TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 328, Number 2, December 1991

THE STEFAN PROBLEM WITH SMALL SURFACE TENSION

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ABSTRACT. The Stefan problem with small surface tension ε is considered. Assuming that the classical Stefan problem (with $\varepsilon = 0$) has a smooth free boundary Γ, we denote the temperature of the solution by θ_0 and consider an approximate solution $\theta_0 + \varepsilon u$ for the case where $\varepsilon \neq 0$, ε small. We first establish the existence and uniqueness of u, and then investigate the effect of u on the free boundary Γ. It is shown that small surface tension affects the free boundary Γ radically differently in the two-phase problem than in the one-phase problem.

0. INTRODUCTION

In the classical formulation of the two-phase Stefan problem the temperatures θ_w and θ_i of water and ice satisfy the following conditions on the interface $\{\Phi(x, t) = 0\}$;

$$\nabla_x\theta_w \cdot \nabla_x\Phi - \nabla_x\theta_i\nabla_x\Phi = \Phi_t, \tag{0.1}$$

$$\theta_w = \theta_i = 0; \tag{0.2}$$

the first condition is the conservation of energy. The functions θ_w, θ_i further satisfy the heat equation in the water and ice sets, respectively, as well as initial and boundary conditions on the fixed portions of the boundary.

For definiteness we shall take in this paper the initial geometry to be as in Figure 1, namely, the fixed boundary (∂D) is surrounded by water (region $G = G_0\backslash\overline{D}$) and the water is surrounded by ice. The fixed boundary does not change in time, but the free boundary (the water-ice interface) will of course change with time.

The one-phase Stefan problem arises when $\theta_i \equiv 0$ in the ice region, or $\theta_w \equiv 0$ in the water region. Taking for definiteness the case $\theta_i \equiv 0$, the interface conditions are

$$\nabla_x\theta_w \cdot \nabla_x\Phi = \Phi_t, \tag{0.3}$$

Received by the editors October 20, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35R35; Secondary 35K20, 35K85.

This work is partially supported by National Science Foundation Grant DMS-86-12880.

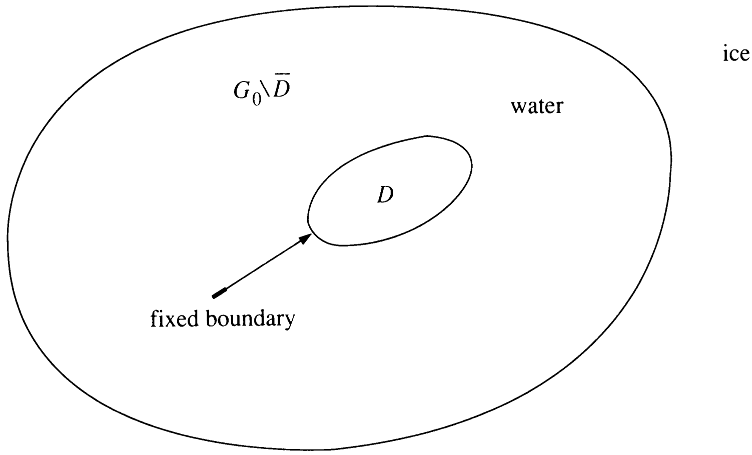

FIGURE 1

$$\theta_w = 0\,. \tag{0.4}$$

Molecular considerations attempting to explain dendritic growth of crystals suggest replacing (0.4) by the Gibbs-Thomson relation

$$\theta_w = \gamma\sigma\kappa \tag{0.5}$$

where

$$\kappa = \text{ mean curvature of the interface } = \frac{\operatorname{div}\vec{\nu}}{n-1} \quad (n = \text{ dimension}), \tag{0.6}$$

$\vec{\nu}$ is the unit normal pointing into the ice (and extended as constant vector along each normal line), σ is the surface tension and γ is a positive constant; for details see [1, 2, 8, 12, 17, 18] and the references given there. Gurtin [9 and 10] has established (0.5) and its counterpart

$$\theta_w = \theta_i = \gamma\sigma\kappa \tag{0.7}$$

for the two-phase case by thermodynamic considerations (see also Langer [14]). The sign of κ in (0.6) is positive (negative) if the intersection of the water region (ice region) with a small ball centered at the interface point is convex.

It is well known that the Stefan problem with interface conditions (0.1), (0.2) or (0.3), (0.4) has a unique global weak solution; see, for instance, [6]. Further, for the one-phase problem the free boundary is known to be smooth (for $t > 0$) under some conditions on the initial geometry and the initial and boundary data (see [7, 6]); smoothness for large time only (i.e. for $t > t_0$) was established by Matano [15] under fairly weak assumptions on the data. For the 1- and 2-phase Stefan problems, smoothness of the free boundary was established by Meirmanov [16] and Hanzawa [11] for small time $(0 \le t \le \delta)$, provided the initial data are smooth and satisfy some compatibility conditions.

On the other hand, for the Stefan problem with surface tension (i.e., (0.2) or (0.4) are replaced by (0.5) or (0.7)) no existence results are known; not even for small time. The nearest results in this direction are due to Duchon and Robert [3] for the one phase Stefan problem in two dimensions, whereby the heat equation $\theta_t - \Delta\theta = 0$ is replaced by the Laplace equation $\Delta\theta = 0$; they established local existence and uniqueness. Visintin [19] proved existence of a weak solution after replacing (0.5) by another condition which is an approximation to the Gibbs-Thomson law.

Added in proof. Another version of a weak solution was studied recently by Luckhaus [20]; he proved existence of a solution.

In this paper we consider the Stefan problem with small surface tension σ and linearize the problem about $\sigma = 0$. The linearized problem turns out to be a nonstandard parabolic problem. We establish existence and uniqueness of a weak solution u and then investigate the effect of u on the original shape of the free boundary of the Stefan problem with zero surface tension.

The one-phase Stefan problem is considered in §§1–7 and the two-phase Stefan problem is considered in §§8, 9.

In §1 we introduce the approximation $\theta = \theta_0 + \gamma\sigma u$ and derive a linear parabolic problem for u; θ_0 is the solution of the Stefan problem with $\sigma = 0$. In §2 we give a weak formulation and in §3 it is proved that a weak solution u exists. Uniqueness of the weak solution is established in §4. In §§6, 7 we investigate the perturbation of the free boundary of θ_0 due to the term u. Assuming that the negative sign holds in (0.6) we prove that

(0.8) small surface tension decreases the water region, for all small times.

We also prove (and this is the main result of §§6, 7) that

(0.9) small surface tension increases the water region for all large times.

More precisely, the free boundary for large times is approximately of the form $|x| = M\sqrt{t}$ $(M > 0)$ in the absence of surface tension and $|x| = M\sqrt{t} + \varepsilon M_0$ in the presence of surface tension (M and M_0 are positive constants).

Some ODE results needed in §§6, 7 are derived in §5.

In §§8, 9 we deal with the two-phase Stefan problem. In §8 we prove the existence and uniqueness of a weak solution $u = (u_1, u_2)$ such that $\theta_0 + \varepsilon u$ is an approximate solution to the two-phase Stefan problem with small surface tension. In §9 we investigate the perturbation of the free boundary due to the term u. Assuming that the water region when $\sigma = 0$ is convex, we find that (0.8) is valid for *all* times; this is radically different (for large times) from the 1-phase situation whereby (0.9) holds.

Our results on existence and uniqueness of a solution to the linearized Stefan problem about $\sigma = 0$ extend to geometries other than the one depicted in Figure 1. However the interesting conclusions on the effect of small surface tension on the free boundary (in §§6, 7 and 9) depend on the geometry of Figure 1, i.e., on the fact that the ice surrounds the water and the water surrounds the fixed boundary.

1. The approximating system

Let D be a bounded domain in $\mathbf{R}^n$ $(n \geq 2)$ with $C^{2,\alpha}$ boundary ∂D. Let G_0 be a bounded domain in $\mathbf{R}^n$ with $C^{2,\alpha}$ boundary such that $\overline{D} \subset G_0$. Set $G = G_0 \backslash \overline{D}$.

For any $T > 0$, set $\partial D_T = \partial D \times (0, T]$. Consider the one-phase Stefan problem with surface tension:

$$\theta_t - \Delta\theta = 0 \quad \text{in } G_T, \tag{1.1}$$

$$\theta = \tilde{\theta} \quad \text{on } \partial D_T \cup (G \times \{0\}), \tag{1.2}$$

$$\theta = \varepsilon\kappa \quad \text{on the free boundary } \Gamma_T, \ \varepsilon > 0, \tag{1.3}$$

$$(X_t + \nabla\theta) \cdot N = 0 \quad \text{on } \Gamma_T. \tag{1.4}$$

Here $\Gamma_T = \bigcup_{0 \leq t \leq T} \Gamma(t)$ and for each $t \in [0, T]$, the free boundary $\Gamma(t)$ (i.e., the interface) is given by $x = X(s, t)$ where s is $(n-1)$-dimensional parameter, $\kappa = \kappa(s, t)$ is the mean curvature of the surface $s \to X(s, t)$ at s, and $N = N(s, t)$ is the outward unit normal of this surface. We easily find that

$$(n-1)\kappa = \operatorname{div}_x N = \text{trace of } M(s, t)$$

where $M(s, t)$ is the matrix in $\mathbf{R}^{n \times n}$ defined by the relations

$$MN = 0, \qquad MX_{s_j} = N_{s_j} \qquad (j = 1, \dots, n-1).$$

In (1.3) $\varepsilon = \gamma\sigma$ where γ, σ are as in (0.5).

In the sequel, the initial and boundary data, designated by $\tilde{\theta}$ in (1.2), are assumed to be positive valued on $\partial D_T \cup (G \times \{0\})$, and $\tilde{\theta}(x, 0) = 0$ on ∂G_0.

The water region at time t is designated by $G(t)$, and $G_T = \bigcup_{0 \leq t \leq T} G(t)$; note that $\partial G(t) = \partial D(t) \cup \Gamma(t)$ where $\partial D(t) = \partial D \times \{t\}$.

Existence theorems for (1.1)–(1.4) have not been established up to now.

Consider next the Stefan problem with zero surface tension:

$$\theta_t - \Delta\theta = 0 \quad \text{in } G_T, \tag{1.5}$$

$$\theta = \tilde{\theta} \quad \text{on } \partial D_T \cup (G \times \{0\}), \tag{1.6}$$

$$\theta = 0 \quad \text{on the free boundary } \Gamma_T, \tag{1.7}$$

$$(z_t + \nabla\theta) \cdot N_0 = 0 \quad \text{on } \Gamma_T \tag{1.8}$$

where $\Gamma_T = \bigcup_{0 \leq t \leq T} \Gamma(t)$ and, for each t, the free boundary $\Gamma(t)$ is given by

$$s \to x = z(s, t), \qquad s = (s_1, \dots, s_{n-1}), \qquad 0 \leq s_i \leq L_i; \tag{1.9}$$

thus $z(s, t)$ is defined in the rectangle

$$L = \prod_{i=1}^{n-1} \{0 \leq s_i \leq L_i\}$$

and is L_i-periodic in s_i for each i. The vector N_0 in (1.8) is an exterior normal, and $G_T = \bigcup_{0\le t\le T} G(t)$, where $G(t)$ is the region occupied by water.

From the results of Hanzawa [11] and Meirmanov [16] it follows that a solution of (1.5)–(1.9) exists for small T and $\Gamma_T \in C^{j+\beta}$ provided

$\tilde{\theta}(x, 0)$ and ∂G_0 are sufficiently smooth, say in $C^{m_j+\beta}$, and $\tilde{\theta}(x, 0)$ satisfies the compatibility conditions of order $m_j - 2$ at ∂G_0.

As for global existence of smooth solutions, the following result was proved by Friedman and Kinderlehrer [7]:

Suppose D is a ball $\{|x| < r_0\}$ and G_0 is star-shaped with respect to any point in $\{|x| < \delta_0\}$ for some $0 < \delta_0 < r_0$. Using polar coordinates $(r, \theta_1, \dots, \theta_{n-1})$ about 0, write

$$\Delta v = r^{1-n}(r^{n-1}v_r)_r + r^{-2}Av$$

and assume that

$$(r^{n-1}\tilde{\theta}(x, 0))_r < 0, \qquad x \in \overline{G},$$
$$\tilde{\theta}(x, t) - \tilde{\theta}(x, 0) - \frac{1}{r_0^2}A\left(\int_0^t \tilde{\theta}(x, s)\,ds\right) > 0, \qquad |x| = r_0,\ 0 < t \le T.$$

Thus the solution of (1.5)–(1.9) exists and the free boundary is given by

$$r = \rho(\theta_1, \dots, \theta_{n-1}, t), \qquad 0 \le t \le T, \tag{1.10}$$

where

$$\rho \in C^\infty \quad \text{in } (\theta_1, \dots, \theta_{n-1}, t), \qquad 0 < t \le T.$$

By combining the local and global results we obtain a solution of (1.5)–(1.8) with the boundary given by (1.10) such that

$$\begin{aligned} \rho \in C^{j+\beta} \quad &\text{if } 0 \le t \le t^* \text{ for some small } t^* > 0, \\ \rho \in C^\infty \quad &\text{if } 0 < t \le T. \end{aligned} \tag{1.11}$$

We shall henceforth denote the solution to (1.5)–(1.8) by (θ_0, z); the free boundary will be denoted by Γ_T and the water region $\{\theta_0 > 0\}$ by G_T. We shall always assume that $z \in C^{3+\beta}$ for $s \in L$, $0 \le t \le T$.

We shall try to find an approximate solution to (1.1)–(1.4) of the form

$$\theta = \theta_0 + \varepsilon u,$$

with free boundary

$$x = z(s, t) + \varepsilon\zeta(s, t).$$

We shall choose an outward normal $N = N_0$ to $x = z(s, t)$ and an outward normal $N = N_0 + \varepsilon N_1$ to $x = z + \varepsilon\zeta$ as follows: N_0 has components

$$N_{0,i} = (-1)^{n+i}\begin{vmatrix} \frac{\partial z_1}{\partial s_1} & \cdots & \frac{\partial z_{i-1}}{\partial s_1} & \frac{\partial z_{i+1}}{\partial s_1} & \cdots & \frac{\partial z_n}{\partial s_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial z_1}{\partial s_{n-1}} & \cdots & \frac{\partial z_{i-1}}{\partial s_{n-1}} & \frac{\partial z_{i+1}}{\partial s_{n-1}} & \cdots & \frac{\partial z_n}{\partial s_{n-1}} \end{vmatrix}$$

and N_1 has components $N_{1,i}$ obtained from

$$\widetilde{N}_{1,i}(\varepsilon) = (-1)^{n+i} \begin{vmatrix} \frac{\partial(z_1+\varepsilon\zeta_1)}{\partial s_1} & \cdots & \frac{\partial(z_{i-1}+\varepsilon\zeta_{i-1})}{\partial s_1} & \frac{\partial(z_{i+1}+\varepsilon\zeta_{i+1})}{\partial s_1} & \cdots & \frac{\partial(z_n+\varepsilon\zeta_n)}{\partial s_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial(z_1+\varepsilon\zeta_1)}{\partial s_{n-1}} & \cdots & \frac{\partial(z_{i-1}+\varepsilon\zeta_{i-1})}{\partial s_{n-1}} & \frac{\partial(z_{i+1}+\varepsilon\zeta_{i+1})}{\partial s_{n-1}} & \cdots & \frac{\partial(z_n+\varepsilon\zeta_n)}{\partial s_{n-1}} \end{vmatrix}$$

by $N_{1,i} = \frac{d}{d\varepsilon}\widetilde{N}_{1,i}(\varepsilon)|_{\varepsilon=0}$.

One can easily deduce the structure

$$N_1 = \sum_{j=1}^{n-1} A_j \frac{\partial \zeta}{\partial s_j} \tag{1.12}$$

where A_j are matrices with elements which depend on $\partial z_i/\partial s_k$ in general. From (1.3) we get

$$\theta_0(z+\varepsilon\zeta, t) + \varepsilon u(z+\varepsilon\zeta, t) = \varepsilon\kappa + O(\varepsilon^2);$$

in view of (1.7), this gives

$$\nabla\theta_0 \cdot \zeta + u = \kappa \quad \text{on } x = z(s, t). \tag{1.13}$$

Next, from (1.4) we have

$$[z_t + \varepsilon\zeta_t + \nabla\theta_0(z+\varepsilon\zeta, t) + \varepsilon\nabla u(z+\varepsilon\zeta, t)] \cdot (N_0 + \varepsilon N_1) = O(\varepsilon^2).$$

Recalling (1.8) we get

$$\begin{aligned} &(z_t + \nabla\theta_0) \cdot N_1 + \zeta_t \cdot N_0 + \nabla u \cdot N_0 \\ &\quad + (\nabla\partial_{x_1}\theta_0 \cdot \zeta, \dots, \nabla\partial_{x_n}\theta_0 \cdot \zeta) \cdot N_0 = 0 \quad \text{on } x = z(s, t). \end{aligned} \tag{1.14}$$

In equations (1.13), (1.14), the mean curvature κ of $s \to z(s, t)$ is known, N_0 and $\nabla\theta_0$ are also known, but u and ζ are unknown (by (1.12) N_1 is known once ζ is known). We wish to eliminate ζ, so as to obtain a single relation for u on the free boundary Γ_T: $x = z(s, t)$. To do this, we write ζ in the form

$$\zeta(s, t) = R(s, t)N_0(s, t). \tag{1.15}$$

Substituting this into (1.13) we get

$$R(s, t) = \frac{\kappa - u}{\nabla\theta_0 \cdot N_0} = h_0(u - \kappa), \qquad h_0 = -\frac{1}{\nabla\theta_0 \cdot N_0} \geq m > 0, \tag{1.16}$$

where m is a constant; the positivity of h_0 follows from the maximum principle applied to θ_0 (θ_0 takes minimum at the free boundary).

Substituting ζ from (1.15) into (1.14) and recalling (1.12), we get

$$\begin{aligned} &(z_t + \nabla\theta_0) \cdot \sum A_j(R_{s_j} N_0 + RN_{0,s_j}) + N_0 \cdot R_t N_0 + N_0 \cdot RN_{0,t} \\ &\quad + \nabla u \cdot N_0 + N_0 \cdot (\nabla\partial_{x_1}\theta_0 \cdot RN_0, \dots, \nabla\partial_{x_n}\theta_0 \cdot RN_0) = 0 \end{aligned}$$

or

$$N_0 \cdot N_0 R_t + \sum (z_t + \nabla\theta_0) \cdot (A_j N_0) R_{s_j}$$
$$+ \{(z_t + \nabla\theta_0) \cdot \sum A_j N_{0,s_j} + N_0 \cdot N_{0,t}$$
$$+ N_0 \cdot (\nabla\partial_{x_1}\theta_0 \cdot N_0, \dots, \nabla\partial_{x_n}\theta_0 \cdot N_0)\} R + \nabla u \cdot N_0 = 0.$$

Substituting R from (1.16) into (1.17), we obtain

$$\begin{aligned} &N_0 \cdot N_0 \frac{d}{dt}(h_0 u - h_0\kappa) + (z_t + \nabla\theta_0) \cdot \sum A_j N_0 \frac{d}{ds_j}(h_0 u - h_0 \kappa) \\ &\quad + \{(z_t + \nabla\theta_0) \cdot \sum A_j N_{0,s_j} + N_0 \cdot N_{0,t} \\ &\qquad + N_0 \cdot (\nabla\partial_{x_1}\theta_0 \cdot N_0, \dots, \nabla\partial_{x_n}\theta_0 \cdot N_0)\}(h_0 u - h_0\kappa) \\ &\quad + \nabla u \cdot N_0 = 0 \quad \text{on } x = z(s, t). \end{aligned} \tag{1.18}$$

The function u also satisfies

$$u_t - \Delta u = 0 \quad \text{in } G_T, \tag{1.19}$$

$$u = 0 \quad \text{on } \partial D_T \text{ and on } G \times \{0\}. \tag{1.20}$$

If u is a solution to (1.18)–(1.20), then defining ζ by (1.15) where R is given by (1.16), the pair

$$\theta = \theta_0 + \varepsilon u, \qquad x = z + \varepsilon\zeta \tag{1.21}$$

will form a first order approximation to problem (1.1)–(1.4).

We wish to study the linear problem (1.18)–(1.20) and to analyze the effect of the term u on the free boundary Γ_T. Note that (1.18) can be written in the form

$$a\frac{du}{dt} + \sum b_j \frac{du}{ds_j} + N_0 \cdot \nabla u + cu = f_0 \tag{1.22}$$

where

$$\begin{aligned} a &= N_0 \cdot N_0 h_0, \qquad b_j = (z_t + \nabla\theta_0) \cdot A_j N_0 h_0, \\ c &= N_0 \cdot N_0 \frac{dh_0}{dt} + (z_t + \nabla\theta_0) \cdot \sum_j A_j N_0 \frac{dh_0}{ds_j} \\ &\quad + \left\{ (z_t + \nabla\theta_0) \cdot \sum_j A_j N_{0,s_j} + N_0 \cdot N_{0,t} \right. \\ &\qquad \left. + N_0 \cdot (\nabla\partial_{x_1}\theta_0 \cdot N_0, \dots, \nabla\partial_{x_n}\theta_0 \cdot N_0) \right\} h_0, \\ f_0 &= N_0 \cdot N_0 \frac{d}{dt}(h_0\kappa) + (z_t + \nabla\theta_0) \cdot \sum A_j N_0 \frac{d}{ds_j}(h_0\kappa) \\ &\quad + \left\{ (z_t + \nabla\theta_0) \cdot \sum A_j N_{0,s_j} + N_0 \cdot N_{0,t} \right. \\ &\qquad \left. + N_0 \cdot (\nabla\partial_{x_1}\theta_0 \cdot N_0, \dots, \nabla\partial_{x_n}\theta_0 \cdot N_0) \right\} h_0\kappa. \end{aligned} \tag{1.23}$$

In (1.22)

$$u = u(z(s, t), t) \quad \text{and} \quad \frac{du}{dt} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial z_i}{\partial t} + \frac{\partial u}{\partial t},$$

$$\frac{du}{ds_j} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial z_i}{\partial s_j}.$$

Similarly one understands the expressions dh_0/dt, dh_0/ds, etc. in (1.23).

Remark 1.1. Since $z \in C^{3+\beta}$, a belongs to $C^{2+\beta}$, b_j belongs to $C^{1+\beta}$, and c, f_0 belong to C^{β}. For the results of §§2–4 it actually suffices that c, f_0 are continuous functions and a, b_j are continuously differentiable.

2. Definition of weak solution

In §1 we derived for $u = [\partial\theta/\partial\varepsilon]_{\varepsilon=0}$ the parabolic system

(2.1) $$u_t - \Delta u = 0 \quad \text{in } G_T,$$

(2.2) $$u = 0 \quad \text{on } \partial D_T \text{ and on } G \times \{0\},$$

(2.3) $$a\frac{du}{dt} + \sum_{j=1}^{n-1} b_j \frac{du}{ds_j} + N_0 \cdot \nabla u + cu = f_0 \quad \text{on } \Gamma_T$$

where a, b_j, c, f_0 are defined in (1.23);

(2.4) $$a \geq a_0 > 0 \qquad (a_0 \text{ constant}).$$

The boundary condition (2.3) is nonstandard. To prove existence (and uniqueness) we shall resort to working with a weak formulation of (2.1)–(2.3).

Let $\varphi(x, t)$ be any smooth function such that $\varphi = 0$ on ∂D_T and on $G(T)$. Formally,

$$0 = \int_{G_T} \varphi(u_t - \Delta u) = \int_0^T \int_{G(t)} \varphi(u_t - \Delta u)\, dx\, dt$$
$$= \int_{G(T)} dx \int_{t(x)}^T \varphi u_t\, dt - \int_0^T \int_{\Gamma(t)} \varphi u_\nu\, d\sigma_t\, dt + \int_0^T \int_{G(t)} \nabla\varphi \cdot \nabla u$$

where ν is the outward unit normal and $t(x) = \min\{t_0; (x, t_0) \in G_T\}$ (recall that the sets $G(t)$ increase with t). Observe that by definition of the surface area $d\sigma_t$, on $\Gamma(t)$, $d\sigma_t = \|N_0\|\, ds$. Hence we get

$$0 = -\int_{G(T)} dx \int_{t(x)}^T \varphi_t u - \int_{G(T)\setminus G(0)} (\varphi u)(x, t(x))\, dx$$
$$+ \int_{G_T} \nabla\varphi \cdot \nabla u - \int_0^T \int_L \varphi u_\nu \|N_0\|\, ds\, dt$$

and $u_\nu \|N_0\| = \nabla u \cdot N_0$. Also

$$\int_{G(T)\setminus G(0)} (\varphi u)(x, t(x))\, dx = \int_0^T \int_L (\varphi u)(z(s, t), t) \left|\frac{\partial z(s, t)}{\partial(s, t)}\right| ds\, dt$$

by the change of variables $x \to z(s, t)$. It follows that

$$(2.5)\quad \begin{aligned} &-\int_{G_T} u\varphi_t + \int_{G_T} \nabla\varphi \cdot \nabla u - \int_0^T \int_L u(z(s,t),t)\varphi(z(s,t),t)\left|\frac{\partial z(s,t)}{\partial(s,t)}\right| \\ &\quad - \int_0^T \int_L \varphi(z(s,t),t)\nabla u \cdot N_0\, ds\, dt = 0. \end{aligned}$$

In the last integral we substitute $\nabla u \cdot N_0$ from (2.3) and integrate by parts in the integrals

$$\iint a\frac{du}{dt}\varphi, \qquad \iint b_j \frac{du}{ds_j}\varphi.$$

We then obtain from (2.5)

$$(2.6)\quad \begin{aligned} &-\int_{G_T} u\varphi_t + \int_{G_T} \nabla\varphi \cdot \nabla u \\ &\quad - \int_0^T \int_L u(z(s,t),t)\left\{\frac{d}{dt}(a\varphi) + \sum \frac{d}{ds_j}(b_j\varphi) - c\varphi + \left|\frac{\partial z(s,t)}{\partial(s,t)}\right|\varphi\right\} ds\, dt \\ &= \int_0^T \int_L \varphi f_0\, ds\, dt. \end{aligned}$$

Set

$$\mathscr{A}_T = \{\varphi \in C^1(\overline{G_T}),\ \varphi = 0 \text{ on } G(T) \cup \partial D_T\}.$$

Definition 2.1. A function u is a *weak solution* of (2.1)–(2.3) if

$$(2.7)\qquad u,\ \nabla u \in L^2(G_T),$$

(2.6) holds for any $\varphi \in \mathscr{A}_T$, and $u = 0$ on ∂D_T and on $G \times \{0\}$ in the usual continuous sense.

Notice that, by (2.7), $u \in L^2(\Gamma_T)$.

Notice also that (2.6) implies that (2.1) holds in G_T and therefore u is a smooth function away from the free boundary Γ_T.

3. Existence of weak solution

The existence of a weak solution depends upon an energy inequality. We first proceed to derive this inequality in a formal manner, assuming that u is smooth up to Γ_T (and thus it satisfies (2.3)). Multiplying (2.1) by u and integrating over $G_\tau \equiv G_T \cap \{t \le \tau\}$ $(\tau \in (0, T))$ we get

$$\begin{aligned}
0 &= 2\int_{G_\tau} u(u_t - \Delta u) = \int_{G(\tau)} \int_{t(x)}^{\tau} (u^2)_t \, dt\, dx \\
&\quad + 2\int_{G_\tau} |\nabla u|^2 - 2\int_0^\tau \int_{\partial G(t)} u u_\nu \, d\sigma_t \, dt \\
&= \int_{G(\tau)} u^2(x, \tau)\, dx - \int_{G(\tau)\setminus G(0)} u^2(x, t(x))\, dx + 2\int_{G_\tau} |\nabla u|^2 \\
&\quad - 2\int_0^\tau \int_L u(\nabla u \cdot N_0)\, ds\, dt \\
&= \int_{G(\tau)} u^2(x, \tau)\, dx + 2\int_{G_\tau} |\nabla u|^2 - \int_0^t \int_L u^2 \left| \frac{\partial z}{\partial(s, t)} \right|^2 \\
&\quad + 2\int_0^t \int_L u \left(a\frac{du}{dt} + \sum b_j \frac{du}{ds_j} + cu - f_0 \right)
\end{aligned}$$

by (2.3). Hence, after integration by parts in the last integral,

$$\begin{aligned}
&\int_{G(\tau)} u^2 + 2\int_{G_\tau} |\nabla u|^2 + \int_L a(z(s, \tau), \tau) u^2(z(s, \tau), \tau)\, ds \\
&\quad = \int_0^\tau \int_L u^2 \left[\left| \frac{\partial z}{\partial(s, t)} \right| + \frac{da}{dt} + \sum_{j=1}^{n-1} \frac{db_j}{ds_j} - 2c \right] ds\, dt \qquad (3.1)\\
&\quad + \int_0^\tau \int_L u f_0 \, ds\, dt
\end{aligned}$$

where the argument in each function in the integrands on the right-hand side is $(z(s, t), t)$. Since the right-hand side of (3.1) is bounded by

$$C\int_0^\tau \int_L u^2 + \int_0^\tau \int_L f_0^2 ,$$

we obtain, after using Gronwall's inequality, the desired energy inequality

$$\begin{aligned}
&\sup_{0<t<T} \int_{G(t)} u^2(x, t)\, dx + \int_{G_T} |\nabla u|^2 \, dx\, dt \\
&\quad + \sup_{0<t<T} \int_L u^2(z(s, t), t)\, ds \le C_T .
\end{aligned} \qquad (3.2)$$

The strategy for constructing a weak solution is to first work with a finite-difference scheme, establish existence and an energy-type inequality analogous to (3.2), and then pass to the limit.

We shall use the finite differences

$$u_t^-(x, k) = \frac{1}{h}(u(x, kh) - u(x, (k-1)h)) \quad \text{(backward)},$$

$$u_t(x, k) = \frac{1}{h}(u(x, (k+1)h) - u(x, kh)) \quad \text{(forward)}$$

where h is any positive number. We introduce a finite-difference version of (2.1)–(2.3):

$$u_{\bar{t}} - \Delta u = 0 \quad \text{in } G(kh) \equiv G^k, \tag{3.3}$$

$$\begin{aligned} &a\frac{1}{h}[u(z(s, kh), kh) - u(z(s, (k-1)h), (k-1)h)] \\ &\quad + \sum b_j \frac{du}{ds_j} + N_0 \cdot \nabla u + cu = f_0 \quad \text{on } \Gamma(kh) \equiv \Gamma^k, \end{aligned} \tag{3.4}$$

$$u = 0 \quad \text{on } \partial D(kh) \equiv \partial D^k \tag{3.5}$$

for $k = 1, \ldots, m$, where $T - h < mh \leq T$. If we set

$$u^k = u(x, kh), \qquad a^k = a(x, kh), \text{ etc.,}$$

then (3.3)–(3.5) read

$$\frac{1}{h}u^k - \Delta u^k = \frac{1}{h}u^{k-1} \quad \text{in } G^k, \tag{3.3$_k$}$$

$$\frac{1}{h}a^k u^k + N_0^k \cdot \nabla u^k + \sum b_j^k \frac{du^k}{ds_j} + c^k u^k = \frac{1}{h}a^k u^{k-1} + f_0^k \quad \text{on } \Gamma^k, \tag{3.3$_k$}$$

$$u^k = 0 \quad \text{on } \partial D^k. \tag{3.5$_k$}$$

Since u^{k-1} is defined only in G^{k-1} and $G^{k-1} \subset G^k$, before we can study the system (3.3_k)–(3.5_k) we must extend the definition of u^{k-1} into all of $G^k \backslash G^{k-1}$. We define this extension using the boundary values of u^{k-1}:

$$u^{k-1}(z(s, t)) = u^{k-1}(z(s, (k-1)h), (k-1)h) \qquad \text{for } s \in L,\ (k-1)h < t < kh. \tag{3.6}$$

Thus, for each $s \in L$, $u^{k-1}(x)$ is constant along the curve $t \to z(s, t)$ passing through $(x, t(x))$.

The system (3.3_k)–(3.5_k) is now a well defined elliptic problem for u^k. Since the boundary condition (3.4_k) has the form

$$\alpha_0 \frac{\partial u}{\partial \nu} + \sum \alpha_j \frac{\partial u^k}{\partial s_j} + \beta u^k = \gamma$$

where ν is the outward unit normal and α_0, β are positive functions, (if h is sufficiently small), the system (3.3_k)–(3.5_k) (or equivalently (3.3)–(3.5)) has a unique solution with $u(x, 0) = 0$.

We proceed to derive an energy inequality for the u^k's, analogous to (3.2). We shall need the identity [13, p. 246]

$$2\alpha_r u_r(u_r - u_{r-1}) = \alpha_r u_r^2 - \alpha_{r-1}u_{r-1}^2 - (\alpha_r - \alpha_{r-1})u_{r-1}^2 + \alpha_r(u_r - u_{r-1})^2. \tag{3.7}$$

Taking $\alpha_r = a^r$ we get

$$2h \sum_{k=1}^{k_0} a^k u^k u_t^-(k) = a^{k_0}(u^{k_0})^2 - a^0(u^0)^2 - h \sum_{k=0}^{k_0-1} a_t(k)(u^k)^2 + h^2 \sum_{k=1}^{k_0} a^k (u_t^-(k))^2 \tag{3.8}$$

where $u_t^-(k)$ is the function $u_t^-(x, k)$, and $a_t(k)$ is $a_t(x, k)$.

Set

$$\lambda(x) = \min\{j ; x \in G^j\}$$

Multiplying (3.3_k) by u^k, integrating over G^k and then summing over k, $1 \le k \le k_0$, we get

$$\begin{aligned} 0 &= 2h \sum_{k=1}^{k_0} \int_{G^k} u_t^- u - 2h \sum_{k=1}^{k_0} \int_{G^k} u\Delta u \\ &= \int_{G^{k_0}} 2h \sum_{k=\lambda(x)}^{k_0} u_t^- u - 2h \sum_{k=1}^{k_0} \int_{G^k} u\Delta u \quad (\text{by (3.8) with } a \equiv 1) \\ &= \int_{G^{k_0}} u^2(x, k_0 h) - \int_{G^{k_0}} u^2(x, (\lambda(x) - 1)h) + h^2 \int_{G^{k_0}} \sum_{k=\lambda(x)}^{k_0} (u_t^-)^2 \\ &\quad - 2h \sum_{k=1}^{k_0} \int_{\partial G^k} u \frac{\partial u}{\partial \nu} + 2h \sum_{k=1}^{k_0} \int_{G^k} |\nabla u|^2 . \end{aligned} \tag{3.9}$$

The second integral on the right-hand side can be written in the form

$$\begin{aligned} &\sum_{k=1}^{k_0} \int_{G^k \setminus G^{k-1}} u^2(x, (k-1)h)\, dx \\ &\quad = \sum_{k=1}^{k_0} \int_{(k-1)h}^{kh} \int_L u^2(z(s, (k-1)h), (k-1)h) \left| \frac{\partial z(s, t)}{\partial (s, t)} \right| ds\, dt \\ &\qquad\qquad (\text{by (3.6) and change of variables}). \end{aligned}$$

Also,

$$\int_{\partial G^k} u \frac{\partial u}{\partial \nu} = \int_L u^k \nabla u^k \cdot N_0^k \, ds ,$$

and in the last integrand we can substitute $\nabla u^k \cdot N_0^k$ from (3.4_k). Using these remarks, we can transform the right-hand side of (3.9) to obtain

$$
\begin{aligned}
0 = & \int_{G^{k_0}} u^2(x, k_0 h) + 2h \sum_{k=1}^{k_0} \int_{G^k} |\nabla u|^2 + h^2 \int_{G^{k_0}} \sum_{k=\lambda(x)}^{k_0} (u_t^-)^2 \\
& - \sum_{k=1}^{k_0} \int_L [u^k(z(s, (k-1)h))]^2 \int_{(k-1)h}^{kh} \left| \frac{\partial z(s, t)}{\partial(s, t)} \right| ds\, dt \\
(3.10) \qquad & + \int_L au^2(z(s, t), t)\, ds \Big|_{t=0}^{t=k_0 h} \\
& - h \sum_{k=0}^{k_0-1} \int_L a_t u^2\, ds + h^2 \sum_{k=1}^{k_0} \int_L (u_t^-)^2 a\, ds \\
& - h \sum_{k=1}^{k_0} \int_L u^2 \sum \frac{db_j}{ds_j} + 2h \sum_{k=1}^{k_0} \int_L cu^2\, ds - 2h \sum_{k=1}^{k_0} \int_L uf_0 .
\end{aligned}
$$

Notice that in the last five integrals the integrand u is evaluated at $z(s, kh)$; the same applies to a, b_j, c, f_0 and the finite difference a_t. In deriving (3.10) we have used (3.8) in order to "integrate by parts" the expression

$$\sum_{k=1}^{k_0} \int_L auu_t^-\, ds .$$

From (3.10) we get

$$(3.11) \qquad \int_{G^{k_0}} u^2(x, k_0 h) + \int_L au^2(z(s, k_0 h), k_0 h)\, ds + 2h \sum_{k=1}^{k_0} \int_{G^k} |\nabla u|^2 \le -J$$

where J represents all the remaining terms in (3.10) with the exception of

$$h^2 \int_{G^{k_0}} \sum (u_t^-)^2 + h^2 \sum \int_L (u_t^-)^2 a$$

which have been dropped. It is easily seen that

$$|J| \le Ch \sum_{k=1}^{k_0} \int_L u^2(z(s, kh), kh)\, ds + h \sum_{k=1}^{k_0} \int_L f_0^2 .$$

Substituting this into (3.11) and using Gronwall's inequality, we find that

$$
\begin{aligned}
& \sup_{1 \le k_0 \le m} \int_{G^{k_0}} u^2(x, k_0 h)\, dx + h \sum_{k=1}^{m} \int_{G^k} |\nabla u|^2\, dx \\
(3.12) \qquad & + \sup_{1 \le k_0 \le m} \int_L u^2(z(s, k_0 h), k_0 h)\, ds \le C_T
\end{aligned}
$$

where C_T is a constant independent of h. This inequality is analogous to (3.2).

Define a function u^h by

$$(3.13) \qquad u^h(x, t) = u(x, kh) \quad \text{for } x \in G^{k+1}, \quad t \in [kh, (k+1)h).$$

Denote by W the completion of the set of smooth functions ψ which vanish on $G \times \{0\}$ and on ∂D_T with the respect to the norm

$$\|\psi\| = \int_{G_T} (\psi^2 + |\nabla_x \psi|^2); \tag{3.14}$$

W is a Hilbert space.

From (3.12) we conclude that there exists a sequence u^h with $h = h_j \to 0$ which is weakly convergent to a function u in W; we can choose the h's so that $T/h = m$, m an integer

Theorem 3.1. *The function* u *is a weak solution of* (2.1)–(2.3).

Proof. It suffices to show that (2.6) is satisfied for any $\varphi \in \mathscr{A}_T$. Indeed, this will imply that (2.1) holds in the usual sense away from Γ_T. Since further $u = 0$ on $(G \times \{0\}) \cup \partial D_T$ in some weak sense, it follows by standard parabolic theory that $u = 0$ on $(G \times \{0\}) \cup \partial D_T$ in the usual continuous sense.

To prove (2.6) we multiply (3.3_k) by φ, integrate over G^k, and sum over k:

$$\begin{aligned} 0 &= h\sum_{k=1}^{m}\int_{G^k}(u^h)_t^-(x, kh)\varphi(x, kh) - h\sum_{k=1}^{m}\int_{G^k}\Delta u^h(x, kh)\varphi(x, kh) \\ &= h\int_{G^m}\sum_{k=\lambda(x)}^{m}(u^h)_t^-(x, kh)\varphi(x, kh) + h\sum_{k=1}^{m}\int_{G^k}\nabla u^h\cdot\nabla\varphi \\ &\quad - h\sum_{k=1}^{m}\int_L(\nabla u^h\cdot N_0^k)\varphi\, ds. \end{aligned} \tag{3.15}$$

Breaking the first integral on the right-hand side into m integrals taken over the sets $\lambda(x) = j$ $(0 \le j \le m)$, we get

$$\begin{aligned} h\int_{G^m}\sum_{k=\lambda(x)}^{m}(u^h)_t^-\varphi &= h\sum_{j=1}^{m}\int_{G^j\setminus G^{j-1}}\sum_{k=j}^{m}(u^h)_t^-\varphi + h\int_{G^0}\sum_{k=1}^{m}(u^h)_t^-\varphi \\ &= \sum_{j=1}^{m}\int_{G^j\setminus G^{j-1}}\Bigg\{[u(x, mh)\varphi(x, mh) - u(x, (j-1)h)\varphi(x, (j-1)h)] \\ &\qquad\qquad - h\sum_{k=j-1}^{m-1}u(x, kh)\varphi_t(x, kh)\Bigg\}\, dx \\ &\quad - h\int_{G^0}\sum_{k=0}^{m-1}u(x, kh)\varphi_t(x, kh)\, dx \\ &= -\sum_{j=1}^{m}\int_{(j-1)h}^{jh}\int_L u(z, (j-1)h)\varphi(z, (j-1)h)\left|\frac{\partial z(s, t)}{\partial(s, t)}\right| ds\, dt \\ &= -h\sum_{k=1}^{m-1}\int_{G^{k+1}}u(x, kh)\varphi_t(x, kh)\, dx, \end{aligned}$$

since $\varphi(x, mh) = \varphi(x, T) = 0$, $u(x, 0) = 0$; here we used (3.13). Using the last computation in (3.15), we obtain

$$\sum_{j=1}^{m}\int_{(j-1)h}^{jh}\int_L u(z, (j-1)h)\varphi(z, (j-1)h)\left|\frac{\partial z(s, t)}{\partial(s, t)}\right| ds\, dt$$
$$+h\sum_{k=1}^{m-1}\int_{G^{k+1}} u(x, kh)\varphi_t(x, kh) - h\sum_{k=1}^{m}\int_{G^k}\nabla u^h\cdot\nabla\varphi \tag{3.16}$$
$$+h\sum_{k=1}^{m}\int_L(\nabla u^h\cdot N_0^k)\varphi(z, kh)\, ds = 0.$$

Since $\int_0^T\int_L|u^h|\le C$, the first sum on the left-hand side of (3.16) is equal to

$$\int_0^T\int_L u^h(z(s, t), t)\varphi(z(s, t), t)\left|\frac{\partial z(s, t)}{\partial(s, t)}\right| + O(h).$$

Next, for any smooth function η,

$$\int_{G_T} u^h\eta = \sum_{j=1}^{m-1}\int_{jh}^{(j+1)h}dt\int_{G(t)}u^h\eta = h\sum_{j=1}^{m-1}\int_{G_{j+1}}u^h(x, jh)\eta(x, jh)$$
$$-\sum_{j=1}^{m-1}\int_{jh}^{(j+1)h}dt\int_t^{(j+1)h}\int_L u^h(z, jh)\eta(z, jh)\left|\frac{\partial z(s, \tau)}{\partial(s, \tau)}\right| ds\, d\tau$$
$$+\sum_{j=1}^{m-1}\int_{jh}^{(j+1)h}dt\int_{G(t)}u^h(x, jh)[\eta(x, t) - \eta(x, jh)]\, dx$$

and the last two sums are $O(h)$ since $\int_0^T\int_L|u^h|\le C$.

Next we obtain

$$h\sum_{k=1}^{m}\int_{G^k}\nabla u^h\cdot\nabla\varphi = \int_{G_T}\nabla u^h\cdot\nabla\varphi + O(h)$$

since $\int_{G_T}|\nabla u^h|\le C$. Finally, to evaluate the last term on the left-hand side of (3.16) we use (3.4) and perform "integration by parts" in the t-variable. This leads to

$$\int_0^T\int_L u^h(z(s, t), t)\left\{\frac{d}{dt}(a\varphi) + \sum\frac{d}{ds_j}(b_j\varphi) - c\varphi\right\}$$
$$+\int_0^T\int_L\varphi f_0\, ds\, dt + O(h).$$

Taking $h\to 0$ in (3.16) and using the above estimates (with $\eta = \varphi_t$) we obtain the relation (2.6). This completes the proof of Theorem 3.1. □

Remark 3.1. Notice that the u_h's satisfy

$$\sup_{0<t<T}\left\{\int_{G(t)}(u^h)^2(x, t)\, dx + \int_{\Gamma(t)}(u^h)^2(x', t)\, d\sigma_t(x')\right\}\le C_T$$

with C_T independent of h. Thus we may conclude that

$$\sup_{0<t<T}\left\{\int_{G(t)} u^2(x\,,\,t) + \int_{\Gamma(t)} u^2(x'\,,\,t)\,d\sigma_t(x')\right\} \le C_T\,. \tag{3.17}$$

Remark 3.2. Multiplying (2.1) by u^{2m-1} (m positive integer) and integrating over G_τ we can formally obtain an energy inequality analogous to (3.2), from which we deduce that

$$\sup_{0<t<T}\left\{\int_{G(t)} u^{2m}(x\,,\,t)\,dx + \int_{\Gamma(t)} u^{2m}(x'\,,\,t)\,d\sigma_t(x')\right\} \le C_{T\,,\,m}\,. \tag{3.18}$$

This can also be proved rigorously by finite differences.

4. Uniqueness

Theorem 4.1. *The weak solution of* (2.1)–(2.3) *is unique.*

Proof. Suppose $u_1\,,\,u_2$ are two solutions and let $u = u_1 - u_2$. From (2.6) we deduce, for every $\eta \in \mathscr{A}_T$,

$$\begin{aligned} &-\int_{G_T} u\eta_t + \int_{G_T} \nabla u\cdot\nabla\eta \\ &-\int_0^T\int_L u\left\{\frac{d}{dt}(a\eta) + \sum\frac{d}{ds_j}(b_j\eta) - c\eta + \left|\frac{\partial z}{\partial(s\,,\,t)}\right|\eta\right\}ds\,dt = 0\,. \end{aligned} \tag{4.1}$$

Let $\tau \in (0\,,\,T)$ and define

$$\eta(x\,,\,t) = \begin{cases} -\int_\tau^t u(x\,,\,t')\,dt' & \text{if } t<\tau\,,\\ 0 \quad \text{if } t>\tau\,. \end{cases} \tag{4.2}$$

Then $\eta = 0$ on $\partial D_T \cup G(T)$ and

$$\nabla\eta(x\,,\,t) = \begin{cases} -\int_\tau^t \nabla u(x\,,\,t')\,dt' & \text{if } t<\tau\,,\\ 0 \quad \text{if } t>\tau\,, \end{cases} \tag{4.3}$$

$$\eta_t = \begin{cases} -u & \text{if } t<\tau\,,\\ 0 & \text{if } t>\tau\,. \end{cases} \tag{4.4}$$

Notice that $\eta\,,\,\nabla\eta\,,\,\eta_t\,,\,\nabla\eta_t \in L^2(G_T)$ and therefore $\eta\,,\,\eta_t\,,\,\nabla\eta \in L^2(\Gamma_T)$.
Now let φ_n be a sequence of smooth functions satisfying:

$$\varphi_n = 0 \quad \text{on } \partial D_T\,,$$
$$\varphi_n = 0 \quad \text{for } t>\tau$$

and

$$\varphi_n \to \eta_t \quad \text{in } W$$

where the norm in W is given by (3.14) (in particular, $\varphi_n \to \eta_t$ in $L^2(\Gamma_T)$).
Then, if

$$\Phi_n(x\,,\,t) = -\int_T^t \varphi_n(x\,,\,t')\,dt'$$

we have $\Phi_n \in \mathscr{A}_T$ so that, by (4.1),

$$-\int_{G_T} u\Phi_{nt} + \int_{G_T} \nabla u \cdot \nabla \Phi_n$$
$$-\int_0^T \int_L u \left\{ \frac{d}{dt}(a\Phi_n) + \sum \frac{d}{ds_j}(b_j\Phi_n) - c\Phi_n + \left| \frac{\partial z}{\partial(s, t)} \right| \Phi_n \right\} = 0.$$

But since $\Phi_n \to \eta$ and $\Phi_{nt} \to \eta_t$ in W, if we let $n \to \infty$ in the above relation we conclude that (4.1) holds also for η.

From (4.1) we get

$$\int_{G_\tau} \eta_t^2 - \int_{G_\tau} \nabla\eta \cdot \nabla\eta_t + \int_0^\tau \int_L \eta_t(z, t) \left\{ a\frac{d\eta}{dt} + \sum b_j \frac{d\eta}{ds_j} + \gamma\eta \right\} = 0 \tag{4.5}$$

where

$$\gamma = \left| \frac{\partial z(s, t)}{\partial(s, t)} \right| + \frac{da}{dt} + \sum \frac{db_j}{ds_j} - c.$$

We next wish to evaluate the expression

$$\begin{aligned} &a\frac{d\eta}{dt} + \sum_{j=1}^{n-1} b_j \frac{d\eta}{ds_j} \\ &\quad = a\eta_t + h_0 \left\{ \|N_0\|^2 \nabla\eta \cdot z_t + \sum_{j=1}^{n-1} (z_t + \nabla\theta_0) \cdot A_j N_0 (\nabla\eta \cdot z_{s_j}) \right\}. \end{aligned} \tag{4.6}$$

Lemma 4.2.

$$(A_j N_0) \cdot N_0 = 0, \tag{4.7}$$

$$(A_j N_0) \cdot z_{s_i} = -\delta_{ij} \|N_0\|^2. \tag{4.8}$$

Assuming the lemma for the moment, we claim that

$$z_t = \frac{z_t \cdot N_0}{\|N_0\|^2} N_0 - \frac{1}{\|N_0\|^2} \sum_{j=1}^{n-1} (z_t \cdot A_j N_0) z_{s_j}. \tag{4.9}$$

Indeed, for $z_t = N_0$ this follows from (4.7) whereas for $z_t = z_{s_i}$ this follows from (4.7) and (4.8). Since the vectors $N_0, z_{s_1}, \dots, z_{s_{n-1}}$ span the entire space $\mathbf{R}^n$, (4.9) holds for any z_t.

From (4.7) it follows that $\nabla\theta_0 \cdot A_j N_0 = 0$ (since $\nabla\theta_0$ is parallel to N_0). Therefore, from (4.6),

$$\begin{aligned} a\frac{d\eta}{dt} + \sum b_j \frac{d\eta}{ds_j} &= a\eta_t + h_0 \|N_0\|^2 \nabla\eta \cdot \left\{ z_t + \sum_{j=1}^{n-1} (z_t \cdot A_j N_0) z_{s_j} \|N_0\|^{-2} \right\} \\ &= a\eta_t + h_0 \|N_0\|^2 \nabla\eta \cdot \frac{z_t \cdot N_0}{\|N_0\|^2} N_0 = a\eta_t + \nabla\eta \cdot N_0 \end{aligned} \tag{4.10}$$

since

$$z_t \cdot N_0 = -\nabla\theta_0 \cdot N_0 = \frac{1}{h_0}.$$

The absence of tangential derivatives of η on the right-hand side of (4.10) is crucial for the proof of uniqueness.

Substituting (4.10) into (4.5) results in

$$\int_{G_\tau} \eta_t^2 - \int_{G_\tau} \nabla\eta \cdot \nabla\eta_t + \int_0^\tau \int_L (a\eta_t^2 + \nabla\eta \cdot N_0 \eta_t + \gamma\eta\eta_t) = 0.$$

By integration by parts,

$$\begin{aligned}-\int_{G_\tau} \nabla\eta \cdot \nabla\eta_t = &-\frac{1}{2}\int_{G(\tau)} |\nabla\eta|^2 + \frac{1}{2}\int_{G(0)} |\nabla\eta|^2(x, 0)\,dx \\ &+ \frac{1}{2}\int_{G(\tau)\setminus G(0)} |\nabla\eta|^2(x, t(x))\,dx\end{aligned}$$

and $\nabla\eta = 0$ on $G(\tau)$. In the last integral we change variables $x = (z(s, t), t)$ to get

$$\int_0^\tau \int_L |\nabla\eta|^2(z(s, t), t)(-\nabla\theta_0 \cdot N_0)\,ds\,dt,$$

since

$$\left|\frac{\partial z(s, t)}{\partial(s, t)}\right| = \left|\det \begin{vmatrix} z_{1,t} & z_{1,s_1} & \cdots & z_{1,s_{n-1}} \\ z_{2,t} & z_{2,s_1} & \cdots & z_{2,s_{n-1}} \\ \vdots & \vdots & & \vdots \\ z_{n,t} & z_{n,s_1} & \cdots & z_{n,s_{n-1}} \end{vmatrix}\right| = z_t \cdot N_0 = -\nabla\theta_0 \cdot N_0.$$

It follows that

$$\begin{aligned}&\int_{G_\tau} \eta_t^2 + \int_0^\tau \int_L \left[\frac{1}{2}|\nabla\eta|^2(-\nabla\theta_0 \cdot N_0) + \frac{|N_0|^2}{(-\nabla\theta_0 \cdot N_0)}\eta_t^2 + \nabla\eta \cdot N_0 \eta_t\right] \\ &\quad \le \int_0^\tau \int_L \left(\varepsilon\eta_t^2 + \frac{C}{\varepsilon}\eta^2\right)\end{aligned} \tag{4.11}$$

for any $\varepsilon > 0$.

By the Cauchy inequality, the expression in brackets in the integrand on the left-hand side of (4.11) is bigger than

$$\frac{1}{6}|\nabla\eta|^2(-\nabla\theta_0 \cdot N_0) + \frac{1}{8}\eta_t^2 \frac{|N_0|^2}{(-\nabla\theta_0 \cdot N_0)}.$$

Choosing ε small we get

$$\int_{G_\tau} \eta_t^2 + \int_0^\tau \int_L \eta_t^2 \le C_\varepsilon \int_0^\tau \int_L \eta^2,$$

and since

$$\eta(x, t) = \int_\tau^t \eta_t$$

we deduce that $\eta = 0$ if τ is small enough. This proves uniqueness for $t < \tau$. We can now proceed step-by-step to prove uniqueness for all $t < T$.

Proof of Lemma 4.2. We first consider a special case.

Lemma 4.3. *Suppose that*

$$\frac{\partial z_i}{\partial s_j}(s, t) = \delta_{ij}, \qquad 1 \le i, j \le n-1,$$
$$\frac{\partial z_n}{\partial s_j}(s, t) = \gamma_j, \qquad 1 \le j \le n-1.$$

Then

(a)

$$N_{1i} = \sum_{\substack{j=1 \\ j \neq i}}^{n-1} \left(\frac{\partial \zeta_j}{\partial s_i}\gamma_j - \frac{\partial \zeta_j}{\partial s_j}\gamma_i \right) - \frac{\partial \zeta_n}{\partial s_i}, \qquad 1 \le i \le n-1,$$

$$N_{1n} = \sum_{j=1}^{n-1} \frac{\partial \zeta_j}{\partial s_j},$$

(b)

$$A_j = (a_{ik}^j) = \begin{cases} -\gamma_i & \text{if } k = j,\ 1 \le i \le n-1,\ i \neq j, \\ \gamma_k & \text{if } i = j,\ 1 \le k \le n-1,\ k \neq i, \\ -1 & \text{if } i = j,\ k = n, \\ 1 & \text{if } i = n, k = j, \\ 0 & \text{otherwise} \end{cases}$$

for $1 \le j \le n-1$;

(c) *For* $1 \le j \le n-1$,

$$A_j N_0 \cdot N_0 = 0, \quad A_j N_0 \cdot z_{s_i} = -\delta_{ij}\|N_0\|^2 \qquad (1 \le i \le n-1),$$

where

$$N_0 = \begin{pmatrix} -\gamma_1 \\ -\gamma_2 \\ \vdots \\ -\gamma_{n-1} \\ 1 \end{pmatrix}.$$

Proof. Recall that N_1 was defined as

$$N_{1i} = (-1)^{n+i} \frac{d}{d\varepsilon} \det \left(\frac{\partial(z_1 + \varepsilon\zeta_1)}{\partial s}, \dots, \left(\frac{\partial(z_i + \varepsilon\zeta_i)}{\partial s}\right)^{\wedge}, \dots, \frac{\partial(z_n + \varepsilon\zeta_n)}{\partial s} \right) \Bigg|_{\varepsilon=0}$$

where $1 \le i \le n$ and

$$\left(\frac{\partial(z_1 + \varepsilon\zeta_1)}{\partial s}, \dots, \left(\frac{\partial(z_i + \varepsilon\zeta_i)}{\partial s}\right)^{\wedge}, \dots, \frac{\partial(z_n + \varepsilon\zeta_n)}{\partial s} \right)$$

denotes the matrix whose columns are

$$\begin{pmatrix} \frac{\partial(z_1+\varepsilon\zeta_1)}{\partial s_1} \\ \vdots \\ \frac{\partial(z_1+\varepsilon\zeta_1)}{\partial s_{n-1}} \end{pmatrix}, \dots, \begin{pmatrix} \frac{\partial(z_{i-1}+\varepsilon\zeta_{i-1})}{\partial s_1} \\ \vdots \\ \frac{\partial(z_{i-1}+\varepsilon\zeta_{i-1})}{\partial s_{n-1}} \end{pmatrix},$$
$$\begin{pmatrix} \frac{\partial(z_{i+1}+\varepsilon\zeta_{i+1})}{\partial s_1} \\ \vdots \\ \frac{\partial(z_{i+1}+\varepsilon\zeta_{i+1})}{\partial s_{n-1}} \end{pmatrix}, \dots, \begin{pmatrix} \frac{\partial(z_n+\varepsilon\zeta_n)}{\partial s_1} \\ \vdots \\ \frac{\partial(z_n+\varepsilon\zeta_n)}{\partial s_{n-1}} \end{pmatrix}.$$

Thus

$$N_{1i} = (-1)^{n+i} \sum_{\substack{j=1 \\ j\neq i}}^{n} \det\left(\frac{\partial z_1}{\partial s}, \dots, \left(\frac{\partial z_i}{\partial s}\right)^{\wedge}, \dots, \frac{\partial z_{j-1}}{\partial s}, \frac{\partial \zeta_j}{\partial s}, \frac{\partial z_{j+1}}{\partial s}, \dots, \frac{\partial z_n}{\partial s}\right).$$

Let

$$B_{ij} = \left(\frac{\partial z_1}{\partial s}, \dots, \left(\frac{\partial z_i}{\partial s}\right)^{\wedge}, \dots, \frac{\partial z_{j-1}}{\partial s}, \frac{\partial \zeta_j}{\partial s}, \frac{\partial z_{j+1}}{\partial s}, \dots, \frac{\partial z_n}{\partial s}\right) \qquad (i \neq j).$$

Then, if $1 \leq i < j \leq n-1$,

$$B_{ij} = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{\partial\zeta_j}{\partial s_1} & 0 & \cdots & 0 & \gamma_1 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{\partial\zeta_j}{\partial s_2} & 0 & \cdots & 0 & \gamma_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \frac{\partial\zeta_j}{\partial s_{i-1}} & 0 & \cdots & 0 & \gamma_{i-1} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{\partial\zeta_j}{\partial s_i} & 0 & \cdots & 0 & \gamma_i \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \frac{\partial\zeta_j}{\partial s_{i+1}} & 0 & \cdots & 0 & \gamma_{i+1} \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & & \cdots & & \cdots \\
\vdots & \vdots & & & & & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & & \cdots & & \cdots & 1 & \frac{\partial\zeta_j}{\partial s_{j-1}} & 0 & \cdots & 0 & \gamma_{j-1} \\
0 & 0 & \cdots & & \cdots & & \cdots & 0 & \frac{\partial\zeta_j}{\partial s_j} & 0 & \cdots & 0 & \gamma_j \\
0 & 0 & \cdots & & \cdots & & \cdots & 0 & \frac{\partial\zeta_j}{\partial s_{j+1}} & 1 & \cdots & 0 & \gamma_{j+1} \\
\vdots & \vdots & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & & \cdots & & \cdots & 0 & \frac{\partial\zeta_j}{\partial s_{n-1}} & 0 & \cdots & 1 & \gamma_{n-1}
\end{pmatrix}$$

that is

$$B_{ij} = \begin{pmatrix}
I_{i-1} & O_{(i-1)\times(j-i-1)} & \frac{\partial\zeta_j}{\partial(s_1, \dots, s_{i-1})} & O_{(i-1)\times(n-j-1)} & \Gamma_1 \\
O_{1\times(i-1)} & O_{1\times(j-i-1)} & \frac{\partial\zeta_j}{\partial s_i} & O_{1\times(n-j-1)} & \gamma_i \\
O_{(j-i-1)\times(i-1)} & I_{j-i-1} & \frac{\partial\zeta_j}{\partial(s_{i+1}, \dots, s_{j-1})} & O_{(j-i-1)\times(n-j-1)} & \Gamma_2 \\
O_{1\times(i-1)} & O_{1\times(j-i-1)} & \frac{\partial\zeta_j}{\partial s_j} & O_{1\times(n-j-1)} & \gamma_j \\
O_{(n-j-1)\times(i-1)} & O_{(n-j-1)\times(j-i-1)} & \frac{\partial\zeta_j}{\partial(s_{j+1}, \dots, s_{n-1})} & I_{n-j-1} & \Gamma_3
\end{pmatrix}$$

where

$$I_k = \text{ identity matrix in } \mathbf{R}^{k\times k},$$
$$O_{k\times l} = \text{ zero matrix in } \mathbf{R}^{k\times l},$$
$$\frac{\partial \zeta_j}{\partial(s_k, \dots, s_l)} = \begin{pmatrix} \frac{\partial\zeta_j}{\partial s_k} \\ \vdots \\ \frac{\partial\zeta_j}{\partial s_l} \end{pmatrix},$$

$$\Gamma_1 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{i-1} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} \gamma_{i+1} \\ \gamma_{i+2} \\ \vdots \\ \gamma_{j-1} \end{pmatrix} \quad \text{and} \quad \Gamma_3 = \begin{pmatrix} \gamma_{j+1} \\ \vdots \\ \gamma_{n-1} \end{pmatrix}.$$

Then, $\det B_{ij} = \det C_{ij}$ where

$$C_{ij} = \begin{pmatrix} O_{1\times(j-i-1)} & \partial\zeta_j/\partial s_i & O_{1\times(n-j-1)} & \gamma_i \\ I_{j-i-1} & \partial\zeta_j/\partial(s_{i+1}, \dots, s_{j-1}) & O_{(j-i-1)\times(n-j-1)} & \Gamma_2 \\ O_{1\times(j-i-1)} & \partial\zeta_j/\partial s_j & O_{1\times(n-j-1)} & \gamma_j \\ O_{(n-j-1)\times(j-i-1)} & \partial\zeta_j/\partial(s_{j+1}, \dots, s_{n-1}) & I_{n-j-1} & \Gamma_3 \end{pmatrix}$$

and, by moving the column with $\partial\zeta_j/\partial s_i$ to the right,

$$\det C_{ij} = (-1)^{n-j-1} \det D_{ij}$$

where

$$D_{ij} = \begin{pmatrix} O_{1\times(j-i-1)} & O_{1\times(n-j-1)} & \partial\zeta_j/\partial s_i & \gamma_i \\ I_{j-i-1} & O_{(j-i-1)\times(n-j-1)} & \partial\zeta_j/\partial(s_{i+1}, \dots, s_{j-1}) & \Gamma_2 \\ O_{1\times(j-i-1)} & O_{1\times(n-j-1)} & \partial\zeta_j/\partial s_j & \gamma_j \\ O_{(n-j-1)\times(j-i-1)} & I_{n-j-1} & \partial\zeta_j/\partial(s_{j+1}, \dots, s_{n-1}) & \Gamma_3 \end{pmatrix}.$$

After moving downward the rows with $\partial\zeta_j/\partial s_i$, $\partial\zeta_j/\partial s_j$, we get

$$\det B_{ij} = (-1)^{n-j-1}(-1)^{n-i-2}(-1)^{n-j-1} \det E_{ij},$$

where

$$E_{ij} = \begin{pmatrix} I_{j-i-1} & O_{(j-i-1)\times(n-j-1)} & \partial\zeta_j/\partial(s_{i+1}, \dots, s_{j-1}) & \Gamma_2 \\ O_{(n-j-i)\times(j-i-1)} & I_{n-j-1} & \partial\zeta_j/\partial(s_{j+1}, \dots, s_{n-1}) & \Gamma_3 \\ O_{1\times(j-i-1)} & O_{1\times(n-j-1)} & \partial\zeta_j/\partial s_i & \gamma_i \\ O_{1\times(j-i-1)} & O_{1\times(n-j-1)} & \partial\zeta_j/\partial s_j & \gamma_j \end{pmatrix}$$

and therefore

$$\det B_{ij} = (-1)^{n-i}\left(\frac{\partial\zeta_j}{\partial s_i}\gamma_j - \frac{\partial\zeta_j}{\partial s_j}\gamma_i\right). \tag{4.12}$$

On the other hand, if $1 \le i < j = n$,

$$B_{ij} = B_{in} = \begin{pmatrix} I_{i-1} & O_{i-1\times(n-i-1)} & \partial\zeta_n/\partial(s_1, \dots, s_{i-1}) \\ O_{1\times(i-1)} & O_{1\times(n-i-1)} & \partial\zeta_n/\partial s_i \\ O_{(n-i-1)\times(i-1)} & I_{n-i-1} & \partial\zeta_n/\partial(s_{i+1}, \dots, s_{n-1}) \end{pmatrix}$$

and, by moving the row with $\partial\zeta_n/\partial s_i$ to the bottom, $\det B_{in} = (-1)^{n-i-1}$ $\partial\zeta_n/\partial s_i$.

Next, if $1 \le j < i = n$,

$$B_{ij} = B_{nj} = \begin{pmatrix} I_{j-1} & \partial\zeta_j/\partial(s_1, \dots, s_{j-1}) & O_{(j-1)\times(n-j-1)} \\ O_{1\times(j-1)} & \partial\zeta_j/\partial s_j & O_{1\times(n-j-1)} \\ O_{(n-j-1)\times(j-1)} & \partial\zeta_j/\partial(s_{j+1}, \dots, s_{n-1}) & I_{n-j-1} \end{pmatrix}$$

and thus

$$\det B_{nj} = \frac{\partial\zeta_j}{\partial s_j}.$$

Finally it is easy to check that formula (4.12) continues to be valid if $1 \le j < i \le n-1$. Hence, if $1 \le i \le n-1$,

$$\begin{aligned} N_{1i} &= (-1)^{n+i} \sum_{\substack{j=1 \\ j\ne i}}^{n} \det B_{ij} \\ &= (-1)^{n+i} \left\{ \sum_{\substack{j=1 \\ j\ne i}}^{n-1} (-1)^{n-i} \left(\frac{\partial\zeta_j}{\partial s_i}\gamma_j - \frac{\partial\zeta_j}{\partial s_j}\gamma_i \right) + (-1)^{n-i-1} \frac{\partial\zeta_n}{\partial s_i} \right\} \\ &= \sum_{\substack{j=1 \\ j\ne i}}^{n-1} \left(\frac{\partial\zeta_j}{\partial s_i}\gamma_i - \frac{\partial\zeta_j}{\partial s_j}\gamma_i \right) - \frac{\partial\zeta_n}{\partial s_i} \end{aligned}$$

and

$$N_{1n} = \sum_{j=1}^{n-1} \det B_{nj} = \sum_{j=1}^{n-1} \frac{\partial\zeta_j}{\partial s_j}.$$

This completes the proof of (a). Next (b) is an immediate consequence of (a) and the relation $N_1 = \sum_{j=1}^{n-1} A_j \partial\zeta/\partial s_j$ which defines the A_j.

Finally, let us prove (c). We have

$$\begin{aligned} (A_j N_0)_i &= \sum_{k=1}^{n} a^j_{ik} N_{0k} = \sum_{k=1}^{n-1} a^j_{ik}(-\gamma_k) + a^j_{in} \\ &= \gamma_i\gamma_j(1-\delta_{in})(1-\delta_{ij}) - \delta_{ij}\sum_{k=1}^{n-1}\gamma_k^2(1-\delta_{ik}) - \delta_{ij} - \delta_{in}\gamma_j \\ &\qquad (1 \le j \le n-1,\ 1 \le i \le n). \end{aligned}$$

Hence

$$A_jN_0 \cdot N_0 = \sum_{i=1}^{n}(A_jN_0)_i \cdot N_{0i}$$

$$= \sum_{i=1}^{n-1}(-\gamma_i)\left\{\gamma_i\gamma_j(1-\delta_{in})(1-\delta_{ij}) - \delta_{ij}\sum_{k=1}^{n-1}\gamma_k^2(1-\delta_{ik}) - \delta_{ij} - \delta_{in}\gamma_j\right\} + (-\gamma_j)$$

$$= (-\gamma_j)\sum_{i=1}^{n-1}\gamma_i^2(1-\delta_{ij}) + \gamma_j\sum_{k=1}^{n-1}\gamma_k^2(1-\delta_{jk}) + \gamma_j - \gamma_j = 0$$

and, for $1 \le i \le n-1$,

$$A_jN_0 \cdot z_{s_i} = \sum_{k=1}^{n}(A_jN_0)_k \cdot \frac{\partial z_k}{\partial s_i}$$

$$= (A_jN_0)_i + (A_jN_0)_n\gamma_i$$

$$= \left\{\gamma_i\gamma_j(1-\delta_{ij}) - \delta_{ij}\sum_{k=1}^{n-1}\gamma_k^2(1-\delta_{ik}) - \delta_{ij}\right\} + \gamma_i(-\gamma_j)$$

$$= -\delta_{ij}\left\{\gamma_i\gamma_j + \sum_{k=1}^{n-1}\gamma_k^2(1-\delta_{ik}) + 1\right\}$$

$$= -\delta_{ij}\left(1 + \sum_{k=1}^{n-1}\gamma_k^2\right) = -\delta_{ij}\|N_0\|^2 .$$

Having completed the proof of Lemma 4.3 we now proceed to prove Lemma 4.2.

Choose a local coordinate system $\tau = \tau(s)$ such that $z = z(\tau)$ satisfies the assumptions in Lemma 4.3. Then, by Lemma 4.3, the corresponding $N_0 = N_0^\tau$, $N_1 = N_1^\tau$ and $A_j = A_j^\tau$ satisfy:

$$A_j^\tau N_0^\tau \cdot N_0^\tau = 0, \tag{4.13}$$

$$A_jN_0^\tau \cdot z_{\tau_j} = -\delta_{i_j}\|N_0^\tau\|^2 . \tag{4.14}$$

Since clearly

$$\left(\frac{\partial(z+\varepsilon\zeta)}{\partial\tau}\right)\left(\frac{\partial\tau}{\partial s}\right) = \left(\frac{\partial(z+\varepsilon\zeta)}{\partial s}\right),$$

we deduce by taking determinants of both sides, differentiating in ε and setting $\varepsilon = 0$,

$$N_1^s = \delta N_1^\tau$$

and, of course, also

$$N_0^s = \delta N_0^\tau$$

where $\delta = \det(\partial\tau/\partial s)$; here N_0^s and N_1^s denote N_0 and N_1 in the s-coordinates. Since

$$\frac{\partial\zeta}{\partial\tau_j} = \sum_\lambda \frac{\partial\zeta}{\partial s_\lambda}\frac{\partial s_\lambda}{\partial\tau_j},$$

we find that

$$\sum_\lambda A_\lambda^s \frac{\partial \zeta}{\partial s_\lambda} = N_1^s = \delta N_1^\tau = \delta \sum_j A_j^\tau \frac{\partial \zeta}{\partial \tau_j} = \delta \sum_{\lambda, j} A_j^\tau \frac{\partial \zeta}{\partial s_\lambda} \frac{\partial s_\lambda}{\partial \tau_j}.$$

It follows that

$$A_\lambda^s = \delta \sum_{j=1}^{n-1} A_j^\tau \frac{\partial s_\lambda}{\partial \tau_j}. \tag{4.15}$$

Consequently

$$\begin{aligned}(A_\lambda^s N_0^s) \cdot N_0^s = \delta^2 A_\lambda^s N_0^\tau \cdot N_0^\tau &= \delta^3 \left(\sum A_j^\tau \frac{\partial s_\lambda}{\partial \tau_j} N_0^\tau \right) \cdot N_0^\tau \\ &= \delta^3 \sum \frac{\partial s_\lambda}{\partial \tau_j} (A_j^\tau N_0^\tau) \cdot N_0^\tau = 0\end{aligned}$$

by (4.13). Also

$$\begin{aligned}A_\lambda^s N_0^s \cdot z_{s_i} &= \delta^2 \sum \frac{\partial s_\lambda}{\partial \tau_j} (A_j^\tau N_0^\tau) \cdot z_{s_i} \\ &= \delta^2 \sum_{j, \mu} \frac{\partial s_\lambda}{\partial \tau_j} (A_j^\tau N_0^\tau \cdot z_{\tau_\mu}) \frac{\partial \tau_\mu}{\partial s_i} \\ &= -\delta^2 \sum_{j, \mu} \frac{\partial s_\lambda}{\partial \tau_j} \|N_0^\tau\|^2 \delta_{j\mu} \frac{\partial \tau_\mu}{\partial s_i} \quad \text{(by (4.14))} \\ &= -\delta^2 \sum_j \frac{\partial s_\lambda}{\partial \tau_j} \frac{\partial \tau_j}{\partial s_i} \|N_0^\tau\|^2 = -\delta_{\lambda i} \delta^2 \|N_0^\tau\|^2 = -\delta_{\lambda i} \|N_0^s\|^2,\end{aligned}$$

and Lemma 4.2 follows.

We conclude this section with several remarks.

Remark 4.1. In the definition of weak solution u we used a specific parametrization s of the smooth free boundary, namely $x = z(s, t)$. If we use another coordinate system τ, then we get another weak solution $u = u^\tau$. Using the transformation (4.15) it is easy to verify that if u is a weak solution with respect to the parametrization s then it is also a weak solution with respect to the parametrization τ.

Remark 4.2. If u is a weak solution in G_T then, as can easily be verified, it is also a weak solution in $G_{T'}$ for any $T' < T$. This remark together with Theorem 4.1 allows us to construct u for all $t > 0$ provided (θ_0, z) is smooth for all $t > 0$. In §§6, 7 we shall take (θ_0, z) smooth for all $t > 0$ and study the free boundary of $\theta_0 + \varepsilon u$, i.e., the surface

$$x = z(s, t) + \varepsilon \zeta(s, t), \quad \text{for } 0 < t < \infty.$$

Remark 4.3. Uniqueness for smooth solutions of (2.1)–(2.3) can be established by the maximum principle, as in [4].

5. Auxiliary results in ODE

In the next two sections we shall be working with solutions to the ODE problem

$$\text{(5.1)} \qquad \mathscr{L}_n w \equiv -w'' - \left(\frac{n-3}{y} + \frac{y}{2}\right) w' + \frac{n-3}{y^2} w = 0, \qquad 0 < y < 1,$$

$$\text{(5.2)} \qquad \frac{1}{2} w'(1) + \frac{2n-3}{4} w(1) = \frac{2n-1}{4},$$

$$\text{(5.3)} \qquad w(0+) = 0.$$

In this section we establish existence and properties of the solution to (5.1)–(5.3).

Theorem 5.1. *If* $n \geq 3$ *there exists a unique solution* w *of* (5.1)–(5.3); *further*

$$\text{(5.4)} \qquad w(y) > 0, \qquad w'(y) > 0 \quad \text{if} \quad 0 < y < 1,$$

$$\text{(5.5)} \qquad w(1) > 1.$$

Theorem 5.2. *If* $n = 2$ *there exists a solution* w *of* (5.1)–(5.3) *satisfying* (5.4), (5.5); w *is the* (*unique*) *minimal nonnegative solution of* (5.1)–(5.3).

Proof of Theorem 5.1. First notice that, if $n = 3$, (5.1)–(5.3) becomes

$$-w'' - \tfrac{y}{2} w' = 0, \qquad 0 < y < 1,$$
$$\tfrac{1}{2} w'(1) + \tfrac{3}{4} w(1) = \tfrac{5}{4},$$
$$w(0+) = 0$$

and this problem can be solved explicitly. Indeed the unique solution w is given by

$$w(y) = \frac{5}{2 + 3\alpha_0} \int_0^y e^{(1-z^2)/4}\, dz$$

where $\alpha_0 = \int_0^1 e^{(1-z^2)/4}\, dz > 1$. In particular, w satisfies (5.4) and (5.5).

Thus we may assume $n \geq 4$. For any $\varepsilon > 0$ consider the problem

$$(P_\varepsilon^\infty) \qquad \begin{gathered} \mathscr{L}_n w = 0, \qquad \varepsilon < y < 1, \\ w(\varepsilon) = 0, \\ \frac{1}{2} w'(1) + \frac{2n-3}{4} w(1) = \frac{2n-1}{4}. \end{gathered}$$

Since $n \geq 4$ the coefficient of w in $\mathscr{L}_n w$ is > 0; hence there exists a unique solution w_ε; furthermore, the maximum principle can be applied. We deduce that $w_\varepsilon(y) > 0$ if $\varepsilon < y \leq 1$ and

$$\text{(5.6)} \qquad w_\varepsilon'(\varepsilon) > 0.$$

We compute that

$$\mathscr{L}_n w_\varepsilon' + \left(\frac{n-3}{y^2} - \frac{1}{2}\right) w_\varepsilon' = (\mathscr{L}_n w_\varepsilon)' + \frac{2(n-3)}{y^3} w_\varepsilon \geq 0$$

and therefore w_ε' cannot take nonpositive minimum in $\varepsilon < y < 1$. If w_ε' takes nonpositive minimum at $y = 1$ then $w_\varepsilon''(1) \leq 0$, $w_\varepsilon'(1) \leq 0$, and (since $w_\varepsilon(1) > 0$) we get $\mathscr{L}_n w_\varepsilon(1) > 0$, a contradiction. Recalling finally (5.6) we deduce that

$$w_\varepsilon'(y) > 0 \quad \text{if } \varepsilon \leq y \leq 1. \tag{5.7}$$

By applying the maximum principle to w_ε we also deduce that $w_\varepsilon(y) \leq w_\varepsilon(1)$, and by (5.2), (5.7),

$$w_\varepsilon(y) \leq \frac{2n-1}{2n-3}, \qquad \varepsilon \leq y \leq 1; \tag{5.8}$$

by comparison we also have

$$w_{\varepsilon'}(y) \leq w_\varepsilon(y) \quad \text{if } \varepsilon < \varepsilon'. \tag{5.9}$$

It follows that $w(y) \equiv \lim_{\varepsilon \to 0} w_\varepsilon(y)$ exists and satisfies $\mathscr{L}_n w = 0$ and the asserted boundary condition (5.2) at $y = 1$, and $w'(y) > 0$ if $0 < y \leq 1$.

Next, for any $0 < \lambda < 1$, $\mathscr{L}_n y^\lambda > 0$ if $0 < y < y_0$ for some $y_0 \in (0, 1)$. Hence,

$$w_\varepsilon(y) \leq C y^\lambda \quad \text{if } \varepsilon < y < y_0 \quad \left(C = \frac{2n-1}{2n-3} y_0^{-\lambda}\right),$$

where (5.8) was used. We conclude that

$$w(y) \leq C y^\lambda, \qquad 0 < y < y_0, \tag{5.10}$$

and, in particular, $w(0) = 0$.

We have proved that w satisfies (5.1)–(5.3). By the maximum principle, the solution to (5.1)–(5.3) is unique.

Observe next that the function $z(y) \equiv y$ satisfies the boundary condition (5.2) and $\mathscr{L}_n z < 0$. Hence, by comparison, $w(y) > z(y)$ if $0 < y \leq 1$ and, in particular, $w(1) > 1$.

Proof of Theorem 5.2. Equations (5.1), (5.2) become

$$\mathscr{L}_2 w \equiv -w'' + \left(\frac{1}{y} - \frac{y}{2}\right) w' - \frac{w}{y^2} = 0, \tag{5.11}$$

$$\tfrac{1}{2} w'(1) + \tfrac{1}{4} w(1) = \tfrac{3}{4}. \tag{5.12}$$

Set

$$\mathscr{L}_0 w = -w'' + \left(\frac{1}{y} - \frac{y}{2}\right) w'.$$

For any $\varepsilon < 1$, ε near 1, by working with

$$\tilde{w}_\varepsilon(y) = \frac{w_\varepsilon((1-\varepsilon)y + \varepsilon)}{1-\varepsilon} \qquad (0 < y < 1)$$

which satisfies: $\tilde{w}_\varepsilon'' = O(1-\varepsilon)$, $\tilde{w}_\varepsilon(0) = 0$, $\tilde{w}_\varepsilon'(1) \approx \frac{3}{2}$, we find that there exists a unique solution $w_\varepsilon(y)$ of (P_ε^∞) (with $n = 2$). We denote by ε_0 the infimum of ε such that for any $\varepsilon \le \varepsilon' < 1$ there exists a unique positive solution $w_{\varepsilon'}$ of $(P_{\varepsilon'}^\infty)$ $(n = 2)$. We first establish some properties of w_ε.

Integrating (5.11) in y, $\varepsilon < y < 1$, and using

$$\int_\varepsilon^1 \left(\frac{1}{y} - \frac{y}{2}\right) w' = \left[\frac{w}{y} - \frac{yw}{2}\right]_\varepsilon^1 + \int_\varepsilon^1 \left(\frac{w}{y^2} + \frac{w}{2}\right) \tag{5.13}$$

and (5.12), we get

$$w_\varepsilon(1) + \frac{1}{2}\int_\varepsilon^1 w_\varepsilon + w_\varepsilon'(\varepsilon) = \frac{3}{2}, \qquad w_\varepsilon'(\varepsilon) \ge 0, \tag{5.14}$$

and, in particular,

$$w_\varepsilon(1) < \tfrac{3}{2}. \tag{5.15}$$

Since

$$\mathscr{L}_0 w_\varepsilon = \frac{w_\varepsilon}{y^2} \ge 0,$$

w_ε cannot take a minimum in $\varepsilon < y \le 1$ (we use here the boundary condition (5.12) and (5.15)) and, further,

$$w_\varepsilon'(\varepsilon) > 0. \tag{5.16}$$

It follows that $w_\varepsilon(y)$ is either monotone increasing in y or else first increases and then decreases. The latter possibility leads to $w_\varepsilon'(1) \le 0$ and then, by (5.12), $w_\varepsilon(1) > 3$ which is a contradiction to (5.15). Consequently,

$$w_\varepsilon'(y) \ge 0 \quad \text{for } \varepsilon < y < 1. \tag{5.17}$$

Next we show that

$$w_\varepsilon(y) > w_{\varepsilon'}(y) \quad \text{if } \varepsilon' \le y \le 1,\ \varepsilon < \varepsilon'. \tag{5.18}$$

Indeed, $w_\varepsilon(y) > \lambda w_{\varepsilon'}(y)$ for $\varepsilon' \le y \le 1$ if λ is positive and small. Denoting by λ_0 the supremum of all λ's for which the inequality holds, it suffices to show that $\lambda_0 > 1$. Suppose $\lambda_0 \le 1$ and consider $\zeta = w_\varepsilon - \lambda_0 w_{\varepsilon'}$. Then

$$\mathscr{L}_0 \zeta = \frac{\zeta}{y^2} \ge 0, \qquad \varepsilon' \le y \le 1,$$

$$\zeta(\varepsilon') > 0, \qquad \tfrac{1}{2}\zeta'(1) + \tfrac{1}{4}\zeta(1) \ge 0.$$

It follows that $\zeta(y) > 0$ if $\varepsilon' \le y \le 1$, and thus $\zeta \ge \delta w_{\varepsilon'}$ for some $\delta > 0$, a contradiction to the definition of λ_0.

By (5.15), (5.17), (5.18), we see that if $\varepsilon_0 > 0$ then $w_0 \equiv \lim_{\varepsilon \to \varepsilon_0} w_\varepsilon$ exists and is a solution of $(P_{\varepsilon_0}^\infty)$, satisfying (5.17). We shall derive a contradiction, thereby proving that ε_0 must be equal to zero.

Let w be the solution of (5.11) under the boundary conditions (5.12) and $w(1) = w_0(1) + \delta$, i.e., $w(1) = w_0(1) + \delta$ and $w'(1) = \frac{3}{2} + \frac{1}{2}w_0(1) - \frac{1}{2}\delta$ for

$\delta > 0$, δ small. Then, by continuity, (using the fact $w_0'(\varepsilon_0) > 0$) $w(y) > 0$ for $\varepsilon < y < 1$, $w(\varepsilon) = 0$, for some ε near ε_0. If $\varepsilon > \varepsilon_0$ then $w = w_\varepsilon$ and we get a contradiction to (5.18) at $y = 1$. Hence $\varepsilon < \varepsilon_0$ and, by varying δ we get positive solutions $w_\varepsilon(y)$ of (P_ε^∞) for all $\varepsilon_1 \le \varepsilon \le \varepsilon_0$, for some $0 < \varepsilon_1 < \varepsilon_0$. We claim that w_ε is the unique nonnegative solution of (P_ε^∞). Indeed, if $\hat{w}_\varepsilon$ is another nonnegative solution then we can proceed as in the proof of (5.18) (noting that $w_\varepsilon - \lambda\hat{w}_\varepsilon \ge 0$ implies $w_\varepsilon'(0) - \lambda w_\varepsilon'(0) > 0$) to show that $w_\varepsilon \ge \lambda\hat{w}_\varepsilon$ for all $0 \le \lambda \le 1$, so that $w_\varepsilon \ge \hat{w}_\varepsilon$, and similarly $\hat{w}_\varepsilon \ge w_\varepsilon$.

Having proved that for each $\varepsilon_1 < \varepsilon \le \varepsilon_0$ there is a unique nonnegative solution of (P_ε^∞), we get a contradiction to the definition of ε_0. It follows that $\varepsilon_0 = 0$.

From (5.15), (5.17), (5.18) and the fact that $\varepsilon_0 = 0$ it follows that

$$w(y) = \lim_{\varepsilon\to 0} w_\varepsilon(y)$$

exists, and it satisfies (5.1), (5.2), (5.4). (If $w'(0) = 0$, $0 < y_0 < 1$, then $w''(y_0) < 0$, by (5.1), so that $w'(y) < 0$ for $y > y_0$, $y - y_0$ small, which is a contradiction.)

To prove (5.3) we integrate (5.11) over $\varepsilon < y < 1$ and use (5.12), and (5.13); we get

$$w(1) + \frac{1}{2}\int_\varepsilon^1 w + \left[w'(\varepsilon) - \frac{w(\varepsilon)}{\varepsilon}\right] + \frac{1}{2}\varepsilon w(\varepsilon) = \frac{3}{2}. \tag{5.19}$$

Taking $\varepsilon \to 0$ in (5.14) and comparing with (5.19), we deduce that

$$\varliminf_{\varepsilon\to 0}\left(w'(\varepsilon) - \frac{w(\varepsilon)}{\varepsilon}\right) \ge 0.$$

If $w(0+) > 0$, this leads to $w'(\varepsilon) > c/\varepsilon$ $(c > 0)$ as $\varepsilon \to 0$, a contradiction. Hence $w(0) = 0$.

To prove that $w(1) > 1$ notice that if $w(1) \le 1$ then, from (5.19) it follows that

$$w'(\varepsilon) - \frac{w(\varepsilon)}{\varepsilon} \ge c > 0, \qquad 0 < \varepsilon < \varepsilon_1,$$

for some constants c, ε_1. Using this in (5.11) we get

$$w''(y) + \frac{y}{2}w'(y) \ge \frac{c}{y}, \qquad 0 < y < \varepsilon_1,$$

or

$$(w'e^{y^2/4})' \ge \frac{c}{y}.$$

Hence, since $w' \ge 0$,

$$w'(y)e^{y^2/4} \ge \int_\varepsilon^y \frac{c}{z}\,dz = c\log\frac{y}{\varepsilon} \to \infty \quad \text{if } \varepsilon \to 0,$$

a contradiction.

We next prove that $w(y)$ is the (unique) minimal nonnegative solution of (5.1)–(5.3). Let $\tilde{w}$ be any nonnegative solution of (5.1)–(5.3). Since $\mathscr{L}_0\tilde{w} \ge 0$,

$\tilde{w}(y) > 0$ if $0 < y < 1$. For any $\varepsilon > 0$, $\tilde{w} \geq \lambda w_\varepsilon$ for $\varepsilon \leq y \leq 1$ if λ is positive and sufficiently small. As before we can argue that the supremum of such λ's must be ≥ 1, so that $\tilde{w} \geq w_\varepsilon$. Taking $\varepsilon \to 0$ we see that $\tilde{w} \geq w$, i.e., w is the minimal solution.

6. The effect of surface tension on the interface (radial case)

If G is a shell $\{r_0 < |x| < r_1\}$ and the data $\tilde{\theta}$ are radially symmetric, then we can try to solve the Stefan problem with surface tension by a function $\theta(r, t)$ where $r = |x|$. The free boundary conditions are

$$\theta = \frac{\varepsilon}{s_\varepsilon(t)}, \qquad -\theta_r = \frac{ds_\varepsilon(t)}{dt} \quad \text{on } r = s_\varepsilon(t).$$

The method of integral equations used in case $\varepsilon = 0$ (e.g. [5]) can be used also for $\varepsilon > 0$ to show that for any large T there is a small $\varepsilon_0 = \varepsilon(T)$ positive such that a solution exists for $0 \leq t \leq T$ if $0 < \varepsilon < \varepsilon_0$; furthermore, the solution depends smoothly on ε.

In this section we consider a special radial solution with free boundary $s(t) = \sqrt{t}$ when $\varepsilon = 0$ and investigate the effect of the ε-approximating u on the free boundary. Our interest is not really in this special radial case, but rather in the method which will be developed to tackle it, for this method will be used in the next section to study the general (nonradial) case. We consider $\theta_0(x, t)$, a function of $|x|$ and t,

$$\theta_0(x, t) = f\left(\frac{|x|}{\sqrt{t}}\right), \qquad t > 1,$$

and the free boundary

$$|x| = \sqrt{t}, \qquad t > 1.$$

One easily finds (see [6, p. 87]) that

$$f(z) = c_0 \int_z^\infty \zeta^{1-n} e^{-\zeta^2/4} d\zeta - c_1,$$

$$c_0 = \frac{1}{2} e^{1/4}, \qquad c_1 = \frac{1}{2} e^{1/4} \int_1^\infty \zeta^{1-n} e^{-\zeta^2/4} d\zeta.$$

We take

$$D = \{|x| < \alpha\} \quad \text{for some } 0 < \alpha < 1$$

and this determines the initial and boundary values $\tilde{\theta}$ of θ_0 on $G \times \{1\}$ ($G = \{\alpha < |x| < 1\}$) and on $\partial D_\infty = \{|x| = \alpha, 1 < t < \infty\}$. Notice that

$$\tilde{\theta} \to \frac{1}{2} e^{1/4} \int_0^1 e^{-\zeta^2/4} \zeta^{n-1} d\zeta \quad \text{as } t \to \infty \text{ on } \partial D_\infty.$$

The free boundary can be represented in the form

$$x = z(s, t) = \sqrt{t}\nu_0, \qquad \nu_0 = \frac{x}{|x|}. \tag{6.1}$$

Clearly (1.22), (1.23) remain true if we replace N_0 by $\gamma(s, t)N_0$ and the A_j's and h_0 accordingly (i.e., A_j by γA_j and h_0 by h_0/γ). Thus, we may take

$$N_0 = z(s, t) = \sqrt{t}\nu_0 . \tag{6.2}$$

Since

$$\begin{aligned} f'(z) &= -\tfrac{1}{2}e^{1/4}z^{1-n}e^{-z^2/4}, \\ f''(z) &= \tfrac{1}{2}e^{1/4}(\tfrac{1}{2}z^{2-n} + (n-1)z^{-n})e^{-z^2/4}, \end{aligned} \tag{6.3}$$

we have

$$f'(1) = -\frac{1}{2}, \qquad f''(1) = \frac{2n-1}{4}. \tag{6.4}$$

Next, on the free boundary,

$$z_t + \nabla\theta_0 = \frac{1}{2\sqrt{t}}\nu_0 + f'(1)\frac{x}{|x|\sqrt{t}} = 0 \tag{6.5}$$

by (6.4); also

$$\nabla\theta_0 \cdot N_0 = f'(1)\frac{x}{|x|\sqrt{t}} \cdot \sqrt{t}\nu_0 = f'(1) = -\frac{1}{2}$$

so that

$$h_0 = -\frac{1}{\nabla\theta_0 \cdot N_0} = 2. \tag{6.6}$$

Thus

$$a = N_0 \cdot N_0 h_0 = 2t. \tag{6.7}$$

Next

$$\begin{aligned} \nabla\partial_{x_i}\theta_0 = \partial_{x_i}\nabla\theta_0 &= \partial_{x_i} f'\left(\frac{|x|}{\sqrt{t}}\right)\frac{x}{|x|\sqrt{t}} \\ &= f''\left(\frac{|x|}{\sqrt{t}}\right)\frac{x_i}{|x|\sqrt{t}}\frac{x}{|x|\sqrt{t}} + f'\left(\frac{|x|}{\sqrt{t}}\right)\frac{1}{\sqrt{t}}\left(\frac{e_i}{|x|} - \frac{xx_i}{|x|^3}\right) \end{aligned}$$

where e_i is the vector with components δ_{ij}. Hence, on $|x| = \sqrt{t}$,

$$\nabla\partial_{x_i}\theta_0 \cdot N_0 = f''(1)\frac{x_i}{|x|\sqrt{t}} + f'(1)\left(\frac{x_i}{|x|^2} - \frac{x_i}{|x|^2}\right) = \frac{2n-1}{4}\frac{x_i}{|x|\sqrt{t}}$$

by (6.4), and

$$\begin{aligned} & N_0 \cdot (\nabla\partial_{x_1}\theta_0 \cdot N_0, \dots, \nabla\partial_{x_n}\theta_0 \cdot N_0) \\ &\quad = \frac{2n-1}{4}\frac{1}{|x|\sqrt{t}}x \cdot \sqrt{t}\frac{x}{|x|} = \frac{2n-1}{4}. \end{aligned} \tag{6.8}$$

From the definition of b_j in (1.23) and (6.5) we see that

$$b_j = 0. \tag{6.9}$$

From the definition of c in (1.23) and (6.5), (6.6), (6.8) we get

$$c = h_0 \left\{ N_0 \cdot N_{0t} + \frac{2n-1}{4} \right\} = 2 \left\{ \sqrt{t} \frac{x}{|x|} \cdot \frac{1}{2\sqrt{t}} \frac{x}{|x|} + \frac{2n-1}{4} \right\},$$

or,

$$c = \frac{2n+1}{2}. \tag{6.10}$$

Next

$$N_0 \cdot \nabla u = \sqrt{t} \frac{x}{|x|} \cdot \nabla u = \sqrt{t} u_r \tag{6.11}$$

where $r = |x|$. Finally, since the mean curvature of $|x| = \sqrt{t}$ is $\kappa = 1/\sqrt{t}$,

$$\begin{aligned} f_0 &= t\frac{d}{dt}(2\kappa) + \left\{ N_0 \cdot N_{0t} + \frac{2n-1}{4} \right\} 2\kappa \\ &= 2t(-\tfrac{1}{2})t^{-3/2} + \frac{2n+1}{2\sqrt{t}} = \frac{2n-1}{2\sqrt{t}}. \end{aligned} \tag{6.12}$$

Combining (6.7), (6.9)–(6.12) we can rewrite (1.19) in the form

$$2t\frac{du}{dt} + \sqrt{t} u_r + \frac{2n+1}{2} u = \frac{2n-1}{2} \frac{1}{\sqrt{t}}. \tag{6.13}$$

Since

$$2t\frac{du}{dt} = 2t\left(u_t + u_r \frac{dr}{dt} \right) = 2tu_t + \sqrt{t} u_r,$$

we can rewrite (6.13) also in the form

$$tu_t + \sqrt{t} u_r + \frac{2n+1}{4} u = \frac{2n-1}{4} \frac{1}{\sqrt{t}} \quad \text{on } |x| = \sqrt{t}. \tag{6.14}$$

The differential equation for u is

$$u_t - u_{rr} - \frac{n-1}{r} u_r = 0. \tag{6.15}$$

We wish to analyze the effect of the approximation $\theta_0 + \varepsilon u$ on the water region. To do this it will be convenient to perform a change of variables

$$t = e^s, \qquad r = ye^{s/2},$$

and work with the dependent function

$$w(y, s) = ru(r, t). \tag{6.16}$$

By direct calculation we find that

$$w_s - w_{yy} - \left(\frac{n-3}{y} + \frac{y}{2} \right) w_y + \frac{n-3}{y^2} w = 0 \quad \text{in } \Omega, \tag{6.17}$$

$$w_s + \frac{1}{2} w_y + \frac{2n-3}{4} w = \frac{2n-1}{4} \quad \text{on } y = 1, \ s > 0, \tag{6.18}$$

$$w = 0 \text{ on the remaining part of } \partial\Omega, \tag{6.19}$$

where

$$\Omega = \{(y, s); \alpha e^{-s/2} < y < 1, 0 < s < \infty\}.$$

Set

$$\Omega_T = \Omega \cap \{s < T\}, \quad \text{for any } T > 0.$$

In the sequel we shall use parabolic Schauder estimates with norm

$$\|w\|_{2+\beta}^{\Omega_T} \equiv \|w_t\|_{\beta}^{\Omega_T} + \sum_{j=0}^{2} \|D_y^j w\|_{\beta}^{\Omega_T} + H_{(1+\beta)/2}^t (D_y w)$$

where $\|v\|_\beta$ is the β-Hölder norm with respect to the distance function

$$d((y, s), (\overline{y}, \overline{s})) = \{|y - \overline{y}|^2 + |s - \overline{s}|\}^{1/2}$$

and $H_{(1+\beta)/2}^t(v)$ is the Hölder coefficient of v in the t variable (with the usual distance function) of exponent $(1+\beta)/2$. We shall also use the "interior" norm $\widehat{\|w\|}_{2+\beta}^{\Omega_T}$ whereby $\|D_y^j w\|_\beta^{\Omega_T}$ is replaced by

$$\widehat{\|D_y^j w\|}_{\beta}^{\Omega_T} = \sup s^j |D_y^j w(y, s)| + \sup s^{j+\beta/2} \frac{|D_y^j w(y, s) - D_y^j w(\overline{y}, \overline{s})|}{|y - \overline{y}|^\beta + |s - \overline{s}|^{\beta/2}}$$

$((y, s) \in \Omega_T, (\overline{y}, \overline{s}) \in \Omega_T$ and $0 < \overline{s} < s)$, and similarly for $D_t w$ (with the factor s^j replaced by s^2) and $H_{(1+\beta)/2}^t(D_y w)$.

We denote the space of functions with finite norm $\| \ \|_{2+\beta}^{\Omega_T}$ by $C^{2+\beta}(\overline{\Omega_T})$.

Theorem 6.1. *There exists a unique solution* w *of* (6.17)–(6.19) *such that*

$$w \in C(\overline{\Omega}_T) \cap C^{2+\beta}(\overline{\Omega}_T \backslash \overline{\Omega}_\lambda)$$

for any $0 < \lambda < T < \infty$.

By going back to the original variables we deduce that the unique weak solutiuon of (2.1)–(2.3) is in $C^{2+\beta}$ in $\overline{G}_T \backslash \overline{G}_\lambda$ for any $0 < \lambda < T < \infty$ and in $C(\overline{G}_T)$.

Proof. Let $\rho(s)$ be a smooth function, $0 \le \rho \le 1$, $\rho(0) = 0$, $\rho(s) = 1$ for $s \ge 1$, $0 \le \rho' \le 2$, and set

$$\rho_\varepsilon(s) = \rho\left(\frac{s}{\varepsilon}\right) \quad \text{for any } \varepsilon > 0.$$

Consider the problem (P_ε) consisting of (6.17), (6.19) and

$$w_s + \frac{1}{2} w_y + \frac{2n-3}{4} w = \frac{2n-1}{4} \rho_\varepsilon(s) \quad \text{on } y = 1, \ s > 0. \tag{6.20}$$

For this problem the consistency condition is satisfied at $y = 1$, $s = 0$.

Let

$$X_\delta = \{w \in C^{2+\beta}(\overline{\Omega}_\delta), \ w = 0 \text{ on } s = 0 \text{ and on } y = \alpha e^{-s/2}\}.$$

For any $w \in X_\delta$ let $\overline{w}$ be the solution of (6.17), (6.19) with

$$\overline{w}(1, s) = E(w)(s) \equiv \int_0^s \left[\frac{2n-1}{4}\rho_\varepsilon(\tau) - \frac{1}{2}w_y(1, \tau) - \frac{2n-3}{4}w(1, \tau)\right] d\tau .$$

Since w_y is Hölder continuous in s of exponent $(1+\beta)/2$, we easily find that

$$\|E(w)\|_{1+\beta/2} \le C_n \left(\frac{\delta^{1-\beta/2}}{\varepsilon} + \delta^{1/2}\|w\|_{2+\beta}^{\Omega_\delta}\right) . \tag{6.21}$$

Hence $\overline{w} \in X_\delta$. Set $\overline{w} = Uw$. If w_1, w_2 belong to X_δ then estimating

$$\|E(w_1) - E(w_2)\|_{1+\beta/2} \le C_n \delta^{1/2}\|w_1 - w_2\|_{2+\beta}^{\Omega_\delta}$$

and using Schauder's estimates [5, 13] we deduce that

$$\|Uw_1 - Uw_2\|_{2+\beta}^{\Omega_\delta} \le C_n\delta^{1/2}\|w_1 - w_2\|_{2+\beta}^{\Omega_\delta}, \qquad C_n \text{ constant (depending on } n\text{)}.$$

Therefore, if δ is sufficiently small then U is a contraction, and consequently, it has a unique fixed point w_ε; w_ε is the solution of (6.17), (6.19), (6.20). To extend the solution for all $t > 0$, set

$$m(t) = \|w_\varepsilon\|_{2+\beta}^{\Omega_t} .$$

Proceeding as in the first step we can derive the estimate

$$m(t+\delta) \le \gamma m(t) + \gamma_0 \delta^{1/2} m(t+\delta) \tag{6.22}$$

where γ, γ_0 are positive constants independent of t, δ. Choosing δ such that $\gamma_0\delta^{1/2} = \frac{1}{2}$ we deduce the estimate $m(t+\delta) \le 2\gamma m(t)$ which allows us to extend the solution w step-by-step for all $t > 0$.

If in the first step we use the interior norm $\widehat{\| \ \|}$ mentioned above, then instead of (6.21) we get

$$\widehat{\|E(w)\|}_{1+\beta/2} \le C + \delta^{1/2}\widehat{\|w_\varepsilon\|}_{2+\beta}^{\Omega_\delta}, \qquad C \text{ independent of } \varepsilon .$$

The mapping U has the unique fixed point w_ε as before, and

$$\widehat{\|w_\varepsilon\|}_{2+\beta}^{\Omega_\delta} \le C_0 , \tag{6.23}$$

where C_0 and δ are independent of ε.

We now let $\varepsilon \to 0$. First observe, by the maximum principle, that

$$0 \le w_{\varepsilon'} \le w_\varepsilon \quad \text{if } 0 < \varepsilon < \varepsilon' . \tag{6.24}$$

Hence $w = \lim w_\varepsilon$ exists. Secondly, by (6.23) and (6.22),

$$\widehat{\|w\|}_{2+\beta}^{\Omega_\delta} \le C_0 , \qquad \|w\|_{2+\beta}^{\Omega_T\setminus\Omega_{\delta/2}} \le C_T \quad \forall T < \infty .$$

Since w clearly satisfies (6.17)–(6.19), it remains to show that $w \in C(\overline{\Omega}_T)$, or just that

$$\lim_{s\to 0} w(y, s) = 0 . \tag{6.25}$$

But the function $(2n-1)s/4$ is a supersolution of (6.17) in any Ω_T for $n \geq 3$ and in Ω_T (T small) if $n = 2$. By the maximum principle we find that

$$w(y, s) \leq \frac{2n-1}{4} s \quad \text{in } \Omega_T,$$

and thus (6.25) follows.

Definition 6.1. We depict two subsets on the free boundary Γ_∞: $\Gamma^+ = \{u > \kappa\}$, $\Gamma^- = \{u < \kappa\}$.

From (1.15), (1.16) we see that $R > 0$ on Γ^+ and $R < 0$ on Γ^-. This means that the approximate free boundary $x = z + \varepsilon\zeta$ expands near Γ^+ and shrinks near Γ^-, i.e.,

(6.26) small surface tension increases the water region near Γ^+ and decreases it near Γ^-.

We now state the main result of this section.

Theorem 6.2. *For any* $0 < \eta < 1$ *the solution* $w(y, s)$ *of* (6.17)–(6.19) *satisfies*

$$\lim_{s\to\infty} w(y, s) = w_\infty(y) \quad \textit{uniformly in } y,\ \eta \leq y \leq 1 \tag{6.27}$$

where w_∞ *is the solution of* (5.1)–(5.3) *asserted in Theorems* 5.1 *and* 5.2.

Corollary 6.3. *The limit* $w_\infty(1) = \lim_{s\to\infty} w(1, s)$ *exists and* $w_\infty(1) > 1$.

In view of (6.16) this means that

$$u > \kappa \text{ on } \Gamma_\infty \backslash \Gamma_{t_0} \quad \text{for some large } t_0 > 0,$$

i.e., the effect of the surface tension is to increase the water region at all times $t > t_0$. The amount of increase is determined by the interface $x = z + \varepsilon\zeta$. Asymptotically,

$$\zeta = 2(u-\kappa)\sqrt{t}\nu_0 \approx 2(w_\infty(1) - 1)\nu_0,$$

which means that the radial increase due to surface tension is, at time t,

$$\approx 2\varepsilon(w_\infty(1) - 1)/\sqrt{t} \quad \text{times the original radius.}$$

Notice also that by (6.25),

$$u < \kappa \quad \text{on } \Gamma_{t_1}$$

for t_1 sufficiently small, i.e., the surface tension decreases the water region at all small times.

For $n \geq 3$ the operator $\mathscr{L}_n$ satisfies the conditions for the maximum principle. This makes the proof of Theorem 6.2 simpler. It will be convenient to first give the proof in this case, and then establish it separately for $n = 2$.

Proof of Theorem 6.2 *in case* $n \geq 3$. Comparing $w(x, s)$ with $w_\infty(x)$ in Ω we immediately find that

$$w(y, s) \leq w_\infty(y) \quad \text{in } \Omega. \tag{6.28}$$

Given any $0 < \varepsilon < 1$, let

$$\widetilde{\Omega}_\varepsilon = \{(x, s); \varepsilon < y < 1, s > s_\varepsilon\}, \qquad s_\varepsilon = \log \frac{\alpha^2}{\varepsilon^2};$$

notice that $\widetilde{\Omega}_\varepsilon = \Omega \cap \{y > \varepsilon, s > s_\varepsilon\}$ and $(\varepsilon, s_\varepsilon) \in \partial\Omega$. Consider the problem

$$(\widetilde{P}_\varepsilon) \qquad \begin{aligned} &\frac{\partial w}{\partial s} + \mathscr{L}_n w = 0 \quad \text{in } \widetilde{\Omega}_\varepsilon, \\ &\frac{\partial w}{\partial s} + \frac{1}{2}\frac{\partial w}{\partial y} + \frac{2n-3}{4} w = \frac{2n-1}{4} \quad \text{on } y = 1, \\ &w = 0 \quad \text{on the remaining boundary of } \widetilde{\Omega}_\varepsilon. \end{aligned}$$

As in the proof of Theorem 6.1, this problem has a unique solution $\tilde{w}_\varepsilon$, continuous on $\overline{\widetilde{\Omega}}_\varepsilon$ and in $C^{2+\beta}$ in $\overline{\widetilde{\Omega}}_\varepsilon \cap \{s > s_\varepsilon\}$. By the maximum principle,

$$(6.29) \qquad 0 \le \tilde{w}_\varepsilon \le w \quad \text{in } \widetilde{\Omega}_\varepsilon.$$

By the maximum principle we also have

$$\tilde{w}_\varepsilon(y + h, s) \ge \tilde{w}_\varepsilon(y, s) \quad (h > 0),$$

so that

$$(6.30) \qquad \frac{\partial \tilde{w}_\varepsilon}{\partial s} \ge 0.$$

We claim that

$$(6.31) \qquad \frac{\partial \tilde{w}_\varepsilon}{\partial s} \le A e^{-\lambda s} \quad \text{for some } A > 0,\ \lambda > 0;$$

A and λ may depend on ε. Once this is proved, it follows that

$$\tilde{w}_\varepsilon(y, s) \to w_\varepsilon(y) \quad \text{as } s \to \infty,$$

uniformly in y, and therefore, by (6.29),

$$\varliminf_{s\to\infty} w(y, s) \ge w_\varepsilon(y)$$

uniformly in y. Recalling that $w_\varepsilon(y) \to w_\infty(y)$, we then deduce that, for any $0 < \eta < 1$,

$$\varliminf_{s\to\infty} w(y, s) \ge w_\infty(y)$$

uniformly in y, $\eta \le y \le 1$. In view of (6.28), the proof of (6.27) is then complete. Thus it remains to prove (6.31).

Note that

$$\frac{\partial \tilde{w}_\varepsilon}{\partial s} \le \widetilde{A}_\varepsilon \quad \text{if } \varepsilon \le y \le 1,\ s = s_\varepsilon + 1.$$

For $n \ge 4$ the function $W \equiv e^{-\lambda s}$ satisfies

$$W_s + \mathscr{L}_n W \ge 0 \quad \text{in } \widetilde{\Omega}_\varepsilon,$$

$$W_s + \frac{1}{2}W_y + \frac{2n-3}{4}W \geq 0 \quad \text{at } y = 1$$

provided $\lambda = \min\{n-3, (2n-3)/4\}$. Applying the maximum principle to

$$\zeta \equiv \frac{\partial \tilde{w}_\varepsilon}{\partial s} - AW, \qquad A = \tilde{A}_\varepsilon e^{\lambda(s_\varepsilon+1)}$$

in $\tilde{\Omega}_\varepsilon \cap \{s > s_\varepsilon + 1\}$ we deduce that $\zeta \leq 0$, and (6.31) follows.

Now let $n = 3$. Setting

$$\alpha_0 = \int_0^1 e^{(1-z^2)/4}\, dz > 1,$$

we know from Theorem 5.1 that

$$\text{(6.32)} \qquad \begin{aligned} w_\infty(y) &= \frac{5}{2+3\alpha_0}\int_0^y e^{(1-z^2)/4}\, dz, \\ w_\infty(1) &= \frac{5\alpha_0}{2+3\alpha_0} > 1. \end{aligned}$$

Further, for small $\gamma > 0$, if W_γ is the solution of

$$\begin{aligned} W_{yy} + \frac{y}{2}W_y + \gamma W &= 0, & 0 < y < 1, \\ \frac{1}{2}W_y + \frac{3-4\gamma}{4}W &= \frac{5}{4}, & y = 1, \\ W(0) &= 0, \end{aligned}$$

then $W_\gamma(y) > 0$ if $\varepsilon \leq y \leq 1$ provided γ is sufficiently small. We can now apply the maximum principle to $\zeta = \partial\tilde{w}_\varepsilon/\partial s - Ae^{-\gamma s}W_\gamma(y)$ and deduce that $\zeta \leq 0$ in $\tilde{\Omega}_\varepsilon \cap \{s > s_\varepsilon + 1\}$, and the assertion (6.31) follows.

Proof of Theorem 5.2 *in case* $n = 2$. The functions

$$e^{-\gamma s}w(y, s) \quad \text{and} \quad e^{-\gamma s}w_\infty(y)$$

satisfy the same parabolic equation in Ω_T:

$$W_s + \mathscr{L}_2 W + \gamma W = 0,$$

with positive lowest order coefficient if

$$\gamma = \frac{1}{y_0^2} \quad \text{where } y_0 = \alpha e^{-T/2}.$$

Applying the maximum principle we conclude that $e^{-\gamma s}w \leq e^{-\gamma s}w_\infty$ in Ω_T, i.e., $w \leq w_\infty$ in Ω_T. Since T is arbitrary,

$$\text{(6.32)} \qquad w \leq w_\infty \quad \text{in } \Omega.$$

Next we define $\tilde{w}_\varepsilon$ as in case $n \geq 3$. Applying the maximum principle to

$$\zeta = e^{-\gamma s}\tilde{w}_\varepsilon$$

and to

$$\zeta = e^{-\gamma s}(\tilde{w}_\varepsilon - w)$$

in $\tilde{\Omega}_\varepsilon$, where $\gamma = 1/\varepsilon^2$, we deduce that

$$0 \le \tilde{w}_\varepsilon \le w \quad \text{in } \tilde{\Omega}_\varepsilon . \tag{6.33}$$

Similarly we deduce that

$$\zeta = \tilde{w}_\varepsilon(y, s+h)e^{-\gamma(s+h)} - \tilde{w}_\varepsilon(y, s)e^{-\gamma s}$$

is ≥ 0 in $\tilde{\Omega}_\varepsilon$ and, consequently,

$$\frac{\partial \tilde{w}_\varepsilon}{\partial s} \ge 0 \quad \text{in } \tilde{\Omega}_\varepsilon . \tag{6.34}$$

Recalling (6.32), it follows that $\partial \tilde{w}_\varepsilon / \partial s \to 0$ "weakly" as $s \to \infty$, i.e.,

$$\int_\Sigma^{\Sigma+1} \frac{\partial \tilde{w}_\varepsilon(y, s)}{\partial s} ds \to 0 \quad \text{if } \Sigma \to \infty, \ \text{for all } \varepsilon \le y \le 1 . \tag{6.35}$$

Set

$$\widetilde{W}_\varepsilon(y, s) = \int_s^{s+1} \tilde{w}_\varepsilon(y, s')\, ds' .$$

Then

$$\begin{aligned} \frac{\partial \widetilde{W}_\varepsilon}{\partial s} + \mathscr{L}_2 \widetilde{W}_\varepsilon &= 0 \quad \text{in } \tilde{\Omega}_\varepsilon , \\ \frac{\partial \widetilde{W}_\varepsilon}{\partial s} + \frac{1}{2}\frac{\partial \widetilde{W}_\varepsilon}{\partial y} + \frac{1}{4}\widetilde{W}_\varepsilon &= \frac{3}{4}, \qquad y = 1,\ s > s_\varepsilon , \\ \widetilde{W}_\varepsilon(\varepsilon, s) &= 0, \qquad s > s_\varepsilon , \\ \widetilde{W}_\varepsilon(y, s_\varepsilon) &\ge 0, \qquad \varepsilon \le y \le 1 . \end{aligned}$$

Further, by (6.32) and $w_\infty(y) \le w_\infty(1) \le \frac{3}{2}$, and (6.35),

$$0 \le \widetilde{W}_\varepsilon \le \frac{3}{2}, \quad 0 \le \frac{\partial \widetilde{W}_\varepsilon}{\partial s} \le \frac{3}{2},$$

$$\frac{\partial \widetilde{W}_\varepsilon}{\partial s} \to 0 \quad \text{as } s \to \infty, \ \text{for each } y \in (\varepsilon, 1) .$$

It follows that $\widetilde{W}_\varepsilon(y, s) \uparrow w_\varepsilon(y)$ as $s \uparrow \infty$ where w_ε is a solution to problems (P_ε^∞) $(n = 2)$, and $w_\varepsilon(y) > 0$ if $\varepsilon < y < 1$. The convergence is uniform in y, by Dini's theorem.

Noting that $\widetilde{W}_\varepsilon(y, s) \le \tilde{w}_\varepsilon(y, s+1)$ and recalling (6.33), we get

$$\varliminf_{s\to\infty} w(y, s) \ge w_\varepsilon(y), \qquad \varepsilon \le y \le 1 .$$

Finally, letting $\varepsilon \to 0$ and recalling also (6.32), the assertion (6.27) follows.

7. The effect of surface tension on the interface (general case)

In this section we shall extend the results of §6 to general shapes of D and G, showing that (small) surface tension increases the water region for large t; the fact that small surface tension decreases the water region for small t is valid whenever u can be shown to be continuous in $\overline{G_{t_*}}$ for some $t_* > 0$.

If

$$0 < c_1 \le \int_{\partial D} \tilde{\theta}(x, t) \le c_2 < \infty \quad \text{for all } t \ge 0 \tag{7.1}$$

then (Matano [15]) there exists a positive number T_0 such that the free boundary has the form (1.10) with $\rho \in C^\infty$ for all $t \ge T_0$. We shall also assume in this section that (1.11) holds for some $T > T_0$. Finally we assume that, for some $T^* > T$,

$$\begin{aligned} &\tilde{\theta}|_{\partial D_\infty} \text{ is independent of } t, \text{ if } t > T^*, \\ &\text{and, if } n = 2,\ \tilde{\theta}|_{\partial D_\infty} \text{ is constant if } t > T^*. \end{aligned} \tag{7.2}$$

Under this additional assumption it was proved by Matano [15] that the function $\rho(\theta, t)$ satisfies

$$|\rho(\theta, t) - Mt^{1/2}| \le C_0 \quad \text{if } t \ge t_0 \tag{7.3}$$

where M, C_0, t_0 are positive constants. From regularity results for the free boundary (cf. [6]) we further deduce that, for any $j > 0$,

$$|D^j(\rho(\theta, t) - Mt^{1/2})| \le C_j \quad \text{if } t > t_0. \tag{7.4}$$

We can therefore perform a change of variable $x \to \xi$ such that, for $t > t_0$, the free boundary becomes $|\xi| = Mt^{1/2}$ whereas the derivatives $\partial x_i/\partial \xi_j$ are $\delta_{ij} + O(t^{-1/2})$. Let us perform another change of variables $\xi \to (y, \varphi_1, \dots, \varphi_{n-1})$, $t \to s$, where

$$t = e^s, \qquad y = |\xi| e^{s/2}$$

and $\varphi_1, \dots, \varphi_{n-1}$ are the spherical coordinates of ξ. Then the function

$$w(y, \varphi_1, \dots, \varphi_{n-1}, s) = |\xi| u$$

satisfies (we take for simplicity $M = 1$),

(7.5)
$$w_s + \widetilde{\mathscr{L}}_n w \equiv w_s - w_{yy} + \mathscr{M} w - \left(\frac{n-3}{y} + \frac{y}{2}\right) w_y + \frac{n-3}{y^2} w = 0 \quad \text{in } \Omega,$$

$$w_s + \frac{1}{2} w_y + \mathscr{N} w + \frac{2n-3}{4} w = \frac{2n-1}{4} \quad \text{at } y = 1 \tag{7.6}$$

where

$$\mathscr{M} w = \sum a_{ij} \frac{\partial^2 w}{\partial \eta_i \partial \eta_j} + \sum b_i \frac{\partial w}{\partial \eta_i},$$

$$\mathcal{N}w = \sum c_i \frac{\partial w}{\partial \eta_i}, \qquad (\eta_1, \dots, \eta_n) = (y, \varphi_1, \dots, \varphi_{n-1}),$$

and

$$\sum |a_{ij}| + \sum |b_i| + \sum |c_i| \le \sigma(s), \qquad \sigma(s) \to 0 \quad \text{if } s \to \infty. \tag{7.7}$$

Equations (7.5), (7.6) are taken of course in the weak sense corresponding to the definition of the weak solution u.

Theorem 7.1. *For any* $0 < \eta < 1$ *the solution* w *satisfies a.e.*

$$w(y, s) \to w_\infty(y) \quad \text{as } s \to \infty, \text{ uniformly in } y, \ \eta \le y \le 1, \tag{7.8}$$

where $w_\infty(y)$ *is the solution of* (5.1)–(5.3), *as described in Theorems* 5.1 *and* 5.2.

Proof. Set

$$\mathcal{L}_n w \equiv -w_{yy} - \left(\frac{n-3}{y} + \frac{y}{2}\right) w_y + \frac{n-3}{y^2} w$$

and consider first the case $n \ge 3$. Let $\tilde{w}_{\varepsilon,\delta}$ $(\delta > 0)$ denote the solution of

$$\frac{\partial w}{\partial s} + \mathcal{L}_n w = -\delta \quad \text{in } \widehat{\Omega}_\varepsilon \equiv \{\varepsilon < y < 1, s > \sigma_\varepsilon\}, \tag{7.9}$$

$$\frac{\partial w}{\partial s} + \frac{1}{2}\frac{\partial w}{\partial y} + \frac{2n-3}{4} w = \frac{2n-1}{4} - \delta \quad \text{on } y = 1, \tag{7.10}$$

$$w = 0 \text{ on } \{y = \varepsilon, s > \sigma_\varepsilon\} \quad \text{and on} \quad \{\varepsilon < y < 1, s = \sigma_\varepsilon\} \tag{7.11}$$

where σ_ε is sufficiently large, depending on δ. As in the proof of (6.31) one can show that

$$\zeta = \pm \frac{\partial \tilde{w}_{\varepsilon,\delta}}{\partial s} - AW \le 0 \quad \text{in } \widehat{\Omega}_\varepsilon$$

and thus

$$\left|\frac{\partial \tilde{w}_{\varepsilon,\delta}}{\partial s}\right| \le Ae^{-\lambda s} \qquad (\lambda > 0).$$

It follows that

$$\tilde{w}_{\varepsilon,\delta}(y, s) \to w_{\varepsilon,\delta}(y) \quad \text{as } s \to \infty. \tag{7.12}$$

From (7.7) we deduce that if σ_ε is sufficiently large then $\tilde{w}_{\varepsilon,\delta}$ satisfies

$$\frac{\partial w}{\partial s} + \widetilde{\mathcal{L}}_n w \le 0 \quad \text{in } \widehat{\Omega}_\varepsilon,$$

$$\frac{\partial w}{\partial s} + \frac{1}{2}\frac{\partial w}{\partial y} + \mathcal{N}w + \frac{2n-3}{4} w \le \frac{2n-1}{4} \quad \text{on } y = 1.$$

If $w(y, s)$ were smooth then by the maximum principle we could deduce that

$$\tilde{w}_{\varepsilon,\delta}(y, s) \le w(y, s) + AW \tag{7.13}$$

where W is the function used in the proof of (6.31), with smaller λ if $n \ge 4$

and slightly modified when $n = 3$:

$$W = e^{-\gamma s}\widehat{W}_\gamma(y)$$

where

$$\begin{gathered}\mathscr{L}_3\widehat{W}_\gamma - \gamma\widehat{W}_\gamma = \gamma, \qquad 0 < y < 1, \\ \widehat{W}_\gamma(0) = 0, \qquad \frac{1}{2}\widehat{W}_\gamma'(1) + \frac{3-4\gamma}{4}\widehat{W}_\gamma(1) = \gamma;\end{gathered}$$

with this modification,

$$\begin{gathered}W_s + \widetilde{\mathscr{L}}_n W \geq 0 \quad \text{in } \widehat{\Omega}_\varepsilon, \\ \frac{\partial W}{\partial s} + \frac{1}{2}\frac{\partial W}{\partial y} + \mathscr{N}W + \frac{2n-3}{4}W \geq 0 \quad \text{on } y = 1\end{gathered}$$

not only for $n \geq 4$ but also for $n = 3$.

From (7.13) we deduce that

$$\varliminf_{s\to\infty} w(y, s) \geq w_{\varepsilon,\delta}(y). \tag{7.14}$$

Since further

$$w_{\varepsilon,\delta}(y) \to w_\varepsilon(y) \quad \text{if } \delta \to 0$$

and

$$w_\varepsilon(y) \to w_\infty(y) \quad \text{if } \varepsilon \to 0,$$

it follows that

$$\varliminf_{s\to\infty} w(y, s) \geq w_\infty(y) \tag{7.15}$$

provided w is smooth.

For the actual weak solution w the inequality (7.13) can be established by working with finite differences, and then (7.15) follows provided "$\varliminf$" is taken in the "essential lim inf" sense.

It remains to prove that

$$\varlimsup_{s\to\infty} w(y, s) \leq w_\infty(y). \tag{7.16}$$

(The uniform convergence in y asserted in Theorem 7.1 follows from the estimates from which (7.15), (7.16) are derived.)

Let $\hat{w}_{\varepsilon,\delta}$ be the solution of

$$\mathscr{L}_n w = \delta, \qquad \varepsilon < y < 1, \tag{7.17}$$

$$\frac{1}{2}w'(1) + \frac{2n-3}{4}w(1) = \frac{2n-1}{4} + \delta, \tag{7.18}$$

$$w(\varepsilon) = 0, \tag{7.19}$$

for any $0 < \delta < 1$. Clearly

$$\hat{w}_{\varepsilon,\delta} \geq 0 \tag{7.20}$$

and we also have an upper bound

$$\hat{w}_{\varepsilon,\delta} \le C \tag{7.21}$$

where C is a constant independent of ε, δ. Indeed, if $n \ge 4$ then $C = 1 + (2n-1)/(2n-3)$ is a supersolution of (7.17)–(7.19) if $\delta < 1$, whereas if $n = 3$ we can take $C - y^2$ as a supersolution provided C is a sufficiently large positive constant.

From (7.20), (7.21) and a compactness argument, $\lim_{\varepsilon\to 0} \hat{w}_{\varepsilon,\delta}(y) = \hat{w}_\delta(y)$ exists $(0 < y \le 1)$ for a sequence $\varepsilon \to 0$, and $\hat{w}_\delta$ satisfies (7.17) with $\varepsilon = 0$, and (7.18). Since $\zeta \equiv Cy^\lambda$ is a supersolution (i.e. $\mathscr{L}_n \zeta \ge \delta$) in $0 < y < y_0$ for some constants $C > 0$, $0 < \lambda < 1$ and small enough y_0, we have (cf. (5.10)) $\hat{w}_\delta(y) \le Cy^\lambda$. It follows that $\lim_{\delta\to 0} \hat{w}_\delta(y) \equiv \hat{w}(y)$ $(0 < y \le 1)$ exists for some sequence $\delta \to 0$, and $\hat{w}$ satisfies (5.1)–(5.3). By uniqueness, $\hat{w} \equiv w_\infty$, i.e.,

$$\hat{w}_\delta(y) \downarrow w_\infty(y) \quad \text{if } \delta \downarrow 0 \qquad (0 < y \le 1). \tag{7.22}$$

Arguing as in the proof of (7.13) we find by comparison that

$$w(y, s) \le \hat{w}_\delta(y) + AW \quad \text{in } \Omega \cap \{s > \hat{\sigma}_\varepsilon\}. \tag{7.23}$$

Letting $s \to \infty$ we get

$$\overline{\lim_{s\to\infty}}\, w(y, s) \le \hat{w}_\delta(y),$$

and together with (7.22), the assertion (7.16) follows in case $n \ge 3$.

The proof for $n = 2$ requires some modifications. For any $\varepsilon \in (0, 1)$, if λ is a sufficiently small positive number then, by standard ODE arguments there exists a positive solution $w = w_{\varepsilon,\lambda}(y)$ of

$$\mathscr{L}_2 w - \lambda w = 0, \qquad \varepsilon < y < 1,$$
$$\tfrac{1}{2}w'(1) + \tfrac{1}{4}w(1) - \lambda w(1) = \tfrac{3}{4},$$
$$w(\varepsilon) = 0.$$

We can then apply the maximum principle to

$$\zeta \equiv \left(\pm\frac{\partial \tilde{w}_{\varepsilon,\delta}}{\partial s} - Ae^{-\lambda s} w_{\varepsilon,\lambda}\right) e^{\gamma s}, \qquad \gamma = \frac{1}{\varepsilon^2},$$

in $\widehat{\Omega}_\varepsilon$ to deduce that $\zeta \le 0$; hence,

$$\left|\frac{\partial \tilde{w}_{\varepsilon,\delta}}{\partial s}\right| \le Ae^{-\lambda s} \qquad (\lambda = \lambda(\varepsilon)).$$

It follows that (7.12) holds and then also (7.14) is valid, for any small $\delta > 0$. The assertion (7.15) now follows from (7.14), as before.

To prove (7.16) we can establish for $n = 2$ the existence of unique positive solution $\hat{w}_{\varepsilon,\delta}$ of (7.17)–(7.19), as in the case $\delta = 0$, and also show that

$$\hat{w}_{\varepsilon,\delta} \uparrow \hat{w}_\delta \quad \text{as } \varepsilon \downarrow 0,$$
$$\hat{w}_\delta \downarrow w_\infty \quad \text{as } \delta \downarrow 0.$$

Thus it remains only to establish an estimate of the form (cf. (7.23))

$$w(y, s) \le \hat{w}_\delta(y) + Ae^{-\lambda s} w(y).$$

This follows by comparison provided we can take w a solution of

$$\mathscr{L}_2 w - \lambda w = 0, \qquad 0 < y < 1,$$
$$\tfrac{1}{2}w'(1) + \tfrac{1}{4}w(1) - \lambda w(1) \ge 0.$$

But w can be constructed as a limit of solutions $w_{\varepsilon,\lambda}$ $(\varepsilon \downarrow 0)$ by the method of proof of §5 for $n = 2$ (with $\lambda = 0$) provided λ is small.

8. The two-phase Stefan problem

In this section we conisder the two-phase Stefan problem with surface tension:

$$
\begin{aligned}
&\partial\theta_1/\partial t - \Delta\theta_1 = 0 \quad \text{in } G_T^1, \\
&\partial\theta_2/\partial t - \Delta\theta_2 = 0 \quad \text{in } G_T^2, \\
&\theta_1 = \tilde{\theta}_1 \quad \text{on } \partial D_T^1 \cup (G_1 \times \{0\}), \\
&\theta_2 = \tilde{\theta}_2 \quad \text{on } \partial D_T^2 \cup (G_2 \times \{0\}), \\
&\theta_1 = \theta_2 = \varepsilon\kappa \quad \text{on the free boundary } \Gamma_T, \\
&(X_t + \nabla(\theta_1 - \theta_2)) \cdot N = 0 \quad \text{on } \Gamma_T
\end{aligned}
\tag{8.1}
$$

where $\tilde{\theta}_1 > 0$, $\tilde{\theta}_2 < 0$. Here D_1, D_2, G_0, G_1, G_2 are smooth domains satisfying:

$$\overline{D}_1 \subset G_0, \quad \overline{G}_0 \subset D_2, \quad G_1 = G_0 \backslash \overline{D}_1, \quad G_2 = D_2 \backslash \overline{G}_0.$$

It is well known that if $\varepsilon = 0$ then the problem has a unique weak solution (cf. [7]) whereas if the initial values are sufficiently smooth and satisfy some consistency conditions at ∂G_0 then there exists a classical solution for some small time interval $0 \le t \le T$ [7, 16]. We shall now assume that a classical solution for $\varepsilon = 0$ exists in some time interval $0 \le t \le T$, and denote it by $(\theta_{01}, \theta_{02}, \Gamma_T)$. We consider for small $\varepsilon > 0$ the approximate solution $(\theta_{01} + \varepsilon u_1, \theta_{02} + \varepsilon u_2)$ with u_1 defined in the water region G_T^1 and u_2 defined in the ice region G_T^2. We shall derive a parabolic system for u_1, u_2 and then prove that it has a unique weak solution.

Introducing the notation (1.9) for Γ_T and denoting by $N = N_0 + \varepsilon N_1$ the normal for the interface $x = z + \varepsilon\zeta$ of the ε-perturbed new free boundary, we find as in §1, that if R is defined by (1.15) then

$$R(s, t) = \frac{\kappa - u_i}{\nabla\theta_{0i} \cdot N_0} = h_i(u_i - \kappa), \qquad h_i = -\frac{1}{\nabla\theta_{0i} \cdot N_0} > 0. \tag{8.2}$$

Further, analogously to (1.17) we get

$$\begin{aligned}(8.3)\qquad & N_0 \cdot N_0 R_t + \sum (z_t + \nabla(\theta_{01} - \theta_{02})) \cdot A_j N_0 R_{s_j} \\ & + \{(z_t + \nabla(\theta_{01} - \theta_{02})) \cdot \sum A_j N_{0,s_j} + N_0 \cdot N_{0t} + N_0 \\ & \cdot (\nabla \partial_{x_1}(\theta_{01} - \theta_{02}) \cdot N_0, \dots, \nabla \partial_{x_n}(\theta_{01} - \theta_{02}) \cdot N_0)\} R \\ & + \nabla(u_1 - u_2) \cdot N_0 = 0.\end{aligned}$$

Setting

$$g_i = \nabla \theta_{0i} \cdot N_0$$

it follows from (8.2) that

$$(8.4)\qquad (-g_2)u_1 = (-g_1)u_2 - \kappa g \qquad (g = g_2 - g_1,\ g_1 < 0,\ g_2 < 0)$$

and (cf. (1.22))

$$(8.5)\qquad a\frac{du_1}{dt} + \sum b_j \frac{du_1}{ds_j} + N_0 \cdot \nabla(u_1 - u_2) + cu_1 = f_0$$

where

$$a = N_0 \cdot N_0 h_1 > 0$$

and b_i, c, f_0 are known functions having a structure similar to (1.23).

Set

$$G_T = G_T^1 \cup \Gamma_T \cup G_T^2$$

and extend g_1, g_2 and κ smoothly into all of $\overline{G}_T$ in such a way that g_1 and g_2 remain negative.

Notice that the function

$$(8.6)\qquad U = \begin{cases} (-g_2)u_1 & \text{in } G_T^1, \\ (-g_1)u_2 - \kappa g & \text{in } G_T^2 \end{cases}$$

is continuous in G_T.

To define a weak solution, we set

$$u = \begin{cases} u_1 & \text{in } G_T^1, \\ u_2 & \text{in } G_T^2 \end{cases}$$

and integrate by parts in

$$\int_{G_T^1 \cup G_T^2} (u_t - \Delta u)\varphi = 0, \qquad \varphi \text{ smooth}.$$

Using (8.4), (8.5) and the integration by parts formulas

$$\begin{aligned}\int_{G_T^1} \varphi_t \psi = & -\int_{G_T^1} \varphi \psi_t - \int_0^T (\varphi\psi)(z, t) z_t \cdot N_0 \, ds \, dt \\ & + \int_{G^1(T)} \varphi\psi \, dx - \int_{G^1(0)} \varphi\psi \, dx,\end{aligned}$$

$$\int_{G_T^2} \varphi_t \psi = -\int_{G_T^2} \varphi \psi_t + \int_0^T (\varphi\psi)(z, t) z_t \cdot N_0 \, ds\, dt + \int_{G^2(T)} \varphi\psi \, dx - \int_{G^2(0)} \varphi\psi\, dx,$$

we find that (cf. (2.6))

$$\begin{aligned} &-\int_{G_T} u\varphi_t + \int_{G_T} \nabla\varphi \cdot \nabla u \\ &- \int_0^T \int_L u_1 \left\{ \frac{d}{dt}(a\varphi) + \sum \frac{d}{ds_j}(b_j \varphi) - c_1 \varphi \right\} ds\, dt \\ &= \int_0^T \int_L \varphi f_1 \, ds\, dt \end{aligned} \tag{8.7}$$

where

$$c_1 = c + \frac{g}{g_1} z_t \cdot N_0, \qquad f_1 = f_0 + \frac{\kappa g}{g_1} z_t \cdot N_0,$$

and φ in any function in $C^1(\overline{G}_T)$ vanishing on $G(T) \cup \partial D_T^1 \cup \partial D_T^2$.

If u_i is smooth then

$$\partial u_i / \partial t - \Delta u_i = 0 \quad \text{in } G_T^i, \tag{8.8}$$

$$u_i = 0 \quad \text{on } \partial D_T^i \quad \text{and} \quad \text{on } G_i \times \{0\}, \tag{8.9}$$

and (8.4), (8.5) hold on Γ_T. We now define a weak solution.

Definition 8.1. A function $u = (u_1, u_2)$ is a weak solution of (8.8), (8.9), (8.4), (8.5) if

$$u_i, \nabla u_i \in L^2(G_T^i) \qquad (i = 1, 2)$$

and if (8.9) holds in the usual continuous sense, (8.4) holds in the trace sense, and (8.7) holds for any test function φ as above.

If we substitute $\varphi = U$ in (8.7) and proceed formally, we obtain, after some integrations by parts (cf. the derivation of (3.2)) an energy estimate

$$\begin{aligned} &\sup_{0<t<T} \int_{G(t)} u^2(x, t)\, dx + \int_{G_T} |\nabla u|^2 \\ &+ \sup_{0<t<T} \int_L u_1^2(z(s, t), t)\, ds \le C_T. \end{aligned} \tag{8.10}$$

To actually use an estimate of this type in order to establish existence of a weak solution, we resort to finite-difference approximations as in §3. Instead of the elliptic problem (3.3) we now have:

$$\frac{1}{h} u_j^k - \Delta u_j^k = \frac{1}{h} u_j^{k-1} \quad \text{in } G^{j,k} \qquad (j = 1, 2), \tag{8.11}$$

where $G^{j,k} = G^j(kh)$, and

(8.12)
$$\frac{1}{h}a^k u_1^k + N_0^k \cdot \nabla(u_1^k - u_2^k) + \sum b_j^k \frac{du_1^k}{ds_j} + c_1^k u_1^k = \frac{1}{h}a^k u^{k-1} + f_1^k \quad \text{on } \Gamma^k,$$

(8.13)
$$(-g_2^k)u_1^k = (-g_1^k)u_2^k - (\kappa g)^k \quad \text{on } \Gamma^k,$$

and

(8.14)
$$u_j^k = 0 \quad \text{on } \partial D^{j,k}$$

where $\partial D^{j,k} = \partial D^j(kh)$.

Setting
$$U^k = \begin{cases} (-g_2)^k u_1^k & \text{in } G^{1,k}, \\ (-g_1)^k u_2^k - (\kappa g)^k & \text{in } G^{2,k}, \end{cases}$$
the system (8.11)–(8.13) can be written in the form

(8.15)
$$-\operatorname{div}(A^k \nabla U^k) + B^k \cdot \nabla U^k + \frac{1}{h}U^k + C^k U^k + F^k = \sum \left(\beta_j^k \frac{\partial U^k}{\partial s_j} + \gamma U^k + \gamma_0\right)\delta$$

where
$$A^k = \begin{cases} (-g_2^k)^{-1} & \text{in } G^{1,k}, \\ (-g_1^k)^{-1} & \text{in } G^{2,k}, \end{cases}$$

and δ is the uniform mass of density 1 distributed over Γ^k.

Formally, if we multiply (8.15) by U^k and integrate over $G^k = G^{1,k} \cup \Gamma^k \cup G^{2,k}$, we get

(8.16)
$$\begin{aligned} &\int_{G^k} A^k |\nabla U^k|^2 + \frac{1}{h}\int_{G^k} (U^k)^2 + \int_L (U^k(z(s, kh), kh))^2\, ds \\ &\quad \le \varepsilon \int_{G^k} |\nabla U^k|^2 + C\int_{G^k} (U^k)^2 + C \\ &\qquad + \int_{\Gamma^k} \gamma (U^k)^2 + \int_{\Gamma^k} \sum \beta_j^k \frac{\partial (U^k)^2}{\partial s_j} \quad (\varepsilon \text{ small}) \end{aligned}$$

and the last integral is equal to
$$-\int_{\Gamma^k} \sum \frac{\partial \beta_j^k}{\partial s_i} (U^k)^2.$$

But for any small $\mu > 0$,
$$\int_{\Gamma^k} (U^k)^2 \le \frac{C}{\mu}\int_{G^k} (U^k)^2 + \mu \int_{G^k} |\nabla U^k|^2.$$

If we substitute this into (8.16) and choose μ small enough and then $h < \mu/C$, we get

$$\int_{G^k} |\nabla U^k|^2 + \frac{1}{h}\int_{G^k} (U^k)^2 + \int_L (U^k(z(s, kh), kh))^2 \, ds \le C_T. \tag{8.17}$$

Notice that if we approximate δ by smooth functions $\delta_1^j(s)\delta_2^j(\lambda)$ where λ is the distance of a point from Γ^k and $\delta_2^j(\lambda) \to \delta(0)$ (=Dirac measure at 0), then (8.15), (8.14) becomes a diffraction problem and by known results (e.g. [13]) it has a unique solution $U^{k,j}$. Since the derivation of (8.17) is valid also for $U^{k,j}$ with a constant C_T independent of j, letting $j \to \infty$ we obtain a solution U^k of (8.11)–(8.14).

We can next derive for the U^k an a priori estimate analogous to (3.12) and use it to conclude that a subsequence of

$$u^h(x, t) = u_j^k(x, kh) \quad \text{if } x \in G^{j,k}, \ \ kh \le t < (k+1)j$$

converges weakly to a weak solution of (8.8), (8.9), (8.4), (8.5).

Theorem 8.1. *There exists a unique weak solution of* (8.8), (8.9), (8.4), (8.5).

Proof. We have already proved existence. To prove uniqueness we take in the definition of weak solution the test function η defined by

$$-\eta_t = U, \qquad \eta(x, T) = 0 \quad (U \text{ as in } (8.6));$$

this can be justified by approximation, as in §4. Then

$$u_1 = -\frac{\eta_t}{(-g_2)}, \qquad u_2 = -\frac{\eta_t}{(-g_1)} + \frac{\kappa g}{(-g_1)},$$

and we can proceed as in §4, making use, in particular, of (4.9). After some integrations by parts, we end up with an inequality similar to (4.11), with the second integral on the left-hand side replaced by

$$\int_0^T \int_L \left\{ \frac{1}{2}|\nabla\eta|^2 g \left[\frac{1}{(-g_2)} - \frac{1}{(-g_1)}\right] + \frac{|N_0|^2}{g_1 g_2}\eta_t^2 + \nabla\eta \cdot N_0 \frac{g}{g_1 g_2}\eta_t \right\}$$

where we have made use of the relation $z_t \cdot N_0 = g$. Since $g = g_2 - g_1$, the integrand is bounded from below by $c\eta_t^2$ $(c > 0)$, and the proof can now be completed by using Gronwall's inequality.

Remark 8.1. In constructing a weak solution we used finite-difference approximation for both the parabolic equation and the boundary conditions. Actually one can use another scheme whereby the finite differencing is performed only with respect to the boundary condition on Γ_T. Then, for the one-phase problem we define the approximating solution by

$$u^h(x, t) = u_k^h(x, t), \qquad (k-1)h \le t < kh,$$

and replace $a\frac{du}{dt}$ by

$$\frac{1}{h}(au_k^h)(z(s, t), t) - \frac{1}{h}a(z(s, t), t)u_k^h(z(s, t-h), t-h),$$

and thus solve a parabolic equation in each interval $(k-1)h \le t \le kh$ with u^h continuous across $t = kh$, for each k. The same procedure can be used for the two-phase problem, finite differencing only (8.5), and applying [13, Chapter IV, §§1–8] to prove that the corresponding parabolic system has a unique solution.

9. The effect of surface tension on the interface

We shall consider in this section the effect of small surface tension on the free boundary. As we shall see, in contrast to the facts established in §§6, 7 for the one-phase Stefan problem, small surface tension, for the two-phase Stefan problem, decreases the water region for *all* time provided the data are radial or "close" to radial.

Let K_i be a solution of

$$\begin{gathered} \partial_t K_i - \Delta K_i \ge 0 \quad \text{on } G_T^i, \\ K_i = l_i \quad \text{on } \partial D_T^i \cup (G_i \times \{0\}), \\ K_i = \kappa \quad \text{on } \Gamma_T, \end{gathered} \tag{9.1}$$

and set $U_i = u_i - K_i$. Then

$$\begin{gathered} \partial_t U_i - \Delta U_i \le 0 \quad \text{in } G_T^i, \\ U_i = -l_i \quad \text{on } \partial D_T^i \cup (G_i \times \{0\}), \\ g_2 U_1 = g_1 U_2 \quad \text{on } \Gamma_T, \\ a\frac{dU_1}{dt} + \sum b_j \frac{dU_1}{ds_j} + cU_1 + (\nabla U_1 - \nabla U_2)\cdot N_0 \\ = (\nabla K_2 - \nabla K_1)\cdot N_0 \quad \text{on } \Gamma_T \end{gathered} \tag{9.2}$$

where a, b_j, c are easily determined by comparing with (8.3); in particular, $a > 0$.

Lemma 9.1. *If $\kappa > 0$ and, for some $l_i > 0$,*

$$(\nabla K_2 - \nabla K_1)\cdot N_0 \le 0 \quad \text{on } \Gamma_T, \tag{9.3}$$

and if u_i is in $C^1(\overline{G_T^i})$, then

$$u_i < \kappa \quad \text{on } \Gamma_T. \tag{9.4}$$

Proof. It suffices to prove that

$$U_i < 0 \quad \text{in } \overline{G_T^i}.$$

If the assertion is not true then, since $U_i < 0$ in $\overline{G_{t_*}^i}$ if t_* is small, there exists a $t_0 > 0$ such that $U_i < 0$ in $\overline{G_t^i}$ for all $t < t_0$ but $U_i(x_0, t_0) = 0$ for some $(x_0, t_0) \in \overline{G_{t_0}^i}$ where $i = 1$ or $i = 2$. By the maximum principle we deduce that (x_0, t_0) must belong to Γ_T, and

$$U_1(x_0, t_0) = U_2(x_0, t_0) = 0,$$

$$\frac{d}{dt}U_1 \ge 0, \qquad \frac{d}{ds_j}U_1 = 0, \qquad (\nabla U_1 - \nabla U_2)\cdot N_0 > 0$$

at (x_0, t_0). But then the last equation in (9.2) implies that $(\nabla K_1 - \nabla \dot{K}_2)\cdot N_0 > 0$ at (x_0, t_0), a contradiction to (9.3).

Consider the radial case where

$$G = \{\alpha < |x| < \beta\}, \qquad \tilde{\theta} = \tilde{\theta}(r, t), \qquad r = |x|,$$

with the free boundary condition

$$\theta = \frac{\varepsilon}{s(t)}, \qquad -\theta_{1,r} + \theta_{2,r} = \frac{ds}{dt}.$$

If $n \geq 3$ then taking $K_1 = K_2 = 1/r$ we have

$$\partial_t K_i - \Delta K_i = \frac{n-3}{r^3} \geq 0.$$

Since also u_i is in $C^1(\overline{G_T^i})$ is in this case (the proof is similar to that of Theorem 6.1), we conclude

Theorem 9.2. *In the radial case, if* $n \geq 3$, *then* (9.4) *holds, i.e., small surface tension decreases the water region for all* $0 \leq t \leq T$.

Notice that (still in the radial case), if K_i^0 is the solution of (9.1) satisfying

$$\partial_t K_i^0 - \Delta K_i^0 = 0 \quad \text{on } G_T^i$$

and

$$0 < l_i \equiv l_i^0 < \frac{1}{r} - \delta \quad \text{on } \partial D_T^i \cup (G_i \times \{0\})$$

where $0 < \delta$ is sufficiently small, then, by comparison with $K_i = \frac{1}{r}$, there exists $\varepsilon = \varepsilon(\delta, T) > 0$ such that

$$(\nabla K_1^0 - \nabla K_2^0) \cdot N_0 > \varepsilon \quad \text{on } \Gamma_T.$$

It then follows that for the nonradial case, if $n \geq 3$ and the data are "close" to radial, we can establish (9.4) for a smooth approximation of u_i (as constructed in Remark 8.1). Hence $u_i \leq \kappa$ on Γ_T.

Consider finally the radial case with $n = 2$; we further specialize to a free boundary $|x| = t^{1/2}$ $(t \geq 1)$, which corresponds to

$$\theta_{0,i} = f_i\left(\frac{|x|}{t^{1/2}}\right) = \alpha_i \int_{|x|/t^{1/2}}^{\infty} \zeta^{-1} e^{-\zeta^2/4}\, d\zeta + \beta_i$$

with appropriate α_i, β_i.

Introducing

$$W_i(y, s) = y e^{s/2} K_i(y e^{s/2}, e^s) \tag{9.5}$$

we find that the system of equations for the K_i becomes

$$\mathscr{L} W_i \equiv \frac{\partial W_i}{\partial s} - \frac{\partial^2 W_i}{\partial y^2} - \left(\frac{y}{2} - \frac{1}{y}\right)\frac{\partial W_i}{\partial y} - \frac{W_i}{y^2} \geq 0 \quad \text{in } \Omega_T^i \tag{9.6}$$

with initial and boundary conditions

$$\text{(9.7)}\qquad \begin{aligned} &W_i = 1 \quad \text{for } y = 1\,, \\ &W_i > 0 \quad \text{on the remaining part of } \partial\Omega_T^i \end{aligned}$$

where

$$\Omega_T^1 = \{(y\,,\,s)\,;\, \alpha e^{-s/2} < y < 1\,,\, 0 < s < T\}\,,$$
$$\Omega_T^2 = \{(y\,,\,s)\,;\, 1 < y < \beta e^{-s/2}\,,\, 0 < s < T\} \qquad (0 < \alpha < \beta < \infty)\,.$$

Let $\widehat{W}_i$ be the solution of $\mathscr{L}\widehat{W}_i = 0$ in $\widehat{\Omega}_T^i$ where

$$\widehat{\Omega}_T^1 = \{(y\,,\,s)\,;\, \alpha e^{-T/2} < y < 1\,,\, 0 < s < T\} \supset \Omega_T^1\,,$$
$$\widehat{\Omega}_T^2 = \{(y\,,\,s)\,;\, 1 < y < \beta\,,\, 0 < s < T\} \supset \Omega_T^2\,,$$

satisfying the initial and boundary conditions:

$$\begin{aligned} &\widehat{W}_i = 1 \quad \text{for } y = 1\,, \\ &\widehat{W}_1 = 0 \quad \text{if } y = \alpha e^{-T/2}\,, \\ &\widehat{W}_2 = 0 \quad \text{if } y = \beta\,, \\ &\widehat{W}_i = 0 \quad \text{if } s = 0\,. \end{aligned}$$

Then, by the maximum principle, $W_i > \widehat{W}_i$ in Ω_T^i (if we choose $W_i = \widehat{W}_i$ on the parabolic part of Ω_T^i) and

$$\text{(9.8)}\qquad \frac{\partial \widehat{W}_2}{\partial y} - \frac{\partial \widehat{W}_1}{\partial y} > \frac{\partial W_2}{\partial y} - \frac{\partial W_1}{\partial y} \quad \text{for } y = 1\,.$$

Applying the maximum principle to $\partial \widehat{W}_i/\partial s$ we deduce that

$$\text{(9.9)}\qquad \frac{\partial \widehat{W}_i}{\partial s} \geq 0\,.$$

We now integrate by parts in

$$\int_{\alpha\varepsilon^{-T/2}}^{1} \mathscr{L}\widehat{W}_1(y\,,\,s)\,dy = 0\,, \qquad \int_1^{\beta} \mathscr{L}\widehat{W}_2(y\,,\,s)\,dy = 0$$

and add the results; we obtain

$$\begin{aligned} \widehat{W}_{1\,,y}(1\,,\,s) - \widehat{W}_{2\,,y}(1\,,\,s) = &\int_{\alpha e^{-T/2}}^{1} \left(\partial_s \widehat{W}_1 + \frac{1}{2}\widehat{W}_1\right) + \int_1^{\beta} \left(\partial_s \widehat{W}_2 + \frac{1}{2}\widehat{W}_2\right) \\ &+ [\widehat{W}_{1\,,y}(\alpha e^{-T/2}\,,\,s) - \widehat{W}_{2\,,y}(\beta\,,\,s)]\,. \end{aligned}$$

Recalling (9.8) and noting that the last expression in brackets is positive (by the maximum principle), it follows that

$$\widehat{W}_{1\,,y}(1\,,\,s) - \widehat{W}_{2\,,y}(1\,,\,s) > 0\,,$$

and from (9.8), (9.5) we then deduce that the condition (9.3) is satisfied. Therefore, by Lemma 9.1:

Theorem 9.3. *If $n = 2$ and the free boundary is given by $|x| = t^{1/2}$ then* (9.4) *holds for all $T > 0$, i.e., small surface tension decreases the water region for all $t \geq 0$.*

Remark 9.1. One can extend Theorem 9.3 also to other radial free boundaries. For instance, if the free boundary has the form

$$|x| = s(t) \quad \text{where } \dot{s}(t) \leq 0 \text{ for } 0 < t < T$$

then, by choosing

$$K_1(x, t) = \frac{1}{s(t)} \quad \text{and} \quad K_2(x, t) = \frac{1}{s(t)},$$

the conditions of Lemma 9.1 are satisfied; consequently (9.4) holds for all $0 < t \leq T$.

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