THE GREEN CORRESPONDENCE FOR THE REPRESENTATIONS OF HECKE ALGEBRAS OF TYPE $A_{r-1}$

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ABSTRACT. We first prove the conjecture mentioned by Leonard K. Jones in his thesis. By applying this conjecture, we obtain that the vertex of an indecomposable $R$-module is an $I$-parabolic subgroup. Finally, we establish the Green correspondence for the representations of Hecke algebras of type $A_{r-1}$.

INTRODUCTION

Let $R$ be a $\mathbb{Q}[u^{1/2}]$-algebra in which $u^{1/2}$ is invertible. Let $(W, S)$ be the symmetric group on $r$ letters where $S$ is the set of basic transpositions. Then the Hecke algebra $H_R$ corresponding to $W$ is a free $R$-module with basis \{\tilde{T}_w; w \in W\} which obey the following multiplication rules (see [Du]):

$$\tilde{T}_w \tilde{T}_s = \begin{cases} 
\tilde{T}_{ws}, & \text{if } w < ws, \\
(u^{-1/2} - u^{1/2})\tilde{T}_w + \tilde{T}_{ws}, & \text{otherwise},
\end{cases}$$

where "<" is the Bruhat order and $w \in W$, $s \in S$.

The study of the representations of the Hecke algebra $H_R$ has turned out many remarkable $q$-analogues of the representations of the symmetric groups (see [DJ1 and 2, Ho and Jo]). In this paper we shall generalize some basic results of Green along the lines of the work of L. Jones (see [Jo]). We organize the paper as follows: After recalling some basic results, we shall prove the conjecture (Theorem 2.7) which has been mentioned in [Jo, 5.3]. The Brauer homomorphism constructed by Jones will play a key role in proving that conjecture. With the aid of this conjecture, we obtain that the vertex of an indecomposable $H_R$ module is an $I$-parabolic subgroup. In the last section, we shall establish the Green correspondence for the representations of Hecke algebra $H_R$.

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1. INDUCED AND INDECOMPOSABLE MODULES

Let $l$ be a positive integer, $l \leq r$, and $\Phi_l(u^{1/2})$ is the $l$th cyclotomic polynomial in $u^{1/2}$. Let $R_l$ be the completion of the polynomial ring in the indeterminate $u^{1/2}$ over $\mathbb{Q}$ localized at the maximal ideal generated by $\Phi_l(u^{1/2})$. Let $K$ be the quotient field of $R_l$ and $F$ the residue class field $R_l/\eta R_l$ where $\eta$ is the generator of the maximal ideal of $R_l$.

We call that $(K, R_l, F)$ is a characteristic 0 modular system. Let $R_\emptyset$ be the completion of the polynomial ring in the indeterminate $u^{1/2}$ over $\mathbb{Q}$ localized at the maximal ideal generated by $\Phi_l(u^{1/2})$.

1.1 Theorem (Mackey's decomposition). Let $\lambda, \mu \vdash r$ and let $N$ be an $\mathbb{H}_\lambda$-module then

$$(N \mathbb{H}_\lambda)_{\mathbb{H}_\mu} \cong \sum_{d \in D_{\lambda\mu}} [(N \otimes \mathbb{H}_\lambda \tilde{T}_d) \otimes \mathbb{H}_{\nu(d)}],$$

where $\nu(d)$ is defined by $W_{\nu(d)} = W_\lambda^d \cap W_\mu$ for all $d \in D_{\lambda\mu}$.

Let $M$ be a finitely generated indecomposable right $\mathbb{H}_\lambda$-module. Then, by [Jo, 3.35], there exists a parabolic subgroup $W_\lambda$ of $W$ unique up to conjugation such that $M$ is relatively $\mathbb{H}_\lambda$-projective (see the definition in [Jo, Chapter 3]) and such that $W_\lambda$ is $W$-conjugate to a parabolic subgroup of any parabolic subgroup $W_\mu$ of $W$ for which $M$ is relatively $\mathbb{H}_\mu$-projective.

We call $W_\lambda$ the vertex of $M$. In the next section we shall see that the vertex of $M$ must be an $l$-parabolic subgroup of $W$.

Mackey's theorem can be used to prove the following result.

1.2 Proposition. Let $W_\tau$ be the vertex of the indecomposable $\mathbb{H}_\tau$-module $M$. Then

(a) There is an indecomposable $\mathbb{H}_\tau$-module $N$ such that $M \mid N \mathbb{H}_\tau$. 


(b) If $N'$ is another indecomposable $\mathcal{H}_r$-module with the property (a), then there is an element

$$d \in N_W(W_t) \cap \mathcal{D}_r$$

such that

$$N \cong N' \otimes_{\mathcal{H}_r} \widetilde{T_d}$$

as $\mathcal{H}_r$-modules.

The notation $X|Y$ means that $X$ is isomorphic to a direct summand of $Y$ and $N_W(W_t)$ denotes the normaliser of $W_t$ in $W$.

**Proof.** Since $M$ is relatively $\mathcal{H}_r$-projective, we have

$$M|M \otimes_{\mathcal{H}_r} \mathcal{H}_R.$$ 

Thus, there is an indecomposable direct summand $N$ of $M_{\mathcal{H}_r}$ such that

$$M|N \otimes_{\mathcal{H}_r} \mathcal{H}_R.$$ 

Hence (a) follows.

Assume that $V$ is an indecomposable $\mathcal{H}_r$-module with

$$M|V \otimes_{\mathcal{H}_r} \mathcal{H}_R.$$ 

Since $N|M_{\mathcal{H}_r}$ we have

$$N|(V \otimes_{\mathcal{H}_r} \mathcal{H}_R)_{\mathcal{H}_r}.$$ 

By Mackey's theorem,

$$(V \otimes_{\mathcal{H}_r} \mathcal{H}_R)_{\mathcal{H}_r} \cong \bigoplus_{d \in \mathcal{D}_r} (V \otimes_{\mathcal{H}_r} \widetilde{T_d} \otimes_{\mathcal{H}_{r(d)}} \mathcal{H}_r)$$

where $\nu(d)$ is defined by $W_{\nu(d)} = W_t^d \cap W_t$. Thus by Krull-Schmidt theorem, there is $d \in \mathcal{D}_r$ such that

$$N|V \otimes_{\mathcal{H}_r} \widetilde{T_d} \otimes_{\mathcal{H}_{r(d)}} \mathcal{H}_r.$$ 

Therefore

$$M|V \otimes_{\mathcal{H}_r} \widetilde{T_d} \otimes_{\mathcal{H}_{r(d)}} \mathcal{H}_R.$$ 

Since $W_t$ is the vertex of $M$ it follows from Higman's criterion [Jo, 3.34] that

$$W_t \subseteq W W_{\nu(d)} = W_t^d \cap W_t.$$ 

Therefore

$$W_t = W_t^d \cap W_t \quad \text{and} \quad d \in N_W(W_t).$$ 

Thus we have

$$N|V \otimes_{\mathcal{H}_r} \widetilde{T_d}.$$ 

Since $V$ is indecomposable, so is $V \otimes_{\mathcal{H}_r} \widetilde{T_d}$, hence

$$N \cong V \otimes_{\mathcal{H}_r} \widetilde{T_d}$$

as $\mathcal{H}_r$-modules. □

The module $N$ is called a source of $M$.

To study the representations of Hecke algebras, the relative norm plays a remarkable role (see [Du, Jo]). Here is the definition of relative norms.
1.3 Definition. Let \( \lambda, \mu \) be compositions of \( r \) such that \( W_\lambda \subseteq W_\mu \). Let \( M \) be an \( \mathcal{H}_\mu \rightarrow \mathcal{H}_\mu \) bimodule and \( b \in M \). Define the relative norm

\[
N_{W_\mu, W_\lambda}(b) = \sum_{w \in \mathcal{S}_r \cap W_\mu} \tilde{T}_{w^{-1}} b \tilde{T}_w.
\]

There are a number of nice properties of relative norms which will be used freely in the subsequent discussion. These results are mostly due to P. Hoefsmit and L. Scott. One can find a complete proof in [Jo, Chapter 3].

Let \( M \) be an \( \mathcal{H}_\lambda \rightarrow \mathcal{H}_\lambda \) bimodule. We define

\[
Z_M(\mathcal{H}_\lambda) = \{ m \in M \mid hm = mh \text{ for all } h \in \mathcal{H}_\lambda \}.
\]

Obviously \( Z(\mathcal{H}_\lambda) = Z_{\mathcal{H}_\lambda}(\mathcal{H}_\lambda) \) is the center of \( \mathcal{H}_\lambda \) and, for \( M = \text{Hom}_R(N, N) \) where \( N \) is a right \( \mathcal{H}_\lambda \)-module,

\[
Z_M(\mathcal{H}_\lambda) = \text{Hom}_{\mathcal{H}_\lambda}(N, N).
\]

One can describe a basis of the center \( Z(\mathcal{H}_\lambda) \) of \( \mathcal{H}_\lambda \) or a basis of the \( q \)-Schur algebra \( S_R(n, r) \) in terms of relative norms (see [Du, Jo]), which give us much more facilities.

Let \( P_k = W_{(k, r-k)} \) where \( k \) satisfies \( r = kl + s \), \( s < l \). In [Jo, 5.3] there is a conjecture as follows:

1.4 Conjecture. \( N_{W, P_k}(1) \) is invertible.

We shall prove this conjecture in the next section.

2. The Invertibility of \( N_{W, P_k}(1) \)

In [Jo] a Brauer-type homomorphism is constructed and the image of certain basis of \( Z(\mathcal{H}_\lambda) \) is discussed modulo a conjecture which has been proved in [Sh, Proposition 11]. The proof of Proposition 2.6 below is motivated by the argument in [Jo].

Let \( \gamma = (l, r - l) \). Then \( \mathcal{H}' \cong \mathcal{H} \otimes \mathcal{H}_{r-1} \) where \( \mathcal{H}' = \mathcal{H}_{(l, r-l)} \) and \( \mathcal{H}_{r-1} = \mathcal{H}_{(l', r-l')} \). Since \( W \) is a disjoint union of the subsets \( W_\gamma \) and \( W'xW' \) for \( x \notin W_\gamma \) and \( W' = W_{(l, r-l)} \), that is,

\[
W = W_\gamma \cup \left( \bigcup_{x \notin W_\gamma} W'xW' \right)
\]

(see the proof of [Jo, 5.1.5]), we have

\[
(2.a) \quad Z(\mathcal{H}_\lambda) \subseteq Z_{\mathcal{H}_\lambda}(\mathcal{H}_\gamma) = Z(\mathcal{H}_\gamma) \oplus Z_M(\mathcal{H}_\gamma)
\]

where \( M_r = \bigoplus_{x \notin W_\gamma} \mathcal{H}_\gamma \tilde{T}_x \mathcal{H}_\gamma \).

Let \( \pi \) be the projection of \( Z(\mathcal{H}_\lambda) \) onto \( Z(\mathcal{H}_\gamma) \) and let \( \sigma \) be the canonical map from \( Z(\mathcal{H}_\gamma) \) onto \( Z(\mathcal{H}_\gamma)/N_{W', 1}(\mathcal{H}_\gamma) \cap Z(\mathcal{H}_\gamma) \). Then the map

\[
f = \sigma \circ \pi : Z(\mathcal{H}_\lambda) \rightarrow Z(\mathcal{H}_\gamma)/N_{W', 1}(\mathcal{H}_\gamma) \cap Z(\mathcal{H}_\gamma)
\]

is an algebraic homomorphism which is called the Brauer homomorphism, following Jones.
2.1 Lemma. Let $\pi$ be as above and let
\[ h = \sum_w a_w \tilde{T}_w \in Z_{\mathcal{H}}(\mathcal{H}). \]
Then $\pi(h) = 0$ if and only if $s_l = (l, l + 1) < w$ for all $w$ with $a_w \neq 0$.

Proof. Immediate from the fact that $w \notin W_{\gamma}$ if and only if $s_l < w$. \hfill $\Box$

2.2 Lemma. $Z(\mathcal{H})/N_{W',1}(\mathcal{H}) \cap Z(\mathcal{H}) \cong [Z(\mathcal{H})/N_{W',1}(\mathcal{H})] \otimes Z(\mathcal{H}_{r-l})$.

Proof. We first claim that
\begin{equation}
N_{W',1}(\mathcal{H}) \otimes Z(\mathcal{H}_{r-l}) = (N_{W',1}(\mathcal{H}) \otimes \mathcal{H}_{r-l}) \cap (Z(\mathcal{H}) \otimes Z(\mathcal{H}_{r-l})).
\end{equation}

Obviously, the left-hand side of (2.b) is contained in the right-hand side. Let \{\(vi, 1 \leq i \leq s\}\ be a basis of $Z(\mathcal{H})$ such that \{\(vi, 1 \leq i \leq t\}\ is a basis of $N_{W',1}(\mathcal{H})$ . If $a = \sum_{i=1}^{t} v_i \otimes a_i$ is an element of right-hand side of (2.b) then it is easy to see $a_i \in Z(\mathcal{H}_{r-l})$ . Hence the claim is proved. Thus by the claim,
\begin{align*}
Z(\mathcal{H})/N_{W',1}(\mathcal{H}) \cap Z(\mathcal{H}) &\cong [Z(\mathcal{H})/N_{W',1}(\mathcal{H}) \otimes \mathcal{H}_{r-l} \cap Z(\mathcal{H}) \otimes Z(\mathcal{H}_{r-l})] \\
&\cong [Z(\mathcal{H})/N_{W',1}(\mathcal{H}) \otimes Z(\mathcal{H}_{r-l})] \\
&\cong [Z(\mathcal{H})/N_{W',1}(\mathcal{H})] \otimes Z(\mathcal{H}_{r-l}).
\end{align*}

Hence the result. \hfill $\Box$

Let $Z = Z(\mathcal{H})/N_{W',1}(\mathcal{H})$ and let $r = kl + s$, $s < l$. For each $m$, $1 \leq m \leq k$ we define
\[ f_m = (\text{id}_Z \otimes \cdots \otimes \text{id}_Z \otimes f) \circ \cdots \circ (\text{id}_Z \otimes f) \circ f. \]
Then $f_m$ is a homomorphism from $Z(\mathcal{H})$ into
\[ Z \otimes \cdots \otimes Z \otimes Z(\mathcal{H}_{r-ml}). \]

If $e$ is a central primitive idempotent of $\mathcal{H}$ then we say that the defect of $e$ is $m$ if $f_m(e) \neq 0$ and $f_{m+1}(e) = 0$.

2.3 Lemma. Let $e$ be a central primitive idempotent of $\mathcal{H}$. Then $eZ(\mathcal{H})$ is a local ring.

Proof. Let
\[ \mathcal{A} = \mathcal{H}^{op} \otimes \mathcal{H}. \]
Then $\mathcal{H}$ is an $\mathcal{A}$-module defined by
\[ h(h_1 \otimes h_2) = h_1 h_2 \]
for $h, h_1, h_2 \in \mathcal{H}$, and the $\mathcal{A}$-submodules of $\mathcal{H}$ are ideals of $\mathcal{H}$. So $e\mathcal{H}$ is an indecomposable $\mathcal{A}$-submodule. Hence $\text{End}_{\mathcal{A}}(e\mathcal{H})$ is a local ring.

Since $\text{End}_{\mathcal{H}}(e\mathcal{H}) = e\mathcal{H}$, it follows that
\[ \text{End}_{\mathcal{A}}(e\mathcal{H}) = Z(e\mathcal{H}) = eZ(\mathcal{H}), \]
hence the result. \hfill $\Box$
2.4 Lemma. Let $e$ be a central primitive idempotent and $f(e) \neq 0$. Then there exist pairwise orthogonal primitive idempotents $\{e_{i1}\}_{1 \leq i \leq s}$, $\{e_{i2}\}_{1 \leq i \leq s}$ of $Z, Z(\mathcal{H}_{r-1})$ respectively such that
\[ f(e) = \sum_{i=1}^{s} e_{i1} \otimes e_{i2}. \]

Moreover, if the defect of $e$ is $m$ then the defect of $e_{i2} \leq m - 1$ for all $i$.

**Proof.** Let $\{e_i\}, \{e'_i\}$ be the pairwise orthogonal primitive idempotents of $Z, Z(\mathcal{H}_{r-1})$ respectively such that
\[ 1_Z = \sum_{i} e_i, \quad 1_{Z(\mathcal{H}_{r-1})} = \sum_{j} e'_j. \]

Then
\[ 1_{Z \otimes Z(\mathcal{H}_{r-1})} = \sum_{i,j} e_i \otimes e'_j \]
is a decomposition of primitive idempotents of the identity of $Z \otimes Z(\mathcal{H}_{r-1})$. Since $f(e)$ is an idempotent of $Z \otimes Z(\mathcal{H}_{r-1})$ we may find an expression of $f(e)$ as desired.

Suppose that there is $t, 1 \leq t \leq s$, such that $e_{t2}$ is of defect $\geq m$. Thus $f_m(e_{t2}) \neq 0$ and therefore
\[ f_{m+1}(e) = \sum_{i=1}^{s} \text{id}_Z(e_{i1}) \otimes f_m(e_{i2}) \neq 0. \]

This is contrary to our assumption. So the defect of $e_{i2} \leq m - 1$ for all $i$. \(\square\)

Let $\iota$ denote the anti-automorphism of $\mathcal{H}_R$ defined by $\iota(\tilde{T}_w) = \tilde{T}_{w^{-1}}$ and $\iota^2 = 1$. Let $P_m = W_{(1^{m}, 1^{m'}, 1^{m}, 1^{m-k})}$, $0 \leq m \leq k$.

2.5 Lemma. Let $e_0$ be a central primitive idempotent of $\mathcal{H}_R$ with defect 0. Then $N_{W, P_k}(1)e_0$ is invertible in $e_0 Z(\mathcal{H}_R)$.

**Proof.** Let $\chi$ be the irreducible character of $\mathcal{H}_R$ over $R = \mathbb{Q}(u)$ associated with $e' = \iota(e_0)$. $d_\chi$ denotes the generic degree of $\chi$. Then
\[ \frac{d_W}{d_\chi} \neq 0 \mod(\Phi_f) \]
(seen [Jo, 5.2.29]). By the proof of [Jo, 3.5] we have
\[ N_{W, 1}(1)e' = \frac{d_W}{d_\chi} \chi(1)e' \neq 0 \mod(\Phi_f). \]

Therefore,
\[ \text{Tr}(N_{W, 1}(1)e') = \frac{d_W}{d_\chi} \chi(1)^2 \neq 0 \mod(\Phi_f). \]

On the other hand, we have
\[ \text{Tr}(N_{W, 1}(1)e') = \text{Tr}(N_{W, P_k}(N_{P_k, 1}(1))e') \\
= \sum_{w \in \mathcal{H}_k} \text{Tr}(\tilde{T}_{w^{-1}}N_{P_k, 1}(1)\tilde{T}_we') \\
= \sum_{w \in \mathcal{H}_k} \text{Tr}(\tilde{T}_w \tilde{T}_{w^{-1}}e'N_{P_k, 1}(1)) \\
= \text{Tr}(\iota(N_{W, P_k}(1))e'N_{P_k, 1}(1)). \]
Since \( t(N_w, p_k(1))e' \) commutes with \( N_{p_k,1}(1) \) we have \( t(N_w, p_k(1)e_0) \) is not nilpotent, hence \( N_w, p_k(1)e_0 \) is not nilpotent. Hence \( N_w, p_k(1)e_0 \) is invertible in \( e_0Z(\mathcal{H}_R) \) since \( e_0Z(\mathcal{H}_R) \) is a local ring. □

We now fix some notation. Let

\[
d_0 = 1, \quad d_m = (l, l + m)(l - 1, l + m - 1) \cdots (l - m + 1, l + 1)
\]

for \( 1 \leq m \leq m(l) = \min\{l, r - l\} \). Then by [Jo, (5.2.2)]

\[
\mathcal{D}_\gamma = \{d_m | 0 \leq m \leq m(l)\}
\]

and

\[
G_m = W^{d_m}_\gamma \cap W_\gamma = W_{(l-m, m, m, r-l-m)}.
\]

2.6 Proposition. Let \( F \) be the Brauer homomorphism,

\[
f: Z(\mathcal{H}_R) \to [Z(\mathcal{H}_I)/N_{W^I,1}(\mathcal{H}_I)] \otimes Z(\mathcal{H}_{r-I}) .
\]

Then

\[
f(N_w, p_k(1)) = kN_{W^I, p_k(1)}.
\]

Proof. By the transitivity of relative norm we have

\[
N_{W, p_k(1)} = N_{W, W_\gamma}(N_{W_\gamma, p_k(1)})
\]

(2.c)

\[
= \sum_{m=0}^{m(l)} N_{W_\gamma, G_m}(\tilde{T}_{d_m}N_{W_\gamma, p_k(1)}(1)\tilde{T}_{d_m}).
\]

Let \( W'' = W_{(1^l, r-l)} \). Then \( G_m = G_{m1} \times G_{m2} \) where \( G_{m1}, G_{m2} \) are the intersections of \( G_m \) with \( W'' \), \( W'' \) respectively. Since

\[
N_{W_\gamma, p_k(1)} = N_{W'', p_k-1(1)} \in Z_{w''}(\mathcal{H}_{G_{m2}})
\]

we may write by (2.a)

\[
N_{W_\gamma, p_k(1)} = N_m + T_m
\]

where \( N_m \in Z(\mathcal{H}_{G_{m2}}) \) and

\[
T_m = \sum_{z \in W''} b_z T_z .
\]

Thus, if \( d_m z d_m \in W_\gamma \) with \( T_z \) involved in \( T_m \), then

\[
d_m z d_m \in W^{d_m}_\gamma \cap W_\gamma = G_m
\]

hence

\[
z \in G^{d_m}_m = G_m .
\]

It follows that \( (l + m, l + m + 1) \in G_m \) since \( z \in W'' \) and \( z \notin G_{m2} \). This is impossible. Therefore \( d_m z d_m \notin W_\gamma \). Thus each term \( \tilde{T}_w \) involved in \( \tilde{T}_{d_m} \tilde{T}_z \tilde{T}_{d_m} \) satisfies \( s_l < w \). Therefore, by 2.1,

\[
\pi(N_{W_\gamma, G_m}(\tilde{T}_{d_m}T_m\tilde{T}_{d_m})) = 0 .
\]

We now examine \( N_m \). By [Jo, 4.33] we express \( N_m \) as a linear combination of the basis of \( Z(\mathcal{H}_{G_{m2}}) : \)

\[
N_m = \sum_{\alpha \in (r-l), W_\gamma \subseteq G_{m2}} a_\alpha N_{G_{m2}, W_\gamma}(\eta_\alpha)
\]

where \( \eta_\alpha \in Z(\mathcal{H}_{W_\gamma}) \).
Let
\[ \eta_\alpha = \sum_{w \in W_\alpha} a_w \tilde{T}_w. \]

Then by [Jo, 5.2.21] we have
\[ \tilde{T}_{dm} \eta_\alpha \tilde{T}_{dm} = \sum_{w \in G_m} a_w \tilde{T}_{dm} \tilde{T}_w \tilde{T}_{dm} = \sum_{w \in G_m} a_w \left( \tilde{T}_{dwxw} + \sum_{x \in W} b_{w} x \tilde{T}_x \right) = \eta_{d_m \alpha} \tilde{T}_{dm} + \sum_{s_l < z} c_{z \tilde{T}_z} \]

where \( \eta_{d_m \alpha} \in Z(\mathcal{H}_{w}^{d_m}) \). Thus if we denote \( S_\alpha = Gm_1 \times W_\alpha \) then by [Jo, 5.2.10] we have
\[ \tilde{T}_{dm} N_{m_2}, w_\alpha (\eta_\alpha) \tilde{T}_{dm} = \tilde{T}_{dm} N_{m_2}, w_\alpha (\eta_\alpha) \tilde{T}_{dm} = \sum_{x \in \mathcal{Z}_m \cap G_m} \tilde{T}_{dwx^{-1}_m} \eta_\alpha \tilde{T}_x \tilde{T}_{dm} = \sum_{x \in \mathcal{Z}_m \cap G_m} \tilde{T}_{x^{-1}} (\tilde{T}_{d_m} \eta_\alpha \tilde{T}_{dm}) \tilde{T}_x = N_{m_2}, \alpha \eta_{d_m \alpha} \tilde{T}_{dm} + \sum_{s_l < z} d_w \tilde{T}_w. \]

Therefore, by (2.1),
\[ \pi \left( N_{W_m, G_m} \left( \sum_{s_l < w} d_w \tilde{T}_w \right) \right) = 0 \]
and if \( 0 < m < l \) then \( N_{Gm_1, m}(1) \) is invertible by [Jo], thus
\[ N_{W_m, G_m}(N_{G_m, S_\alpha}(\eta_{d_m \alpha} \tilde{T}_{dm})) = N_{W_m, G_m}(1) N_{W_m, G_m}(\eta_{d_m \alpha} \tilde{T}_{dm}) = N_{W_m, 1} \left( \frac{1}{N_{Gm_1, m}(1)} N_{W_m, G_m}(\eta_{d_m \alpha} \tilde{T}_{dm}) \right) \]
which lies in the kernel of \( \sigma \).

By observing (2.c) and the above arguments we obtain that
\[ f(N_{W_m, p_k}(1)) = \begin{cases} N_{W_m, p_k}(1), & \text{if } m(l) < l, \\ \sum_{m=0, l} f(N_{W_m, G_m}(\tilde{T}_{d_m} N_{W_m, p_k}(1) \tilde{T}_{dm})), & \text{if } m(l) = l. \end{cases} \]

In particular, if \( k = 1 \) then \( m(l) < l \) and
\[ f(N_{W_m, p_k}(1)) = N_{W_m, p_k}(1). \]

So the assertion is true for \( k = 1 \). Assume now that \( k > 1 \). Then \( m(l) = l \) and we have
\[ f(N_{W_m, p_k}(1)) = N_{W_m, p_k}(1) + f(N_{W_m, G_l}(\tilde{T}_{d_l} N_{W_m, p_k}(1) \tilde{T}_{d_l})). \]
By induction we get

$$NW_{r', p_{k-1}}(1) = (k-1)NW_{r', p_{k-1}}(1) + \sum_{w \in W'' \atop (2l, 2l+1) < w} f_w T_w$$

where $W_{r'} = W_{(1', l, r-2l)}$. Thus similar reason as before shows that

$$\pi\left(N_{W_{r'}, G_i}\left(\tilde{T}_{d_l} \sum_{w \in W'' \atop (2l, 2l+1) < w} f_w T_w T_{d_l}\right)\right) = 0.$$ 

So we eventually obtain that

$$f(N_{W_{r'}, p_k}(1)) = N_{W_{r'}, p_k}(1) + \sum_{w \in W'' \atop 2l < w} g_w T_w$$

since $P_{k}^{d_l} = P_k$ and

$$\tilde{T}_{d_l}^2 = \tilde{T}_1 + \sum_{w \in W \atop 2l < w} g_w \tilde{T}_w$$

by [Jo, 5.2.20].

2.7 Theorem. Let $e$ be a central primitive idempotent of $H_R$. Then $NW_{r, p_k}(1)e$ is invertible in $eZ(H_R)$. Therefore $NW_{r, p_k}(1)$ is invertible in $H_R$.

Proof. The first statement is true if $e$ is of defect 0 by (2.5). Assume that $e$ is of defect $m > 0$. By (2.4)

$$f(e) = \sum_{i=1}^{s} e_{i1} \otimes e_{i2}$$

where $e_{i1}, e_{i2}$ are the central primitive idempotents of $Z, Z(H_{r-1})$ respectively, and the defect of $e_{i2} < m$. Thus by the previous proposition,

$$f(N_{W_{r}, p_k}(1)e) = f(N_{W_{r}, p_k}(1))f(e)$$

$$= kN_{W_{r}, p_k}(1) \sum_{i=1}^{s} e_{i1} \otimes e_{i2}$$

By induction we have that $NW_{r', p_{k-1}}(1)e_{i2}$ is invertible in $e_{i2}Z(H_{r-1})$ for all $i$, and $\{e_{i1} \otimes e_{i2}\}_{1 \leq i \leq s}$ is orthogonal pairwise. So $f(N_{W_{r}, p_k}(1)e)$ is not nilpotent. Therefore, $N_{W_{r}, p_k}(1)e$ is not nilpotent. So it is invertible in $eZ(H_R)$ since $eZ(H_R)$ is a local ring.

Let $1 = \sum_{i=1}^{s} e_i$ be a decomposition of central primitive idempotents of the identity of $H_R$. Then

$$NW_{r, p_k}(1) = \sum_{i=1}^{s} NW_{r, p_k}(1)e_i.$$
Since \( N_{W,R}(1)e_i \) is invertible in \( e_i\mathcal{Z}(\mathcal{H}_R) \) for all \( i \) we have \( N_{W,R}(1) \) is invertible.

3. Green correspondence

In this section we shall present a couple of applications of Theorem 2.7.

Recall from §1 that if \( M \) is an indecomposable \( \mathcal{H}_R \)-module then there is a "minimal" parabolic subgroup \( W_\lambda \) of \( W \) such that \( M \) is relatively \( \mathcal{H}_\lambda \)-projective. Such a group \( W_\lambda \) is called a vertex of \( M \). Now we can say more about the vertex.

3.1 Theorem. Let \( M \) be a finitely generated indecomposable \( \mathcal{H}_R \)-module. Then the vertex of \( M \) is an \( I \)-parabolic subgroup of \( W \).

Proof. Let \( H \) be the vertex of \( M \) and \( P \) the maximal \( I \)-parabolic subgroup of \( H \). Then, by Higman's criterion,

\[
N_{W,H}(\text{Hom}_{\mathcal{H}_R}(M,M)) = \text{Hom}_{\mathcal{H}_H}(M,M).
\]

Since \( N_{H,P}(1) \) is invertible, we have

\[
N_{W,H}(\text{Hom}_{\mathcal{H}_R}(M,M)) = N_{W,P} \left( \frac{1}{N_{H,P}(1)} \text{Hom}_{\mathcal{H}_H}(M,M) \right)
\subseteq N_{W,P}(\text{Hom}_{\mathcal{H}_P}(M,M))
\]

therefore

\[
N_{W,P}(\text{Hom}_{\mathcal{H}_P}(M,M)) = \text{Hom}_{\mathcal{H}_P}(M,M).
\]

By Higman's criterion again we get \( M \) is relatively \( \mathcal{H}_P \)-projective. This forces \( P = H \).

In the modular representation theory of finite groups, the Green correspondence connects indecomposable modules for the group \( G \) with modules for its local subgroups (see \([\text{AI, F}]\)). Now, we start to establish a \( q \)-analogue of the Green correspondence for the representations of Hecke algebras. Such a generalization becomes apparent as soon as 3.1 is set up. First of all we need a couple of lemmas.

3.2 Lemma. Let \( M \) be an indecomposable \( \mathcal{H}_R \)-module with vertex \( W_\tau \) and \( W_P \) is a parabolic subgroup containing \( W_\tau \). Then there is an indecomposable \( \mathcal{H}_P \)-module \( N \) satisfying any two of the following statements:

(a) \( N \mid M_{\mathcal{H}_P} \);
(b) \( M \mid N \otimes_{\mathcal{H}_P} \mathcal{H}_R \);
(c) \( N \) has vertex \( W_\tau \).

Proof. Since \( W_\tau \) is a vertex of \( M \) we have

\[
M \mid M \otimes_{\mathcal{H}_P} \mathcal{H}_R.
\]

Thus,

\[
M \mid (M \otimes_{\mathcal{H}_P} \mathcal{H}_P) \otimes_{\mathcal{H}_P} \mathcal{H}_R
\]

since \( W_\tau \subseteq W_P \). Hence, \( M \) is relatively \( \mathcal{H}_P \)-projective by Higman's criterion. Therefore

\[
M \mid M \otimes_{\mathcal{H}_P} \mathcal{H}_R.
\]
Thus, there is an indecomposable summand $N$ of $M_{\mathcal{H}_p}$ such that
$$M|N \otimes_{\mathcal{H}_p} \mathcal{H}_R.$$ So (a) and (b) hold.

Let $V$ be a source of $M$. Then $M|V \otimes_{\mathcal{H}_i} \mathcal{H}_R$ and hence, $M|(V_{\mathcal{H}_p})_{\mathcal{H}_R}$. Thus there is an indecomposable summand $N$ of $V_{\mathcal{H}_p}$ with $M|N_{\mathcal{H}_R}$. We claim that $N$ has vertex $W'_{\tau}$.

Since $N|V_{\mathcal{H}_p}$, we have $N$ is relatively $\mathcal{K}_i$-projective, so there is a vertex $W'_{\tau}$ of $N$ with $W'_{\tau} \subseteq W_{\tau}$. Let $V'$ be a $\mathcal{H}_i$-module with $N|(V'_{\mathcal{H}_p})_{\mathcal{H}_R}$. Then $N_{\mathcal{H}_R}|((V'_{\mathcal{H}_p})_{\mathcal{H}_R}$ and $((V'_{\mathcal{H}_p})_{\mathcal{H}_R} = (V'_{\mathcal{H}_p})_{\mathcal{H}_R}$,

hence
$$M|N_{\mathcal{H}_R}|V' \otimes_{\mathcal{H}_i} \mathcal{H}_R.$$ That is $M$ is relatively $\mathcal{K}_i$-projective. Thus $W'_{\tau}$ contains a conjugate of $W_{\tau}$.

Since $W'_{\tau} \subseteq W_{\tau}$ we have $W'_{\tau} = W_{\tau}$. (b) and (c) are true.

By the proof of 1.2 there is an indecomposable $\mathcal{H}_i$-module $V$ such that $V|M_{\mathcal{H}_i}$ and $M|V_{\mathcal{H}_R}$. Hence there is an indecomposable $\mathcal{H}_p$-module $N$ with $N|M_{\mathcal{H}_p}$ and $V|N_{\mathcal{H}_R}$. We shall prove that $N$ has vertex $W_{\tau}$.

Since $N|M_{\mathcal{H}_p}$ we have $N|(V_{\mathcal{H}_p})_{\mathcal{H}_p}$. By Mackey's theorem there exists $d \in \mathcal{D}_{\tau p}$ such that
$$N|V \otimes_{\mathcal{H}_i} \tilde{T}_d \otimes_{\mathcal{H}_i(d)} \mathcal{H}_p.$$ Hence $N$ has a vertex $W_{\tau}$ with $W_{\tau} \subseteq W_{\tau}^d \cap W_p = W_{\tau}(d)$. Assume that $V'$ is a source of $N$, $N|V' \otimes_{\mathcal{H}_i} \mathcal{H}_p$. Thus
$$V|(V'_{\mathcal{H}_p})_{\mathcal{H}_i}.$$ By Mackey's theorem we see that $V$ is relatively $\mathcal{K}_{W_{\tau}^d \cap W_{\tau}}$-projective for some $z$. Thus $M$ is also relatively $\mathcal{K}_{W_{\tau}^d \cap W_{\tau}}$-projective, hence $W_{\tau} \subseteq W_{\tau}^d \cap W_{\tau}$. Therefore,
$$W_{\tau'} = W_{\tau}^d \quad (d \in \mathcal{D}_{\tau p})$$ since $W'_{\tau} \subseteq W_{\tau}^d$. Hence $W_{\tau}$ is a vertex of $N$. (a) and (c) hold. □

3.3 Lemma. Let $\tau$, $\rho$ be as in 3.2. If $N$ is a relatively $\mathcal{H}_i$-projective $\mathcal{H}_p$-module, then
$$(N \otimes_{\mathcal{H}_i} \mathcal{H}_R)_{\mathcal{H}_p} \cong N \oplus Y$$ where every indecomposable summand of $Y$ is relatively projective for a subgroup of the form
$$W_{\tau}^d \cap W_p, \quad for \ d \in \mathcal{D}_{\tau p}, \ d \neq 1.$$ Proof. Since $N$ is $\mathcal{H}_i$-projective we have $N|V \otimes_{\mathcal{H}_i} \mathcal{H}_p$ for some $\mathcal{H}_i$-module $V$. Thus
$$V \otimes_{\mathcal{H}_i} \mathcal{H}_p = N \oplus T$$ for some $\mathcal{H}_p$-module $T$.

Now,
$$V \otimes_{\mathcal{H}_i} \mathcal{H}_R \cong (N \otimes_{\mathcal{H}_p} \mathcal{H}_R) \oplus (T \otimes_{\mathcal{H}_p} \mathcal{H}_R)$$ and
$$(V \otimes_{\mathcal{H}_i} \mathcal{H}_R)_{\mathcal{H}_p} \cong N \oplus Y \oplus T \oplus X$$
where

\[(N \otimes_{\mathcal{H}_p} \mathcal{H}_R)_{\mathcal{H}_p}^p = N \oplus Y, \quad (T \otimes_{\mathcal{H}_p} \mathcal{H}_R)_{\mathcal{H}_p}^p = T \oplus X\]

for suitable $\mathcal{H}_p$-modules $X, Y$ by Mackey's theorem.

On the other hand, by Mackey's theorem again,

\[(V \otimes_{\mathcal{H}_p} \mathcal{H}_R)_{\mathcal{H}_p}^p = \bigoplus_{d \in \mathcal{D}_p} (V \otimes_{\mathcal{H}_p} \mathcal{H}_{v(d)} \mathcal{H}_p^d) = V \otimes_{\mathcal{H}_p} \mathcal{H}_p \oplus U\]

where $d = 1$ gives $V \otimes_{\mathcal{H}_p} \mathcal{H}_p^1$, the $U$ is the direct sum of all terms for $d \neq 1$ and so each indecomposable summand of $U$ is relatively projective for a subgroup of the form $W_{t^d} \cap W_\theta$, $d \in \mathcal{D}_p$, $d \neq 1$. The Krull-Schmidt theorem implies that $X \oplus Y \cong U$. So $Y$ is as claimed. □

From now on we assume that $W_\theta = W_{(ml,m-ml)}$, $W_\theta = W_{(l^m,m-ml)}$. Then

\[W_\theta \supseteq N_{W}(W_\theta) \supseteq W_{(l^m,m-ml)} \supseteq W_\theta .\]

Let $\mathcal{P}$ be the collection of all parabolic subgroups of $W$. If $\mathcal{S}$ is a collection of parabolic subgroups of $W$, then $P \in W \mathcal{S}$ for $P \in \mathcal{P}$ means $P = Hx$ for some $H \in \mathcal{P}$, $x \in W$.

We say that $\mathcal{H}_R$-module $M$ is relatively $\mathcal{S}$-projective if $M = \bigoplus_i M_i$, and each $M_i$ is relatively projective for a group in $\mathcal{S}$.

Let

\[\mathcal{X} = \{H \in \mathcal{P} | H \subseteq W_\theta^d \cap W_\theta \text{ for some } d \in W, d \notin W_\rho \},\]

\[\mathcal{Y} = \{H \in \mathcal{P} | H \subseteq W_\theta^d \cap W_\rho \text{ for some } d \in W, d \notin W_\rho \},\]

\[\mathcal{Z} = \{P \in \mathcal{P} | P \subseteq W_\theta \text{ is } l \text{-parabolic}, P \notin W \mathcal{X} \} .\]

Observe that $W_\rho \supseteq N_{W}(W_\theta)$, $\mathcal{X}$ consists of proper subgroups of $W_\theta$, but $W_\theta \in \mathcal{Z}$.

3.4 Lemma. If $W_t$ is an $l$-parabolic subgroup of $W_\theta$ then the following assertions are equivalent:

(a) $W_t \in W \mathcal{X}$;
(b) $W_t \in \mathcal{X}$;
(c) $W_t \in \mathcal{Y}$;
(d) $W_t \in W_\rho \mathcal{Y}$.

Proof. (a) $\Rightarrow$ (b) If (a) holds then there exists $x \in W$ with

\[W_t \subseteq (W_\theta^d \cap W_\theta)^x\]

for some $d \in W, d \notin W_\rho$. Since either $dx$ or $x$ is not in $W_\rho$ we have $W_t \in \mathcal{X}$.

(b) $\Rightarrow$ (c) Obvious, since $\mathcal{X} \subseteq \mathcal{Y}$.

(c) $\Rightarrow$ (d) Obvious.

(d) $\Rightarrow$ (a) Suppose that (d) holds. Then there exist $x \in W_\rho$, $d \in W, d \notin W_\rho$ with

\[W_t \subseteq (W_\rho^d \cap W_\rho)^x .\]

Thus,

\[W_t \subseteq W_\rho^{dx} \cap W_\rho\]

and $dx \notin W_\rho$. Hence, $W_t \in W \mathcal{X}$.
3.5 Corollary. (a) If $M$ is relatively $\mathcal{K}$-projective $\mathcal{H}_R$-module then $M_{\mathcal{K}_p}$ is $\mathcal{Y}'$-projective.

(b) If $N$ is relatively $\mathcal{Y}'$-projective $\mathcal{H}_p$-module with a vertex contained in $W_\theta$ for each indecomposable summand of $N$, then $N_{\mathcal{K}_p}$ is relatively $\mathcal{K}$-projective.

Proof. If $L$ is an indecomposable summand of $M$ then $L$ is relatively projective for a parabolic subgroup $W_\lambda$ of the group $W_\theta^d \cap W_\theta$, for $d \in W$, $d \notin W_\rho$. Thus we have

$$L|L \otimes_{\mathcal{K}_p} \mathcal{H}_R.$$

By Mackey's theorem, $L_{\mathcal{K}_p}$ is relatively projective for the collection $\mathcal{P}$,

$$\mathcal{P} = \{Q \in \mathcal{P} | Q \subseteq W_\lambda \cap W_\rho \text{ for some } z \in \mathcal{D}_{kp}\}.$$

Since $W_\lambda \cap W_\rho \subseteq (W_\theta^d \cap W_\theta)^z \cap W_\rho$ and either $dz$ or $z$ is not in $W_\rho$ we have $\mathcal{P} \subseteq \mathcal{Y}'$. Therefore $L_{\mathcal{K}_p}$ and $M_{\mathcal{K}_p}$ are relatively $\mathcal{Y}'$-projective.

On the other hand, if $L$ is an indecomposable summand of $N$ and $L$ has a vertex $W_\tau$ with $W_\tau \in \mathcal{Y}$, $W_\tau \subseteq W_\theta$, then $L_{\mathcal{K}_p}$ is relatively $\mathcal{K}$-projective and $W_\tau \in \mathcal{H}'$ by 3.4. So $L_{\mathcal{K}_p}$ and $N_{\mathcal{K}_p}$ is relatively $\mathcal{K}$-projective. $\square$

We now prove our main result in this section, which is a $q$-analogue of the Green correspondence in the representation theory of finite groups.

3.6 Theorem. There is a one to one correspondence between isomorphic classes of indecomposable $\mathcal{H}_R$-modules with vertex in $\mathcal{I}$ and isomorphic classes of indecomposable $\mathcal{H}_p$-modules with vertex in $\mathcal{I}$, which can be characterized as follows:

(a) Let $M$ be an indecomposable $\mathcal{H}_R$-module with vertex $W_\tau$ in $\mathcal{I}$. Then $M_{\mathcal{K}_p}$ has a unique indecomposable direct summand $f(M)$ with $W_\tau$ as vertex. Furthermore,

$$M_{\mathcal{K}_p} \cong f(M) \oplus \bigoplus N_i$$

where a vertex $N_i$ lies in $\mathcal{Y}$ for all $i$.

(b) Let $N$ be an indecomposable $\mathcal{H}_p$-module with vertex $W_\tau$ in $\mathcal{I}$. Then $N_{\mathcal{K}_p}$ has a unique indecomposable direct summand $g(N)$ with $W_\tau$ as vertex. Furthermore,

$$N_{\mathcal{K}_p} = g(N) \oplus \bigoplus M_j$$

where $M_j$ has a vertex in $\mathcal{K}$ for all $j$.

(c) In particular, $g(f(M)) = M$ and $f(g(N)) = N$.

Proof. (a) By 3.2 there is an indecomposable $\mathcal{H}_p$-module $V$ with vertex $W_\tau$ and

(3.a)$$M|V \otimes_{\mathcal{K}_p} \mathcal{H}_R.$$

Applying 3.3 we obtain

$$(V \otimes_{\mathcal{K}_p} \mathcal{H}_R)_{\mathcal{K}_p} = V \oplus Y_1$$

where $Y_1$ is $\mathcal{K}$-projective. Thus, $M_{\mathcal{K}_p}$ is isomorphic to $V \oplus Y$ or $Y$ for some summand $Y$ of $Y_1$. However, again by 3.2, $M_{\mathcal{K}_p}$ has an indecomposable summand $V'$ with vertex $W_\tau$. Now we claim that $V'$ cannot be isomorphic
to a summand of $Y$. Otherwise, $W_t \in W_\rho \mathcal{Y}$ by [Jo, 3.35], hence $W_t \in W \mathcal{H}$ by 3.4. This is contrary to $W_t \in \mathcal{Z}$. Hence $V' \cong V$ and

$$M_{\mathcal{F}_\rho} \cong V \oplus Y$$

just as claimed. The argument also shows $V$ is unique up to isomorphism. Let $f(M) = V$. Then $f$ is well defined.

(b) Let

$$N_{\mathcal{F}_\rho} = M_1 \oplus M_2 \oplus \cdots \oplus M_t$$

be a direct sum of indecomposable $\mathcal{F}_R$-module. Since, by 3.3,

$$(N_{\mathcal{F}_\rho})_{\mathcal{F}_\rho} \cong N \oplus Y$$

where $Y$ is relatively $\mathcal{Y}$-projective, we have, after renumbering, that

$$M_1 \mathcal{F}_\rho \cong N \oplus Y_1, \quad (M_i)_{\mathcal{F}_\rho} \cong Y_i, \quad 2 \leq i \leq t,$$

where the $Y_i$'s are $\mathcal{F}_\rho$-modules and

$$Y \cong Y_1 \oplus Y_2 \oplus \cdots \oplus Y_t.$$

We claim that $M_1$ has a vertex in $\mathcal{Z}$ and that $M_2, \ldots, M_t$ are $\mathcal{F}$-projective. Indeed, since $M_1|N \otimes \mathcal{F}_R$ and $N|N \otimes \mathcal{F}_R$ we have

$$M_1|N \otimes \mathcal{F}_R.$$

Hence $M_1$ has a vertex contained in $W_\theta$. Let $W_{t'} \subseteq W_\theta$ be a vertex of $M_1$. Suppose that $M_1$ is relatively $\mathcal{F}$-projective. Then 3.5 implies that $(M_1)_{\mathcal{F}_\rho} \cong N \oplus Y_1$ is relatively $\mathcal{Y}$-projective. It follows that $W_t \in W_\rho \mathcal{Y}$ since the vertex of $N$ is $W_t$, hence $W_t \in W \mathcal{H}$ by 3.4, a contradiction. So $M_1$ is not $\mathcal{F}$-projective, that is $W_{t'} \notin W \mathcal{H}$ by 3.4 again. Hence $W_{t'} \in \mathcal{Z}$, as claimed.

Moreover, if $M_i$, $(i > 1)$ was not relatively $\mathcal{F}$-projective then, by 3.5, $(M_i)_{\mathcal{F}_\rho}$ would not be relatively $\mathcal{Y}$-projective since $M_i$ has a vertex contained in $W_\theta$, a contradiction. Hence $M_1$ is indeed relatively $\mathcal{F}$-projective. Let $g(N) = M_1$. We have seen that $g(N)$ is unique up to isomorphism, so $g$ is well defined.

It remains to check (c). By (3.a) and (3.c) we have

$$M|f(M)_{\mathcal{F}_\rho}, \quad f(M)_{\mathcal{F}_\rho} \cong g(f(M)) \oplus X$$

where $X$ is relatively $\mathcal{F}$-projective. Hence, $g(f(M)) \cong M$. Similarly, (3.b) and (3.d) imply $f(g(N)) \cong N$.

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