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EXTENDING CELLULAR COHOMOLOGY TO C^*-ALGEBRAS

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ABSTRACT. A filtration on the K-theory of C^*-algebras is introduced. The relative quotients define groups $H_n(A)$, $n \geq 0$, for any C^*-algebra A, which we call the spherical homology of A. This extends cellular cohomology in the sense that

$$H_n(C(X)) \otimes \mathbf{Q} \cong H^n(X; \mathbf{Q})$$

for X a finite CW-complex. While no extension of cellular cohomology which is derived from a filtration on K-theory can be additive, Morita-invariant, and continuous, H_n is shown to be infinitely additive, Morita invariant for unital C^*-algebras, and continuous in limited cases.

1. INTRODUCTION

When we say that a contravariant functor F from algebraic topology has been extended to C^*-algebras, we mean that a covariant functor G has been found which is defined for C^*-algebras and that there is a natural isomorphism

$$G(C(X)) \cong F(X)$$

for any compact space X for which F is defined. Here $C(X)$ denotes the C^*-algebra of complex-valued, continuous functions on X. The classical example of this is the extension of K-theory K^* to C^*-algebras. In this case, the extended functor K_* is almost as well behaved as the original. In [6], the second author has shown that it is impossible to extend the standard cohomology H^* to C^*-algebras if one insists that the extension behave too much like the original. Therefore, we will consider extensions of cohomology to C^*-algebras which are not so well behaved.

For a compact space X, there is a grading induced on $K^*(X) \otimes \mathbf{Q}$ by the Chern character isomorphism

$$\mathrm{ch}\colon K^n(X) \otimes \mathbf{Q} \to \bigoplus_{k \geq 0} H^{n+2k}(X; \mathbf{Q}), \qquad n = 0, 1.$$

One way to extend $H^*(-; \mathbf{Q})$ to C^*-algebras would be to extend the grading of $K^*(-) \otimes \mathbf{Q}$ to a grading of $K_*(-) \otimes \mathbf{Q}$. Unfortunately, no nontrivial gradings

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of $K_*(-) \otimes \mathbf{Q}$ exist [6, Proposition 5.1]. We must turn instead to filtrations of K-theory.

We would like to define decreasing filtrations

$$F_0K_0 = K_0 \supseteq F_2K_0 \supseteq F_4K_0 \supseteq \cdots ,$$
$$F_1K_1 = K_1 \supseteq F_3K_1 \supseteq F_5K_1 \supseteq \cdots ,$$

such that, for any n,

$$F_nK_n(C(X)) \otimes \mathbf{Q} = \mathrm{ch}^{-1}\left(\bigoplus_{k=n,\, n+2,\, \ldots} H^k(X\,;\mathbf{Q}) \right) . \tag{1.1}$$

Suppose we are given such a filtration. If b denotes the Bott element in $K_n(C(S^n))$, then (1.1) implies that some multiple of b is in $F_nK_n(C(S^n))$; we may as well assume $b \in F_nK_n(C(S^n))$. For A a unital C^*-algebra, we identify $K_*(M_k \otimes A)$ with $K_*(A)$ in the usual way. If

$$\phi\colon C(S^n) \to M_k \otimes A$$

is a unital *-homomorphism, then it is natural to require $\phi_*(b) \in F_nK_n(A)$. In fact, the subset of $K_n(A)$ consisting of such push-forwards of the Bott element is a subgroup, so we take this as the definition of the spherical filtration in the unital case.

The spherical homology groups are defined as the relative quotients

$$H_n(A) = F_nK_n(A)/F_{n+2}K_n(A) .$$

Theorem 4.1 shows that (1.1) holds for X a finite CW-complex. Therefore

$$H_n(C(X)) \otimes \mathbf{Q} \cong H^n(X\,;\mathbf{Q}) . \tag{1.2}$$

Unfortunately (1.2) does not hold with integer coefficients. Nevertheless, we will often call $H_n(A)$ the homology of A, even though this is misleading and the term spherical homology is preferred.

If F_*K_* is any filtration for which (1.1) holds, then by [6, Proposition 5.3] the relative quotients cannot be simultaneously additive, Morita-invariant, and continuous. In light of this fact, the spherical filtration does as well as can be expected; it is infinitely-additive, Morita-invariant in the unital case, and continuous in the commutative case. It is not, in general, continuous.

Our calculation of $H_n(C(X))$ depends heavily on Rosenberg's work in [10] which, in turn, is based on Segal's work in [12]. If A is a C^*-algebra, define $k_n(A)$ as

$$k_n(A) = \varinjlim_k [C_0(\mathbf{R}^{n+k}),\, C_0(\mathbf{R}^k) \otimes \mathscr{K} \otimes A]_0 .$$

(See below for an explanation of notation.) Rosenberg shows that k_n is an extension of reduced connective K-theory. Cuntz has proposed defining a homology for C^*-algebras as

$$h_n(A\,;\mathbf{Q}) = (k_nA)/\beta(k_{n+2}(A))) \otimes \mathbf{Q} ,$$

where $\beta\colon k_{n+2} \to k_n$ is a "Bott periodicity map."

Because the spherical filtration is defined via K-theory rather than homotopy theory, it should be easier to compute than k_n and h_n. For example, consider

the UHF algebra M_{2^∞}. It is easy to show (see Lemma 2.3) that $H_2(M_{2^\infty}) = 0$. Calculating $h_2(M_{2^\infty}; \mathbf{Q})$ seems to involve difficult homotopy questions, such as whether $[C(S^2), M_{2^\infty}]$ is trivial. Work in progress with George Elliott has shown that

$$F_2K_0((M_{2^\infty} \otimes \mathscr{K})^\sim) = 0,$$

and we suspect that calculating $[C(S^2), (M_{2^\infty} \otimes \mathscr{K})^\sim]$ is beyond our reach. (Curiously, $F_2K_0((M_{2^\infty} \otimes \widetilde{\mathscr{K}})) \cong \mathbf{Z}$. See §6.4.)

§§2–4 are aimed at defining H_n and showing this to be an extension of cellular homology. In particular, §3 includes a summary of those parts of [10] that we shall need.

The last three sections are concerned with the behavior of H_n in the noncommutative case. General properties of H_n are in §5; for example, the behavior of H_n on short exact sequences and free products is discussed. §6 computes examples. The final section describes a conjecture regarding generalized determinants which, if true, would provide a good tool for calculating F_3K_1 in several examples.

The appendix describes some AF embeddings related to those that Pimsner constructs in [8]. We prove that these embeddings are homotopically trivial. This is used in §6 where it is shown that H_2 can fail to be middle-exact in cases where the surjection in a short exact sequence is not a cofibration.

We now introduce some conventions. Let A be a C^*-algebra. The set of selfadjoint elements of A will be denoted A_{sa}. The suspension and unitization of A will be denoted SA and $\widetilde{A}$ respectively. (Definitions may be found in [2, §§3.2 and 8.2].) The k by k matrices over A will be denoted by either $M_k \otimes A$ or $M_k(A)$. We shall use $\mathscr{K}$ to denote the algebra of compact operators on an infinite-dimensional, separable Hilbert space.

If B is a second C^*-algebra and $f\colon A \to B$ is a homomorphism (by which we mean star-homomorphism) then Sf and $\tilde{f}$ will denote the induced star-homomorphisms

$$Sf\colon SA \to SB, \qquad \tilde{f}\colon A \to B.$$

We will not make a blanket assumption of a unit except in §7. If A and B are unital, then $\mathrm{Hom}(A, B)$ (respectively $[A, B]$) will denote the collection of unital homomorphisms (respectively homotopy classes of unital homomorphisms) from A to B. In the not necessarily unital case, we will use the notation $\mathrm{Hom}(A, B)_0$ and $[A, B]_0$.

By the word trace we shall mean a bounded trace, i.e., a continuous linear map $\tau\colon A \to \mathbf{C}$ such that $\tau(ab) = \tau(ba)$ and $\tau(a^*) = \overline{\tau(a)}$ for $a, b \in A$. We do not assume positivity. Given a trace $\tau\colon A \to \mathbf{C}$, we also use τ to denote the trace $\tau\colon M_k \otimes A \to \mathbf{C}$ defined by $\tau((a_{ij})) = \sum \tau(a_{ii})$.

Assuming A is unital, the group of unitary elements in $M_k(A)$ will be denoted by $U_k(A)$, or just U_k if $A = \mathbf{C}$. By the infinite unitary group of A we mean

$$U_\infty(A) = \varinjlim U_k(A).$$

The connected component of the identity in these groups will be denoted $U_k(A)_0$ and $U_\infty(A)_0$.

2. The spherical filtration

In this section we define a filtration on K-theory which we call the spherical filtration. As is done when defining K-theory, we first give a definition for the unital case, then show that adding a new unit to a unital algebra has the predicted effect, and finally define the filtration for a nonunital C^*-algebra A in terms of the filtration of $K_*(\widetilde{A})$.

Let b_n, or simply b, denote the element of $K_n(C(S^n))$, $n \geq 1$, which has Chern character equal to the fundamental class. We call b, somewhat inaccurately, the Bott element. Another way to describe b is as a generator of $K_1(C(S^n)) \cong \mathbf{Z}$ if n is odd, and if n is even, as a generator of $K_0(C_0(\mathbf{R}^n))$ regarded as a subgroup of $K_0(C(S^n))$. For example, $b_2 = [e] - [1]$, where

$$e = \frac{1}{2}\begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix} \tag{2.1}$$

and $x, y, z \in C(S^2)$ denote the coordinate functions of $\mathbf{R}^3$ restricted to

$$S^2 = \{v \in \mathbf{R}^3 | \, \|v\| = 1\}.$$

A formula for a representative of b_3 is given in §7.

The nth level of the spherical filtration will consist of push-forwards of b by homomorphisms from $C(S^n)$ to the matrices over a given C^*-algebra. In the usual way, we identify $K_*(M_k \otimes A)$ with $K_*(A)$. We also identify K_{n+2} with K_n via Bott periodicity. Thus, we will write either $F_4K_0(A)$ or $F_4K_4(A)$, whichever is more convenient.

Definition 2.1. Let A be a unital C^*-algebra. The *spherical filtration* $F_*K_*(A)$ of $K_*(A)$ is defined as follows:

$$F_0K_0(A) = K_0(A), \qquad F_1K_1(A) = K_1(A),$$
$$F_nK_n(A) = \{\phi_*(b) | \phi \in \mathrm{Hom}(C(S^n), M_k \otimes A)\}, \qquad n \geq 2.$$

We could also have defined $F_1K_1(A)$ as push-forwards of b_1. This is because, for every $u \in U_k(A)$, sending $e^{2\pi i x}$ to u defines a unital homomorphism of $C(S^1)$ to $M_k \otimes A$.

Notice also that, even if $\phi\colon C(S^n) \to M_k \otimes A$ is not unital, $\phi_*(b) \in F_nK_n(A)$. To see this, let $x_0 \in S^n$ be the north pole, and define $\psi\colon C(S^n) \to M_k \otimes A$ by

$$\psi(f) = \phi(f) + f(x_0)(1 - \phi(1)).$$

Then ψ is unital, and $\phi_*(b) = \psi_*(b)$.

Proposition 2.2. *The sequence of subsets $F_nK_n(A)$ is a natural filtration of $K_*(A)$ by subgroups.*

Proof. It is obvious that $F_nK_n(A)$ is a subgroup. It is natural because, if $\gamma \in \mathrm{Hom}_0(A, B)$, then

$$\gamma_*(\phi_*(b)) = ((1 \otimes \gamma) \circ \phi)_*(b)$$

and so

$$\gamma_*(F_nK_n(A)) \subseteq F_nK_n(B).$$

One may easily see that b_3 can be represented by a unitary $\mathscr{E} \in C(S^3, M_2)$ which evaluates to the identity at the north pole. The induced map

$$C_0(\mathbf{R}^1) \to M_2 \otimes C_0(\mathbf{R}^3),$$

can be suspended, and unitized, to produce homomorphisms

$$C(S^n) \to M_2 \otimes C(S^{n+2})$$

which send b_n to b_{n+2}. This implies immediately that

$$F_{n+2}K_n(A) \subseteq F_nK_n(A). \quad \square$$

Lemma 2.3. *For a unital C^*-algebra A,*

$$F_2K_0(A) \subseteq \bigcap \ker(\tau_*\colon K_0(A) \to \mathbf{R}),$$

where the intersection is taken over all traces on A.

Lemma 2.4. *Let A and B denote unital C^*-algebras. Then*

(1) $F_nK_n(A \oplus B) \cong F_nK_n(A) \oplus F_nK_n(B)$.
(2) *The inclusion $A \to \widetilde{A}$ induces an isomorphism $F_nK_n(A) \cong F_nK_n(\widetilde{A})$ for $n \geq 1$.*

These two lemmas are easy to prove. The second justifies making the following definition.

Definition 2.5. Let A be a nonunital C^*-algebra. The spherical filtration $F_*K_*(A)$ of $K_*(A)$ is defined as follows:

$$F_0K_0(A) = K_0(A), \qquad F_nK_n(A) = F_nK_n(\widetilde{A}), \quad n \geq 1.$$

Recall that, by definition, $K_n(A) \subseteq K_n(\widetilde{A})$. One may easily check that $F_nK_n(A) \subseteq K_n(A)$. It is also easy to see that Proposition 2.2 and Lemmas 2.3 and 2.4 hold for nonunital C^*-algebras.

There is a more subtle point regarding nonunital C^*-algebras, namely the use of $C_0(\mathbf{R}^n)$ in place of $C(S^n)$ in the definition of $F_*K_*(A)$. This is really a homotopy question, so we will discuss it in the next section.

3. Matricially stable homotopy

Many results on the spherical filtration of a C^*-algebra A can be deduced from the study of $[C(S^n), M_k \otimes A]$. We collect in this section the homotopy results that we shall need later. Matricially stable homotopy is an interesting subject in its own right. We refer the reader to [10] for a fuller account of the subject. Some results from [10] have been restated here in a form that is convenient for our purposes.

For a compact space X with base point x_0 and A a unital C^*-algebra, we specify a base point $\phi_0 \in \mathrm{Hom}(C(X), A)$ by $\phi_0(f) = f(x_0)1_A$. Taking a direct sum with ϕ_0 specifies a sequence of mappings

$$\cdots \to [C(X), M_k \otimes A] \to [C(X), M_{k+1} \otimes A] \to \cdots$$

whose limit we denote by $[[C(X), A]]$, the *matricially stable homotopy set* of homomorphisms from $C(X)$ to A. Two unital homomorphisms $\psi_i \in \mathrm{Hom}(C(X), M_{k_i} \otimes A)$, $i = 1, 2$, are called *stably homotopic* if $\psi_1 \oplus \phi_0 \oplus \cdots \oplus \phi_0$ is homotopic to $\psi_2 \oplus \phi_0 \oplus \cdots \oplus \phi_0$ for some numbers of copies of ϕ_0.

In general, $[[C(X), A]]$ is an abelian semigroup with unit. The sum of $\phi\colon C(X) \to M_k \otimes A$ and $\psi\colon C(X) \to M_l \otimes A$ is the map

$$\begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix} : C(X) \to M_{k+l} \otimes A$$

which we will denote by $\phi \oplus \psi$.

Of course, $[[C(X), -]]$ is a covariant functor.

Proposition 3.1. *Let A be a unital C^*-algebra, and let $n \geq 1$ be an integer. Then*

(1) $[[C(S^n), A]]$ *is a group,*
(2) *the map $\phi \mapsto \phi_*(b)$ defines a natural transformation $[[C(S^n), -]] \to K_n$ whose image is $F_n K_n$.*

Proof. Part (2) is clear, while part (1) is perhaps already well known. We include a proof of (1) for completeness.

Regarding S^n as a subset of $\mathbf{R}^{n+1}$, we define $T\colon S^n \to S^n$ by

$$T(x_1, \dots, x_n, x_{n+1}) = T(x_1, \dots, x_n, -x_{n+1}).$$

Let $\tau\colon C(S^n) \to C(S^n)$ be the corresponding homomorphism. To prove the existence of inverses, it suffices to prove that the homomorphism $\Gamma\colon C(S^n) \to M_2 \otimes C(S^n)$, defined by $\Gamma(f) = \begin{pmatrix} f & \\ & f \circ T \end{pmatrix}$, is homotopic to $\phi_0 \oplus \phi_0$.

Conjugating $\Gamma(f)$ by a rotation matrix on just the lower hemisphere provides a homotopy of Γ with the map

$$f \mapsto \begin{pmatrix} f(x_1, \dots, x_{n-1}, |x_n|) & \\ & f(x_1, \dots, x_{n-1}, -|x_n|) \end{pmatrix}.$$

The latter homomorphism is clearly homotopic to $\phi_0 \oplus \phi_0$. □

There are other compact spaces X for which $[[C(X), A]]$ is a group. For example, if $\phi\colon C(\mathbf{T}^2) \to M_k \otimes A$ is specified by a pair U, V of commuting unitaries, then a triple application of the Whitehead lemma [2, Proposition 3.4.1] shows that the commuting unitary pair

$$\begin{pmatrix} U^{-1} & \\ & 1 \end{pmatrix}, \qquad \begin{pmatrix} V & \\ & V^{-2} \end{pmatrix}$$

defines the inverse to ϕ in $[[C(\mathbf{T}^2), A]]$. Can the compact spaces for which $[[C(X), A]]$ is a group be classified?

For Y a compact space, $[[C(S^n), C(Y)]]$ is essentially known since Rosenberg [10] has shown it to be isomorphic to $b\tilde{u}^n(Y)$, the reduced, connective K-theory of Y. (This is not exactly how the result is stated in [10]. To obtain this isomorphism from [10, Corollary 4.10], one must "de-loop" to eliminate the infinite suspensions.) Since most operator algebraists are not familiar with connective K-theory, we shall derive the two results we need from [10, Theorem 4.9] without mention of connective K-theory. The curious reader is referred to [1, p. 105; 12].

We need a little more notation. Suppose X and Y are spaces with base points. We let $[X, Y]_+$ denote the base point preserving continuous functions modulo base point preserving homotopies. For the base point free counterpart we use the notation $[X, Y]$. Finally, ΩX shall denote the base point fixed loop space of X.

Lemma 3.2. *Let $n \leq m$ be positive integers. Then*

(1) $[[C(S^n), C(S^m)]] = \begin{cases} \mathbf{Z} & \text{if } m-n \text{ is even}, \\ 0 & \text{if } m-n \text{ is odd}, \end{cases}$

(2) $\phi, \psi\colon C(S^n) \to M_k \otimes C(S^m)$ *are stably homotopic if and only if* $K_*(\phi) = K_*(\psi)$.

Proof. [10, Theorem 4.9] states that $\Omega^{n-1} \varinjlim_k \operatorname{Hom}(C(S^n), M_k)$ is homotopy equivalent to the infinite unitary group U_∞. It will follow from Lemma 3.5 that

$$\begin{aligned}[C(S^n), C(S^m) \otimes M_k] &= [C_0(\mathbf{R}^n), C_0(\mathbf{R}^m) \otimes M_k]_0 \\ &= [S^m, \operatorname{Hom}(C(S^n), M_k)]_+,\end{aligned}$$

so we have

$$\begin{aligned}[[C(S^n), C(S^m)]] &= \varinjlim_k [S^m, \operatorname{Hom}(C(S^n), M_k)]_+ \\ &= \left[S^m, \varinjlim_k \operatorname{Hom}(C(S^n), M_k)\right]_+ \\ &= \left[S^{m-n+1}, \Omega^{n-1} \varinjlim_k \operatorname{Hom}(C(S^n), M_k)\right]_+ \\ &= [S^{m-n+1}, U_\infty] = \begin{cases} \mathbf{Z} & \text{if } m-n \text{ is even}, \\ 0 & \text{if } m-n \text{ is odd}. \end{cases}\end{aligned}$$

The last equality is Bott periodicity.

Part (2) follows easily from (1). □

Lemma 3.3. *If Y is a finite CW-complex, then* $[[C(S^n), C(Y)]] = 0$ *for* $n > \dim Y$.

Proof. [10, Theorem 4.9] also states that $\varinjlim_k \operatorname{Hom}(C(S^n), M_k)$ is $(n-1)$-connected. Since

$$[C(S^n), C(Y) \otimes M_k] = [Y, \operatorname{Hom}(C(S^n), M_k)],$$

we have $[[C(S^n), C(Y)]] = 0$ whenever $\dim Y \leq n-1$. □

We finish this section with a proposition which, while being technical in nature, is useful when working with nonunital C^*-algebras. From now on, we shall regard $b = b_n$ as either an element of $K_n(C_0(\mathbf{R}^n))$ or an element of $K_n(C(S^n))$, whichever is more convenient.

Proposition 3.4. *For any C^*-algebra A,*

$$F_n K_n(A) = \{x \in K_n(A) | x = \phi_*(b) \text{ for some } \phi\colon C_0(\mathbf{R}^n) \to M_k \otimes A\}$$

for all $n \geq 1$.

Proof. This is trivial for $n = 1$. For $n \geq 2$, this follows from the next lemma. □

Given a C^*-algebra A, let ε denote the surjection in the split exact sequence for $\widetilde{A}$,

$$0 \to A \to \widetilde{A} \xrightarrow{\varepsilon} \mathbf{C} \to 0.$$

We also denote by ε the amplifications of ε to $M_k \otimes \widetilde{A} \to M_k$. Since $(M_k \otimes A)^\sim$ is naturally contained in $M_k \otimes \widetilde{A}$, we may regard the unitization $\tilde{\phi}$ of a homomorphism $\phi\colon C_0(\mathbf{R}^n) \to M_k \otimes A$ as a mapping $\tilde{\phi}\colon C(S^n) \to M_k \otimes \widetilde{A}$.

For $k = 1$, and perhaps all k, the following is well known. However, the reader is cautioned that it can happen that $[A, B]_0 \ncong [\widetilde{A}, \widetilde{B}]$ for C^*-algebras A and B. The simplest example is $A = \mathbf{C}$ and $B = 0$.

Lemma 3.5. *The unitization of homomorphisms induces a bijection*

$$[C_0(\mathbf{R}^n), M_k \otimes A]_0 \cong [C(S^n), M_k \otimes \widetilde{A}]$$

for any C^*-algebra A and $n \geq 2$.

Proof. To prove injectivity, it suffices to show that, given a path $\psi_t\colon C(S^n) \to M_k \otimes \widetilde{A}$ such that

$$\varepsilon \circ \psi_0 = \varepsilon \circ \psi_1 = \phi_0 \oplus \cdots \oplus \phi_0,$$

this path can be deformed, without changing endpoints, so that

$$\varepsilon \circ \psi_t = \phi_0 \oplus \cdots \oplus \phi_0, \qquad t \in [0, 1].$$

We leave it to the reader to show that the points in S^n corresponding to the one-dimensional subrepresentations of $\varepsilon \circ \psi_t$ may be assumed to vary smoothly. (One way to show this is to apply the following argument to short subpaths.) A smooth path in S^n, $n \geq 2$, must miss an open disk U, so we have $C_0(U)$ contained in the kernel of $\varepsilon \circ \psi_t$ for all t.

Let $\eta_t\colon S^n \to S^n$ be a homotopy from id_{S^n} to a map η_1 which sends the complement of U to the north pole. Let η_t also denote the induced map on $C(S^n)$. The concatenation of the paths

$$t \mapsto \psi_0 \circ \eta_t, \quad t \mapsto \psi_t \circ \eta_1, \quad t \mapsto \psi_0 \circ \eta_{1-t},$$

is the desired deformation of ψ_t.

The proof of surjectivity is similar, and easier, so we omit it. □

4. The skeletal filtration

Let X be a finite CW-complex, and let X^n denote the n-skeleton of X. The K-theory elements which vanish on X^n form a subgroup, and the sequence of these subgroups forms a filtration of $K^*(X) \cong K_*(C(X))$. This section is devoted to proving that this filtration, known as the skeletal filtration, coincides with the spherical filtration.

The proof we give is decidedly "low-tech." An algebraic topologist could derive this result quickly and directly from Rosenberg's work, using spectral sequences for example. We take the longer route because it is the one that we, and the average functional analyst, can follow. Also, our proof depends only on Lemmas 3.2(2) and 3.3, results we feel are easy to believe, if hard to prove directly.

Theorem 4.1. *Let X be a finite CW-complex and $\rho_n\colon C(X) \to C(X^n)$ the restriction to the n-skeleton X^n of X. For $n \geq 1$,*

$$F_n K_n(C(X)) = \ker K_n(\rho_{n-1}),$$

i.e., the skeletal and spherical filtrations are identical. Moreover, there is a natural isomorphism

$$H_n(C(X)) \otimes \mathbf{Q} \cong H^n(X\,;\,\mathbf{Q})\,.$$

Proof. Unless stated otherwise, ρ_{n-1*} will denote the map

$$\rho_{n-1*}\colon K_n(C(X)) \to K_n(C(X^{n-1}))\,,$$

i.e., $K_n(\rho_{n-1})$.

Since we know by Lemma 3.3 that F_nK_n vanishes above the dimension of a CW-complex, we have

$$\rho_{n-1*}(F_nK_n(C(X))) \subseteq F_nK_n(C(X^{n-1})) = 0$$

and thus, for all n,

$$F_nK_n(C(X)) \subseteq \ker(\rho_{n-1*})\,.$$

The real work to be done is proving the reverse inclusion for $2 \le n \le \dim X$, this being trivial for other values of n. We show this first for $n = \dim X$, and then prove it in general by induction on the number of cells.

If X is n-dimensional, there is an exact sequence

$$0 \to \bigoplus C_0(\mathbf{R}^n) \xrightarrow{i} C(X) \xrightarrow{\rho_{n-1}} C(X^{n-1}) \to 0\,,$$

where there is one copy of $C_0(\mathbf{R}^n)$ for each top-dimensional cell. The middle-exactness of K_* and Proposition 3.4 imply that

$$\ker(\rho_{n-1*}) = \operatorname{Im}(i_*) \subseteq F_nK_n(C(X))\,.$$

We now assume that $2 \le n < d = \dim X$ and that the theorem is true for all spaces with fewer cells, of dimension greater than one, than X. Let Σ denote X minus the interior of one top-dimensional cell, so that

$$X = \Sigma \cup_f B^d\,,$$

where $f\colon \partial B^d \to \Sigma$ is the attaching map for the cell B^d. For each k there is a pull-back diagram:

$$\begin{tikzcd}
 & C(X) \otimes M_k \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\
C(\Sigma) \otimes M_k \arrow[dr, "j_1"'] & & C(B^d) \otimes M_k \arrow[dl, "j_2"] \\
 & C(S^{d-1}) \otimes M_k &
\end{tikzcd}$$

Suppose $e \in K_n(C(X))$ and $\rho_{n-1*}(e) = 0$. The $(n-1)$-skeletons of X and Σ coincide, so the induction hypothesis implies that there exists $\phi\colon C(S^n) \to C(\Sigma) \otimes M_k$ with $\phi_*(b) = i_{1*}(e)$. Since e vanishes on X^{n-1}, it vanishes on a point and also on B^d, i.e., $i_{2*}(e) = 0$. Therefore,

$$(j_1 \circ \phi)_*(b) = j_{2*}i_{2*}(e) = 0\,.$$

Lemma 3.2 implies that, by increasing k, we may assume that $j_1 \circ \phi$ is homotopic to point evaluation. The map

$$S^{d-1} \to \operatorname{Hom}(C(S^n)\,,\, M_k)$$

corresponding to $j_1 \circ \phi$ is null-homotopic, and so extends to a map

$$B^d \to \mathrm{Hom}(C(S^n), M_k).$$

This, in turn, corresponds to a homomorphism $\gamma\colon C(S^n) \to C(B^d) \otimes M_k$ which is a lift, by j_2, of $j_1 \circ \phi$. By the pull-back property, the pair (ϕ, γ) determines a map $\psi\colon C(S^n) \to C(X) \otimes M_k$ such that $i_1 \circ \psi = \phi$.

Let $f = \psi_*(b)$. If $d - n$ is odd, then i_{1*} is injective. By construction, $i_{1*}(e) = i_{1*}(f)$, so $e = f \in F_n K_n(C(X))$ and we are done. If $d - n$ is even, then notice that e and f agree on $X^{d-1} \subset \Sigma$. We have already shown that

$$\ker(\rho_{d-1*}) = F_d K_d(C(X)),$$

so

$$f - e \in F_d K_d(C(X)) \subseteq F_n K_n(C(X))$$

which implies that $e \in F_n K_n(C(X))$.

The last sentence of the theorem now follows easily from the fact that the Chern character is rationally an isomorphism. □

5. Some properties of the filtration

In this section we explore a few properties that hold for the spherical filtration in the noncommutative situation. We begin with a proposition which shows where this filtration stands with respect to the three axioms that [6, Proposition 5.3] shows cannot all be satisfied, namely additivity, continuity, and matricial stability.

Proposition 5.1. *The spherical filtration $F_* K_*$ satisfies the following properties:*

(1) *Matricial Stability: $F_n K_n(A) \cong F_n K_n(M_k \otimes A)$ for any C^*-algebra A, $k > 0$.*
(2) *Infinite-Additivity: $F_n K_n(\bigoplus_1^\infty A_k) = \bigoplus_1^\infty F_n K_n(A_k)$ for C^*-algebras A_k.*
(3) *Commutative Continuity: $F_n K_n(\varinjlim A_k) \cong \varinjlim F_n K_n(A)$ for any inductive system $(A_k)_k$ of commutative C^*-algebras.*

Proof. The first property was built into the definition of the spherical filtration and the second has an obvious proof.

It suffices to verify the third property in the unital case, so let $X = \varprojlim X_k$ for compact spaces X_k. Theorem 1.3.2 in [7] states that

$$[\varprojlim X_k, Z] = \varinjlim [X_k, Z]$$

whenever Z is an absolute neighborhood retract. Since $\mathrm{Hom}(C(S^n), M_j)$ is a finite CW-complex, and so an ANR,

$$\begin{aligned}[C(S^n), C(X) \otimes M_j] &= [X, \mathrm{Hom}(C(S^n), M_j)] \\ &= \varinjlim_k [X_k, \mathrm{Hom}(C(S^n), M_j)] \\ &= \varinjlim_k [C(S^n), C(X_k) \otimes M_j].\end{aligned}$$

The third property now follows. □

Corollary 5.2. *Let X be a compact Hausdorff space, and let $H^n(X;\mathbf{Q})$ denote the Čech cohomology of X. Then, for $n \geq 0$,*

$$H_n(C(X)) \otimes \mathbf{Q} \cong H^n(X;\mathbf{Q}).$$

Improving slightly on Proposition 5.1(1), we can handle (strong) Morita equivalence between unital C^*-algebras.

Lemma 5.3. *If pAp is a full corner of a unital C^*-algebra A, with $\psi: pAp \to A$ the inclusion, there is an embedding $\phi: A \to M_k(pAp)$, for some k, of A as a full corner, such that*

$$\phi_*: K_n(A) \to K_n(pAp), \qquad \psi_*: K_n(pAp) \to K_n(A)$$

are inverse to each other.

Proof. This follows easily from the proof of [9, Proposition 2.4]. □

Theorem 5.4. *If X is an equivalence bimodule inducing a Morita equivalence between unital C^*-algebras A and B, then the induced isomorphisms*

$$\Phi_X: K_n(A) \to K_n(B), \qquad n = 0, 1,$$

preserve the spherical filtration, i.e.,

$$\Phi_X(F_nK_n(A)) = F_nK_n(B), \qquad n \geq 0.$$

Proof. By definition, $\Phi_X = \phi_*^{-1} \circ \theta_*$, where $\theta: A \to L$ and $\phi: B \to L$ are embeddings as full corners into the linking algebra. By the last lemma, there exists a homomorphism $\psi: L \to M_k \otimes B$ such that $\psi_* = \phi_*^{-1}$. Therefore, $\Phi_X = (\psi \circ \theta)_*$ must respect the filtration. By symmetry, so does Φ_X^{-1}. □

Theorem 5.5. *Suppose A, B, and C are C^*-algebras and C is a common subalgebra of A and B with inclusion maps $j_1: C \to A$ and $j_2: C \to B$. Suppose also that there are retracts $\psi_1: A \to C$ and $\psi_2: B \to C$, i.e., assume $\psi_1 \circ j_1 = \psi_2 \circ j_2 = \mathrm{id}$. Then there is a natural isomorphism, for $n \geq 0$,*

$$F_nK_n(A *_C B) \cong F_nK_n(D),$$

where $D = \{(a, b) \in A \oplus B | \psi_1(a) = \psi_2(b)\}$.

Proof. In [3], Cuntz shows that there is a homomorphism $k: A *_C B \to D$ such that the induced map k_* is an isomorphism on K-theory. The inverse to k_* is

$$(j_* - \psi_*): K_*(D) \to K_*(A *_C D),$$

where $j: D \to M_2(A *_C B)$ and $\psi: D \to A *_C B$ are certain homomorphisms. Therefore, k_* and $(k_*)^{-1}$ both respect the spherical filtration. □

Theorem 5.6. *Suppose $0 \to I \to A \xrightarrow{\pi} B \to 0$ is exact and π is a cofibration. Then the K-theory boundary $\partial: K_n(B) \to K_{n+1}(I)$ respects the spherical filtration in the sense that, for $n \geq 0$,*

$$\partial(F_nK_n(B)) \subseteq F_{n+1}K_{n+1}(I).$$

Proof. This is trivial for $n = 0$, so suppose $n \geq 1$. Let B^{n+1} denote the closed $(n+1)$-ball. By choosing the correct orientation for the inclusion $\mathbf{R}^{n+1} \to B^{n+1}$, we obtain an exact sequence

$$0 \to C_0(\mathbf{R}^{n+1}) \to C(B^{n+1}) \xrightarrow{\rho} C(S^n) \to 0$$

such that $\partial(b_n) = b_{n+1}$.

It suffices to consider the case where A, B, and π are unital. Assume that $x = \phi_*(b) \in F_nK_n(B)$, where $\phi\colon C(S^n) \to M_k \otimes B$ is unital. The composition $\phi \circ \rho$ is homotopic to $\phi_0 \oplus \cdots \oplus \phi_0$ simply because $C(B^{n+1})$ is contractible. Since

$$\phi_0 \oplus \cdots \oplus \phi_0\colon C(B^{n+1}) \to M_k \otimes B$$

lifts to

$$\phi_0 \oplus \cdots \oplus \phi_0\colon C(B^{n+1}) \to M_k \otimes A,$$

and since $1 \otimes \pi$ is a cofibration whenever π is [11, Proposition 1.11], $\rho \circ \phi$ lifts to some $\eta\colon C(B^{n+1}) \to M_k \otimes A$. Let ψ denote the restriction of η to $C_0(\mathbf{R}^{n+1})$.

The naturality of ∂, applied to the diagram

$$\begin{tikzcd}
0 \arrow[r] & I \otimes M_k \arrow[r] & A \otimes M_k \arrow[r] & B \otimes M_k \arrow[r] & 0 \\
0 \arrow[r] & C_0(\mathbf{R}^{n+1}) \arrow[r] \arrow[u, "\psi"'] & C(B^{n+1}) \arrow[r] \arrow[u, "\eta"'] & C(S^n) \arrow[r] \arrow[u, "\phi"'] & 0
\end{tikzcd}$$

implies that

$$\partial x = \partial \phi_*(b_n) = \psi_* \partial(b_n) = \psi_*(b_{n+1}) \in F_{n+1}K_{n+1}(I). \quad \square$$

It is unknown to us to what extent the sequence

$$\cdots \to F_nK_n(I) \to F_nK_n(A) \to F_nK_n(B) \xrightarrow{\partial} F_{n+1}K_{n+1}(I) \to \cdots$$

is exact when $\pi\colon A \to B$ is a cofibration. As examples will demonstrate, F_nK_n need not be middle-exact and ∂ need not respect the filtration when π is not a cofibration.

6. Examples

6.1. Continuous trace C^*-algebras. If A is a unital, continuous-trace C^*-algebra with trivial Dixmier-Douady invariant $\delta(A) \in H^3(\widehat{A}, \mathbf{Z})$, then A is Morita equivalent to $C(\widehat{A})$ and

$$H_n(A) \otimes \mathbf{Q} \cong H^n(\widehat{A}; \mathbf{Q}).$$

If, in addition, $\widehat{A}$ has the homotopy-type of a finite CW-complex, then $F_*K_*(A)$ is isomorphic to $K^*(\widehat{A})$ with the skeletal filtration. In particular, let A_θ denote the rotation algebra for $0 \le \theta < 1$. If θ is rational, then

$$H_n(A_\theta) = \begin{cases} \mathbf{Z} & \text{for } n = 0, \\ \mathbf{Z} \oplus \mathbf{Z} & \text{for } n = 1, \\ \mathbf{Z} & \text{for } n = 2, \\ 0 & \text{for } n \ge 3. \end{cases}$$

If θ is irrational, the unique normalized trace $\tau\colon A_\theta \to \mathbf{C}$ is faithful on K_0, so Lemma 2.3 implies that $F_2K_0(A_\theta) = 0$ and

$$H_n(A_\theta) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{for } n = 0, \\ 0 & \text{for } n > 0 \text{ even}. \end{cases}$$

We cannot calculate $F_3K_1(A_\theta)$ for θ irrational, but conjecture that it is zero.

6.2. Infinite homotopy groups. We define the infinite even sphere $S^{2\infty}$ to be $\varinjlim C(S^{2n}) \otimes M_{2^n}$. The connecting maps are $\beta_{2n} \otimes 1$, where β_{2n} is as defined in the proof of Proposition 2.2. Clearly

$$K_n(S^{2\infty}) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{for } n = 0, \\ 0 & \text{for } n = 1. \end{cases}$$

There is a trace τ on $S^{2\infty}$ defined by the traces

$$C(S^{2n}) \otimes M_{2^n} \xrightarrow[2^{-n}\phi_0 \otimes \mathrm{Tr}]{} \mathbf{C}.$$

Using this trace, one easily shows that

$$H_n(S^{2\infty}) = \begin{cases} \mathbf{Z} & \text{for } n = 0, \\ 0 & \text{for } n > 0. \end{cases}$$

One may define, in a similar fashion, an infinite odd sphere and conclude the same result on homology.

This example illustrates the need to define *infinite even* and *infinite odd homology groups.* For any C^*-algebra A, we define

$$H_{2\infty}(A) = \bigcap_{n \geq 0} F_{2n}K_0(A), \qquad H_{2\infty+1}(A) = \bigcap_{n \geq 0} F_{2n+1}K_1(A).$$

For example, we have

$$H_{2\infty}(S^{2\infty}) = H_{2\infty+1}(S^{2\infty+1}) = \mathbf{Z}.$$

6.3. A poorly behaved boundary map. Consider the exact sequence

$$0 \to \mathscr{K}(\mathscr{H}) \to \mathscr{B}(\mathscr{H}) \to \mathscr{Q}(\mathscr{H}) \to 0.$$

The surjection here is not a cofibration, and we will show that ∂ does not respect the spherical filtration.

A consequence of the spectral theorem for compact operators is that

$$F_nK_n(\mathscr{K}) = \begin{cases} \mathbf{Z} & \text{for } n = 0, \\ 0 & \text{for } n > 0, \end{cases}$$

and the universal coefficient theorem can be used to show that

$$F_nK_n(\mathscr{Q}(\mathscr{H})) = \begin{cases} 0 & \text{for } n \text{ even}, \\ \mathbf{Z} & \text{for } n \text{ odd}. \end{cases}$$

The K-theory boundary is an isomorphism in this case, so for $n \geq 1$,

$$\partial(F_{2n-1}K_1(\mathscr{Q}(\mathscr{H}))) \not\subseteq F_{2n}K_0(\mathscr{K}).$$

6.4. AF algebras. The next sequence of examples will show that F_2K_0 is not, in general, middle-exact or continuous. In [6], a certain embedding $\psi\colon C(\mathbf{T}^2) \to A$ is proven to be injective on K_0, where A is the unital AF algebra with $K_0(A) = \mathbf{Z}[1/2] \oplus \mathbf{Z}$, with the lexicographic order and order unit $(1, 0)$. An immediate consequence of this result is that $F_2K_0(A) \neq 0$. (In fact, $F_2K_0(A) = 0 \oplus \mathbf{Z}$.) Therefore, F_2K_0 is not continuous because F_2K_0 is zero on any finite-dimensional C^*-algebra.

There is an exact sequence

$$0 \to \mathscr{K} \to A \xrightarrow{p} M_{2\infty} \to 0,$$

where M_{2^∞} denotes the type 2^∞ UHF algebra. We saw above that $F_2K_0(\mathscr{K}) = 0$, and Lemma 2.3 implies that $F_2K_0(M_{2^\infty}) = 0$. Therefore,

$$F_2K_0(\mathscr{K}) \to F_2K_0(A) \to F_2K_0(M_{2^\infty})$$

is not exact.

Fortunately, the surjection p is not a cofibration. It is still conceivable that there is a long exact sequence in F_*K_* arising from a short exact sequence containing a cofibration. The fact that p is not a cofibration follows from a result in the appendix concerning Pimsner's AF embedding construction. This will show that $p \circ \psi$ is homotopic to $p \circ \phi_0 = \phi_0$, while K-theory is an obstruction by ψ and ϕ_0 being homotopic.

Other embeddings of $C(\mathbf{T}^2)$, and hence $C(S^2)$, into AF algebras can be constructed that are injective on K_0. Work in progress by the second author and Elliott shows that, for many AF algebras, one may find embeddings of $C(\mathbf{T}^2)$ which induce all possible maps on K_0. In particular, it can be shown that

$$F_2K_0(A \otimes M_{2^\infty}) = \bigcap \ker(\tau_*\colon K_0(A \otimes M_{2^\infty}) \to \mathbf{R})$$

for any unital AF algebra A. The intersection is over all traces on $A \otimes M_{2^\infty}$.

We have, as yet, little knowledge about F_4K_0 of AF algebras, and no knowledge about H_2. We do know that $F_4K_0(A)$ can be nonzero for some AF algebras, specifically $A \otimes B$, where A and B are AF algebras with F_2K_0 nonzero.

7. Determinants and the three-sphere

An important tool for calculating $F_2K_0(A)$ is Lemma 2.3 which relates $\mathrm{Hom}(C(S^2, A))$ to traces on A. When calculating $F_3K_1(A)$, one would like a similar result relating $\mathrm{Hom}(C(S^3, A))$ to determinants. This section will review the subject of generalized determinants for C^*-algebras, state some conjectures, and show how $F_3K_1(A)$ can be calculated in examples if one assumes these conjectures.

N.B. In this section, we will assume that all C^*-algebras, and homomorphisms, are unital. Also, we will continue to use topological K_1.

We will need to select a representative $\ell \in U_2(C(S^3))$ of the Bott element $b \in K_1(C(S^3))$. Considering S^3 as the unit sphere in $\mathbf{C}^2$, let $m, n \in C(S^3)$ denote the two coordinate functions. We let

$$\ell = \begin{pmatrix} m & -n^* \\ n & m^* \end{pmatrix}.$$

This is easily seen to be a generator of $K_1(C(S^3))$, and so $[\ell] = \pm b$. If we are off by a minus sign, we replace m by m^* and so assume $[\ell] = b$.

We will be interested in the set

$$\{\phi(\ell) | \phi \in \mathrm{Hom}(C(S^3), A)\},$$

where A is an arbitrary C^*-algebra. This is equal to the set

$$\left\{ \begin{pmatrix} M & -N^* \\ N & M^* \end{pmatrix} \middle| \begin{array}{l} M, N \in A \text{ are commuting normals} \\ \text{such that } MM^* + NN^* = 1 \end{array} \right\}.$$

It would appear that these unitaries should have determinant one. While this is true for trivial reasons when A is commutative, it is unknown in general.

A determinant on a C^*-algebra is only defined after one has specified a trace τ on A. In [5], de la Harpe and Skandalis define a homomorphism

$$\Delta_\tau\colon U_\infty(A)_0 \to \mathbf{R}/\tau_*(K_0(A)).$$

This is called the determinant associated to τ because

$$\Delta_\tau(e^{2\pi i h}) = q(\tau(h)), \qquad h \in M_n(A)_{\mathrm{sa}}.$$

(Throughout this section $q\colon \mathbf{R} \to \mathbf{R}/\tau_*(K_0(A))$ will denote the canonical quotient map.)

In [4], the first author has shown how to extend this determinant to all of $U_\infty(A)$, but only when $\tau_*(K_0(A)) = \mathbf{Z}$. It is easy to remove this restriction. The following basic results on determinants are proven by straightforward modifications of the proofs in [4].

Definition 7.1. We say that a group homomorphism

$$\det\colon U_\infty(A) \to \mathbf{R}/\tau_*(K_0(A))$$

is a *determinant function* associated to the trace τ on the C^*-algebra A if

$$\det(e^{2\pi i h}) = q(\tau(h)), \qquad h \in M_n(A)_{\mathrm{sa}}.$$

We will tighten this phrase to say det is a determinant associated to (A, τ). A determinant associated to a trace is not unique, but any two are related in a simple way.

Theorem 7.2. *Let τ be a trace on a C^*-algebra A. Then A admits a determinant associated to τ, and given a determinant* $\det_0$, *all others are given by*

$$\det(u) = \det_0(u) + \eta([u]), \qquad u \in U_\infty(A),$$

where $\eta\colon K_1(A) \to \mathbf{R}/\tau_*(K_0(A))$ is a group homomorphism.

Lemma 7.3. *If* det *is a determinant associated to (A, τ), then*

(1) *If $u \in U_\infty(A)$ is in the connected component of the identity, then*

$$\det(u) = q\left(\frac{1}{2\pi i}\int_0^1 \tau(x_t' x_t^{-1})\,dt\right),$$

where x_t is any smooth path in $\mathrm{GL}_\infty(A)$ *with $x_0 = 1$ and $x_1 = u$.*

(2) *If γ is any smooth path in* $\mathrm{GL}_\infty(A)$, *with unitary endpoints, then*

$$\det(\gamma(1)) - \det(\gamma(0)) = q\left(\frac{1}{2\pi i}\int_\gamma \tau(\gamma'\gamma^{-1})\right).$$

One may consider determinants defined on $\mathrm{GL}_\infty(A)$ with values in $\mathbf{C}/\tau_*(K_0(A))$. For our purposes, however, this offers no advantage.

Now we turn to the commutative case, $A = C(X)$. Here we have the usual A-valued determinant and trace on $M_n(A) = C(X, M_n)$. We denote these by Det_n and T_n. Of course, these can be computed pointwise.

Theorem 7.4. *Suppose X is a compact space and* det *is a determinant associated to $(C(X), \tau)$. If $\phi\colon C(S^3) \to M_k \otimes C(X)$ is a homomorphism, then* $\det(\phi(\mathscr{E}))$ *depends only on ϕ_*, the induced map on K_1.*

Proof. If det is one determinant associated to $(C(X), \tau)$, then

$$u \mapsto \det(\mathrm{Det}_n(u)), \qquad u \in U_n(C(X)),$$

defines another. By Theorem 7.2, there is a homomorphism

$$\eta\colon K_1(C(X)) \to \mathbf{R}/\tau_*(K_0(C(X)))$$

such that

$$\det(u) = \eta([u]) + \det(\mathrm{Det}_n(u))$$

for $u \in U_n(C(X))$. Recall that when A, B, C, D are in $M_k(\mathbf{C})$ and $AB = BA$, we have the block-determinant rule

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(DA - CB).$$

Applied pointwise, this rule shows that $\mathrm{Det}_{2k}(\phi(\mathscr{E})) = 1$, and so

$$\det(\phi(\mathscr{E})) = \eta([\phi(\mathscr{E})]) + \det(1) = \eta(\phi_*(b)). \quad \square$$

Returning to the noncommutative situation, we state two versions of our conjecture.

Conjecture 7.5. Suppose det is a determinant associated to (A, τ). If $\phi\colon C(S^3) \to A$ is a homomorphism, then $\det(\phi(\mathscr{E}))$ depends only on $\phi_*(\mathscr{E})$. In particular, if $\phi_*(b) = 0$, then $\det(\phi(\mathscr{E})) = 0$.

Conjecture 7.6. Suppose det is a determinant associated to (A, τ). If $\phi_0, \phi_1\colon C(S^3) \to A$ are homotopic, then $\det(\phi_0(\mathscr{E})) = \det(\phi_1(\mathscr{E}))$.

Suppose $\phi_t\colon C(S^3) \to A$ is a path such that $M_t = \phi_t(m)$ and $N_t = \phi_t(n)$ happen to be *smooth*. Then

$$\begin{aligned}\tau(\phi_t(\mathscr{E})'\phi_t(\mathscr{E})^{-1}) &= \tau\begin{pmatrix} M_t'M_t^* + N_t^{*\prime}N_t & M_t'N_t^* - N_t^{*\prime}M_t \\ N_t'M_t^* - M_t^{*\prime}N_t & N_t'N_t^* + M_t^{*\prime}M_t \end{pmatrix} \\ &= \tau((M_tM_t^* + N_tN_t^*)') = \tau((1)') = 0.\end{aligned}$$

Therefore, the conjecture is true in this special case. One consequence of this is that $\det(\phi(\mathscr{E})) = 0$ if $\phi\colon C(S^3) \to A$ is not injective.

The most obvious application of this conjecture would be to suspensions. Let A be a C^*-algebra. Recall that there is an isomorphism $\sigma\colon K_1(SA) \to K_0(A)$ which sends $[e^{2\pi i t p}]$ to $[p]$ for any projection $p \in M_k \otimes A$.

Lemma 7.7. *Suppose* det *is a determinant associated to (A, τ) and that $\tau_*(K_0(A))$ is countable. If $u = (u_t)$ is an element of $U_n((SA)^\sim)$ and* $\det(u_t) = 0$ *for all $t \in [0, 1]$, then $\tau_* \circ \sigma([u]) = 0$.*

Proof. If $t \mapsto u_t$ is piecewise smooth, then the result is trivial because we have the formula

$$\tau_* \circ \sigma([u]) = \frac{1}{2\pi i}\int_0^1 \tau(u_t'u_t^*)\,dt$$

(see [2, p. 76]).

Suppose $t \mapsto u_t$ is simply continuous. Choose points $t_0 = 0 < t_1 < \cdots < t_n = 1$ such that, for any k,

$$\|u_x - u_y\| < \tfrac{1}{2} \quad \text{for } t_k \le x, y \le t_{k+1}.$$

Fix k. Applying a branch of log, we find a continuous, selfadjoint path $h(s)$ such that $h(t_k) = 0$ and

$$u_{t_k}^{-1} u_s = e^{2\pi i h(x)} \quad \text{for } t_k \le s \le t_{k+1}.$$

It follows from our hypothesis that $\tau(h(s)) \in \tau_*(K_0(A))$. Since $\tau(h(t_k)) = 0$ and $\tau(h(s))$ varies continously in a countable subset of $\mathbf{R}$, we have $\tau(h(t_{k+1})) = 0$ as well. Let $h_k = h(t_{k+1})$. Let v denote the path

$$v_t = u_{t_k} e^{2\pi i s h_k},$$

where $t \in [t_k, t_{k+1}]$, $t = t_k + s(t_{k+1} - t_k)$. Therefore,

$$\tau_* \circ \sigma([u]) = \tau_* \circ \sigma([v]) = 0$$

because $t \mapsto v_t$ is piecewise smooth and $\det(v_t) = \det(u_0) = 0$. □

Consider the UHF algebra $M_{2\infty}$. The trace on $M_{2\infty}$ induces a faithful map on $K_0(M_{2\infty})$. If Conjecture 7.6 were true for $M_k \otimes M_{2\infty}$, $k \ge 1$, then Lemma 7.7 would imply that $F_3K_1(SM_{2\infty}) = 0$.

The introduction of generalized rotation numbers is necessary for our other applications. We will simply state the definition of a rotation number map, referring the reader to [4] for details.

Suppose α is an automorphism of a C^*-algebra A and τ is a trace on A. Let $K_1(A)^\alpha$ denote the α-invariant subgroup of $K_1(A)$. The rotation number map is the homomorphism

$$\rho_\alpha^\tau \colon K_1(A)^\alpha \to \mathbf{R}/\tau_*(K_0(A))$$

defined by $\rho_\alpha^\tau([u]) = \det(\alpha(u^*)u)$, where u is a unitary representing some element of $K_1(A)^\alpha$ and det is any determinant associated to τ.

Proposition 7.8. *If Conjecture* 7.6 *is true, then*

$$F_3K_1(A_\theta) = F_3K_1(C_r^*(F_2)) = 0,$$

where A_θ denotes any irrational rotation algebra and $C_r^*(F_2)$ denotes the reduced C^*-algebra of the free group on two generators.

Proof. Let A denote either A_θ or $C_r^*(F_2)$, with τ the canonical trace and u, v the canonical unitary generators. Choose $r, s \in \mathbf{R}$ so that $\{1, \theta, r, s\}$ are linearly independent over $\mathbf{Q}$, and consider the action α on A defined by

$$\alpha(u) = e^{2\pi i r} u, \qquad \alpha(v) = e^{2\pi i s} v.$$

There is a homotopy of α to the identity, so $K_1(A)^\alpha = K_1(A)$. Clearly $\rho_\alpha^\tau([u]) = r$ and $\rho_\alpha^\tau([v]) = s$, and since $[u]$ and $[v]$ generate $K_1(A)$, the rotation number map is injective.

Suppose we are given $\phi \colon C(S^3) \to M_k \otimes A$. Then ϕ and $\alpha \circ \phi$ are homotopic, so by the conjecture,

$$\rho_\alpha^\tau([\phi(\ell)]) = -\det(\alpha \circ \phi(\ell)) + \det(\phi(\ell)) = 0,$$

hence $[\phi(\ell)] = 0$. □

It is perhaps reasonable to expect that $F_3K_1(C_r^*(F_2)) = 0$ because, in the unreduced case, we have, by Theorems 4.1 and 5.5,

$$F_3K_1(C^*(F_2)) \cong F_3K_1(C(S^1) *_{\mathbf{C}} C(S^1)) \cong F_3K_1(C(X)) = 0,$$

where X is the figure-eight.

Appendix: Homotopy properties of certain AF embeddings

Pimsner [8] describes a method for embedding certain crossed products $C(X) \rtimes \mathbf{Z}$ into AF algebras. When the action of $\mathbf{Z}$ on X is trivial, one certainly does not need Pimsner's construction to find such an embedding. Nevertheless, Pimsner's construction, applied in this trivial case, produces a very interesting embedding [6].

This appendix is devoted to proving that certain AF embeddings, which are closely related to Pimsner's embeddings, are homotopically trivial. This ties up a loose end left in §6.4.

Suppose that T denotes the trivial automorphism of a compact, metrizable space X. Adopting all of Pimsner's notation from [8], we suppose that $(\mathscr{V}_n)_n$, $(m_n)_n$, and $(F_n)_n$ are choices of open covers, multiplicities, and decomposition maps for which the embedding $\rho\colon C(X) \rtimes \mathbf{Z} \to A$ of [8, Theorem 7] can be constructed. Recall that A is the limit of the system

$$\cdots \to A_n \xrightarrow{\phi_n} A_{n+1} \to \cdots,$$

where $A_n = \bigoplus_{\omega\in\Omega} M_\omega$ and $\phi_n = \bigoplus_{\omega\in\Omega_{n+1}} \phi_\omega$. The covariant form (π, U) of ρ is given by $\pi = \lim \pi_n$ and $U = \lim U_n$, where

$$U_n = \bigoplus_{\omega\in\Omega} U_\omega, \qquad \pi_n = \bigoplus_{\omega\in\Omega_n} \pi_\omega\colon C(X) \to A_n.$$

There is an easily identified quotient of A corresponding to the set of trivial pseudo-orbits. By a *trivial pseudo-orbit* we mean a pseudo-orbit ω with principal period $p(\omega) = 1$. The triviality of T implies that $\mathscr{V}_n^{(m_n)} = \mathscr{V}_n$, so Ω consists of periodic $\mathscr{V}_n$ pseudo-orbits of $T^{m_n} = T$. Let

$$\Delta_n = \{\omega \in \Omega_n | p(\omega) = 1\}, \qquad B_n = \bigoplus_{\omega\in\Delta_n} A_\omega.$$

A trivial pseudo-orbit can only be decomposed into trivial pseudo-orbits. By the definition of the ϕ_ω, we have $\phi_n(B_n) \subseteq B_{n+1}$. We obtain an AF algebra B, and a quotient map $p\colon A \to B$, by taking the limit of the diagram:

$$\begin{array}{ccccc} \cdots & \to A_n & \xrightarrow{\phi_n} A_{n+1} & \to \cdots \\ & \downarrow & \downarrow & \\ \cdots & \to B_n & \xrightarrow{\phi_n|B_n} B_{n+1} & \to \cdots \end{array}$$

Theorem A.1. *Suppose X is a compact, connected, metrizable absolute neighborhood retract and $\mathbf{Z}$ acts trivially on X. Let $\rho\colon C(X) \rtimes \mathbf{Z} \to A$ denote Pimsner's AF embedding corresponding to $(\mathscr{V}_n)_n$, $(m_n)_n$, and $(F_n)_n$. If $p\colon A \to B$ denotes the surjection corresponding to the collection of all trivial pseudo-orbits, then the composition $p \circ \rho\colon C(X) \rtimes \mathbf{Z} \to B$ is homotopic to point-evaluation ϕ_0.*

Proof. If $i \in I_n$, let σ_i denote the trivial pseudo-orbit $\ldots, i, i, i, \ldots$. Recall that there are functions $f_n \colon I_{n+1} \to I_n$ such that $\mathscr{V}_{n+1} \preccurlyeq_{f_n} \mathscr{V}_n$. If $\sigma_i \in \Delta_{n+1}$, then σ_i must decompose as m_{n+1}/m_n copies of $\sigma_{f_n(i)} \in \Delta_n$. The Bratteli diagram for B has one vertex v_i for each $\sigma_i \in \Delta_{n+1}$ and v_i is connected to $v_{f_n(i)}$ with multiplicity m_{n+1}/m_n. Comparison of Bratteli diagrams shows that $B \cong C(\Delta) \otimes M_{m_\infty}$, where Δ is the inverse limit of $\cdots \leftarrow \Delta_n \leftarrow \Delta_{n+1} \leftarrow \cdots$, $\sigma_{f_n(i)} \mapsfrom \sigma_i$, and M_{m_∞} is the UHF algebra corresponding to the generalized integer

$$m_\infty = \prod_{n=1}^{\infty} m_{n+1}/m_n .$$

Let (π_0, U_0) be the covariant form of $p \circ \rho$. If σ is a trivial pseudo-orbit, π_σ maps $C(X)$ into the scalars of M_σ. Therefore,

$$\pi_0(C(X)) \subseteq C(\Delta) \otimes 1 .$$

The assumption that X is a connected ANR implies that

$$[\Delta, X] = \varinjlim [\Delta_n, X] = *,$$

so there is a homotopy π_t from π_0 to ϕ_0 with $\pi_t(C(X)) \subseteq C(\Delta) \otimes 1$. The unitary group of an AF algebra is connected; let U_t be any path of unitaries from U_0 to 1. Since $C(\Delta) \otimes 1$ is in the center of $C(\Delta) \otimes M_{2^\infty}$, it follows that (π_t, U_t) is a path of covariant homomorphisms from (π_0, U_0) to $(\phi_0, 1)$. □

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