SYMOMETRIC LOCAL ALGEBRAS WITH 5-DIMENSIONAL CENTER

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Dedicated to Hiroyuki Tachikawa on the occasion of his sixtieth birthday

Abstract. We prove that a symmetric split local algebra whose center is 5-dimensional has dimension 5 or 8. This implies that the defect groups of a block of a finite group containing exactly five irreducible Frobenius characters and exactly one irreducible Brauer character have order 5 or are nonabelian of order 8.

Let $F$ be a field, and let $A$ be a finite-dimensional associative unitary $F$-algebra with center $Z$ and radical $J$. Then $A$ is called split local if $\dim A/J = 1$, and $A$ is called symmetric if there is a linear map $\lambda : A \to F$ whose kernel contains all Lie commutators $[x, y] := xy - yx$ ($x, y \in A$) but no nonzero ideal of $A$. Suppose now that $A$ is symmetric and split local. In [6] the second author proved that $A$ is necessarily commutative if $\dim Z \leq 4$. This incorporated earlier results by R. Brauer and J. Brandt [1]. In this paper we are dealing with the next case.

Theorem. Let $F$ be a field, and let $A$ be a symmetric split local $F$-algebra with center $Z$. If $\dim Z = 5$ then $\dim A \in \{5, 8\}$.

The group algebra of a group of order 5 over a field of characteristic 5 is an example for the case $\dim A = 5$, and the group algebra of a nonabelian group of order 8 over a field of characteristic 2 is an example for the case $\dim A = 8$.

Corollary. Let $F$ be an algebraically closed field, let $G$ be a finite group, let $P$ be an indecomposable projective $FG$-module, and set $A := \text{End}_{FG}(P)$. If the center of $A$ has dimension 5 then $\dim A \in \{5, 8\}$.

Proof. We choose a primitive idempotent $i$ in $FG$ such that $P$ is isomorphic to $FGi$. Then $A$ is isomorphic to $iFGi$. Since $FG$ is a symmetric $F$-algebra, so are $iFGi$ and $A$. Since $P$ is indecomposable and $F$ is algebraically closed, $A$ is split local. Hence the corollary follows from the theorem.

We have the following application to block theory.

Proposition. Let $F$ be an algebraically closed field, let $G$ be a finite group, and let $B$ be a block of $FG$ containing exactly 5 irreducible complex characters.
and exactly one irreducible Brauer character. Then the defect groups of \( B \) have order 5 or are nonabelian of order 8.

**Proof.** Let \( P \) denote the only indecomposable projective \( FG \)-module in \( \beta \), and set \( A := \text{End}_{FG}(P) \). By Lemma B in [4], \( B \) is isomorphic to a complete matrix algebra over \( A \); in particular, \( A \) and \( B \) have isomorphic centers. By \((2G)\) in [2], the dimension of the center of \( B \) coincides with the number of irreducible complex characters in \( B \), so the center of \( A \) has dimension 5. By the corollary, \( A \) has dimension 5 or 8. On the other hand, Lemma B in [4] shows that the dimension of \( A \) coincides with the order of a defect group \( D \) of \( B \). Hence \( D \) has order 5 or 8. Assume now that \( D \) is abelian of order 8. Then \( B \) cannot be nilpotent in the sense of [3]; for otherwise \( B \) would contain 8 irreducible complex characters by the main result of [3]. Thus \( D \) must be elementary abelian. But in this case we obtain a contradiction using the results in [7].

The remainder of this paper consists of a proof of the theorem. Let \( A \) be a symmetric split local algebra over a field \( F \) and denote by \( Z \) the center and by \( J \) the radical of \( A \). We may and do assume that \( F \) is algebraically closed. For a subset \( X \) of \( A \), we denote by \( FX \) the linear subspace of \( A \) spanned by \( X \). The subspace \( K := F\{[x, y] : x, y \in A\} \) will be particularly important for us. Since \( A = F1 + J \) we have \( K = [J, J] \subset J^2 \). We fix a linear map \( \lambda : A \to F \) the kernel of which contains \( K \) but no nonzero ideal of \( A \). Then 0 is the only ideal of \( A \) contained in \( K \). For any linear subspace \( U \) of \( A \), \( U^\perp := \{a \in A : \lambda(aU) = 0\} \) is a linear subspace of \( A \) such that \( \dim A = \dim U + \dim U^\perp \) and \((U^\perp)^\perp = U \). We have \( Z^\perp = K \) (see [5]); in particular, \( \dim Z = \dim A/K \). Moreover,

\[ I^\perp = \{a \in A : aI = 0\} = \{a \in A : Ia = 0\} \]

for any ideal \( I \) of \( A \); in particular, \( I^\perp \) is an ideal of \( A \). Furthermore, \( \dim J^\perp = \dim A/J = 1 \). Hence, if \( J^n = 0 \) for some positive integer \( n \) then \( J^{n-1} \subset J^\perp \); in particular, \( \dim J^{n-1} \leq \dim J^\perp = 1 \). We will often use this fact without special reference.

### 1. Preliminary results

From now on we suppose that \( \dim Z = 5 \). We may and will assume that \( \dim A \geq 6 \); for otherwise we are done.

**(1.1) Lemma.** We have \( \dim A \geq 8 \).

**Proof.** Assume that \( \dim A \leq 7 \). Then there are elements \( a, b \in A \) such that \( A = Z + Fa + Fb \). Therefore \( K = F[a, b] \); in particular, \( \dim K \cap Z \leq \dim K \leq 1 \). Now Lemma D in [6] implies that \( A \) is commutative, so \( \dim A = \dim Z = 5 \), a contradiction.

If \( \dim A = 8 \), then the theorem is proved, so we may and will assume that \( \dim A \geq 9 \). We are then looking for a contradiction.

**(1.2) Lemma.** We have \( \dim A/K + J^3 = 4 \), and one of the following occurs:

- \( \dim J/J^2 = 2 \), \( \dim J^2/J^3 = 2 \), \( \dim J^3/J^4 \geq 2 \), \( \dim J^4/J^5 \geq 1 \), \( K + J^3 = K + J^4 \).
(1.4) $\dim J/J^2 = 3$, $\dim J^2/J^3 = 2$, $\dim J^3/J^4 \geq 2$, $\dim J^4/J^5 \geq 1$, $J^2 = K + J^3 = K + J^4$.

(1.5) $\dim J/J^2 = 3$, $\dim J^2/J^3 = 3$, $\dim J^3/J^4 \geq 2$, $\dim J^4/J^5 \geq 1$, $J^2 = K + J^3 = K + J^4$.

Proof. Since $\dim J \geq 8$ we have $J^2 \neq 0$. Thus Nakayama’s Lemma implies that $J^2 \neq J^3$. Furthermore, $J \not\subseteq Z$, so $\dim J^2/J^3 \geq 2$ by Lemma G in [6]; in particular, $\dim J/J^2 \geq 2$ by Lemma E in [6], and $J^3 \neq 0$. Hence $J^3 \neq J^4$ by Nakayama’s Lemma, and $J^3 \not\subseteq K$. Thus

$$\dim A/J^2 \leq \dim A/K + J^3 < \dim A/K = \dim Z = 5;$$

in particular, $\dim J/J^2 \in \{2, 3\}$, so $\dim J^2 \geq 5$. This means that $J^2 \not\subseteq Z$ which implies by Lemma G in [6] that $\dim J^3/J^4 \geq 2$. Hence $J^4 \neq 0$, and $J^4 \neq J^5$ by Nakayama’s Lemma again. Moreover, $J^4 \not\subseteq K$, so $\dim A/K + J^3 \leq \dim A/K + J^4 < \dim A/K = 5$.

Suppose first that $\dim J/J^2 = 2$ and write $J = F\{a, b\} + J^2$ with elements $a, b \in J$. Then $A = F\{1, a, b\} + J^2$ and $K \subseteq F\{a, b\} + J^3$; in particular, $\dim K + J^3/J^4 \leq 1$, so $\dim A/J^3 \leq 5$ and $\dim J^2/J^3 = 2$. Thus $\dim A/J^3 = 5$ and $\dim A/K + J^3 = 4$. Hence also $\dim A/K + J^4 = 4$.

Finally, suppose that $\dim J/J^2 = 3$ and write $J = F\{a, b, c\} + J^2$ with elements $a, b, c \in J$. Then $K \subseteq F\{a, b\}, [a, c], [b, c]\} + J^3$; in particular, $\dim K + J^3/J^4 \leq 3$. Thus $\dim A/J^3 \leq 7$ and $\dim J^2/J^3 \in \{2, 3\}$. Since $4 = \dim A/J^2 \leq \dim A/K + J^3 \leq \dim A/K + J^4 \leq 4$ the result follows.

We will deal with these cases in §§2, 3 and 4, respectively. The following results will be useful later on.

(1.6) Lemma. There is an element $x \in J$ such that $x^2 \not\in J^3$.

Proof. By (1.2) we have $\dim J/J^2 \leq 3$. We write $J = F\{a, b, c\} + J^2$ with elements $a, b, c \in J$. If $x^2 \in J^3$ for $x \in J$ then $ab + ba = (a+b)^2 - a^2 - b^2 \in J^3$. Thus $a = a - b (mod J^3)$. Similarly, $c = c - a (mod J^3)$ and $b = c - (-bc) (mod J^3)$. Therefore $J^2 = F\{a^2, ab, ac, ba, b^2, bc, ca, cb, c^2\} + J^3 = F\{ab, ac, bc\} + J^3$;

in particular, $\dim J^2/J^3 \leq 3$. Now we apply Lemma E in [6] to obtain $J^3 = F\{a^2b, a^2c, abc, bab, bac, b^2c\} + J^4 = Fabc + J^4$ and $J^4 = F a^2b c + J^5 = J^5$ contradicting (1.2).

(1.7) Lemma. There are elements $a, b \in J$ such that $a^2 + J^3, ab + J^3$ or $a^2 + J^3, ba + J^3$ are linearly independent in $J^2/J^3$.

Proof. By (1.6), there is an element $a \in J$ such that $a^2 \not\in J^3$; in particular, $a \not\in J^2$. By (1.2) there are therefore elements $b, c \in J$ such that $J = F\{a, b, c\} + J^2$. We may assume that $ab, ba, ac, ca \in F a^2 + J^3$; for otherwise the result is proved. Then $K + J^3 = F\{a, b\}, [a, c], [b, c]\} + J^3 \subseteq F\{a^2, [b, c]\} + J^3$; in particular, $\dim K + J^3/J^3 \leq 2$. Hence, by (1.2), $\dim J^2/J^3 = 2$.

Now consider the case where $b^2 \not\in F a^2 + J^3$; in particular, $b^2 \not\in J^3$. Then we can interchange the roles of $a$ and $b$ and therefore assume that
ab, ba, bc, cb ∈ Fb^2 + J^3. Since a^2 + J^3 and b^2 + J^3 form a basis of J^2/J^3 this implies that ab, ba ∈ J^3. Thus (a + b)^2 + J^3 = a^2 + b^2 + J^3 and (a + b)b + J^3 = b^2 + J^3 are linearly independent, and the result follows in this case.

Therefore we may also assume that b^2 ∈ Fa^2 + J^3 and, similarly, c^2 ∈ Fa^2 + J^3. Then

J^2 = F\{a^2, ab, ac, ba, b^2, bc, ca, cb, c^2\} + J^3 = F\{a^2, bc, cb\} + J^3.

Thus J^2 = F\{a^2, bc\} + J^3 or J^2 = F\{a^2, cb\} + J^3; we may assume that J^2 = F\{a^2, bc\} + J^3. Then Lemma E in [6] implies that

J^3 = F\{a^3, abc, ba^2, b^2c\} + J^4 = F\{a^3, a^2c\} + J^4 = Fa^3 + J^4;

in particular, dim J^3/J^4 ≤ 1 contradicting (1.2).

We now choose elements a, b ∈ J as in (1.7). By symmetry we may assume that ab ∈ Fa^2 + J^3; in particular, a ∈ J^2 and b ∈ Fa + J^2. Thus a + J^2, b + J^2 are linearly independent in J/J^2. By (1.2), we can find an element c ∈ J such that J = F\{a, b, c\} + J^2.

2. THE CASE (1.3)

In this section we use the same hypothesis and notation as before, but we assume in addition that (1.3) holds. Then J = Fa + Fb + J^2 and J^2 = Fa^2 + Fab + J^3. Thus Lemma E in [6] implies that J^3 = Fa^2 + Fa^2b + J^4, J^4 = Fa^4 + Fa^3b + J^5 and J^5 = Fa^5 + Fa^4b + J^6; in particular, dim J^3/J^4 = 2. Thus a^3 + J^4 and a^2b + J^4 form a basis of J^3/J^4. Moreover, dim J^4 ≥ 2; in particular, J^5 ≠ 0. Thus J^5 ⊈ K and 4 = dim A/K + J^4 ≤ dim A/K + J^5 < dim A/K = dim Z = 5 by (1.2). We conclude that dim A/K + J^5 = 4.

Furthermore, A = F\{1, a, b, a^2, ab, a^3, a^2b\} + J^4, so

K ⊂ F\{[a, b], [a, ab], [a, a^2b], [b, a^2], [b, ab], [b, a^3], [b, a^2b], [a^2, ab]\} + J^5.

Since J^2 = F\{a^2, ab, a^3, a^2b\} + J^4 there are elements α_i, β_i, γ_i, δ_i ∈ F (i = 1, 2) such that

\[ ba ≡ α_1 a^2 + β_1 ab + γ_1 a^3 + δ_1 a^2 b \pmod{J^4}, \]

\[ b^2 ≡ α_2 a^2 + β_2 ab + γ_2 a^3 + δ_2 a^2 b \pmod{J^4}. \]

We have to distinguish between two cases.

Case 1. β_1 ≠ 1. In this case we set ζ := α_1/(1 − β_1) and b' := b − ζa. Then J = Fa + Fb' + J^2, J^2 = Fa^2 + Fab' + J^3 and

\[ b' a ≡ ba - ζ a^2 ≡ (α_1 - ζ) a^2 + β_1 ab \]

\[ ≡ (α_1 - ζ + β_1 ζ) a^2 + β_1 ab' ≡ β_1 ab' \pmod{J^3}. \]

Thus we may replace b by b' and therefore assume that α_1 = 0. Then

\[ 0 ≡ (b^2) a - b(ba) ≡ α_2 a^3 + β_2 a^2 b - β_1 bab ≡ α_2 a^3 + β_1 β_2 a^2 b - β_1^2 ab^2 \]

\[ ≡ (α_2 - α_2 β_1^2) a^3 + (β_1 β_2 - β_1^2 β_2) a^2 b \pmod{J^4}. \]
and, similarly,
\[ 0 \equiv (b^2)b - b(b^2) \equiv (\alpha_2\beta_2 - \alpha_2\beta_1\beta_2)a^3 + (\alpha_2 + \beta_2^2 - \alpha_2\beta_1^2 - \beta_1\beta_2^2)a^2b \pmod{J^4}. \]

Since \( a^3 + J^4 \) and \( a^2b + J^4 \) form a basis of \( J^3/J^4 \) we conclude that
\[
(2.1) \quad 0 = \alpha_2 - \alpha_2\beta_1^2, \quad (2.2) \quad 0 = \beta_1\beta_2 - \beta_1^2\beta_2, \\
(2.3) \quad 0 = \alpha_2\beta_2 - \alpha_2\beta_1\beta_2, \quad (2.4) \quad 0 = \alpha_2 + \beta_2^2 - \alpha_2\beta_1^2 - \beta_1\beta_2^2.
\]

Subtracting (2.1) from (2.4) we obtain \( \beta_2^2 = \beta_1\beta_2^2 \). Since \( \beta_1 \neq 1 \) this implies \( \beta_2 = 0 \). From (2.1) we also conclude that \( \alpha_2 = 0 \) or \( \beta_1^2 = 1 \). We assume first that \( \alpha_2 = 0 \). Then
\[
[a, ab] = a^2b - aba \equiv (1 - \beta_1)a^2b \pmod{J^4}, \\
[b, a^2] = ba^2 - a^2b \equiv \beta_1aba - a^2b \equiv (\beta_1^2 - 1)a^2b \pmod{J^4}, \\
[b, ab] = bab - ab^2 \equiv \beta_1ab^2 \equiv 0 \pmod{J^4}.
\]

This shows that \( K \subset F[a, b] + Fa^2b + J^4 \); in particular, \( \dim K + J^4/J^4 \leq 2 \). Thus \( \dim A/J^4 \leq 6 \) by (1.2), a contradiction.

Hence we must have \( \alpha_2 \neq 0 \) and \( \beta_1^2 = 1 \). Since \( \beta_1 \neq 1 \) this implies \( \beta_1 = -1 \) and \( \text{char } F \neq 2 \). It is now easy to check that
\[
[a, a^2b] = 2a^3b \pmod{J^5}, \quad [b, a^2] = -2\delta_1a^3b \pmod{J^5}, \\
[b, a^3] = -2a^3b \pmod{J^5}, \quad [b, a^2b] = [a^2, ab] = 0 \pmod{J^5}.
\]

Thus \( K \subset F\{[a, b], [a, ab], [b, ab], a^3b\} + J^5 \); in particular, \( \dim K + J^5/J^5 \leq 4 \). Hence \( \dim A/J^5 \leq 8 \) and \( \dim J^4/J^5 = 1 \). By Lemma G in [6], this implies that \( J^3 \subset Z \); in particular, \( a^2b \in Z \). Thus \( a^3b \equiv a^2ba \equiv -a^3b \pmod{J^5} \). Since \( \text{char } F \neq 2 \) this implies \( a^3b \in J^5 \). Therefore
\[
K \subset F\{[a, b], [a, ab], [b, ab]\} + J^5;
\]
in particular, \( \dim K + J^5/J^5 \leq 3 \). Hence \( \dim A/J^5 \leq 7 \), a contradiction.

Case 2. \( \beta_1 = 1 \). Assume first that \( \alpha_1 = 0 \). Then \( [a, b] \in J^3 \) and \( K \subset J^3 \), so \( \dim A/K + J^3 = \dim A/J^3 = 5 \) contradicting (1.2). Thus we must have \( \alpha_1 \neq 0 \).

Now we set \( a' := \alpha_1a \). Then \( J = Fa' + Fb + J^2 \), \( J^2 = F(a')^2 + Fa'b + J^3 \) and
\[
ba' \equiv \alpha_1ba \equiv \alpha_1^2a^2 + \alpha_1ab \equiv (a')^2 + a'b \pmod{J^3}.
\]

Hence we may replace \( a \) by \( a' \) and therefore assume that \( \alpha_1 = 1 \). As in Case 1 we compute
\[
0 \equiv (b^2)a - b(ba) \equiv (\beta_2 - 2)a^3 - 2a^2b \pmod{J^4}.
\]

Since \( a^3 + J^4 \) and \( a^2b + J^4 \) form a basis of \( J^3/J^4 \) this implies that \( \text{char } F = 2 \) and \( \beta_2 = 0 \). Hence
\[
[a, a^2b] = a^4 \pmod{J^5}, \quad [b, a^2] = \delta_1a^4 \pmod{J^5}, \\
[b, a^3] = a^4 \pmod{J^5}, \quad [b, a^2b] = [a^2, ab] \equiv 0 \pmod{J^5}.
\]

Therefore \( K \subset F\{[a, b], [a, ab], [b, ab], a^4\} + J^5 \); in particular, \( \dim K + J^5/J^5 \leq 4 \). Hence \( \dim A/J^5 \leq 8 \) and \( \dim J^4/J^5 = 1 \). By Lemma G in [6], this implies that \( J^3 \subset Z \); in particular, \( a^2b \in Z \). Thus \( a^3b \equiv a^2ba \equiv a^4 + a^3b \pmod{J^5} \).
(mod \(J^5\)). Therefore \(a^4 \in J^5\) and \(J^5 = Fa^5 + Fa^4b + J^6 = J^6\). Hence \(J^5 = 0\) by Nakayama's Lemma, a contradiction.

3. The case (1.4)

In this section we assume hypothesis and notation from §1. In addition, we assume that (1.4) holds. Then \(J^2 = Fa^2 + Fab + J^3\) and \(J^3 = Fa^3 + Fa^2b + J^4\) by Lemma E in [6]; in particular, \(\dim J^3/J^4 = 2\). Hence \(a^3 + J^4, a^2b + J^4\) form a basis of \(J^3/J^4\). There are elements \(\alpha, \beta \in F\) such that \(ac \equiv \alpha a^2 + \beta ab \pmod{J^3}\). Setting \(c' := c - \alpha a - \beta b\) we then have \(J = F[a, b, c'] + J^2\) and \(ac' \equiv ac - \alpha a^2 - \beta ab \equiv 0 \pmod{J^3}\). Hence we may replace \(c\) by \(c'\) and therefore assume that \(ac \in J^3\). We choose elements \(\alpha_i, \beta_i \in F\) \((i = 1, 2, 3, 4)\) such that

\[
bc \equiv \alpha_1 a^2 + \beta_1 ab \pmod{J^3}, \quad ca \equiv \alpha_2 a^2 + \beta_2 ab \pmod{J^3}, \\
\]

\[
\alpha_3 a^2 + \beta_3 ab \pmod{J^3}, \quad c^2 \equiv \alpha_4 a^2 + \beta_4 ab \pmod{J^3}.
\]

Then

\[
0 \equiv (ac)a \equiv a(ca) = a_2a^3 + \beta_2 a^2b \pmod{J^4},
\]

\[
0 \equiv (ac)b \equiv a(cb) = a_3a^3 + \beta_3 a^2b \pmod{J^4},
\]

\[
0 \equiv (ac)c \equiv a(c^2) = a_4a^3 + \beta_4 a^2b \pmod{J^4}.
\]

Hence \(a_2^2 = \beta_2 = a_3 = \beta_3 = a_4 = \beta_4 = 0\); in particular, \(ca, cb, c^2 \in J^3\). Thus

\[
0 \equiv b(c^2) \equiv (bc)c \equiv a_1a^2\beta + \beta_1 a_2bc \equiv a_1\beta_1 a^3 + \beta_1^2 a^2b \pmod{J^4},
\]

and we obtain \(\beta_1 = 0\). Thus \(0 \equiv b(cb) \equiv (bc)b \equiv a_1 a^2b \pmod{J^4}\). Therefore \(a_1 = 0\); in particular, \(bc \in J^3\). Thus \([a, c], [b, c] \in J^3\) and \(K \subset F[[a, b], [a, c], [b, c]] + J^3 \subset F[a, b] + J^3\); in particular, \(\dim K + J^3/J^3 \leq 1\). Thus \(\dim A/J^3 \leq 5\) by (1.2), a contradiction.

4. The case (1.5)

In this section we assume hypothesis and notation from §1. In addition, we assume that (1.5) holds. Since \(J = F[a, b, c] + J^2\) we have \(J^2 = F[a^2, ab, ac, ba, b^2, bc, ca, cb, c^2] + J^3\). Since \(\dim J^2/J^3 = 3\) we must have \(J^2 = F[a^2, ab, d] + J^3\) for some element \(d \in \{ac, ba, b^2, bc, ca, cb, c^2\}\). Since \(J^2 = K + J^4\) we obtain

\[
J^2 = F[[a, b], [a, c], [a, ab], [a, d], [b, c], [b, a^2], [b, ab], [b, d], [c, a^2], [c, ab], [c, d]] + J^4.
\]

We choose elements \(\alpha_i, \beta_i, \gamma_i \in F\) \((i = 1, 2, \ldots, 7)\) such that

\[
ac \equiv \alpha_1 a^2 + \beta_1 ab + \gamma_1 d \pmod{J^3}, \quad ba \equiv \alpha_2 a^2 + \beta_2 ab + \gamma_2 d \pmod{J^3},
\]

\[
b^2 \equiv \alpha_3 a^2 + \beta_3 ab + \gamma_3 d \pmod{J^3}, \quad bc \equiv \alpha_4 a^2 + \beta_4 ab + \gamma_4 d \pmod{J^3},
\]

\[
ca \equiv \alpha_5 a^2 + \beta_5 ab + \gamma_5 d \pmod{J^3}, \quad cb \equiv \alpha_6 a^2 + \beta_6 ab + \gamma_6 d \pmod{J^3},
\]

\[
c^2 \equiv \alpha_7 a^2 + \beta_7 ab + \gamma_7 d \pmod{J^3}.
\]

(4.1) Lemma. We may assume that \(d = ac\) or \(d = ba\).

Proof. Case 1. \(d = ac\). In this case there is nothing to prove.
Case 2. $d = ba$. In this case there is nothing to prove either.

Case 3. $d = b^2$. In this case we may assume that $ba \in Fa^2 + Fab + J^3$; for otherwise we are in Case 2. Similarly, we may assume that $ba \in Fb^2 + Fab + J^3$; for otherwise we interchange $a$ and $b$ and are in Case 2 again. Hence $ba \in Fab + J^3$, and we may write $ba \equiv \alpha ab \pmod{J^3}$ for some element $\alpha \in F$.

Now we set $b' := a + b$. Then we have
\[
ab' = a^2 + ab, \quad (b')^2 \equiv a^2 + (1 + \alpha)ab + b^2 \pmod{J^3};
\]
in particular, $J = F\{a, b', c\} + J^2$ and $J^2 = F\{a^2, ab', (b')^2\} + J^3$. Hence we may similarly assume that $b' a \in Fab' + J^3$. We write $b' a \equiv \beta ab'$ (mod $J^3$) with some element $\beta \in F$. Then
\[
\beta a^2 + \beta ab \equiv \beta ab' \equiv b' a \equiv (a + b)a \equiv a^2 + ba \equiv a^2 + \alpha ab \pmod{J^3}.
\]
Since $a^2 + J^3$ and $ab + J^3$ are linearly independent this means that $\alpha = \beta = 1$; in particular, $[a, b] \in J^3$, and $J^2 = F[a, c] + F[b, c] + J^3$ contradicting the fact that $\dim J^2/J^3 = 3$.

Case 4. $d = bc$. In this case we may assume that $ac, ba, b^2 \in Fa^2 + Fab + J^3$; for otherwise we are in Cases 1, 2 or 3 again. Then we replace $c$ by $c - \alpha x a - \beta x b$ and may therefore assume that $0 = \alpha x = \beta x$. Moreover, we may assume that $a^2 + J^3, ab + J^3, bc + J^3$ form a basis of $J^2/J^3$ this implies that
\[
0 \equiv \begin{vmatrix} 1 + \alpha_x & \beta_x & 0 \\ \alpha_3 x & 1 + \beta_3 x & 0 \\ 0 & 0 & \xi \end{vmatrix} = \xi + (\alpha x + \beta x)\xi^2 + (\alpha_3 x - \beta_3 x)\xi^3
\]
for $\xi \in F$. Since $F$ is infinite this is impossible.

Case 5. $d = ca$, i.e., $\alpha_3 = \beta_3 = 0, \gamma_3 = 1$. We may assume that $\gamma_i = 0$ for $i = 1, 2, 3, 4$; for otherwise we are in Cases 1, 2, 3, 4, respectively. Then we replace $c$ by $c - \alpha a - \beta b$ and may therefore assume that $0 = \alpha x = \beta x$. Moreover, $\beta_2 = 0$; for otherwise we are in Case 1. Similarly, we may assume that $\alpha_3 = 0$; for otherwise we interchange $a$ and $b$ and are then in Case 4 for the opposite algebra of $A$. Now we replace $b$ by $b - \gamma_6 a$ and may then assume that $\gamma_6 = 0$. Furthermore, we may assume that $\gamma_7 = 0$; for otherwise we replace $(a, b, c)$ by $(b, c, a)$ and are then in Case 4 again. Finally, we may assume that $\beta_7 = 0$; for otherwise we interchange $b$ and $c$ and are then in Case 3 for the opposite algebra of $A$. As in Case 4, we may assume
\[
J^2 \neq F\{((\xi a + \eta b + c)a, (\xi a + \eta b + c)b, (\xi a + \eta b + c)c) + J^3
= F\{(\xi + \alpha_2 \eta)a^2 + ca, \alpha_6 a^2 + (\xi + \beta_3 \eta + \beta_6)ab,
\alpha_4 \eta a^2 + \beta_4 \eta ab + \gamma_6 ca\} + J^3
\]
for $\xi, \eta \in F$. Since $a^2 + J^3, ab + J^3, ca + J^3$ form a basis of $J^2/J^3$ we may compute the corresponding determinant and obtain
\[
0 = \gamma_6 \xi^2 + (\beta_3 \gamma_7 + \alpha_4 \beta_7 - \alpha_4 \xi)\xi + \beta_6 \gamma_6 \xi + (\alpha_2 \beta_3 \gamma_7 - \alpha_4 \beta_6)\xi^2
+ (\alpha_2 \beta_6 \gamma_7 + \alpha_6 \beta_4 - \alpha_4 \beta_6)\eta
\]
for \( \xi, \eta \in F \). Since \( F \) is infinite this implies that all coefficients on the right-hand side vanish; in particular, \( 0 = \gamma_7 = \alpha_4 \). Then, similarly, we may assume that

\[
J^2 \neq F \{ a(a + \eta b + c), b(a + \eta b + c), c(a + \eta b + c) \} + J^3
\]

\[
= F \{ a^2 + \eta ab, \alpha_2 a^2 + (\beta_3 \eta + \beta_4)ab, \alpha_6 \eta a^2 + \beta_6 \eta ab + ca \} + J^3
\]

for \( \eta \in F \). Computing the corresponding determinant we obtain \( 0 = (\beta_3 - \alpha_2) \eta + \beta_4 \) for \( \eta \in F \). As before this implies that \( \beta_3 = \alpha_2 \) and \( \beta_4 = 0 \). Finally, we may assume that

\[
J^2 \neq F \{ (\xi a + b + c)^2; (\xi a + b + c)a, a(\xi a + b + c) \} + J^3
\]

\[
= F \{(\xi^2 + \alpha_2 \xi + \alpha_6)a^2 + (\xi + \alpha_2 + \beta_6)ab + \xi ca,
(\xi + \alpha_2)a^2 + ca, \xi a^2 + ab \} + J^3
\]

for \( \xi \in F \); for otherwise we replace \((a, b)\) by \((\xi a + b + c, a)\) and are in Case 2 again. Computing the corresponding determinant we obtain \( 0 = \xi^2 + (\alpha_2 + \beta_6)\xi - \alpha_6 \) for \( \xi \in F \) which is impossible.

**Case 6.** \( d = cb \), i.e. \( \alpha_6 = \beta_6 = 0, \gamma_6 = 1 \). We may assume that \( \gamma_i = 0 \) for \( i = 1, 2, \ldots, 5 \); for otherwise we are in Cases 1, 2, \ldots, 5, respectively. Then we replace \( c \) by \( c - \alpha_1 a - \beta_1 b \) and may therefore assume that \( 0 = \alpha_1 = \beta_1 \). We may also assume that \( \beta_2 = 0 \); for otherwise we are in Case 4 for the opposite algebra of \( A \). Similarly, we may assume that \( \alpha_3 = 0 \); for otherwise we interchange \( a \) and \( b \) and are then in Case 1. As in the previous cases we may assume

\[
J^2 \neq F \{ (\xi a + \eta b + c)a, (\xi a + \eta b + c)b, (\xi a + \eta b + c)c \} + J^3
\]

\[
= F \{ (\xi + \alpha_2 \eta + \alpha_5)a^2 + \beta_5 ab, (\xi + \beta_3 \eta)ab + cb,
(\alpha_4 \eta + \alpha_7)a^2 + (\beta_4 \eta + \beta_7)ab + \gamma_7 cb \} + J^3
\]

for \( \xi, \eta \in F \). We work out the corresponding determinant and obtain

\[
0 = \gamma_7 \xi^2 + (\beta_3 \gamma_7 + \alpha_2 \gamma_7 - \beta_4)\xi \eta + (\alpha_5 \gamma_7 - \beta_7)\xi + (\alpha_2 \beta_3 \gamma_7 - \alpha_2 \beta_4)\eta^2
\]

\[
+ (\alpha_5 \beta_3 \gamma_7 - \alpha_2 \beta_7 - \alpha_5 \beta_4 + \alpha_4 \beta_5)\eta + (\alpha_7 \beta_5 - \alpha_5 \beta_7)
\]

for \( \xi, \eta \in F \). Therefore all coefficients on the right-hand side vanish; in particular, \( 0 = \gamma_7 = \beta_4 = \beta_7 \). Similarly, we have

\[
J^2 \neq F \{ a(\xi a + b + c), b(\xi a + b + c), c(\xi a + b + c) \} + J^3
\]

\[
= F \{ a^2 + ab, (\alpha_2 \xi + \alpha_4)a^2 + \beta_3 ab, (\alpha_5 \xi + \alpha_7)a^2 + \beta_5 \xi ab + cb \} + J^3
\]

for \( \xi \in F \). Computing the corresponding determinant we obtain \( 0 = (\beta_3 - \alpha_2)\xi - \alpha_4 \) for \( \xi \in F \) which again implies that \( \beta_3 = \alpha_2 \) and \( \alpha_4 = 0 \). We may also assume that

\[
J^2 \neq F \{ (\xi a + \eta b + c)^2; (\xi a + \eta b + c)a, a(\xi a + \eta b + c) \} + J^3
\]

\[
= F \{(\xi^2 + \alpha_2 \xi + \alpha_5 \xi + \alpha_7)a^2 + (\xi \eta + \beta_5 \xi + \alpha_2 \eta^2)ab + \eta cb,
(\xi + \alpha_2 \eta + \alpha_5)a^2 + \beta_5 ab, \xi a^2 + \eta ab \} + J^3
\]

for \( \xi, \eta \in F \); for otherwise we replace \((a, b, c)\) by \((\xi a + \eta b + c, a, b)\) and are then in Case 2 again. Working out the corresponding determinant we obtain \( 0 = \xi \eta^2 + \alpha_2 \eta^3 - \beta_3 \xi \eta + \alpha_5 \eta^2 \) for \( \xi, \eta \in F \) which is impossible.
Case 7. $d = c^2$. In this case we may assume that $ac, ba, b^2, bc, ca, cb \in Fa^2 + Fab + J^3$; for otherwise we are in Cases 1, 2, \ldots, 6, respectively. Then $J^2 = F\{[a, b], [a, c], [b, c]\} + J^3 \subset Fa^2 + Fab + J^3$; in particular, $\dim J^2/J^3 \leq 2$ contradicting (1.5).

(4.2) Lemma. We may assume that $d = ac$.

Proof. We assume the contrary. Then we may assume that $d = ba$, by (4.1); in particular, $a_2 = b_2 = 0, g_2 = 1$. We have $g_1 = 0$. After replacing $c$ by $c - a_1a - b_1b$ we may even assume $0 = a_1 = b_1$. Similarly, we may assume $b_5 = 0$. Moreover, after replacing $b$ by $b - g_3a$ we may also assume that $g_3 = 0$. We then have

$$J^2 \not= (\xi a + \eta b + \zeta c)J + J^3$$

$$= F\{(\xi a + \eta b + \zeta c)a, (\xi a + \eta b + \zeta c)b, (\xi a + \eta b + \zeta c)c\} + J^3$$

$$= F\{(\xi + a_5\zeta)a^2 + (\eta + g_5\zeta)ab, (a_3\eta + a_6\zeta)a^2 + (\xi + b_3\eta + b_6\zeta)ab$$

$$+ g_5\zeta ba, (a_4\eta + a_7\zeta)a^2 + (b_4\eta + b_7\zeta)ab + (g_4\eta + g_7\zeta)ba\} + J^3$$

for $\xi, \eta, \zeta \in F$. Since $a^2 + J^3, ab + J^3, ba + J^3$ form a basis of $J^2/J^3$ this implies that

$$0 = \begin{vmatrix}
\xi + a_5\zeta & 0 & \eta + g_5\zeta \\
0 & \xi + a_5\zeta & \eta + g_5\zeta \\
a_3\eta + a_6\zeta & (a_4\eta + a_7\zeta)a^2 + (b_4\eta + b_7\zeta)ab + (g_4\eta + g_7\zeta)ba
\end{vmatrix}$$

$$= g_4\zeta^2\eta + g_7\zeta^2\xi + (b_3\eta - a_4\xi)\eta^2$$

$$+ (b_3\eta + b_6\eta + a_5\zeta - b_4\xi - a_7 - a_4\eta)\xi\eta$$

$$+ (b_6\eta + a_5\zeta - b_7\zeta - a_7\eta)\xi^2 + (a_3\beta_4 - a_4\beta_3)\eta^3$$

$$+ (a_5\beta_3\eta + a_3\beta_7 + a_6\beta_4 + a_4\beta_5\eta - a_4\beta_6 - a_7\beta_3 - a_4\beta_3\eta)\xi^2$$

$$+ (a_5\beta_3\eta + a_5\beta_6\eta - a_5\beta_6\xi + a_6\beta_7 + a_3\beta_7\eta + a_6\beta_4\xi$$

$$- a_7\beta_6 - a_4\beta_6\eta - a_7\beta_3\eta)\eta^2$$

$$+ (a_5\beta_6\eta - a_5\beta_7\eta + a_6\beta_7\eta - a_3\beta_6\zeta)\zeta^3$$

for $\xi, \eta, \zeta \in F$. Since $F$ is infinite this implies that all coefficients on the right-hand side have to vanish; in particular, $0 = \gamma_4 = \gamma_7 = a_4 = a_3 \beta_4$ and $a_7 = -b_4\gamma_6$. Then, similarly, we have

$$J^2 \not= F\{a(\xi a + \eta b + \zeta c), b(\xi a + \eta b + \zeta c), c(\xi a + \eta b + \zeta c)\} + J^3$$

$$= F\{\alpha a^2 + \gamma ab, a_3\eta a^2 + (b_3\eta + b_4\zeta)ab + \xi ba,$$

$$(a_5\xi + a_6\eta + a_7\zeta)a^2 + (b_6\eta + b_7\zeta)ab + (g_5\xi + g_6\eta)ba\} + J^3$$

for $\xi, \eta, \zeta \in F$. As before, we work out the corresponding determinant and obtain

$$0 = (b_3\gamma_5 - b_6 + a_5)\xi^2\eta + (b_4\gamma_5 - b_7)\xi^2\zeta + (b_3\gamma_6 - a_3\gamma_5 + a_6)\eta^2 - a_3\gamma_6\eta^3$$

for $\xi, \eta, \zeta \in F$. Again, this implies that $b_6 = b_3\gamma_5 + a_5, b_7 = b_4\gamma_5, a_6 = a_3\gamma_5 - b_3\gamma_6, 0 = a_3\gamma_6$. On the other hand,

$$J^2 = F\{[a, b], [a, c], [b, c]\} + J^3$$

$$= F\{ab - ba, a_5a^2 + g_5ba, (b_3\gamma_6 - a_3\gamma_5)a^2$$

$$+ (b_4 - a_5 - b_3\gamma_5)ab - b_6ba\} + J^3.$$
Since \( a^2 + J^3 \), \( ab + J^3 \) and \( ba + J^3 \) form a basis of \( J^2/J^3 \) a computation of the corresponding determinant yields
\[
0 \neq \alpha_5 \gamma_6 + \beta_3 \gamma_5 \gamma_6 - \alpha_3 \gamma_5^2 - \alpha_5 \beta_4 + \alpha_5^2 + \alpha_5 \beta_3 \gamma_5^5.
\]
Moreover, since \( J^2 = F\{a^2, ab, ba\} + J^3 \), Lemma E in [6] implies that \( J^3 = F\{a^3, a^2b, aba, ba^2, bab, b^2a\} + J^4 = F\{a^3, a^2b, aba, ba^2, bab\} + J^4 \).

Now we distinguish two cases.

Case 1. \( \alpha_5 \neq 0 \). In this case we replace \( a \) by \( \alpha_5 a \) and may then assume that \( \alpha_5 = 1 \).

Thus
\[
0 \equiv a(ca) - (ac)a \equiv a^3 + \gamma_5 aba \pmod{J^4},
\]
\[
0 \equiv b(ca) - (bc)a \equiv (\beta_3 \gamma_5 - \alpha_3 \gamma_5^2 - \beta_4)aba + ba^2 \pmod{J^4}.
\]

Now we distinguish two more cases.

Case 1.1. \( \beta_4 \neq 0 \). In this case we have \( \alpha_3 = 0 \) since \( 0 = \alpha_3 \beta_4 \).

Moreover,
\[
0 \equiv (b^2)c - b(bc) \equiv \beta_3 \beta_4 a^2 b - \beta_4 aba \pmod{J^4},
\]
\[
0 \equiv a(c^2) - (ac)c \equiv \beta_4 \gamma_5 a^2 b + \beta_4 \gamma_5 b a b \pmod{J^4};
\]
in particular, \( J^3 = F a^2 b + F aba + J^4 \). Hence \( a^2 b + J^4 \) and \( aba + J^4 \) are linearly independent. Then \( \gamma_5 = 0 \), and we obtain the contradiction
\[
0 \equiv a(cb) - (ac)b \equiv a^2 b + \gamma_6 aba \pmod{J^4}.
\]

Case 1.2. \( \beta_4 = 0 \). Here we have to distinguish two more cases.

Case 1.2.1. \( \beta_3 \neq 0 \). In this case we replace \( b \) by \( \beta_3^{-1} b \) and may then assume that \( \beta_3 = 1 \).

Then
\[
0 \equiv (b^2)b - b(b^2) \equiv (1 + \alpha_3) a^2 b - \alpha_3 \gamma_5 aba - bab \pmod{J^4},
\]
\[
0 \equiv a(cb) - (ac)b \equiv (1 + \gamma_5) a^2 b + (\gamma_5 \gamma_6 - \alpha_3 \gamma_5^2 + \gamma_6) b a b \pmod{J^4};
\]
in particular, \( J^3 = F a^2 b + F aba + J^4 \). Thus \( a^2 b + J^4 \) and \( aba + J^4 \) are linearly independent. Then \( \gamma_5 = -1 \) and \( \alpha_3 = 0 \). But now we obtain the contradiction
\[
\alpha_5 \gamma_6 + \beta_3 \gamma_5 \gamma_6 - \alpha_3 \gamma_5^2 - \alpha_5 \beta_4 + \alpha_5^2 + \alpha_5 \beta_3 \gamma_5 = 0.
\]

Case 1.2.2. \( \beta_3 = 0 \). Here we have
\[
0 \equiv a(cb) - (ac)b \equiv a^2 b + (\gamma_6 - \alpha_3 \gamma_5^2) aba \pmod{J^4},
\]
\[
0 \equiv b(cb) - (bc)b \equiv \alpha_3^2 \gamma_5 aba + bab \pmod{J^4};
\]
in particular, \( J^3 = F aba + J^4 \), a contradiction.

Case 2. \( \alpha_5 = 0 \). Then
\[
0 \neq \alpha_5 \gamma_6 + \beta_3 \gamma_5 \gamma_6 - \alpha_3 \gamma_5^2 - \alpha_5 \beta_4 + \alpha_5^2 + \alpha_5 \beta_3 \gamma_5 = \beta_3 \gamma_5 \gamma_6 - \alpha_3 \gamma_5^2;
\]
in particular, \( \gamma_5 \neq 0 \). Now we replace \( b \) by \( \gamma_5 b \) and may therefore assume that \( \gamma_5 = 1 \).

Hence
\[
0 \equiv a(ca) - (ac)a \equiv aba \pmod{J^4}.
\]

We distinguish two more cases.
Case 2.1. $\alpha_3 \neq 0$. In this case $\beta_4 = \gamma_6 = 0$ since $0 = \alpha_3 \beta_4 = \alpha_3 \gamma_6$. We now replace $a$ by $\sqrt{\alpha_3}a$ and may therefore assume that $\alpha_3 = 1$. Then

\[
0 \equiv b(ca) - (bc)a \equiv a^3 \pmod{J^4},
\]

\[
0 \equiv b(cb) - (bc)b \equiv ba^2 + \beta_3 bab \pmod{J^4},
\]

\[
0 \equiv a(cb) - (ac)b \equiv \beta_3 a^2 b \pmod{J^4},
\]

\[
0 \equiv (b^2)b - b(b^2) \equiv a^2 b \pmod{J^4};
\]

in particular, $J^3 = Fbab + J^4$, a contradiction.

Case 2.2. $\alpha_3 = 0$. In this case we have $0 \neq \beta_3 \gamma_6 - \alpha_3 = \beta_3 \gamma_6$, i.e. $\beta_3 \neq 0 \neq \gamma_6$. We now replace $a$ by $\beta_3 a$ and may then assume that $\beta_3 = 1$. We compute

\[
0 \equiv a(cb) - (ac)b \equiv a^2 b - \gamma_6 a^3 \pmod{J^4},
\]

\[
0 \equiv (b^2)b - b(b^2) \equiv \gamma_6 a^3 - bab \pmod{J^4},
\]

\[
0 \equiv (bc)c - b(c^2) \equiv (\beta_4 \gamma_6 - \beta_4 \gamma_6)a^3 + \beta_4 \gamma_6 ba^2 \pmod{J^4};
\]

in particular, $J^3 = Fa^3 + Fba^2 + J^4$. Thus $a^3 + J^4$ and $ba^2 + J^4$ are linearly independent. Then $\beta_4 = 0$ since $\gamma_6 \neq 0$. But now we obtain the contradiction

\[
0 \equiv b(cb) - (bc)b \equiv \gamma_6 a^3 - \gamma_6 ba^2 \pmod{J^4}.
\]

In the remainder of this paper we may and will assume that $J^2 = F\{a^2, ab, ac\} + J^3$. Then $J^3 = F\{a^3, a^2b, a^2c\} + J^4$ and $J^4 = F\{a^4, a^3b, a^3c\} + J^5$ by Lemma E in [6]; in particular, $\dim J^3/J^4 \in \{2, 3\}$. Since $J^4 \neq J^5$ we have $a^3 \notin J^4$.

(4.3) Lemma. The elements $a$, $b$, $c$ can be chosen such that one of the following holds:

(4.4) $0 = \alpha_2 = \beta_2 = \alpha_5$, $\gamma_2 = 1$, $\alpha_6 = \alpha_4 - 1$; $\beta_5 + \gamma_5 \neq 1$;

(4.5) $0 = \alpha_2 = \beta_2$, $\gamma_2 = \alpha_5 = 1$, $\gamma_5 = 1 - \beta_5$, $\beta_6 - \beta_4 + \gamma_6 - \gamma_4 \neq 0$;

(4.6) $0 = \alpha_2 = \gamma_2 = \alpha_5 = \beta_5$, $\gamma_5 = \beta_2 \neq 1$, $\alpha_4 = 1 \neq \alpha_6$.

Proof. We distinguish between two cases.

Case 1. $\gamma_2 \neq 0$. In this case we replace $c$ by $\alpha_2 a + \beta_2 b + \gamma_2 c$ and may therefore assume that $0 = \alpha_2 = \beta_2$ and $\gamma_2 = 1$. Now we distinguish two more cases.

Case 1.1. $\beta_5 + \gamma_5 \neq 1$. In this case we set $\xi := \alpha_5/(\beta_5 + \gamma_5 - 1)$ and replace $b$ by $b + \xi a$ and $c$ by $c + \xi a$. Then we have $\alpha_5 = 0$. Hence

\[
J^2 = F\{[a, b], [a, c], [b, c]\} + J^3
= F\{ab - ac, \beta_5 ab + (\gamma_5 - 1)ac, (\alpha_5 - \alpha_6)a^2
+ (\beta_4 - \beta_6)ab + (\gamma_4 - \gamma_6)ac\} + J^3;
\]

in particular, $\alpha_4 \neq \alpha_6$. Now we replace $a$ by $(\alpha_4 - \alpha_6)^{1/2}a$ and may then assume that $\alpha_6 = \alpha_4 - 1$.

Case 1.2. $\beta_5 + \gamma_5 = 1$. In this case we have

\[
J^2 = F\{[a, b], [a, c], [b, c]\} + J^3
= F\{ab - ac, \alpha_5 a^2 + \beta_5 ab - \beta_5 ac, (\alpha_4 - \alpha_6)a^2
+ (\beta_4 - \beta_6)ab + (\gamma_4 - \gamma_6)ac\} + J^3.
\]
Since \(a^2 + J^3, ab + J^3, ac + J^3\) form a basis of \(J^2/J^3\) we work out the corresponding determinant and obtain \(0 \neq \alpha_5(\beta_6 - \beta_4 + \gamma_6 - \gamma_4)\), so \(\beta_6 - \beta_4 + \gamma_6 - \gamma_4 \neq 0 \neq \alpha_5\). Then we replace \(a\) by \(\alpha_5a\) and may therefore assume that \(\alpha_5 = 1\).

**Case 2.** \(\gamma_2 = 0\). In this case we may assume that \(\beta_5 = 0\); for otherwise we interchange \(b\) and \(c\) and are then in Case 1 again. Similarly, we may assume that \(\gamma_5 = \beta_2\); otherwise we replace \(b\) by \(b + c\) and are then in Case 1 again. Hence

\[
J^2 = F\{[a,b], [a,c], [b,c]\} + J^3
\]

\[
= F\{\alpha_2a^2 + (\beta_2 - 1)ab, \alpha_5a^2 + (\beta_2 - 1)ac, (\alpha_4 - \alpha_6)a^2 + (\gamma_4 - \gamma_6)ab + (\gamma_4 - \gamma_6)ac\} + J^3.
\]

Since \(\dim J^2/J^3 = 3\) this implies that \(\beta_2 \neq 1\). Now we replace \(b\) by \(b + \alpha_2(\beta_2 - 1)^{-1}a\) and \(c\) by \(c + \alpha_5(\beta_2 - 1)^{-1}a\) and may then assume that \(0 = \alpha_2 = \alpha_5\). In this situation we have \(\alpha_4 \neq 0\) or \(\alpha_6 \neq 0\). If necessary, we interchange \(b\) and \(c\) and may then assume that \(\alpha_4 \neq 0\). Finally we replace \(b\) by \(\alpha_4^{-1}b\) and may therefore assume that \(\alpha_4 = 1\).

Now we treat the cases above separately.

**Lemma.** The case (4.4) does not occur.

**Proof.** We assume the contrary and distinguish two cases.

**Case 1.** \(\dim J^3/J^4 = 3\). In this case the elements \(a^3 + J^4, a^2b + J^4, a^2c + J^4\) form a basis of \(J^3/J^4\). Since

\[
0 \equiv (b^2)a - b(ba) \equiv (\alpha_3 - \alpha_7)a^3 + (\beta_5\gamma_3 - \beta_7)a^2b + (\beta_3 + \gamma_3\gamma_5 - \gamma_7)a^2c \quad (\text{mod } J^4)
\]

we conclude that \(\alpha_7 = \alpha_3\), \(\beta_7 = \beta_5\gamma_3\) and \(\gamma_7 = \beta_3 + \gamma_3\gamma_5\). Similarly, using the fact that \(0 = (bc)a - b(ca) + c(ba) - (cb)a + J^4\) we obtain \(\beta_5 = -1\), so \(\gamma_5 \neq 2\). This also shows that \(\gamma_6 = \beta_4 - \beta_6 + \gamma_4\) and \(0 = (2 - \gamma_5)(\beta_4 - \beta_6)\). Since \(\gamma_5 \neq 2\) this implies that \(\beta_4 = \beta_6\) and \(\gamma_6 = \gamma_4\). Then, using the fact that \(0 = (b^2)b - b(b^2) + J^4\) and \(0 = (bc)b - b(cb) + J^4\) we see that \(0 = (\alpha_3 - \alpha_4 + 1)(\beta_3 - \gamma_3) = (\alpha_3 - \alpha_4 + 1)(\beta_4 - \gamma_4)\). Now we distinguish two cases.

**Case 1.1.** \(\alpha_4 \neq \alpha_3 + 1\). Then \(\gamma_3 = \beta_3\) and \(\gamma_4 = \beta_4\). Moreover, the fact that \(0 = (bc)a - b(ca) + J^4\) implies that \(0 = \beta_3\gamma_5\). We distinguish two more cases.

**Case 1.1.1.** \(\gamma_5 \neq 0\), \(\beta_5 = 0\). In this case we use the fact that \(0 = (bc)b - b(cb) + J^4\) to obtain \(0 = \gamma_5(1 - \alpha_4)\), so \(\alpha_4 = 1\). But this leads to a contradiction using the fact that \(0 = (bc)b - b(cb) + J^4\) again.

**Case 1.1.2.** \(\gamma_5 = 0\). In this case we use the fact that \(0 = (bc)b - b(cb) + J^4\) to obtain \(2\alpha_4 = 1\); in particular, \(\text{char } F \neq 2\). Then we use the fact that \(0 = (bc)a - b(ca) + J^4\) to conclude that \(\beta_4 = 0\), we use the fact that \(0 = (c^2)a - c(ca) + J^4\) to see that \(\beta_3 = 0\), and we use the fact that \(0 = (bc)c - b(c^2) + J^4\) to show that \(\alpha_4 = 0\). But this contradicts the fact that \(0 = (bc)b - b(cb) + J^4\).

**Case 1.2.** \(\alpha_4 = \alpha_3 + 1\). In this case the fact that \(0 = (bc)a - b(ca) + J^4\) implies
that $0 = 2\alpha_3 + 1 - \alpha_3\gamma_5$. Thus
\[ J^2 = F\{[a, b], [a, c], [b, c], [a, ab], [a, ac], [b, a^2], [b, ab], [b, ac], [c, a^2], [c, ab], [c, ac]\} + J^4 \]
\[ \subset F\{[a, b], [a, c], [b, c], a^2b, a^2c\} + J^4 \]
as is easily checked. But this is a contradiction since $\dim J^2/J^4 = 6$.

Case 2. $\dim J^3/J^4 = 2$. Here we distinguish two more cases.

Case 2.1. $a^2b \in F a^3 + J^4$. In this case we have $J^3 = F\{a^3, a^2b, a^2c\} + J^4 = F\{a^3, a^2c\} + J^4$ and write $a^2b \equiv \delta a^3$ (mod $J^4$) with some element $\delta \in F$. Then $a^3c \equiv a^2ba \equiv \delta a^4$ (mod $J^5$), so $J^4 = F\{a^4, a^3c\} + J^5 = Fa^4 + J^5$. Since $J^4 \neq J^5$ this implies that $\dim J^4/J^5 = 1$. By Lemma G in [6], $J^3 \subset Z$; in particular, $a^2c \in Z$. Hence
\[ 0 \equiv (a^2c)a - a(a^2c) \equiv a^2(ca) - a^3c \equiv (\beta_5 + \gamma_5 - 1)\delta a^3 \pmod{J^4}. \]
Since $\beta_5 + \gamma_5 \neq 1$ and $a^3 \notin J^4$ we conclude that $\delta = 0$. But now
\[ 0 \equiv a^2(ba) - (a^2b)a \equiv \alpha_4a^4 \pmod{J^5}, \]
\[ 0 \equiv a^2(c) - (a^2c)b \equiv \alpha_5a^4 \pmod{J^5}, \]
\[ 0 \equiv (b^2)a - b(ba) \equiv (\alpha_3 - \alpha_7)a^3 + (\beta_3 + \gamma_3\gamma_5 - \gamma_7)a^2c \pmod{J^4}, \]
\[ 0 \equiv (ab)a - c(ba) \equiv (\alpha_4 - 1 - \alpha_4\beta_5 - \alpha_7\gamma_5)a^3 + (\alpha_5 + \gamma_5\gamma_6 - \beta_5\gamma_4 - \gamma_5\gamma_7)a^2c \pmod{J^4}. \]
This leads to the contradiction $0 = \alpha_4 = \alpha_3 = \alpha_7 = -1$.

Case 2.2. $a^2b \notin F a^3 + J^4$. Since $a^3 \notin J^4$ and $\dim J^3/J^4 = 2$ the elements $a^3 + J^4$ and $a^2b + J^4$ form a basis of $J^3/J^4$ in this case. We write $a^2c \equiv \delta a^3 + ea^2b$ (mod $J^4$) with elements $\delta, e \in F$. Since $J^4 = Fa^4 + Fa^3b + J^5$ and $J^4 \neq J^5$ we have $\dim J^4/J^5 \in \{1, 2\}$. Let us distinguish the corresponding cases.

Case 2.2.1. $\dim J^4/J^5 = 1$. In this case $J^3 \subset Z$ by Lemma G in [6]; in particular, $a^2b, a^2c \in Z$. Hence
\[ 0 \equiv (a^2b)a - a(a^2b) \equiv a^2(ba) - a^3b \equiv a^3c - a^3b \equiv (\alpha_3 - \alpha_7) a^3 \pmod{J^5}, \]
\[ 0 \equiv (a^2c)a - a(a^2c) \equiv a^2(ca) - a^3c \equiv (\gamma_5 - 1)\delta a^4 + (\beta_5 + \gamma_5 e - e)a^3b \equiv (\beta_5 + \gamma_5 - 1)a^3b \pmod{J^5}. \]
Since $\beta_5 + \gamma_5 \neq 1$ this implies that $a^3b \in J^5$. Hence $J^4 = Fa^4 + J^5$ and $\delta a^4 \in J^5$. Since $\dim J^4/J^5 = 1$ we must have $\delta = 0$. Therefore
\[ 0 \equiv (b^2)a - b(ba) \equiv (\alpha_3 - \alpha_7) a^3 \]
\[ + (\beta_5\gamma_3 - \beta_7 + \beta_3 e + \gamma_3\gamma_5 e - \gamma_7 e)a^2b \pmod{J^4}; \]
in particular, $\alpha_7 = \alpha_3$. Similarly, using the fact that $0 = (bc)a - b(ca) + c(ba) - (cb)a + J^4$ we see that $\beta_5 = -1$; in particular, $\gamma_5 \neq 2$. Hence
\[ 0 \equiv a^2(cb) - (a^2c)b \equiv (\alpha_4 - 1 - \alpha_3\gamma_5) a^4 \pmod{J^5}, \]
\[ 0 \equiv a^2(c^2) - (a^2c)c \equiv (\alpha_3 - \alpha_4 e) a^4 \pmod{J^5}. \]
Since \( a^4 \not\in J^5 \) this implies that \( \alpha_3 = \alpha_4 \epsilon \) and \( \alpha_4 - \alpha_4 \epsilon^2 = 1 \); in particular, \( \alpha_4 \neq 0 \) and \( \epsilon^2 \neq 1 \). But since \( a^2c \in Z \) we have
\[
0 \equiv (a^2c)b - (a^2c)(ac) \equiv \epsilon (a^4 - (a^4 + 1 - \gamma_5 \epsilon)) \quad (\text{mod } J^5),
\]
\[
0 \equiv (a^2c)b - b(a^2c) \equiv (2 - \gamma_5 \epsilon) \alpha_4 \epsilon^2 a^4 \quad (\text{mod } J^5).
\]

Hence \( \gamma_5 \epsilon = 2 \) and \( \epsilon^2 = 1 \), a contradiction.

**Case 2.2.2.** \( \dim J^4/J^5 = 2 \). In this case
\[
0 \equiv (a^2c)a - a^2(ca) \equiv (\delta + \delta \epsilon - \gamma_5 \delta) a^4 + (\epsilon^2 - \beta_5 - \gamma_5 \epsilon)a^3 b \quad (\text{mod } J^5),
\]
\[
0 \equiv (a^2c)b - a^2(cb) \equiv (\alpha_3 \epsilon + \gamma_3 \delta \epsilon - \alpha_4 + 1 - \gamma_6 \epsilon) a^4
\]
\[
+ (\delta + \beta_3 \epsilon + \gamma_3 \epsilon^2 - \beta_6 - \gamma_6 \epsilon)a^3 b \quad (\text{mod } J^5),
\]
\[
0 \equiv (a^2c)c - a^2c^2 \equiv (\alpha_4 \epsilon + \gamma_4 \delta \epsilon - \alpha_7 - \gamma_7 \delta + \delta^2) a^4
\]
\[
+ (\delta + \beta_4 \epsilon + \gamma_4 \epsilon^2 - \beta_7 - \gamma_7 \epsilon)a^3 b \quad (\text{mod } J^5).
\]

Since \( a^4 + J^5 \) and \( a^3 b + J^5 \) form a basis of \( J^4/J^5 \) this implies that all coefficients on the right-hand side vanish; in particular, \( 0 = \delta + \delta \epsilon - \gamma_5 \delta \).

Assume that \( \delta \neq 0 \). Then \( \epsilon = \gamma_5 - 1 \) and we obtain the contradiction \( 0 = \epsilon^2 - \beta_5 - \gamma_5 \epsilon = 1 - \beta_5 - \gamma_5 \). Hence we must have \( \delta = 0 \). Therefore
\[
0 \equiv (b^2)a - b(ba) \equiv (\alpha_3 - \alpha_7) a^3
\]
\[
+ (\beta_5 \gamma_3 - \beta_7 + \beta_3 \epsilon + \gamma_3 \gamma_5 \epsilon - \gamma_7 \epsilon)a^2 b \quad (\text{mod } J^4);
\]

in particular, \( \alpha_7 = \alpha_3 \). Similarly, using the fact that \( 0 = (bc)a - b(ca) + c(ba) - (cb)a + J^4 \) we see that \( \beta_5 = -1 \). Hence \( \epsilon^2 - \gamma_5 \epsilon = -1 \); in particular, \( \epsilon \neq 0 \).

Therefore \( 0 = \alpha_3 \epsilon + \gamma_3 \delta \epsilon - \alpha_4 + 1 - \gamma_6 \epsilon = \alpha_3 \epsilon - \alpha_4 + 1 \), and \( \alpha_4 = \alpha_3 \epsilon + 1 \). Hence \( 0 = \alpha_4 \epsilon + \gamma_4 \delta \epsilon - \alpha_7 - \gamma_7 \delta + \delta^2 = \alpha_3 \epsilon^2 + \epsilon - \alpha_3 \); in particular, \( \alpha_3 \neq 0 \) and \( \epsilon^2 \neq 1 \). But this leads to the contradiction
\[
0 \equiv b(a^2c) - (ba)ac
\]
\[
\equiv -\epsilon^2 a^4 + (\beta_6 \gamma_5 \epsilon - \beta_3 \epsilon + \gamma_5 \gamma_6 \epsilon^2 - \gamma_3 \epsilon^2 - \beta_7 \gamma_5 + \beta_4 - \gamma_5 \gamma_7 \epsilon + \gamma_4 \epsilon)a^3 b 
\]
\[
\quad (\text{mod } J^5).
\]

(4.8) **Lemma.** *The case (4.5) does not occur.*

**Proof.** We assume the contrary and distinguish two cases.

**Case 1.** \( \dim J^3/J^4 = 3 \). In this case we have
\[
0 \equiv (bc)a - b(ca)
\]
\[
\equiv (\alpha_4 + \gamma_4 - 1 - \alpha_6 \beta_5 - \alpha_7 + \alpha_7 \beta_5 \epsilon) a^3 + (\beta_5 \gamma_4 - \beta_5 - \beta_5 \beta_6 - \beta_7 + \beta_5 \beta_7) a^2 b
\]
\[
+ (\beta_4 + \gamma_4 - \beta_5 \gamma_4 - 1 + \beta_5 - \beta_5 \gamma_6 - \gamma_7 + \beta_5 \gamma_7) a^2 c \quad (\text{mod } J^4).
\]

Since \( a^3 + J^4 \), \( a^2 b + J^4 \), \( a^2 c + J^4 \) form a basis of \( J^3/J^4 \) we obtain
\[
(4.9) \quad 0 = \alpha_4 + \gamma_4 - 1 - \alpha_6 \beta_5 - \alpha_7 + \alpha_7 \beta_5 ,
\]
\[
(4.10) \quad 0 = \beta_5 \gamma_4 - \beta_5 - \beta_5 \beta_6 - \beta_7 + \beta_5 \beta_7 ,
\]
\[
(4.11) \quad 0 = \beta_4 + \gamma_4 - \beta_5 \gamma_4 - 1 + \beta_5 - \beta_5 \gamma_6 - \gamma_7 + \beta_5 \gamma_7 .
\]
Similarly, using the fact that \(0 \equiv (ca)b - c(ab) \pmod{J^4}\) we obtain the following equations:

\[
\begin{align*}
(4.12) & \quad 0 = \alpha_6 + \gamma_6 - \alpha_4 \beta_5 - \alpha_7 + \alpha_7 \beta_5, \\
(4.13) & \quad 0 = \beta_5 \gamma_6 - \beta_4 \beta_5 - \beta_7 + \beta_5 \beta_7, \\
(4.14) & \quad 0 = \beta_6 + \gamma_6 - \beta_5 \gamma_6 - 1 - \beta_5 \gamma_4 - \gamma_7 + \beta_5 \gamma_7.
\end{align*}
\]

Now we add (4.10) and (4.11) and subtract (4.13) and (4.14) from the result to obtain \(0 = (\beta_5 + 1)(\beta_4 - \beta_6 + \gamma_4 - \gamma_6)\). Hence \(\beta_5 = -1\). Then we subtract (4.12) from (4.9) and obtain \(0 = \gamma_4 - \gamma_6 - 1\). Hence \(\gamma_6 = \gamma_4 - 1\). Next we subtract (4.14) from (4.11) and obtain \(0 = \beta_4 - \beta_6\). Hence \(\beta_6 = \beta_4\). Then we use the fact that \(b(ba) \equiv (b^2)a \pmod{J^4}\) to obtain that \(\alpha_7 = \alpha_3 + \gamma_3, \beta_7 = -\gamma_3\) and \(\gamma_7 = \beta_3 + 2\gamma_3\). Now (4.10) implies that \(\beta_4 = \gamma_4 - 1 - 2\gamma_3\). Using the fact that \(0 \equiv (c^2)a - c(ca) \pmod{J^4}\) we obtain the following equations:

\[
\begin{align*}
(4.15) & \quad 0 = \beta_3 - \gamma_3 - 3 - 4\alpha_3 + 2\alpha_4 + 2\alpha_6, \\
(4.16) & \quad 0 = 4\gamma_4 - 2\beta_3 - 1 - 6\gamma_3;
\end{align*}
\]

in particular, \(\text{char } F \neq 2\). Now (4.11) forces \(0 = 4 + 2\beta_3 + 6\gamma_3 - 4\gamma_4\), so \(\beta_3 = 2\gamma_4 - 3\gamma_3 - 2\). Next we multiply (4.9) by 2 and subtract (4.15) to obtain \(0 = 3\), so \(\text{char } F = 3\). Thus

\[
J^2 = F\{[a, b], [a, c], [b, c], [a, ab], [a, ac], [b, a^2], \\
[b, ab], [b, ac], [c, a^2], [c, ab], [c, ac]\} + J^4
\]

\[
\subset F\{[a, b], [a, c], [b, c], a^3, a^2b - a^2c\} + J^4
\]

as is easily checked. But this contradicts the fact that \(\dim J^2/J^4 = 6\).

**Case 2.** \(\dim J^3/J^4 = 2\). We distinguish two more cases.

**Case 2.1.** \(a^2b \in F a^3 + J^4\). In this case \(J^3 = F\{a^3, a^2b, a^2c\} + J^4 = F\{a^3, a^2c\} + J^4\), and \(a^2b \equiv \delta a^4 \pmod{J^4}\) for some element \(\delta \in F\). Since \(a^3c \equiv a^2b a \equiv \delta a^4 \pmod{J^5}\) we see that \(J^4 = Fa^4 + Fa^3c + J^5 = Fa^4 + J^5\). Since \(J^4 \not\equiv J^5\) this implies that \(\dim J^4/J^5 = 1\). Now Lemma G in [6] shows that \(J^3 \subset Z\); in particular, \(a^2c \in Z\). But this leads to the contradiction \(0 \equiv (a^2c)a - a(a^2c) \equiv a^2(c(a)) - a^3c \equiv a^4 \pmod{J^5}\).

**Case 2.2.** \(a^2b \not\in F a^3 + J^4\). Since \(a^3 \not\in J^4\) and \(\dim J^3/J^4 = 2\) the elements \(a^3 + J^4\) and \(a^2b + J^4\) form a basis of \(J^3/J^4\) in this case. We write \(a^2c \equiv \delta a^3 + \epsilon a^2b \pmod{J^4}\) with elements \(\delta, \epsilon \in F\). Since \(J^4 = Fa^4 + Fa^3b + J^5\) and \(J^4 \not\equiv J^5\) we have \(\dim J^4/J^5 \in \{1, 2\}\). Let us distinguish the corresponding cases.

**Case 2.2.1.** \(\dim J^4/J^5 = 2\). In this case the elements \(a^4 + J^5\) and \(a^3b + J^5\) form a basis of \(J^4/J^5\). Since

\[
0 \equiv (a^2c)a - a^2(ca) \equiv (\delta \epsilon + \beta_5 \delta - 1)a^4 + (\epsilon - 1)(\epsilon + \beta_5)a^3b \pmod{J^5}
\]

this implies that \(\delta \epsilon + \beta_5 \delta - 1 = 0\) and \((\epsilon - 1)(\epsilon + \beta_5) = 0\). The first equation forces \(\epsilon = -\beta_5\), so \(\epsilon = 1\) by the second equation. Then, using the fact that \(0 \equiv (bc)a - b(ca) + c(ba) - (cb)a \pmod{J^4}\) we obtain the contradiction \(0 = (1 + \beta_5)(\beta_4 - \beta_6 + \gamma_4 - \gamma_6)\).
Case 2.2.2. \( \dim J^4/J^5 = 1 \). In this case \( J^3 \subset Z \) by Lemma G in [6]; in particular, \( a^2b, a^2c \in Z \). Hence \( a^2b \equiv a^2ba \equiv a^3c \quad (\text{mod } J^5) \) and \( a^2b \equiv a^3c \equiv a^2ca \equiv a^4 + a^3b \quad (\text{mod } J^5) \), so \( a^4 \in J^3 \) and \( J^4 = Fa^3b + J^5 \). Furthermore, since \( a^3b \equiv a^3c \equiv ea^3b \quad (\text{mod } J^5) \) we must have \( e = 1 \). Using the fact that \( 0 \equiv (bc)a - b(ca) + (ca)b - (cb)a \quad (\text{mod } J^4) \) we obtain \( 0 \equiv (1 + \beta_5)(\beta_4 - \beta_6 + \gamma_4 - \gamma_6) \), so \( \beta_5 = -1 \). Similarly, using the fact that \( 0 \equiv (b^2)a - b(ba) \quad (\text{mod } J^4) \) we obtain \( \gamma_7 = \beta_3 + \gamma_3 - \beta_4 - \gamma_4 \). Then \( 0 \equiv (a^2c)b - b(a^2c) \equiv (a^2c)b - (ba)ac \quad (\text{mod } J^5) \) implies that \( \delta = 1 + \beta_3 + \gamma_3 - \beta_4 - \gamma_4 \). But now the fact that \( 0 \equiv a^2(c^2) - (a^2c)c \quad (\text{mod } J^5) \) leads to a contradiction.

\[(4.17) \quad \text{Lemma. The case (4.6) does not occur.}\]

\[\text{Proof. We assume the contrary and distinguish two cases.}\]

Case 1. \( \dim J^3/J^4 = 3 \). In this case the elements \( a^3 + J^4, a^2b + J^4, a^2c + J^4 \) form a basis of \( J^3/J^4 \). Since \( \beta_2 \neq 1 \) and

\[0 \equiv (bc)a - b(ca) \equiv (1 - \beta_2^2)a^3 + \beta_4(\beta_2 - \beta_2^2)a^2b + \gamma_4(\beta_2 - \beta_2^2)a^2c \quad (\text{mod } J^4)\]

this implies that \( \beta_2 = -1 \); in particular, \( \text{char } F \neq 2 \). Hence \( 0 = 2\beta_4 = 2\gamma_4 \), so \( 0 = \beta_4 = \gamma_4 \). Then, using similarly the fact that \( 0 \equiv c(ba) - (cb)a \quad (\text{mod } J^4) \) we obtain \( 0 = \beta_6 = \gamma_6 \). But now the fact that \( 0 \equiv (bc)b - b(cb) \quad (\text{mod } J^4) \) leads to a contradiction.

Case 2. \( \dim J^3/J^4 = 2 \). We distinguish two more cases.

Case 2.1. \( a^2b \in F a^3 + J^4 \). In this case we have \( J^3 = F\{a^3, a^2b, a^2c\} + J^4 = Fa^3 + Fa^2c + J^4 \) and \( J^4 = Fa^4 + Fa^3c + J^5 \). Assume that \( a^4 \in J^5 \). Then \( J^4 = Fa^3c + J^5 \); in particular, \( \dim J^4/J^5 = 1 \) since \( J^4 \neq J^5 \). Hence Lemma G in [6] implies that \( J^3 \subset Z \); in particular, \( a^2c \in Z \). But this leads to the contradiction \( a^3c \equiv a^2ca \equiv \beta_2a^3c \quad (\text{mod } J^5) \).

We write \( a^2b \equiv \delta a^3 \quad (\text{mod } J^4) \) with some element \( \delta \in F \). Then \( \delta a^4 \equiv a^2ba \equiv \beta_2a^2b \equiv \beta_2^2a^4 \quad (\text{mod } J^5) \). Since \( a^4 \notin J^5 \) and \( \beta_2 \neq 1 \) this implies that \( \delta = 0 \). As in Case 1, we now use the fact that \( 0 \equiv (bc)a - b(ca) \quad (\text{mod } J^4) \) to obtain that \( \beta_2 = -1 \), char \( F \neq 2 \) and \( \gamma_4 = 0 \). Similarly, using the fact that \( 0 \equiv (cb)a - c(ba) \equiv (b^2)a - b(ba) \quad (\text{mod } J^4) \) we obtain \( 0 = 2\gamma_6 = 2\gamma_3 \), so \( 0 = \gamma_6 = \gamma_3 \). But this yields a contradiction using the fact that \( 0 \equiv (bc)c - b(c^2) \quad (\text{mod } J^4) \).

Case 2.2. \( a^2b \notin F a^3 + J^4 \). Since \( a^3 \notin J^4 \) and \( \dim J^3/J^4 = 2 \) the elements \( a^3 + J^4 \) and \( a^2b + J^4 \) form a basis of \( J^3/J^4 \) in this case, and \( J^4 = Fa^4 + Fa^3b + J^5 \). Assume that \( a^4 \in J^5 \). Then \( J^4 = Fa^3b + J^5 \); in particular, \( \dim J^4/J^5 = 1 \) since \( J^4 \neq J^5 \). Hence Lemma G in [6] implies that \( J^3 \subset Z \); in particular, \( a^2b \in Z \). But now we obtain the contradiction \( a^2b \equiv a^2ba \equiv \beta_2a^3b \quad (\text{mod } J^5) \).

Hence \( a^4 \notin J^5 \), and we write \( a^2c \equiv \delta a^3 + ea^2b \quad (\text{mod } J^4) \) with elements \( \delta, e \in F \). Then \( 0 \equiv (a^2c)a - a^2(ca) \equiv (1 - \beta_2)\delta a^4 \quad (\text{mod } J^5) \), so \( \delta = 0 \) since \( \beta_2 \neq 1 \) and \( a^4 \notin J^5 \). As in Case 1, we now use the fact that \( 0 \equiv (bc)a - b(ca) \quad (\text{mod } J^4) \) to obtain \( \beta_2 = -1 \) and char \( F \neq 2 \). Then we distinguish two more cases.

Case 2.2.1. \( \dim J^4/J^5 = 2 \). In this case the elements \( a^4 + J^5 \) and \( a^3b + J^5 \) form a basis of \( J^4/J^5 \). Using the fact that \( 0 \equiv b(a^2c) - (ba)ac \quad (\text{mod } J^5) \)
we obtain \( \alpha_3 \varepsilon = 1 \). But this leads to a contradiction using the fact that \( 0 \equiv (a^2c)b - a^2(cb) \pmod{J^4} \).

Case 2.2.2. \( \dim J^4/J^5 = 1 \). In this case we have \( J^4 = Fa^4 + J^5 \) since \( a^4 \notin J^5 \). Moreover, Lemma G in [6] implies that \( J^3 \subset Z \); in particular, \( a^2b \in Z \). Thus \( a^3b \equiv a^2ba \equiv -a^3b \pmod{J^5} \), so \( a^3b \in J^5 \) since \( \text{char } F \neq 2 \). This, however, leads to a contradiction using the fact that \( 0 \equiv (a^2c)b - b(a^2c) \equiv a^2(cb) - (ba)ac \pmod{J^5} \).

References


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