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ON WEAK CONVERGENCE IN DYNAMICAL SYSTEMS TO SELF-SIMILAR PROCESSES WITH SPECTRAL REPRESENTATION

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ABSTRACT. Let (X, μ, T) be an aperiodic dynamical system. Set $S_m f = f + \cdots + f \circ T^{m-1}$, where f is a measurable function on X. Let $Y(t)$ be one of a class of self-similar process with a "nice" spectral representation, for instance, either a fractional Brownian motion, a Hermite process, or a harmonizable fractional stable motion. We show that there is an f on X, and constants $A_m \to +\infty$ so that

$$A_m^{-1} S_{[mt]} f \stackrel{d}{\Rightarrow} Y(t),$$

the convergence being understood in the sense of weak convergence of all finite dimensional distributions in t.

1. INTRODUCTION

Let $(X, \mathscr{A}, \mu, T)$ be a dynamical system, which is to say that $(X, \mathscr{A}, \mu)$ is a Lebesgue space with $\mu(X) = 1$, and $T: X \to X$ preserves μ-measure. For a measurable function f on X, set $S_m f = f + \cdots + f \circ T^{m-1}$. We are interested in the weak convergence of the sums $S_m f$, as a function of m. To be more precise, the problem we consider is this: Start with a stochastic process $Y(t)$. To avoid some trivial cases, assume that for all $t > 0$, $Y(\cdot)$ is continuous in probability at t, and the distribution of $Y(t)$ is nondegenerate. We then ask: is there a measurable function f on X, and constants $A_m \to +\infty$ so that

$$A_m^{-1} S_{[mt]} f \stackrel{d}{\Rightarrow} Y(t)? \tag{1.1}$$

Here, convergence is in the sense of weak convergence of all finite dimensional distributions in t.

Previous results of this type have been in specific kinds of dynamical systems, for instance ergodic Markov shifts in [GL] and certain flows on manifolds in [S], to cite two well-known references. Moreover, the limiting process $Y(t)$ has been a Brownian motion. The goal of this paper is to show that in any aperiodic dynamical system, (1.1) holds for a reasonably broad class of stochastic

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processes, which include the Brownian motion. These results generalize (and were motivated by) the interesting Central Limit Theorem of [BD].

Before stating our results, we note that (1.1) already imposes some necessary conditions on the stochastic process $Y(t)$. Namely, $Y(0)=0$; $Y(t)$ has stationary increments; and [Lam] $Y(t)$ enjoys the following time rescaling property: for some $H>0$,

$$a^{-H}Y(at) \stackrel{d}{=} Y(t), \qquad a>0. \tag{1.2}$$

Similar to (1.1), equality here is understood to be with respect to all finite dimensional distributions. Such a process is referred to in the literature as a self-similar of index H, stationary increment process. We shall refer to this briefly as H-ss si. The interest in these processes is due in part to their usefulness in modeling various physical phenomena, especially those with long-range dependence. We refer the reader to the surveys [T2 and M].

A last necessary condition for (1.1) is $A_m = m^H L(m)$, where $L(m)$ is slowly varying at infinity. [Lam] again. Also, if $Y(t)$ is square integrable, normalize it so that $EY^2(1)=1$. Then $EY^2(t)=t^{2H}$, so that one has $0<H\le 1$. The case $H=1$ is trivial, as $Y(t)$ can be realized as $t\cdot Y(1)$, and so is excluded from consideration in this paper.

We show that several kinds of processes can arise in (1.1). Three examples are the fractional Brownian motions, briefly fBm [MvN]; the Hermite processes, of index $1/2<H<1$ [Dob, Maj]; and the harmonizable fractional stable motions [CM, T1]. The unifying element behind these three classes of processes is an especially nice spectral representation. In this section, we shall only state a result in the easiest case of the fBm as the other two cases are almost corollaries of the proof in that case. The modifications needed are sketched in §4 below.

A process $B_H(t)$ is a standard fBm of index H if $B_H(t)$ is H-ss si; is a real-valued Gaussian process; and $EB_H(t)^2 = 1$. Note that $B_{1/2}(t)$ is a standard Brownian motion.

Recall that $(X, \mathscr{A}, \mu, T)$ is aperiodic if for all $A\in\mathscr{A}$ with $\mu(A)>0$, and integers m, there is a measurable $B\subset A$ so that $\mu(T^m B\cap B)<\mu(B)$.

Theorem 1.1. *Let* $(X, \mathscr{A}, \mu, T)$ *be aperiodic, and* $0<H<1$. *Then there is an* $f\in L^2(\mu)$ *so that*

$$m^{-H}S_{[mt]}f \stackrel{d}{\Rightarrow} B_H(t).$$

In fact, we can prove a stronger almost everywhere result, see §3. By taking $t=1$ in this theorem, we obtain the Central Limit Theorem of [BD] mentioned above. We mention that the techniques of [BD] do *not* allow the deduction of a functional limit theorem, like this one.

One would like to apply the weak convergence results of this paper to dynamical systems, just as previous results, with the Brownian motion as a limit, have found applications (See [DK, PT, PUZ, and the survey D].) First however, information about the kinds of functions satisfying our theorems must be developed.

2. Spectral representation

Fix $0 < H < 1$, and consider the stationary sequence $\xi_k = B_H(k+1) - B_H(k)$, for $k = 0, 1, 2, \dots$. This sequence admits a representation

$$\xi_k = \int_{-\pi}^{\pi} e^{ik\lambda} W(d\lambda),$$

where W is a Gaussian random spectral measure, with control measure G. $W(\cdot)$ can be defined this way: For all $A, B \subset (0, \pi)$

(i) $W(A)$ is a complex Gaussian random variable of mean zero and variance $G(A)$;

(ii) $EW(A)\overline{W(B)} = G(A \cap B)$;

(iii) $W(-A) = \overline{W(A)}$;

(iv) $W(\bigcup_1^n A_i) = \sum_{i=1}^n W(A_i)$ for mutually disjoint $A_1, \dots, A_n$.

Further, for any $f \in L^2(G)$,

$$\left\| \int_{-\pi}^{\pi} f(\lambda) W(d\lambda) \right\|_2 = \left(\int_{-\pi}^{\pi} |f(\lambda)|^2 G(d\lambda) \right)^{1/2}. \tag{2.1}$$

We also note that as $H < 1$, $\frac{1}{m} B_H(m) \to 0$ a.s. so that in particular $G(\{0\}) = 0$.

Set $K_m(\lambda) = \sum_{k=0}^{m-1} e^{ik\lambda} = (e^{im\lambda} - 1)/(e^{i\lambda} - 1)$. Then

$$B_H(m) = \int_{-\pi}^{\pi} K_m(\lambda) W(d\lambda).$$

For the proof of Theorem 1.1, it is convenient to work with a discrete spectral measure, whose properties are collected together in the next lemma.

Let $\{a_j : j \geq 0\}$ be decreasing to 0, with $a_0 = \pi$. Then the intervals $I_j = [a_j, a_{j-1})$, $j \geq 1$, partition $(0, \pi)$. Let $\lambda_j \in I_j$ be rational with respect to π. Define a new Gaussian random spectral measure $V(\cdot)$ on $(0, \pi)$ by $V(\{\lambda_j\}) = W(I_j)$. This measure is defined on the same probability space as W. Set $l(m) = \lambda_{[\log m]}$, for $m \geq 4$.

Lemma 2.1. *The intervals I_j above can be chosen so that the following three conditions hold.*

$$l(m) \geq (\log m)^{-1}, \qquad m \geq 4, \tag{2.2}$$

$$\left| \int_{l(m)}^{\pi} K_m(\lambda) W(d\lambda) \right|, \qquad \left| \int_{l(m)}^{\pi} K_m(\lambda) V(d\lambda) \right| = O((\log m)^2) \quad \textit{a.s.}, \tag{2.3}$$

and

$$\left| \int_0^{l(m)} K_m(\lambda)(W - V)(d\lambda) \right| = O((\log m)^2) \quad \textit{a.s.} \tag{2.4}$$

Proof. To satisfy (2.2), choose the intervals I_j so that

$$I_{[\log m]} \subset ((\log m)^{-1}, \pi), \qquad m \geq 4,$$

which can clearly be accomplished. In addition, require that

$$\sup_{j>\log m} \operatorname{length}(I_j) \le m^{-2}. \tag{2.5}$$

To see (2.3) for $W(\cdot)$, observe that by (2.1),

$$\left\| \int_{l(m)}^{\pi} K_m(\lambda) W(d\lambda) \right\|_2 \le C \sup_{l(m)\le\lambda<\pi} |K_m(\lambda)| \le C(\log m) \tag{2.6}$$

Here, we have used (2.2) and the following inequality for K_m.

$$|K_m(\lambda)| \le C/|\lambda|, \qquad 0 < |\lambda| \le \pi. \tag{2.7}$$

(C is a positive constant, which might change from line to line.) Using the exponential-squared tail of Gaussian random variables, one easily sees that (2.6) implies (2.3), for $W(\cdot)$. The proof for $V(\cdot)$ is completely similar.

To see (2.4), use (2.5) and (2.1) to see that

$$\begin{aligned} \left\| \int_0^{l(m)} K_m(\lambda)(W - V)\, d\lambda \right\|_2 &\le C \sup_{j\ge\log m} \sup_{\lambda, \lambda' \in I_j} |K_m(\lambda) - K_m(\lambda')| \\ &\le C \|K_m'\|_\infty \sup_{j\ge\log m} \operatorname{length}(I_j) \le C. \end{aligned}$$

This is more than enough to prove (2.4). □

3. ASIP

We can in fact prove a stronger form of Theorem 1.1 by suitably enlarging the dynamical system $(X, \mathscr{A}, \mu, T)$. For $0 < H < 1$, let $B_H(t)$ be a standard fBm, with underlying probability space $(\Omega, \mathscr{F}, P)$. Define a new dynamical system $(\overline{X}, \overline{\mathscr{A}}, \overline{\mu}, \overline{T})$ by

$$\overline{X} = X \otimes \Omega, \qquad \overline{\mathscr{A}} = \mathscr{A} \otimes \mathscr{F}, \qquad \overline{\mu} = \mu \otimes P, \qquad \overline{T} = T \otimes \mathrm{Id}_\Omega.$$

For a measurable function from $\overline{X}$, set $\overline{S}_m f = \sum_{k=0}^{m-1} f \circ \overline{T}^k$.

Theorem 3.1. *Let $(X, \mathscr{A}, \mu, T)$ be aperiodic, and fix $0 < H < 1$. Then there is a standard fBm $\overline{B}_H$ defined on $(\overline{X}, \overline{\mathscr{A}}, \overline{\mu})$, and an $\overline{f} \in L^2(\overline{\mu})$ so that*

$$|\overline{S}_m \overline{f} - \overline{B}_H(m)| = O((\log m)^2) \quad \textit{a.e. } (\overline{X})$$

This result is known in probability theory as an Almost Sure Invariance Principle (ASIP). Its strength lies in the fact that it not only proves Theorem 1.1 (on $\overline{X}$), it also shows that a wide range of almost sure fluctuation results for B_H are inherited by the sums $\overline{S}_m \overline{f}$. This point is explained in the introduction to [PS], in the case of the Brownian motion. (Yet another almost sure result inherited by $\overline{S}_m \overline{f}$ is seen from Theorem 1 of [LP].) The only known results of this type for fBm concern generalizations of the law of the iterated logarithm. See [TC, O, and Or].

Also, it appears that Theorem 3.1 is the first ASIP with the fBm as a limit. Every similar result known to the author approximates partial sums by a process with independent increments.

Noting that (1.1) is an isomorphic invariant of the dynamical system, we see that in many cases there is no loss of generality in working with $\overline{X}$. Moreover, it will be clear that much of our proof could be carried out on X alone, at the cost of obscuring the proof with more detail. See comments at the end of this section.

Proof. We divide the proof into three steps.

Step 1. Let J be an integer, and $\varepsilon > 0$. A set $F \in \mathscr{A}$ is called an (J, ε) *tower* if $F, TF, \ldots, T^{J-1}F$ are disjoint, and $\mu(\bigcup_{k=0}^{J-1} T^k F) > 1 - \varepsilon$. Rochlin's lemma states that there is an (J, ε) tower in X for all pairs (J, ε).

Let λ be rational with respect to π. With towers we will be able to construct a special function g so that $S_m g(x)$ "acts like" $K_m(\lambda)$ for an arbitrarily large number of m's. (This is sufficient in view of (2.3) above.) It is done this way: let J and N be integers, and $\varepsilon > 0$. The second integer N should further satisfy $e^{iN\lambda} = 1$. Let F be a (JN, ε) tower. Call the following function *special for* $(\lambda, F, J, N, \varepsilon)$.

$$g(x) = \begin{cases} e^{ik\lambda}, & x \in T^k F,\ 0 \le k < JN, \\ 0, & \text{otherwise}. \end{cases}$$

The properties of g which we need are

$$S_m(g)(x) = g(x) K_m(\lambda), \qquad x \in \bigcup_{0}^{(J-1)N-1} T^k F,\ m \le N, \tag{3.1}$$

$$S_N(g)(x) = 0, \qquad x \in \bigcup_{0}^{(J-1)N-1} T^k F;$$

and

$$|S_m(g)(x)| \le C/|\lambda|, \quad \text{all } m \text{ and } x. \tag{3.2}$$

The first property follows immediately from the definition of g; the second follows from the choice of N; and the last follows from the first two, and (2.7).

Step 2. We will now construct the function $\overline{f}$, and the process $\overline{B}_H$. Let $\{\lambda_i\}$ be as in Lemma 2.1. We will need a special function for each λ_j, the parameters of which are determined in this way. Choose integers N_j so that

$$e^{iN_j\lambda_j} = 1, \quad \text{all } j$$

and

$$N_{[\log m]} > m, \qquad m \ge 4. \tag{3.3}$$

Choose integers J_j so that $\sum_j J_j^{-1} < +\infty$. Let $\varepsilon_j = J_j^{-1}$, let F_j be a (J_jN_j, ε_j) tower, and g_j be special for $(\lambda_j, F_j, J_j, N_j, \varepsilon_j)$. Let

$$R_j = X \setminus \bigcup_0^{(J_j-1)N_j-1} T^k F_j.$$

Then

$$\sum_j \mu(R_j) < +\infty. \tag{3.4}$$

Let $\overline{f} = \sum_j g_j \otimes V_j + \overline{g_j \otimes V_j}$. Here we have set $V_j = V(\{\lambda_j\})$. ($V(\cdot)$ is the spectral measure of Lemma 2.1.) To see that $\overline{f} \in L^2(\overline{\mu})$, note that the V_j are independent random variables, moreover, $e^{i\lambda}V_j \stackrel{d}{=} V_j$ for $0 \le \lambda < 2\pi$. Thus for each $x \in X$, $\overline{f}(x, \cdot)$ is a Gaussian series, with

$$E(\overline{f}(x, \cdot))^2 \le 2\sum_j EV_j^2 = E\left(\int_0^\pi V(d\lambda)\right)^2 < +\infty. \tag{3.5}$$

Thus, the sum $\sum g_j(x)V_j$ converges a.s. for all x. Hence $\overline{f}$ is well-defined a.e. $(\overline{X})$ and is in $L^2(\overline{X})$.

We now define the process $\overline{B}_H$. It is enough to define a version of the spectral measure W, introduced in §2. In turn, it is enough to define W on $(0, \pi)$. Recall the intervals I_j of Lemma 2.1 partition $(0, \pi)$. Then for a measurable $A \subset (0, \pi)$, set

$$\overline{W}(A \cap I_j)(x, \omega) = \begin{cases} g_j(x)W(A \cap I_j)(\omega), & g_j(x) \ne 0, \\ W(A \cap I_j)(\omega), & g_j(x) = 0. \end{cases}$$

As W is a σ-finite measure almost surely, this defines $\overline{W}$. Also note that applying Lemma 2.1 to $\overline{W}$, we see that

$$\overline{V}_j = \overline{V}(\{\lambda_j\})(x, \omega) = \begin{cases} g_j(x)W(I_j)(\omega), & g_j(x) \ne 0, \\ W(I_j)(\omega), & g_j(x) = 0. \end{cases}$$

Step 3. We can now conclude the proof of Theorem 3.1. Set

$$h_m = \sum_{j \le \log m} g_j \otimes V_j \quad \text{and} \quad h^m = \sum_{j > \log m} g_j \otimes V_j.$$

By Lemma 2.1 it suffices to prove the following:

$$\overline{S}_m(h_m) = O((\log m)^2) \quad \text{a.e. } (\overline{X}), \tag{3.6}$$

and

$$\overline{S}_m(h^m) - \sum_{j \ge \log m} K_m(\lambda_j)\overline{V}_j = O(1) \quad \text{a.e. } (\overline{X}). \tag{3.7}$$

For the first result, fix $x \in X$. Then $\overline{S}_m h_m(x, \cdot)$ is a Gaussian random variable. Moreover,

$$(E|\overline{S}_m h_m(x, \cdot)|^2)^{1/2} \leq c \sup_{j \leq \log m} |S_m g_j(x)| \leq c(\log m),$$

by (2.2) and (3.2). Then (3.6) follows from this fact just as (2.3) follows from (2.6) above.

To see (3.7), for almost all $x \in X$ there is an m so that $x \in X \setminus \bigcup_{j \geq \log m} R_j$ by (3.4). Then, by (3.1) and (3.3), we have

$$\begin{aligned} \overline{S}_m h^m(x, \cdot) &= \sum_{j \geq \log m} S_m g_j(x) \otimes V_j \\ &= \sum_{j \geq \log m} g_j(x) K_m(\lambda_j) \otimes V_j \\ &= \sum_{j \geq \log m} K_m(\lambda_j) \overline{V}_j(x, \cdot), \end{aligned}$$

thus (3.7) holds. □

The proof above can be carried out on X, instead of $\overline{X}$. The only new step needed is to discretize the random variables V_j, to get random variables V_j'. Do this in such a way that

$$e^{ik\lambda_j} V_j' \overset{d}{=} V_j', \qquad k = 1, 2, \ldots.$$

As the V_j are independent random variables, the V_j' will be independent. Moreover, if the discretization is fine enough, one can work with the V_j', up to a $O((\log m)^2)$ a.e. error term. Then one constructs special functions g_j so that

$$g_j|_{F_j} \overset{d}{=} V_j',$$

$$g_j(T^k x) = e^{ik\lambda_j} g_j(x), \qquad x \in F_j, \ 0 \leq k < J_j N_j,$$

and $g_j(x) = 0$ for $x \in X \setminus \bigcup_0^{J_j N_j - 1} T^k F_j$. Lemma 4 of [BD] should be used to define the g_j. The rest of the details are left to the interested reader.

4. Further processes

We discuss some extensions of the proof of Theorem 3.1 mentioned in the Introduction. These examples below are not complete, but suggest the range of processes that can arise as a limit. Other examples can be found in §3 of [GS].

4.a. Hermite processes. The Hermite processes are square-integrable ss si processes, which can be represented as multiple Wiener-Itô integrals. We follow [Dob and Maj] in our presentation below. Let M be a complex Gaussian random spectral measure, with control measure G satisfying

$$G(cA) = c^q G(A), \qquad A \subset \mathbf{R},$$

for some q. Set $\phi_t(x) = (e^{itx} - 1)/(ix)$, fix an integer k, and let $0 < q < 1/k$. Then define

$$(4.1) \qquad Y(t) = C_{qk} \int_{\mathbf{R}^k}' \phi_t(x_1 + \cdots + x_k)|x_1|^q \cdots |x_k|^q M(dx_1) \cdots M(dx_k).$$

This is a multiple Wiener-Itô integral, where the prime on the integral sign indicates that integration is not performed over the hyperplanes $|x_i| = |x_j|$, $1 \le i < j \le k$. $Y(t)$ is H-ss si, with $\frac{1}{2} < H = 1 - kq/2 < 1$. For $k = 1$, it is a fBm, and for $k \neq 1$, it has a nonclassical distribution. (For $k = 2$, the characteristic function can be found in [R, T3].) The constant C_{qk} is chosen so that $EZ^2(1) = 1$.

Theorem 4.1. *For each k and $0 < q < 1/k$ there is an $f \in L^2(X)$ so that*

$$(4.2) \qquad m^{1-kq/2} S_{[mt]} f \overset{d}{\Rightarrow} Y(t).$$

We indicate how the proof would go. First, fix a stationary Gaussian sequence ξ_j so that

$$E\xi_j \xi_0 \sim j^{-q}, \qquad j \to +\infty,$$

and write $\xi_j = \int_{-\pi}^{\pi} e^{ijx} W(dx)$. (The ξ_j can be taken to be $B_H(j+1) - B_H(j)$, where $H = 1 - q/2$.) It is then known that the "sums"

$$(4.3) \qquad Y_m = C_{qk} \int_{(-\pi,\pi)^k} K_m(x_1 + \cdots + x_k) W(dx_1) \cdots W(dx_k)$$

satisfy (4.2). (See §4 of [Mae] for a nice summary of this; §8 of [Maj] for a proof.) A critical point in the proof of this fact, which we also use, is that the representations (4.1) and (4.3) are increasing concentrated near the origin. (This is how (2.2) and (2.3) are used in the proof of Theorem 3.1.)

The methods of §3 allow one to easily approximate (4.3): Using the notation of that section, set

$$\overline{f}_k = \sum_{(j_1, \ldots, j_k)}' g_{j_1} \cdots g_{j_k} \otimes V_{j_1} \cdots V_{j_k} + \overline{g_{j_1} \cdots g_{j_k} \otimes V_{j_1} \cdots V_{j_k}}$$

where the prime on the sum indicates that the sum is *not* performed over the hyperdiagonals $|j_u| = |j_V|$, $1 \le u < v \le k$.

Observe that by modifying the definition of the special functions g_j, one can insure that the difference

$$S_m(g_{j_1} \cdots g_{j_k})(x) - g_{j_1}(x) \cdots g_{j_k}(x) K_m(\lambda_{j_1} + \cdots + \lambda_{j_k})$$

can be made arbitrarily small, in measure and in $L^2(X)$, for arbitrarily many m. (The only change is that the "junk sets" R_j, in (3.4) above, have to be made smaller in measure.)

The analogue of (2.1) is also needed. For any function $f: \mathbf{R}^k \to \mathbf{C}$, which is symmetric with respect to permutations of its arguments,

$$\left\| \int_{\mathbf{R}^k}' f(x_1, \dots, x_k) W(dx_1) \cdots W(dx_k) \right\|_2 .$$
$$= C_k \left(\int_{\mathbf{R}^k} |f(x_1, \dots, x_k)|^2 dx_1 \cdots dx_k \right)^{1/2} .$$

With these observations, the proof of Theorem 3.1 is easily modified to yield Theorem 4.1.

4.b. Harmonizable fractional stable motion. We consider a class of ss si processes without second moment, obtained by replacing the Gaussian spectral measure used thus far, by a stable spectral measure. The resulting processes are special in that they do have a spectral representation, for such a representation need not exist for a stable process [H]. To define these processes, let $l(\cdot)$ denote Lebesgue measure on $\mathbf{R}$, fix $0 < \alpha \le 2$, and $0 < H < 1$. Let $M(dx)$ be a complex-valued, isotropic (i.e. rotationally invariant) independently scattered α-stable random measure on $\mathbf{R}_+$, with control measure l. A more detailed description of M is this: for any two Borel sets $A, B \subset \mathbf{R}_+$, with $l(A)$, $l(B) < +\infty$, $M(A)$ and $M(B)$ are complex-valued random variables satisfying

$$e^{it} M(A) \stackrel{d}{=} M(A), \qquad t \in \mathbf{R},$$
$$E \exp\{it(\theta_1 \operatorname{Re}(M(A) - M(B)) + \theta_2 \operatorname{Im}(M(A) - M(B))\}$$
$$= \exp(-l(A \Delta B)|t(\theta_1, \theta_2)|^\alpha), \qquad t \in \mathbf{R}, \ (\theta_1, \theta_2) \in \mathbf{R}^2,$$

and $M(A)$ and $M(B)$ are independent if $A \cap B = 0$. Extended M to all of $\mathbf{R}$ by $M(-A) = \overline{M(A)}$.

For $0 < H < 1$, set

$$Y_{\alpha, H}(t) = \int_{-\infty}^{\infty} \frac{e^{itx} - 1}{ix} |x|^{1-H-1/\alpha} M(dx) .$$

This is an H-ss si process, and for $\alpha = 2$ it is a fractional Brownian motion. For $0 < \alpha < 2$, $Y_{\alpha, H}(t)$ is a symmetric α-stable random variable, so in particular $P(|Y_{\alpha, H}(t)| > x) \sim C_{\alpha, H} t^{\alpha H} x^{-\alpha}$, as $x \to +\infty$. Thus $Y_{\alpha, H} \in \text{weak}-L^\alpha = L^{\alpha\infty}$, and for $0 < \alpha \le 1$ is not integrable. Define the $L^{\alpha\infty}$ "norm" by

$$\|Z\|_{\alpha\infty} = \sup_{z>0} z(\mu(|z| > z))^{1/\alpha} .$$

For $\alpha \ge 1$, $\|\cdot\|_{\alpha\infty}$ is equivalent to a norm; for $0 < \alpha < 1$, this fails.

Theorem 4.2. *Let $(X, \mathscr{A}, \mu, T)$ be aperiodic. Fix $1 \le \alpha < 2$, and $0 < H < 1$. Then there is an $f \in L_{\alpha\infty}(\mu)$ so that*

$$m^{-H} S_{[mt]} f \stackrel{d}{\Rightarrow} Y_{\alpha, H}(t) .$$

The proof involves only trivial modifications of the proof of Theorem 3.1. First observe that $Y_{\alpha, H}(m)$, $m = 1, 2, \dots,$ admits a spectral representation on $(-\pi, \pi)$:

$$Y_{\alpha, H}(m) = \int_{-\pi}^{\pi} K_m(\lambda) N(d\lambda),$$

with control measure η. Then $\eta(\{0\}) = 0$, and for all $f \in L^{\alpha}(\eta)$,

$$\left\| \int_{-\pi}^{\pi} f(\lambda) N(d\lambda) \right\|_{\alpha\infty} = C_{\alpha} \left(\int_{-\pi}^{\pi} |f(\lambda)|^{\alpha} \, d\lambda \right)^{1/\alpha}.$$

With these observations, the remainder of the proof is easily checked.

4.3. Stationary sequences. One has the feeling that the range of limit behavior present in an aperiodic dynamical system should be rich. A rigorous result in this direction would be to show (1.1) for any square-integrable ss si process $Y(t)$. In fact, a result can be stated without any reference to self-similarity, which we do now. Let

$$\xi_k = \int_{-\pi}^{\pi} e^{ik\lambda} W(d\lambda)$$

be a square-integrable stationary sequence. Set $S_m = \xi_0 + \cdots + \xi_{m-1}$, and $s_m^2 = ES_m^2$. Assume that $W(\{0\}) = 0$ a.s., and $m^{\varepsilon} \le s_m^2 \le m^{2-\varepsilon}$, for some $\varepsilon > 0$ and all large m. These two conditions imply that $\frac{1}{m} S_m \to 0$ a.s. Further assume that W satisfies the following "isotropic" property: for any finite number of pairwise disjoint Borel measurable sets $A_1, A_2, \ldots, A_n \subset (0, \pi)$, and $\lambda_1, \lambda_2, \ldots, \lambda_n \subset (0, 2\pi)$,

$$\{e^{i\lambda i} W(A_i)\} \stackrel{d}{=} \{W(A_i)\}.$$

Theorem 4.3. *Let $(X, \mathscr{A}, \mu, T)$ be aperiodic, and ξ_k as above. Then there is an $f \in L^2(\mu)$ so that*

$$\sup_{a \in \mathbf{R}} |\mu(S_m f > a) - P(S_m > a)| \to 0, \qquad m \to +\infty.$$

This appears to be the most general result available from the method of proof used in this paper. Note that it contains Theorem 1.1 as an immediate consequence, but this appears to be the only corollary of Theorem 4.3 of importance. We have stated this result to advertise the problem of generalizing it.

The only part of our proof that needs modification is Lemma 2.1. Instead, with the notation of that lemma, one must prove.

Lemma 4.1. *The intervals I_j can be chosen so that the following four conditions hold*:

$$l(m) \ge (\log m)^{-1};$$

$$\left\| \int_{l(m)}^{\pi} K_m(\lambda) W(dx) \right\|_2, \left\| \int_{l(m)}^{\pi} K_m(\lambda) V(d\lambda) \right\|_2 = O((\log m));$$

$$\left\| \int_0^{l(m)} K_m(\lambda)(W - V)(d\lambda) \right\|_2 = O(\log m)$$

and for all complex numbers z_i, *of modulus* 1, *the sum* $\sum z_i V(I_i)$ *converges a.e.*

Notice that the middle two properties are *not* almost sure statements. The last condition is needed to make sure that $\overline{f}$ is well defined. (See the argument concerning (3.5) above.) The Rademacher-Menchoff Theorem [Z, vol. II, p. 193] is ideally suited to obtain this last condition. Use it, together with the following observation, to make a judicious choice of the intervals I_j. For all large m,

$$m^2 EW([0, 1/m))^2 \leq CE\left|\int_{-\pi}^{\pi} K_m(\lambda)W(d\lambda)\right|^2 \leq Cm^{2-\varepsilon}$$

so that $EW([0, 1/m))^2 \leq Cm^{-\varepsilon}$.

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