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HEEGAARD DIAGRAMS OF 3-MANIFOLDS

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Dedicated to Professor Fujitsugu Hosokawa on his sixtieth birthday

ABSTRACT. For a 3-manifold $M(L)$ obtained by an integral Dehn surgery along an n-bridge link L with n-components we define a concept of planar Heegaard diagrams of $M(L)$ using a link diagram of L. Then by using Homma-Ochiai-Takahashi's theorem and a planar Heegaard diagram of $M(L)$ we will completely determine if $M(L)$ is the standard 3-sphere in the case when L is a 2-bridge link with 2-components.

1. INTRODUCTION

It is well known that every closed connected orientable 3-manifold is a 3-manifold $M(L)$ obtained by integral Dehn surgery along an n-bridge link L with n-components (see [5, 12, 6, 7]). Birman and Powell found the concept of a special Heegaard diagram of $M(L)$, making use of pure $2n$-plat representations of L in [1 and 3]. In this paper, we will consider a method to directly construct 3-manifolds to make their Heegaard diagrams using the bridge diagrams of n-bridge links with n-components. The fundamental groups of such 3-manifolds have good presentations associated with the Heegaard diagrams (see Theorem 1). "Good" means that using such presentations, we can easily determine if $M(L)$ is the standard 3-sphere S^3 by Whitehead's procedure. It is not known in general whether the Whitehead conjecture is true for planar Heegaard diagrams of S^3 (see [11, 16], and the last section in this paper).

The author proved in [9] that all 3-manifolds obtained by nontrivial Dehn surgery along nontrivial 2-bridge knots are not the standard 3-sphere S^3. Using Homma-Ochiai-Takahashi's theorem [11] (see Theorem 2), a similar result will be proved in the case when surgery curves are 2-bridge links with 2-components, other than the torus link.

In this paper, we work in piecewise linear category. S^n and D^n denote n-sphere and n-disk, respectively. Let X be a manifold and Y a submanifold properly embedded in X. $N(X, Y)$ denotes a regular neighborhood of Y in

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X. Closure, interior and boundary over one symbol $\cdot$ are denoted by $\mathrm{cl}(\cdot)$, $\mathrm{int}(\cdot)$ and $\partial(\cdot)$, respectively.

2. Heegaard diagrams obtained by Dehn surgery along links

A Heegaard splitting $(H_1, H_2; F)$ of a 3-manifold M is a representation of M as $H_1 \cup H_2$, where H_1 and H_2 are handlebodies of some fixed genus n and $H_1 \cap H_2 = \partial H_1 = \partial H_2 = F$, a Heegaard surface. A properly embedded 2-disk D in a handlebody H of genus n is called a meridian disk of H if $\mathrm{cl}(H - N(D, H))$ is a handlebody of genus $n-1$. Moreover a collection of mutually disjoint disks $D_1, D_2, \dots, D_n$ in H is called a complete system of meridian disks of H if $\mathrm{cl}(H - \bigcup N(D_i, H))$ is a 3-ball. A collection of n mutually disjoint circles on the boundary of H is called a complete system of meridians of H (or ∂H) if it bounds a complete system of meridian disks of H.

Let $v = \{v_1, v_2, \dots, v_n\}$ and $w = \{w_1, w_2, \dots, w_n\}$ be complete systems of meridians of H_1 and H_2, respectively. The triplet $\{F; v, w\}$ is called a Heegaard diagram of a Heegaard splitting $(H_1, H_2; F)$ of M (or simply M).

Let H_1 be a handlebody of genus n standardly embedded in R^3. Let $v = \{v_1, v_2, \dots, v_n\}$ be a standard complete system of meridians of $\partial H_1 = F$ and let $x = \{x_1, x_2, \dots, x_n\}$ be a collection of mutually disjoint circles on F such that x_i intersects v_j transversely at only one point if $i = j$ and are disjoint from v_j if $i \neq j$. Assume further that each x_i bounds a disk in the complement of H_1 in R^3. The pair $\{v; x\}$ is called a standard meridian-longitude system of H_1. Let L be a collection of mutually disjoint circles included in $F \subset R^3$ (or S^3). Then L is called an n-bridge link if each connected component of L always intersects some x_i transversely at only one point, and $L \cap x$ is n points $(i = 1, 2, \dots, n)$.

Let L be an n-bridge link with n-components $K_1, K_2, \dots, K_n$. Then we can assume that each K_i intersects x_i in one point p_i, that p_i is disjoint from $v_i \cap x_i$, that K_i is disjoint from x_j if $i \neq j$, and that $K_{p_i} \cap (v_i \cup x_i) = p_i$, where K_{p_i} is the closure of the connected component including p_i among $K_i - v$. We call such a link canonical. From now on, we assume that all bridge links to be considered are canonical. Let $b(x_i, K_i)$ denote $N(x_i \cup K_i, F)$ and u_i be $\partial b(x_i, K_i)$. Then u_i is a circle on F and there exist n mutually disjoint disks $U_1', U_2', \dots, U_n'$ outside H_1 in S^3 such that $u_i = \partial U_i'$.

Lemma 1. *Let E be $H_1 \cup (\bigcup U_i' \times I)$. Then $\mathrm{cl}(S^3 - E)$ consists of one 3-ball B'^3 and n solid tori $V_1', V_2', \dots, V_n'$. Moreover $E \cup B'^3$ is homeomorphic to $S^3 - \mathrm{int}(N(L, S^3))$.*

Proof. Without loss of generality, we can assume that the torus $N(x_i \cup K_i, F) \cup (\bigcup U_i')$ does bound V_i'. Then each x_i bounds a disk in V_i' and so we can assume that each V_i' is a regular neighborhood of K_i in S^3.

Let w_i' be circles on $b(x_i, K_i)$ $(i = 1, 2, \ldots, n)$ and $w' = \{w_1', w_2', \ldots, w_n'\}$. Then w_i' gives a framing for V_i'. Let L be an oriented link in S^3 and $M(L; w')$ denote the 3-manifold obtained by the Dehn surgery determined by the framing. Then we have

Lemma 2. *$(F; v, w')$ gives a Heegaard diagram of the 3-manifold $M(L; w')$. Conversely all 3-manifolds obtained by Dehn surgery along L have such Heegaard diagrams.*

From now on, we denote $M(L; w')$ as $M(L; a_1/b_1, \ldots, a_n/b_n)$ when w_i' gives surgery coefficients a_i/b_i $(i = 1, \ldots, n)$ and later we abbreviate $M(L; a_1/b_1, \ldots, a_n/b_n)$ to $M(L)$.

Let H_2 be another handlebody of genus n standardly embedded in R^3 and $\{w; y\}$ be a standard meridian-longitude system of H_2. Then $M(L; w')$ has a Heegaard splitting $H_1 \cup H_2$ such that w_i is identified with w_i' $(i = 1, 2, \ldots, n)$. It is well known that any closed connected 3-manifold can be obtained by integral Dehn surgery along some n-bridge link with n components (see [5, 12, 6, 7]). And so, by Lemma 1 and Lemma 2, we have

Proposition 1. *Any closed connected orientable 3-manifold has a Heegaard diagram $(F; v, w)$ such that*

(1) *it is naturally obtained by integral Dehn surgery along an n-bridge link L with n-components,*

(2) *each meridian w_i intersects x_i transversely at only one point and it is disjoint from x_j, if $i \neq j$, and*

(3) *each longitude x_i is identified with y_i $(i = 1, 2, \ldots, n)$. Moreover, the dual diagram $(F; w, v)$ has the same property also.*

It will be noted that Proposition 1 was found by Birman and Powell in [1 and 3], because every pure $2n$-plat gives an n-bridge link with n-components and that a theorem similar to Proposition 1 in the case when L is a knot was also done by the author in [9].

Let L be a canonical n-bridge link with n-components. Then the number of points in $L \cap v$ is called the complexity of L or $(F; v, w)$, a Heegaard diagram mentioned above.

Next we introduce notations and defining terms for presentations of fundamental groups of 3-manifolds. Let $(F; v, w)$ be a Heegaard diagram of a 3-manifold M. Then orienting F and all the circles in v and w, we can get cyclic words $w_1(v), w_2(v), \ldots, w_n(v)$ from $w_1, w_2, \ldots, w_n$, respectively, by travelling each w_i once (in the given orientation) and reading v_k or v_k^{-1} for each crossing with v_k when w_i does cross v_k from left to right or from right to left, respectively. Then we have the following theorem.

Theorem 1. *Let M be a closed connected orientable 3-manifold and G be the fundamental group of M. Then G has the following presentation:*

$$\begin{array}{ll} \textit{generators}: & v_1, v_2, \ldots, v_n \\ \textit{relators}: & v_1^{s_1} = v_{11}^{e_{11}} \cdot v_{12}^{e_{12}} \cdots v_{1q1}^{e_1 q_1} \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & v_n^{s_n} = v_{n1}^{e_{n1}} \cdot v_{n2}^{e_{n2}} \cdots v_{nq_n}^{e_n q_n} \end{array}$$

where $s_1, s_2, \ldots, s_n$ *are integers,* e_{ij} *is* 1 *or* -1, v_{ij} *is some* v_k $(k = 1, 2, \ldots, n)$ *for all* i *and* j $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, q_i)$, *and* v_{ij} *is different from* v_{ij+1} $(j = 1, 2, \ldots, q_i - 1)$ *and both* v_{i1} *and* v_{iq_i} *are different from* v_i, *for all* i $(i = 1, 2, \ldots, n)$.

Proof. Let $(F; v, w)$ be a Heegaard diagram of $M(L; w')$ given by Proposition 1. Then each w_i induces the cyclic word $w_i(v)$ which gives the ith relator of G. Let us suppose that $(F; v, w)$ is a Heegaard diagram with the minimal complexity among Heegaard diagrams given by Proposition 1 and that both v_{cj} and v_{cj+1} are the same as v_k for some k. Then two cases happen.

Case 1. $e_{cj}e_{cj+1} = -1$; This case is illustrated in Figure 1. Let τ_1 be the (v_{cj}, v_{cj+1})-section of w_i. Then $\mathrm{int}(\tau_1)$ is disjoint from v and so along τ_1 v_k can be changed to the new meridian v_k' and the new Heegaard diagram obtained by the change of meridians has smaller complexity than $(F; v, w)$ (see Figure 1).

Case 2. $e_{cj}e_{cj+1} = 1$; This case is illustrated in Figure 2. The (v_{cj}, v_{cj+1})-section S of v_k, which does not include the point p_k, always includes at least two crossing points, because the section S must intersect w_i and w_k. Let τ_2 be the (v_{cj}, v_{cj+1})-section of w_i. Then $\mathrm{int}(\tau_2)$ is disjoint from v and so along τ_2 v_k can be changed to the new meridian v_k' and the new Heegaard diagram obtained by the change of meridians has the complexity which is at least one less than the one of $(F; v, w)$ (see Figure 2).

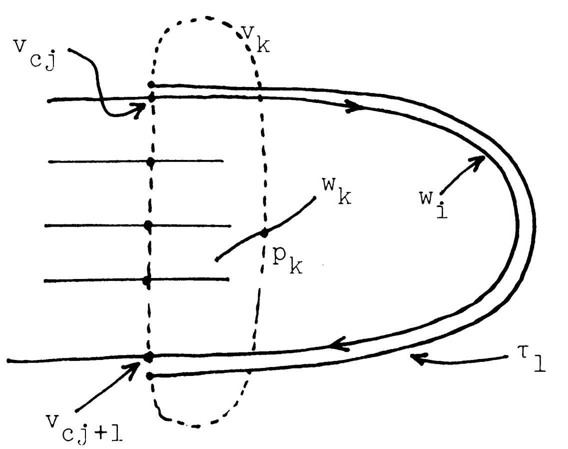


FIGURE 1

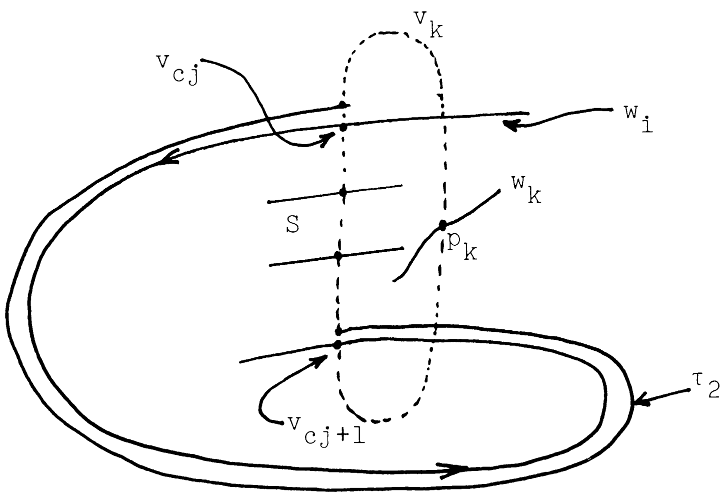


FIGURE 2

In the discussion mentioned above, every word $w_i(v)$ may change to a new word if $w_i \cap v_k \neq \varnothing$ but all $w_1, w_2, \dots, w_n$ remain to be fixed. Since both $\mathrm{int}(\tau_1)$ and $\mathrm{int}(\tau_2)$ are disjoint from all x_i and v_j $(i, j = 1, 2, \dots, n)$, the new Heegaard diagram obtained by the change of meridians induces a link diagram of L with smaller crossing points.

Thus all the meridians in w have no such arcs τ_1 and τ_2, since $(F; v, w)$ has the minimal complexity.

Note that by a similar method, we can verify that all meridians in v have no arcs similar to such τ_1 and τ_2.

Later in group G with the presentation is called the characteristic group of $M(L; w')$ and let G^* denote the group given by the dual diagram $(F; w, v)$ from Theorem 1.

It will be noticed that the work $v_i^{s_i}$ is induced from the part of w_i in $N(x_i, F)$ and the one $v_{i1}^{ei_1} \cdot v_{i2}^{ei_2} \cdots v_{iq_i}^{e_i q_i}$ is induced from the part of w_i in $w_i - N(x_i, F)$ $(i = 1, 2, \dots, n)$ and that from now on we can assume that all Heegard diagrams to be considered, including their dual diagrams, satisfy these conditions.

Let τ be an (oriented) arc on F such that $\mathrm{int}(\tau) \cap (v \cup w) = \varnothing$, $\partial\tau \cap (v \cup w - v \cap w) = a \cup b$ are two disjoint points, and $(a \cup b) \subset v_i$ (resp. w_i). Then τ is called a wave of type 0 for v_i (resp. w_i) if it intersects v_i (resp. w_i) with the opposite sign and is also called a wave of type 1 for v_i (resp. w_i) if it intersects v_i (resp. w_i) and $\tau \subset F - \bigcup N(x_i, F)$. Note that $F - \bigcup N(x_i, F)$ is a planar surface and we can assume that $N(x_i, F) = N(y_i, F)$. If a Heegaard diagram has such arcs as τ_1 and τ_2 mentioned in the proof of Theorem 1, then it also has a wave of type 1 at these arcs. Later a wave of type 0 is simply called a wave (see the definition in [11]).

Let M be $M(L; w')$ and A be the relation matrix for $H_1(M, Z)$ induced from G by abelianizing the relators of G given by Theorem 1. Then we have

Corollary 1.1. *The matrix* $A = (a_{ij})$ *is a symmetric, integer matrix. If* M *is* $M(L; b_1, b_2, \dots, b_n)$, *then* $a_{ii} = b_i$ $(i = 1, 2, \dots, n)$.

Proof. If $n = 2$, then L is a 2-bridge link and so A is symmetric (see the next section). Otherwise, $K_i \cup K_j$ is a 2-bridge link and so $a_{ij} = a_{ji}$, where $i \neq j$ and $i, j = 1, 2, \dots, n$. Moreover $a_{ii} = b_i$, because $b_i = 1$ if and only if $e(v_i) = 1$, where $e(v_i)$ is the exponent sum of v_i in $w_i(v)$ $(i = 1, 2, \dots, n)$.

3. Dehn surgery along a 2-bridge link along 2-components

Let L be a 2-bridge link with 2-components, $L = K_1 \cup K_2$, let $w' = w_1' \cup w_2'$ be frame curves of a Dehn surgery along L and let $(F; v, w)$ be a Heegaard diagram of genus 2 given by Proposition 1 from w'. Furthermore let M be $M(L; w')$ and G be the characteristic group. Then by Theorem 1, G has the following presentation:

$$G = \{a, b; a^m = b^{e_{11}} a^{e_{12}} \cdots b^{e_{1q_1}}, b^n = a^{e_{21}} b^{e_{22}} \cdots a^{e_{2q_2}}\}.$$

Since L is a 2-bridge link, L has a Schubert's normal form of type (p, q) as a 2-bridge diagram, where p is even, q is odd with $0 < q < p$ and p, q are relatively prime (see [13]). Thus $q_1 = q_2$ and $e_{1i} = e_{2i}$ $(i = 1, 2, \dots, q_1)$. Moreover, there exists an involution f on F such that $f(v_i) = v_i$, $f(x_i) = x_i$, $f(w_i') = w_i'$, and that p_i is a fixed point among the six fixed points of f, where $p_i = x_i \cap w_i'$ and $i = 1, 2$. Thus we have that $e_i = e_{r-i+1}$, where $i = 1, 2, \dots, [r/2]$, $[r/2]$ means the greatest integer less than or equal to $r/2$ and $r = q_1$ (see [2]). Then G is as follows:

$$\begin{aligned} G = \{a, b; a^m &= b^{e_1} a^{e_2} b^{e_3} \cdots a^{e_{r-1}} b^{e_r}, \\ b^n &= a^{e_1} b^{e_2} a^{e_3} \cdots b^{e_{r-1}} a^{e_r}, \\ & r \text{ is an odd integer}, \ e_i \text{ is } \pm 1, \text{ and} \\ & \qquad e_i = e_{r-i+1} \ (i = 1, 2, \dots, [r/2])\}. \end{aligned}$$

Moreover the dual presentation G^* of G given by $(F; w, v)$ is as follows:

$$\begin{aligned} G^* = \{c, d; c^m &= d^{f_1} c^{f_2} d^{f_3} \cdots c^{f_{r-1}} d^{f_r}, \\ d^n &= c^{f_1} d^{f_2} c^{f_3} \cdots d^{f_{r-1}} c^{f_r}, \\ & r \text{ is an odd integer}, \ f_i \text{ is } \pm 1, \text{ and} \\ & \qquad f_i = f_{r-i+1} \ (i = 1, 2, \dots, [r/2])\}. \end{aligned}$$

Let us suppose that $1 < q < p/2$ and let s be the smallest integer which satisfies $sq < p < (s+1)q$. Then $s \geq 2$. Thus, if $1 < q < p/2$ (or even if $p/2 < q < p-1$), then we can assume that $e_1 = 1$, $e_2 = 1, \dots, e_s = 1$ and $e_{s+1} = -1$.

Here we study the case when M is S^3. Let $r = 2k-1$ and let us suppose that all $e_1, e_2, \dots, e_r$ are $+1$. In this case, we have that $f_i = +1$ $(i = 1, 2, \dots, r)$

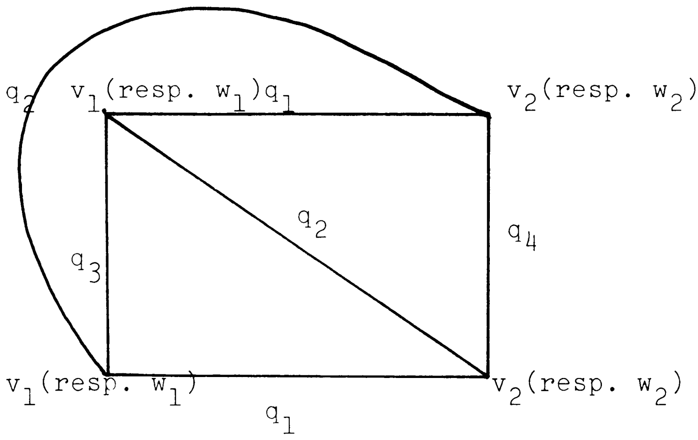


FIGURE 3

and L is a $(k, \pm 1)$ torus link with 2-components because either $q = 1$ or $q = p - 1$.

Case 1.1. $r = 1$; L is the Hopf link. Then M is S^3 if and only if either $mn = 0$ or $mn = 2$.

Case 1.2. $r \geq 3$; Let q_1, q_2, q_3 and q_4 (resp. q_1', q_2', q_3' and q_4') be non-negative integers which give the numbers of edges of the Whitehead graph of $(F; v, w)$ (resp. $(F; w, v)$) illustrated in Figure 3. If either $m \geq 1$ or $n \geq 1$, then all q_1, q_2, q_1' and q_2' are positive. If $m, n \leq -2$, then all $q_1, q_3, q_4, q_1', q_3'$ and q_4' are positive. Thus if either $m, n \leq -2$ or $m \geq 1$ or $n \geq 1$, then $(F; v, w)$ has no waves. Then M is not S^3 by Homma-Ochiai-Takahashi's theorem in [11]. If $m = 0$, then $a = b^{n+1}$ and $G = \{b; b^{k(n+2)-(n+1)} = 1\}$. Thus G is trivial if and only if either $n = -2$ or $k = 2$, $n = -4$ or $k = 3$, $n = -3$. If $m = -1$, then $G = \{a, b; (ab)^k = b^{n+1} = 1, k > 1\}$ and so $H_1(M, Z)$ is nontrivial. Thus M is S^3 in the case when $m = 0$ and either $n = -2$ or $k = 2$, $n = -4$ or $k = 3$, $n = -3$. It will be noted that to verify M to be S^3 we can use the reduction procedure of Heegaard diagrams through waves (see [11]).

Next let us suppose that some e_i is -1 $(i = 1, 2, \dots, r)$. If all e_i $(i = 1, 2, \dots, r)$ are -1, then this case reduces to the above-mentioned case. And so another e_j $(j \neq i)$ is $+1$. Since the case when either $e_{2i-1} = 1$, $e_{2i} = -1$ or $e_{2i-1} = -1$, $e_{2i} = 1$ $(i = 1, 2, \dots, k+1)$ reduces to the first case, we can assume that either $e_1 = 1$, $e_2 = 1, \dots, e_s = 1$, $e_{s+1} = -1$ or $e_1 = -1$, $e_2 = -1, \dots, e_s = -1$, $e_{s+1} = 1$, where $s \geq 2$. Then $(F; v, w)$ (resp. $(F; w, v)$) has as the Whitehead graph the graph illustrated in Figure 3, where $q_1, q_2 > 0$ (resp. $q_1', q_2' > 0$), and so it has no waves. By the theorem in [11], M is not S^3. Hence, by Corollary 1.1 we have

Theorem 2. *Let M be a closed 3-manifold obtained by nontrivial integral Dehn surgery along a nontrivial 2-bridge link with 2-components. Then M is not S^3*

if L is not a $(k, \pm 1)$-torus link with 2-components. If M is S^3, then we have the following cases:

(1) *L is a Hopf link, and either $M = M(L; m, 0)$ or $M = M(L; 2, 1)$ or $M = M(L; -2, -1)$.*

(2) *L is a $(2, \pm 1)$-torus link, and $M = M(L; 1, 5)$.*

(3) *L is a $(3, \pm 1)$-torus link, and $M = M(L; 2, 5)$.*

(2) *L is a $(k, \pm 1)$-torus link $(k > 1)$, and $M = M(L; k-1, k+1)$.*

In particular, if L is a nontrivial 2-bridge link, $M(L; \pm 1, \pm 1)$ is not S^3.

Let G_1 be the characteristic group of $M(L_1 \cup L_2; m_1/n_1, m_2/n_2)$ and G_1^* be the dual presentation of G_1. Then G_1 and G_1^* are as follows (see the presentations of G and G^*);

$$\begin{aligned} G_1 = \{a, b;\ & a^{-i_1} = Aa^{i_2}A \cdots a^{i_{n1}}A, \\ & b^{-j_1} = Bb^{j_2}B \cdots b^{j_{n-2}}B, \\ & A = b^{e_1}a^{e_2}b^{e_3} \cdots a^{e_{r-1}}b^{e_r}, \\ & B = a^{e_1}b^{e_2}a^{e_3} \cdots b^{e_{r-1}}a^{e_r}\}, \\ G_1^* = \{c, d;\ & c^{-i_0} = d^{e_1 n_2}c^{e_2 n_1}d^{e_3 n_2} \cdots c^{e_{r-1}n_1}d^{e_r n_2}, \\ & d^{-j_0} = c^{e_1 n_1}d^{e_2 n_2}c^{e_3 n_1} \cdots d^{e_{r-1}n_2}c^{e_r n_1}\} \end{aligned}$$

where $i_2 = i_3 = \cdots = i_{n_1}$, $i_1 = i_2 \pm 1$, $j_2 = j_3 = \cdots = j_{n_2}$, $j_1 = j_2 \pm 1$, $i_0 = i_1 + i_2 + \cdots + i_{n_1}$ and $j_0 = j_1 + j_2 + \cdots + j_{n_2}$. Let θ be a 9-tuple $(k, n_1, i_0, i_1, i_2, n_2, j_0, j_1, j_2)$. Then we have the following theorem:

Theorem 3. *Let M_1 be $M(L_1 \cup L_2; m_1/n_1, m_2/n_2)$ and let $n_1 \geq 2$ and $n_2 \geq 1$. Then M_1 is not S^3, if $L = L_1 \cup L_2$ is not a $(k, \pm 1)$-torus link. If M_1 is S^3, then L is a $(k, \pm 1)$-torus link and θ is as follows*:

Case 1. $k = 1$, $n_1 \geq 2$, $n_2 = 1$:

$(1, n_1, n_1+1, 2, 1, 1, 1, 1, *)$, $(1, n_1, 1, 1, 0, 1, n_1+1, n_1+1, *)$,
$(1, 3, -2, 0, -1, 1, -2, -2, *)$, $(1, 2, -1, 0, -1, 1, -3, -3, *)$,
$(1, 2, -3, -1, -2, 1, -1, -1, *)$, $(1, 2, 1, 0, 1, 1, 1, 1, *)$,
$(1, 3, 2, 0, 1, 1, 2, 2, *)$, $(1, 2, 1, 0, 1, 1, 3, 3, *)$,
$(1, n_1, -1, -1, 0, 1, -n_1 - 1, -n_1 - 1, *)$,
$(1, n_1, -n_1 - 1, -2, -1, 1, -1, -1, *)$,
$(1, n_1, 1, 1, 0, 1, n_1 - 1, n_1 - 1, *)$,
$(1, n_1, -n_1 + 1, 0, -1, 1, -1, -1, *)$,
$(1, n_1, n_1 - 1, 0, 1, 1, 1, 1, *)$,
$(1, n_1, -1, -1, 0, 1, -n_1 + 1, -n_1 + 1, *)$.

Case 2. $k = 1$, $n_1 \geq 2$, $n_2 \geq 2$;

$(1, n_1, 1, 1, 0, 2, 2n_1 + 1, n_1, n_1 + 1)$,
$(1, n_1, 1, 1, 0, 2, 2n_1 - 1, n_1, n_1 - 1)$,
$(1, n_1, 1, 1, 0, n_2, n_1 n_2 + 1, n_1 + 1, n_1)$,

$(1, n_1, 1, 1, 0, n_2, n_1n_2 - 1, n_1 - 1, n_1)$,
$(1, n_1, -1, -1, 0, 2, -2n_1 + 1, -n_1, -n_1 + 1)$,
$(1, n_1, -1, -1, 0, 2, -2n_1 - 1, -n_1, -n_1 - 1)$,
$(1, n_1, -1, -1, 0, n_2, -n_1n_2 - 1, -n_1 - 1, -n_1)$,
$(1, n_1, -1, -1, 0, n_2, -n_1n_2 + 1, -n_1 + 1, -n_1)$,
$(1, n_1, n_1 - 1, 0, 1, n_1 - 2, n_1 - 1, 2, 1)$,
$(1, 2, 1, 0, 1, n_2, 2n_2 + 1, 3, 2)$,
$(1, 2, 1, 0, 1, 2, 5, 2, 3)$, $(1, 4, 3, 0, 1, 2, 3, 1, 2)$,
$(1, 3, 2, 0, 1, 3, 5, 1, 2)$, $(1, n_1, n_1 - 1, 0, 1, n_1, n_1 + 1, 2, 1)$,
$(1, 2, 1, 0, 1, n_2, 2n_2-1, 1, 2)$, $(1, 2, -1, 0, -1, 2, -5, -2, -3)$,
$(1, 3, -2, 0, -1, 3, -5, -1, -2)$,
$(1, n_1, 1 - n_1, 0, -1, n_1 - 2, 1 - n_1, -2, -1)$,
$(1, 2, -1, 0, -1, n_2, -2n_2 - 1, -3, -2)$,
$(1, 2, -1, 0, -1, n_2, -2n_2 + 1, -1, -2)$,
$(1, n_1, 1 - n_1, 0, -1, n_1, -n_1 - 1, -2, -1)$.

Case 3. $k \geq 2$, $n_1 \geq 2$, $n_2 = 1$;

$(2, n_1, 3n_1+1, 4, 3, 1, 0, 0, *)$, $(k, k-2, 2k-3, 3, 2, 1, 0, 0, *)$,
$(3, 3, 8, 2, 3, 1, 0, 0, *)$, $(k, k, 2k + 1, 3, 2, 1, 0, 0, *)$,
$(k, k, 1, 1, 0, 1, 2, 2, *)$, $(k, k + 2, 1, 1, 0, 1, 2, 2, *)$,
$(2, 2, 1, 0, 1, 1, 2, 2, *)$.

Case 4. $k \geq 2$, $n_1 \geq 2$, $n_2 \geq 2$:

$(k, 2, 1, 1, 0, 2, 3, 2, 1)$, $(3, 2, 1, 1, 0, 3, 5, 1, 2)$,
$(2, 2, 1, 1, 0, 4, 7, 1, 2)$,
$(2, 2, 1, 0, 1, 4, 7, 1, 2)$, $(3, 2, 1, 0, 1, 3, 5, 1, 2)$,
$(k, 2, 1, 0, 1, 2, 3, 1, 2)$,
$(2, 3, 2, 0, 1, 2, 3, 1, 2)$, $(k, n_1, n_1 - 1, 0, 1, n_1, n_1 + 1, 2, 1)$,
$(2, n_2 + 1, n_2, 0, 1, n_2, n_2 + 1, 2, 1)$,
$(k, n_1, n_1 + 1, 2, 1, n_1, n_1 - 1, 0, 1)$,
$(2, n_1, n_1 + 1, 2, 1, n_1 + 1, n_1, 0, 1)$.

Proof. If $1 < q < p - 1$, then by the proof of Theorem 2 all q_1, q_2, q_1' and q_2' are positive and so $M(L_1 \cup L_2; m_1/n_1, m_2/n_2)$ is not S^3. Let us suppose that either $q = 1$ or $q = p - 1$. Then $e_1 = e_2 = \cdots = e_r = 1$ and there exist four cases: Case 1, Case 2, Case 3, Case 4. Let λ be the determinant of the relation matrix of G_1.

Case 1. $\lambda = i_0j_0 - n_1$; $\lambda = \pm 1$ and $i_0 = i_2n_1 \pm 1$. Then there are the four following cases:

1.1. $\lambda = 1$, $i_0 = i_2n_1 + 1$.
1.2. $\lambda = 1$, $i_0 = i_2n_1 - 1$.
1.3. $\lambda = -1$, $i_0 = i_2n_1 + 1$.
1.4. $\lambda = -1$, $i_0 = i_2n_1 - 1$.

In the case 1.1, we have that $n_1(i_2 j_0 - 1) = 1 - j_0$. If $i_2 = 0$, then $j_0 = 1$, $i_1 = i_2 + 1 = 1$, $i_0 = 1$. If $i_2 \neq 1$, then $i_2 j_0 > 1$, $1 > j_0$ and so $0 > j_0$, $0 > i_2$. Thus θ is as follows:

$$(1, n_1, n_1+1, 2, 1, 1, 1, 1, *), \quad (1, n_1, 1, 1, 0, 1, n_1+1, n_1+1, *)$$

$$(1, 3, -2, 0, -1, 1, -2, -2, *), \quad (1, 2, -1, 0, -1, 1, -3, -3, *),$$

$$(1, 2, -3, -1, -2, 1, -1, -1, *).$$

A similar method mentioned above is applied to the cases 1.2, 1.3 and 1.4.

Case 2. $\lambda = i_0 j_0 - n_1 n_2$; $\lambda = \pm 1$, $i_0 = i_2 n_1 \pm 1$. Since $i_0 j_0 > 1$, we can assume that $|i_0| \leq n_1$. If $i_2 = 0$, then $i_0 = \pm 1$. If $i_2 > 0$, then $i_0 > 0$ and so $i_2 = 1$, $i_0 = n_1 - 1$. If $i_2 < 0$, then $i_0 < 0$ and so $i_2 = -1$, $i_0 = -n_1 + 1$.

2.1. $i_2 = 0$, $i_0 = 1$,
2.2. $i_2 = 0$, $i_0 = -1$,
2.3. $i_2 = 1$, $i_0 = n_1 - 1$,
2.4. $i_2 = -1$, $i_0 = -n_1 + 1$.

In the case 2.1, θ is as follows:

$$(1, n_1, 1, 1, 0, n_2, n_1 n_2 + 1, n_1 + 1, n),$$
$$(1, n_1, 1, 1, 0, n_2, n_1 n_2 - 1, n_1 - 1, n_1),$$
$$(1, n_1, 1, 1, 0, 2, 2n_1 + 1, n_1, n_1 + 1),$$
$$(1, n_1, 1, 1, 0, 2, 2n_1 - 1, n_1, n_1 - 1).$$

In the case 2.3, if $\lambda = j_0(n_1 - 1) - n_1 n_2 = 1$, then $j_0 > 0$. Let $s = j_0 - n_2$ (> 0). Then $j_0 = n_1 s - 1$, $n_2 = n_1 s - s - 1$. Since $n_1 \geq 2$, $n_2 \geq s - 1$. If $j_0 = n_2 + s = n_2 j_2 + 1$, then $n_2(j_2 - 1) = s - 1$. If $s = 1$, then $j_2 = 1, j_1 = 2$, $j_0 = n_2 + 1$, $n_2 = n_1 - 2$. If $s > 1$, then $n_2 \leq s - 1$ and so $n_2 = s - 1$. Thus $j_0 = 2n_2 + 1$, $j_2 = 2$, $j_1 = 3$, $n_1 = 2$. If $j_0 = n_2 + s = n_2 j_2 - 1$, then $n_2(j_2 - 1) = s + 1$. Since $s > 0$, then $n_2 \leq s + 1$. If $n_2 = s - 1$, then $n_1 = 2$ and so $2s - j_2(s - 1)$. Since $s > 2$, $s = 3$ and so $j_2 = 3$, $j_1 = 2$, $n_2 = 2$. If $n_2 = s$, then $2s + 1 = n_1 s$. But this is impossible, because $s > 1$. If $n_2 = s + 1$, then $n_1 s = 2s + 2$. Since $s > 0$, $s = 1$ or $s = 2$. If $s = 1$, then $n_1 = 4$, $n_2 = 2$, $j_0 = 3$, $j_1 = 1$, $j_2 = 2$. If $s = 2$, then $n_1 = 3$, $n_2 = 3$, $j_0 = 5$, $j_1 = 1$, $j_2 = 2$. A similar method mentioned above is applied to the case of $\lambda = -1$.

Case 3. $\lambda = \{(k-1) + j_0\}\{(k-1)n_1 + i_0\} - k^2 n_1$: In this case, if either $i_2 \leq -1$ or $j_0 \leq -1$ or $i_2 \geq 2$, $j_0 \geq 2$, then by the Main Theorem in [11] M_1 is not S^3. Thus we have the following cases:

3.1. $i_2 = 0$,
3.2. $i_2 = 1$,
3.3. $j_0 = 0$,
3.4. $j_0 = 1$.

Since $k \geq 2$ and $n_2 = 1$, the case 3.4 is impossible. A similar method mentioned in Case 2 can be applied to the other cases. Note that if j_0 $(= j_1) = 2$, then M_1 is a 3-manifold of genus 1 but that M_1 is not a homology 3-sphere if $i_2 \geq 2$.

Case 4. $\lambda = \{(k-1)n_2 + j_0\}\{(k-1)n_1 + i_0\} - k^2 n_1 n_2$: In this case, if either $i_2 \leq -1$ or $j_2 \leq -1$ or $i_2 \geq 2$, $j_2 \geq 2$, then by the Main Theorem in [11] M_1 is not S^3. A similar method mentioned in Case 3 can be applied to the case.

4. Dehn surgery along an n-bridge link with n-components

Let L be an n-bridge link with n-components, M be $M(L)$ and G be the characteristic group of $M(L)$. Moreover, let $A = (a_{ij})$ be the relation matrix for $H_1(M, Z)$ induced from G.

Proposition 2. *Suppose $M = M(L)$ is a homology 3-sphere. If the matrix A satisfies $a_{ij} = 0$ if $i = n$ and $j = 1, 2, \ldots, n-1$, or if $j = n$ and $i = 1, 2, \ldots, n-1$, and that $a_{nn} = p$, and v_{nj} $(j = 1, 2, \ldots, q_n)$ are different from v_n, then M is $M(L')$, where L' is an $(n-1)$-bridge link with $(n-1)$-components.*

Proof. Since $H_1(M, Z) = 0$, we have that $p = \pm 1$. Then w_n intersects v_n transversely at only one point and so we can cancel all the intersections of v_n and w_i $(i = 1, 2, \ldots, n-1)$ using band sum operations (see [11, 15]) and later we get that $v_n \cap w = v_n \cap w_n$ is one point. Thus we can cancel the 3-ball $N(D_n \cup D'_n, M)$, where D_n (resp. D'_n) is a meridian disk in H_1 (resp. H_2) with $v_n = \partial D_n$ (resp. $w_n = \partial D'_n$) and so get a Heegaard diagram of genus $n-1$.

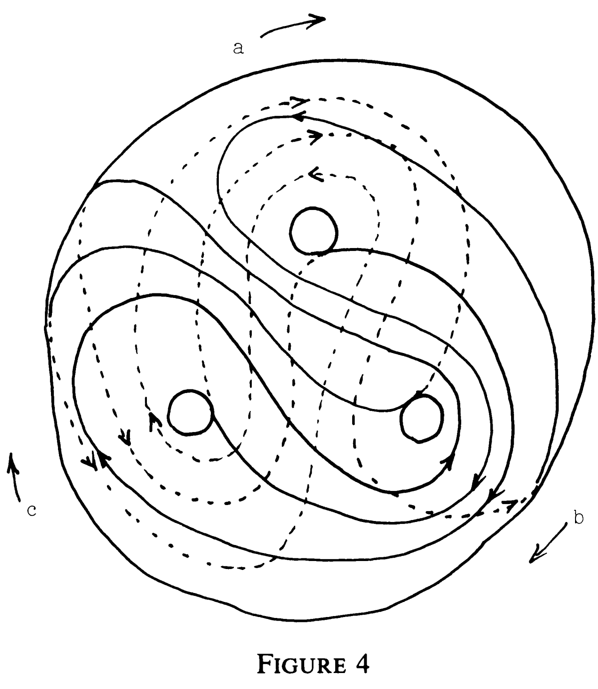


FIGURE 4

It will be noted that if M is not a homology 3-sphere, Proposition 2 is false, because all the 3-manifolds obtained by Dehn surgery along the Borromian link 6_2^3 satisfy the condition of Proposition 2 and the class of such the manifolds contains the 3-dimensional torus (see Figure 4) but it has no Heegaard diagram of genus 2.

Proposition 3. *Let M be $M(L)$, where L is 6_2^3 or 8_5^3. Then M is not S^3.*

Proof. Let us suppose that M is a homology 3-sphere. Then M satisfies the hypothesis of Proposition 2, because 6_2^3 and 8_5^3 are 3-bridge links with 3-components, and G is as follows:

$$G = \{a, b, c; a^m = bcb^{-1}c^{-1}, b^n = c^{-1}a^{-1}ca, c^1 = ab^{-1}a^{-1}b\} \quad \text{if } L \text{ is } 6_5^3,$$

$$\begin{aligned} G = \{a, b, c; a^m &= b^{-1}a^{-1}cbc^{-1}b^{-1}, \\ b^n &= a^{-1}b^{-1}a^{-1}bc^{-1}b^{-1}abc, \\ c^1 &= abab^{-1}a^{-1}b^{-1}a^{-1}b\} \quad \text{if } L \text{ is } 8_5^3. \end{aligned}$$

Then M has a Heegaard diagram of genus 2 $(F_2; v^2, w^2)$ by Proposition 2 and has G^2 as the characteristic group of the diagram such that

$$G^2 = \{a, b; a^m = bab^{-1}a^{-1}b^{-1}ab, b^n = aba^{-1}b^{-1}a^{-1}ba\}$$

if L is 6_5^3 and $p = 1$,

$$\begin{aligned} G^2 = \{a, b; a^m &= b^{-1}a^{-1}b^{-1}bababa^{-1}b^{-1}a^{-1}b^{-1}, \\ b^n &= a^{-1}b^{-1}a^{-1}ababab^{-1}a^{-1}b^{-1}a^{-1}\} \end{aligned}$$

if L is 8_5^3 and $p = 1$.

Note that in the case of $p = -1$ G^2 has also the similar presentation to that mentioned above. And so by Theorem 2, M is never S^3.

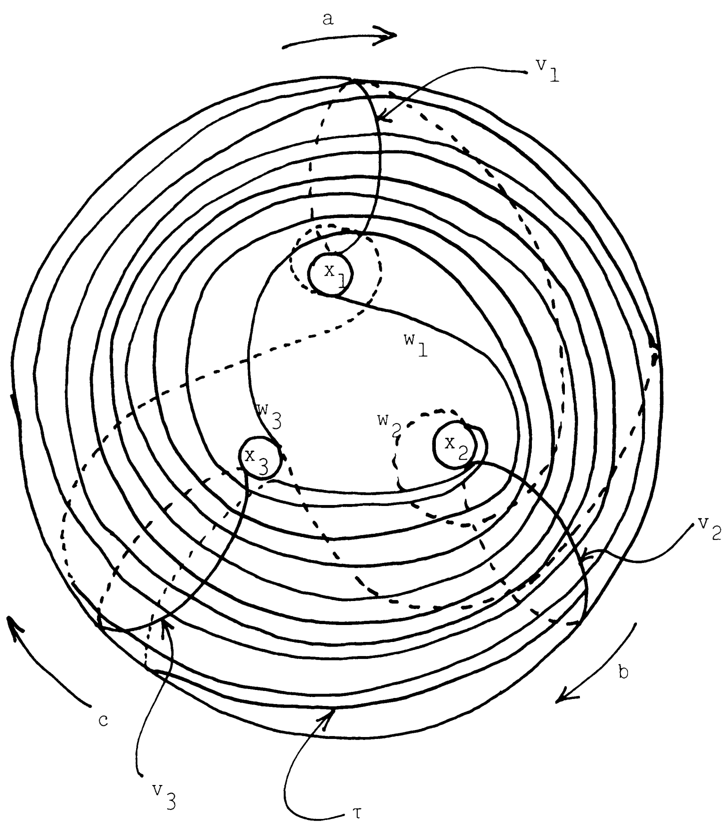
Next let L be an n-bridge $(k, 1)$-torus link with n-components, M be $M(L)$, and $A = (a_{ij})$ be the relation matrix.

Proposition 4. *Suppose that $a_{ij} = k$ if $i \neq j$ $(i, j = 1, 2, \dots, n)$ and that either $a_{11} = 1$, $a_{ii} = k + 1$ if i is even and $a_{ii} = k - 1$ if i is odd $(1 < i \leq n)$, in the case when n is odd, or $a_{ii} = k + 1$ if i is odd, and $a_{ii} = k - 1$ if i is even $(1 \leq i \leq n)$, in the case when n is even. Then M is S^3.*

Proof. Since the Heegaard diagram $(F; v, w)$ has always a wave of type 0 (but not type 1) we can reduce it to another one which has Heegaard genus $n - 1$ by cancelling a handle through the wave and continuing this process (see Figure 5). Finally we get a Heegaard diagram of genus one and it is easily seen that G is trivial and so M is S^3.

Note that if the matrix A does not satisfy the hypothesis of Proposition 4, then many nontrivial homology 3-spheres may be obtained (see [4]).

Next let $L(t_1, t_2, \dots, t_{[n/2]})$ be an n-bridge $(k + 1, 1)$-torus link with n-components with full twists $t_1, t_2, \dots, t_{[n/2]}$ (see Figure 6) and M be



G = { a, b, c; a = bcabcabc,
b^{-2} = cabcabca,
1 = abcabcab } (k = 3)

FIGURE 5

$M(L(t_1, t_2, \dots, t_{[n/2]}))$, where $n \geq 3$ and all of the $t_1, t_2, \dots,$ and $t_{[n/2]}$ are integer. Then G is as follows:

$$\begin{aligned} G = \{a, b, c, d;\ a^{m_1} &= bcd(abcd)^k(ab)^{t_1}, \\ b^{m_2} &= cd(abcd)^k a(ba)^{t_1}, \\ c^{m_3} &= d(abcd)^k ab(cd)^{t_2}, \\ d^{m_4} &= (abcd)^k abc(dc)^{t_2}\} \quad \text{if } n = 4, \end{aligned}$$

$$\begin{aligned} G = \{a, b, c, d, e;\ a^{m_1} &= bcde(abcde)^k, \\ b^{m_2} &= cde(abcde)^k a(bc)^{t_1}, \\ c^{m_3} &= de(abcde)^k ab(cb)^{t_1}, \\ d^{m_4} &= e(abcde)^k abc(de)^{t_2}, \\ e^{m_5} &= (abcde)^k abcd(ed)^{t2}\} \quad \text{if } n = 5. \end{aligned}$$

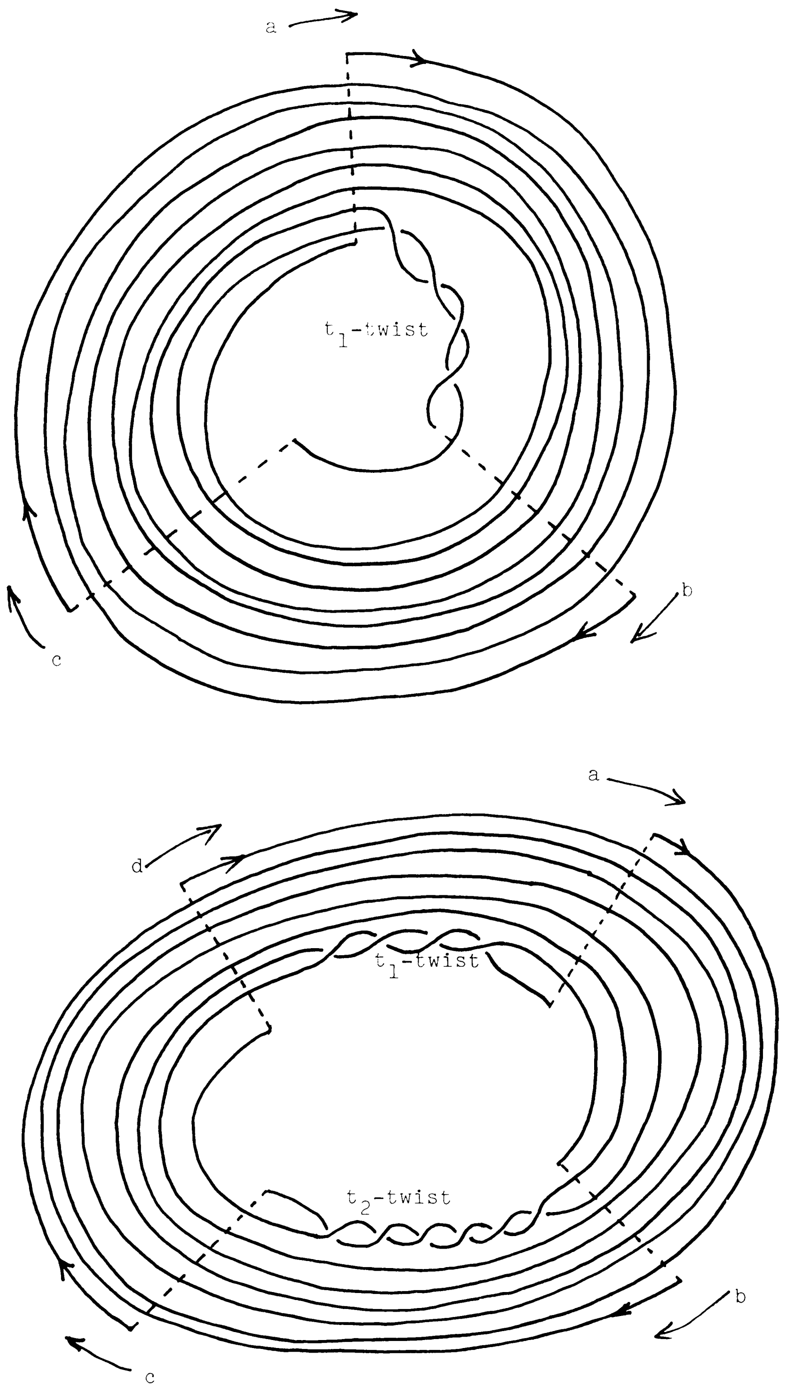


FIGURE 6

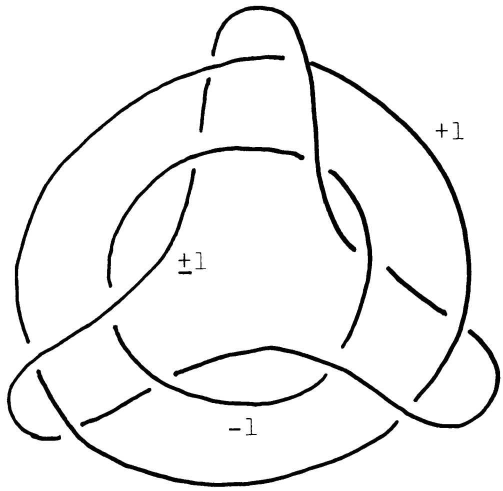


FIGURE 7

Case 1: $n = 4$ (n is even). In this case, if $m_1 = -2$, $m_2 = 0$, $m_3 = -2$, and $m_4 = 0$, then M is S^3 (see the proof of Proposition 4).

Case 2: $n = 5$ (n is odd). In this case, if $m_1 = k \pm 1$, $m_2 = -2$, $m_3 = 0$, $m_4 = -2$, and $m_5 = 0$, then M is S^3 (see the proof of the Proposition 4).

Then by Corollary 1.1 we have

Proposition 5. *If* $t_1 = t_2 = \cdots = t_{[n/2]}$, *and* $k = -t_1 - 1$, *then*

$$M(L(t_1, t_2, \dots, t_{[n/2]}); \pm 1, 1, -1, 1, -1, \dots, -1, 1)$$

is S^3 *in the case when* n *is odd, and*

$$M(L(t_1, t_2, \dots, t_{[n/2]}); 1, -1, 1, -1, \dots, 1, -1)$$

is also S^3 *in the case when* n *is even.*

By the way, we have examined many Heegaard diagrams of S^3 given by Proposition 1 and cannot find the nontrivial one without waves. Heegaard diagrams of 3-manifolds which have the properties (2) and (3) described in Proposition 1 are called planar Heegaard diagrams. Finally, we will propose the following conjecture: 'Nontrivial planar Heegaard diagrams of S^3 always have waves." This conjecture probably has the affirmative answer in the case when the bridge number is three. It will be noticed that the conjecture is an alternative version of Whitehead's conjecture with restricted conditions and that the original one has counterexamples (see [14, 8, 10, 16]). Let L^0, L^1 and $L(t_1)$ be the 3-bridge links with 3-components as illustrated in Figure 7. Then we will conjecture that if L is a nontrivial 3-bridge link with 3-components and $M(L; \pm 1, \pm 1, \pm 1)$ is S^3, then L is one of the three links L^0, L^1 and $L(t_1)$. Notice that for the 3-bridge link L, illustrated in Figure 8 with k full twists, $M(L; k, k+1, k+2)$ is S^3 but $M(L; \pm 1, \pm 1, \pm 1)$ is not a homology 3-sphere if $k > 0$.

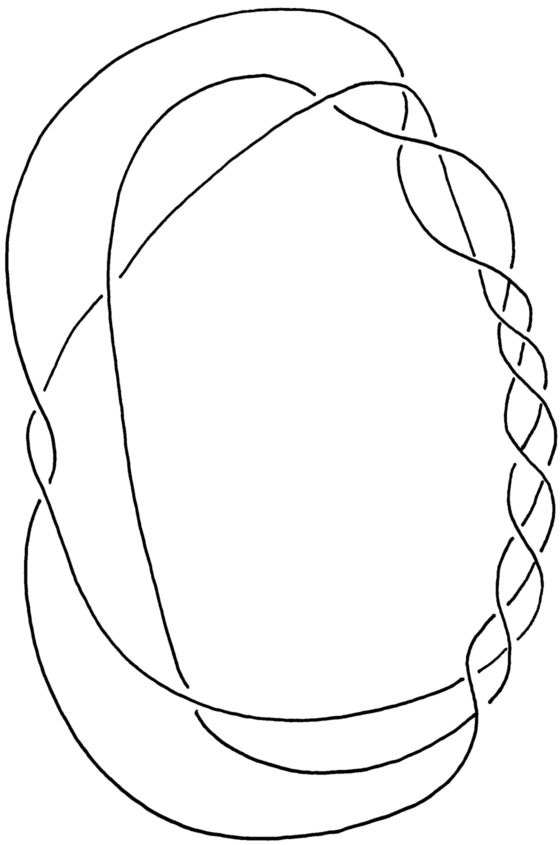


FIGURE 8

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