\bigcap_{p>1} A_p$, and hence (P9) $h' \in \bigcap_{r<\infty} RH_r$ and $(h^{-1})' \in \bigcap_{p>1} A_p$. This defines $T_\gamma^{-1}w = w \circ h^{-1} \cdot (h^{-1})'^{\gamma}$.

The case $\gamma > 1/r$ is treated similarly.

8. Theorem 2.2 for $p_0 = 1$ and multipliers from A_p to A_q

The purpose of this section is to present a $p_0 = 1$ version of Theorem 2.2 and study pointwise multipliers from A_p to A_q.

Theorem 8.1. *For* $1 < q_0 < \infty$ *and* $\gamma \in \mathbf{R}$ *the following statements are equivalent*:

(1) $A_{q_0}(T_{h,\gamma}w) \prec A_1(w)$, $w \in A_1$, *where* $T_{h,\gamma}w = w \circ h \cdot h'^{\gamma}$.

(2) $h'^{1+(\gamma-1)p} \in A_{pq_0}$, $p \geq 1$, *and* $h' \in A_\infty$.

Proof. (1) $\to$ (2). Here we use Theorem 4.4 with $r_0 = 1$. We have

$$\int Mf^{q_0} w \circ h \cdot h'^{\gamma} \leq c \int f^{q_0} w \circ h \cdot h'^{\gamma}, \qquad c \prec A_1(w).$$

Change variables via $t = h(x)$ and proceed as in the proof of Theorem 2.1 to obtain

$$\int Tg^{q_0} w \leq c \int g^{q_0} w, \qquad w \in A_1,\ c \prec A_1(w)$$

where $f = h'^{(1-\gamma)/q_0} g \circ h$ and

$$Tg = \frac{1}{h'^{(1-\gamma)/q_0}} [M\{h'^{(1-\gamma)/q_0} g \circ h\} \circ h^{-1}].$$

By Theorem 4.4,

$$\int Tg^{p} w \leq c \int g^{p} w, \qquad p \geq q_0,\ w \in A_1.$$

We let now $w \equiv 1$ and obtain in terms of f

$$\int Mf^p h'^{1+(\gamma-1)p/q_0} \le c \int f^p h'^{1+(\gamma-1)p/q_0}.$$

Hence $h'^{1+(\gamma-1)p/q_0} \in A_p$, $p \ge q_0$, i.e.

$$h'^{1+(\gamma-1)p} \in A_{pq_0}, \qquad p \ge 1.$$

From (P11) we have that $h' \in A_\infty$.

(2) → (1). The proof is similar to the proof of the corresponding implication of Theorem 2.2 and we will be brief.

Let $w \in A_1$. Then there is $\tau > 1$ such that $w^\tau \in A_1$. Let

$$L = \frac{1}{|I|}\int_I w \circ h \cdot h'^\gamma \left(\frac{1}{|I|}\int_I w \circ h \cdot h'^{1-q_0'} \cdot h'^{\gamma(1-q_0')}\right)^{q_0-1} \equiv K \cdot H.$$

If $t = h(x)$, $J = h(I)$, we estimate as before

$$K \le \frac{|J|}{|I|}\left(\frac{1}{|J|}\int_J w^\tau\right)^{1/\tau}\left(\frac{1}{|J|}\int_J (h^{-1})'^{(1-\gamma)\tau'}\right)^{1/\tau'}$$

and

$$H \le \sup_J \left(\frac{1}{w}\right) \cdot \left(\frac{|J|}{|I|}\right)^{q_0-1}\left(\frac{1}{|J|}\int_J (h^{-1})'^{1-\gamma(1-q_0')}\right)^{q_0-1}.$$

Hence

$$L \le A_1(w^\tau)^{1/\tau}\left\{\frac{\left[\frac{1}{|J|}\int_J (h^{-1})'^{(1-\gamma)\tau'}\right]^{1/(1-\gamma)\tau'}}{|I|/|J|}\right\}^{1-\gamma}$$
$$\cdot \left\{\frac{\left[\frac{1}{|J|}\int_J (h^{-1})'^{1-\gamma(1-q_0')}\right]^{1/[1-\gamma(1-q_0')]}}{|I|/|J|}\right\}^{q_0-1+\gamma}.$$

It is to be noted that whenever $0 < (1-\gamma)\tau' \le 1$ or $0 < 1-\gamma(1-q_0') \le 1$, the corresponding expression { } can be dropped by Hölder's inequality.

Case 1: $\gamma > 1$. By (P4), since $h'^{1+(\gamma-1)p/q_0} \in A_p$, $p \ge q_0$, $h' \in A_{p(q+\gamma-1)}$. Since

$$\frac{p(q_0+\gamma-1)}{q_0+(\gamma-1)p} \ge \frac{q_0+\gamma-1}{\gamma}, \qquad p \ge q_0,$$

we see that

$$h' \in \bigcap_{r \ge (q_0+\gamma-1)/\gamma} A_r$$

or from (P12)

$$(h^{-1})' \in \bigcap_{r \le (q_0+\gamma-1)/(q_0-1)} A_r.$$

Since also $h' \in \bigcap_{r<\infty} RH_r$ we get that $(h^{-1})' \in \bigcap_{p>1} A_p$. The result follows.

Case 2. $\gamma = 1$. Then $h' \in A_{q_0}$ or $(h^{-1})' \in RH_{q_0'}$ and again we are done.

Case 3. $1 - q_0 \le \gamma < 1$. Since $1 + (\gamma - 1)p < 0$ for large p, we obtain by (P3) that $1/h' \in \bigcap_{r<\infty} RH_r$. Since also $h' \in A_\infty$, by (P9), $h' \in \bigcap_{p>1} A_p$, and hence $(h^{-1})' \in \bigcap_{r<\infty} RH_r$ by (P12). Since $1 - \gamma(1 - q_0') \ge 0$ and $1 - \gamma > 0$ for $1 - q_0 \le \gamma < 1$, the result for this case follows.

Case 4. $\gamma < 1 - q_0$. Again $(h^{-1})' \in \bigcap_{r<\infty} RH_r$. Since $1/h'^{-\gamma} \in A_{q_0}$, and $1/h' \in A_\infty$ by (P4), $1/h' \in A_{1+(q_0-1)/(-\gamma)}$, and this gives us $h' \in RH_{-\gamma/q_0-1}$ or $(h^{-1})' \in A_{\gamma/(q_0+\gamma-1)}$. The proof is now complete.

The technique treating the case $p_0 = q_0 = 1$ is slightly different from the one used above.

Theorem 8.2. *Let* $\gamma \in \mathbf{R}$.

(1) *If* $\gamma > 1$, *then* $A_1(T_{h,\gamma} w) \prec A_1(w)$, $w \in A_1$, *if and only if* $h' \in \bigcap_{r<\infty} A_1^r$.

(2) *If* $0 < \gamma \le 1$, *then* $A_1(T_{h,\gamma} w) \prec A_1(w)$, $w \in A_1$, *if and only if* $h' \in A_1$.

(3) *If* $\gamma = 0$, *then* $A_1(w \circ h) \prec A_1(w)$, $w \in A_1$, *if and only if* $h' \in \bigcap_{p>1} A_p$.

(4) *If* $\gamma < 0$, *then* $A_1(T_{h,\gamma} w) \prec A_1(w)$, $w \in A_1$, *if and only if* $1/h' \in \bigcap_{r<\infty} A_1^r$ *and* $h' \in A_\infty$.

Proof. (1) For the necessity observe that $h'^{\gamma} \in A_1$, and by Theorem 8.1, for $q_0 > 1$, $h'^{1+(\gamma-1)p} \in A_{pq_0}$, $p \ge 1$. Hence $h'^r \in A_\infty$, $r < \infty$, and since $h'^{r\gamma/r} \in A_1$, by (P4), $h'^r \in A_1$.

To prove that the condition is sufficient, we first note that by [6, 2.9] $A_1(w \circ h) \prec A_1(w)$ and by [6, 2.14], h'^{γ} is a pointwise multiplier from A_1 to A_1.

The cases (2) and (3) are in [6, 2.8 and 2.9].

For the necessity of (4), note that $1/h'^{-\gamma} \in A_1$, and for $q_0 > 1$ by Theorem 8.1, $h'^{1+(\gamma-1)p} \in A_{pq_0}$. Hence $1/h'^r \in A_1$, $r < \infty$. As in (1) by (P4), $1/h'^r \in A_1$. Conversely, since clearly $1/h' \in \bigcap_{r<\infty} RH_r$ we have by (P9) that $h' \in \bigcap_{p>1} A_p$ since $h' \in A_\infty$. Hence, again $A_1(w \circ h) \prec A_1(w)$ and $1/h'^{\gamma}$ is a pointwise multiplier from A_1 to A_1.

We come now to the multiplier problem for A_p. We first remark that if $A_{q_0}(w\varphi) \prec A_{p_0}(w)$, $w \in A_{p_0}$, where $\varphi : \mathbf{R} \to \mathbf{R}_+$, then $q_0 \ge p_0$. The proof of this is similar to the proof of Theorem 2.1 and we will point this out in the implication $(1) \to (2)$ of the theorem below.

Theorem 8.3. *For* $1 < p_0 \le q_0 < \infty$ *and* $\varphi : \mathbf{R} \to \mathbf{R}_+$, *the following statements are equivalent*:

(1) $A_{q_0}(w\varphi) \prec A_{p_0}(w)$, $w \in A_{p_0}$.

(2) $\varphi^{p/p_0} \in A_{pq_0/p_0}$, $1 < p < \infty$.

(3) $\varphi \in \bigcap_{p>1+q_0-p_0} A_p \cap \bigcap_{r<\infty} RH_r$.

Proof. (1) → (2). By (P7) we have $\int Mf^{q_0} w\varphi \le c \int f^{q_0} w\varphi$, $c \prec A_{p_0}(w)$. Let $\sigma_0 = q_0/p_0$ and let $g = f^{\sigma_0}\varphi^{1/p_0}$ or $f = \varphi^{-1/\sigma_0 p_0} g^{1/\sigma_0}$. If

$$Tg = M\{\varphi^{-1/\sigma_0 p_0} g^{1/\sigma_0}\}^{\sigma_0} \cdot \varphi^{1/p_0},$$

then

$$\int Tg^{p_0} w \le c \int g^{p_0} w, \qquad w \in A_{p_0},\ c \prec A_{p_0}(w).$$

We can apply now Rubio de Francia's extrapolation theorem and obtain for $1 < p < \infty$ and $w \in A_p$,

$$\int Tg^p w \le c \int g^p w.$$

If $w \equiv 1$, then in terms of f we get

$$(*) \qquad \int (Mf)^{\sigma_0 p} \varphi^{p/p_0} \le c \int f^{\sigma_0 p} \varphi^{p/p_0},$$

which implies that $\varphi^{p/p_0} \in A_{\sigma_0 p}$, $1 < p < \infty$.

We remark that up to $(*)$ the fact that $q_0 \ge p_0$ was not needed. Thus, if $q_0 < p_0$, then we could let $p = p_0/q_0$ in $(*)$ and arrive, as in Theorem 2.1, at the contradiction $\varphi \equiv 0$. This shows that (1) implies that $q_0 \ge p_0$.

(2) → (3). Since $\varphi \in A_\infty$ and $\varphi^{p/p_0} \in A_{\sigma_0 p}$, by (P4) we have that $\varphi \in A_{1+p_0(\sigma_0 p-1)/p}$, $1 < p < \infty$. Since $1 + p_0(\sigma_0 p - 1)/p > 1 + q_0 - p_0$, we get $\varphi \in \bigcap_{p>1+q_0-p_0} A_p$. From $\varphi^{p/p_0} \in A_{\sigma_0 p} \subset A_\infty$ we obtain, using (P3), that $\varphi \in \bigcap_{r<\infty} RH_r$.

(3) → (1). Let $w \in A_{p_0}$ and choose $\tau > 1$ such that $w^\tau \in A_{p_0}$. If we set

$$L = \frac{1}{|I|}\int_I w\varphi \left(\frac{1}{|I|}\int_I w^{1-q_0'} \cdot \varphi^{1-q_0'}\right)^{q_0-1} \equiv K \cdot H,$$

then

$$K \le \left(\frac{1}{|I|}\int_I w^\tau\right)^{1/\tau} \left(\frac{1}{|I|}\int_I \varphi^{\tau'}\right)^{1/\tau'} \le c\left(\frac{1}{|I|}\int_I w^\tau\right)^{1/\tau} \frac{1}{|I|}\int_I \varphi.$$

In H we use Hölder's inequality with indices

$$\rho = \frac{p_0' - 1}{q_0' - 1}\tau = \frac{q_0 - 1}{p_0 - 1}\tau$$

and ρ' to estimate

$$H \le \left\{\left(\frac{1}{|I|}\int_I w^{\tau(1-p_0')}\right)^{(p_0-1)/(q_0-1)\tau} \cdot \left(\frac{1}{|I|}\int_I \varphi^{\rho'(1-q_0')}\right)^{1/\rho'}\right\}^{q_0-1}.$$

Hence

$$L \le cA_{p_0}(w^\tau)^{1/\tau} \frac{1}{|I|}\int_I \varphi \left(\frac{1}{|I|}\int_I \varphi^{\rho'(1-q_0')}\right)^{(q_0-1)/\rho'}.$$

We complete the proof by showing that $\sigma > 1+q_0-p_0$, where $\sigma'-1 = \rho'(q_0'-1)$. Since

$$\sigma - 1 = \frac{1}{\rho'(q_0' - 1)} = \frac{q_0\tau - \tau - p_0 + 1}{\tau},$$

we get

$$\sigma = \frac{q_0\tau - p_0 + 1}{\tau} > 1 + q_0 - p_0,$$

and hence $L \le cA_{p_0}(w^\tau)^{1/\tau}A_\sigma(\varphi)$.

The case $p_0 = 1$ of Theorem 8.3 requires a slightly different argument.

Theorem 8.4. *For* $1 < q_0 < \infty$ *and* $\varphi : \mathbf{R} \to \mathbf{R}_+$ *the following statements are equivalent*:

(1) $A_{q_0}(w\varphi) \prec A_1(w)$, $w \in A_1$.

(2) $\varphi^p \in A_{pq_0}$, $p \ge 1$.

Proof. (1) $\to$ (2). As before we have $\int Mf^{q_0}w\varphi \le c\int f^{q_0}w\varphi$, $c \prec A_1(w)$. We let now $g = f\cdot\varphi^{1/q_0}$ or $f = \varphi^{-1/q_0}g$ and $Tg = M(\varphi^{-1/q_0}g)\cdot\varphi^{1/q_0}$. Then

$$\int Tg^{q_0}w \le c\int g^{q_0}w, \qquad c \prec A_1(w).$$

We can apply now Theorem 4.4 with $r_0 = 1$ and obtain that for $p \ge q_0$ and $w \in A_1$,

$$\int Tg^p w \le c\int g^p w.$$

If $w \equiv 1$, this gives us $\int Mf^p\varphi^{p/q_0} \le c\int f^p\varphi^{p/q_0}$, or $\varphi^{p/q_0} \in A_p$, $p \ge q_0$, which is (2).

(2) $\to$ (1). Let $w \in A_1$ and choose $\tau > 1$ with $w^\tau \in A_1$. If $L = K\cdot H$ is as in Theorem 8.3, then we estimate

$$K \le \left(\frac{1}{|I|}\int_I w^\tau\right)^{1/\tau}\left(\frac{1}{|I|}\int_I \varphi^{\tau'}\right)^{1/\tau'} \le c\left(\frac{1}{|I|}\int_I w^\tau\right)^{1/\tau}\frac{1}{|I|}\int_I\varphi$$

since $\varphi \in \bigcap_{r<\infty} RH_r$ by (P3), and

$$H \le \sup_I\left(\frac{1}{w}\right)\cdot\left(\frac{1}{|I|}\int_I \varphi^{1-q_0'}\right)^{q_0-1}.$$

From this we see that

$$L \le cA_1(w^\tau)^{1/\tau}A_{q_0}(\varphi)$$

and the proof is complete since $\varphi \in A_{q_0}$.

Remark. The pointwise multipliers from A_1 to A_1, i.e., the case $p_0 = q_0 = 1$, have been characterized in [6, 2.14] and are precisely those $\varphi : \mathbf{R} \to \mathbf{R}_+$ which satisfy $\varphi \in \bigcap_{r<\infty} A_1^r$.

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