TRACE FUNCTIONS IN THE RING OF FRACTIONS OF POLYCYCLIC GROUP RINGS

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Dedicated to the memory of I. N. Herstein

Abstract. Let $KG$ be the group ring of a polycyclic-by-finite group $G$ over a field $K$ of characteristic zero, $R$ be the Goldie ring of fractions of $KG$, $S$ be an arbitrary subring of $R_{n\times n}$. We prove that the intersection of the commutator subring $[S, S]$ with the center $Z(S)$ is nilpotent. This implies the existence of a nontrivial trace function in $R_{n\times n}$.

1

Let $G$ be a polycyclic-by-finite group, $K$ be a commutative field of characteristic zero. (Throughout this paper the term "field" is used in the sense of "skew field.") It is well known that the group ring $KG$ is semiprime Noetherian and hence has a Goldie ring of fractions which we denote by $R$. Let $S$ be a subring of the matrix ring $R_{n\times n}$, $Z(S)$ be its center and $[S, S]$ be the $K$-subalgebra of $R_{n\times n}$ generated by all the commutators $[x, y] = xy - yx$, $x, y \in S$. Our first main result is the following theorem which is motivated by R. Snider's article [1].

The intersection

\[(1.1) \quad [S, S] \cap Z(S)\]

is a nilpotent ring (see Theorem 3). (It is known that (1.1) is a subring; the proof of this fact is easy.)

We obtain immediately from Theorem 3 an affirmative answer to the question, posed by R. Snider in [1]: Let $G$ be a poly-$Z$-group, $K$ be a commutative field of characteristic zero, $D$ be the field of fractions of $KG$. Does

\[(1.2) \quad [D, D] \neq D?\]

In particular, does

\[(1.3) \quad 1 \notin [D, D].\]

We see thus that the relations (1.2) and (1.3) do hold in $D$. Furthermore, this result implies that there exists a nontrivial trace function $t: D \to D/[D, D]$, defined by

\[t(d) = d + [D, D]\]
and this function can be extended to a function \( T: D_{n \times n} \to D/[D, D] \) by

\[
t(d_{ij}) = \sum_i t(d_{ii}),
\]

where \( (d_{ij}) \) is an arbitrary matrix from \( D_{n \times n} \) (see [1–3]). Snider proved in [1] the relation (1.3) and hence the existence of nontrivial trace functions in the case when \( G \) is abelian-by-{infinite cyclic}.

The proof of Theorem 3 will be based on the following result (see Theorem 2):

Let \( K \) be an arbitrary commutative field and \( R \) be the ring of fractions of \( KG \) and

\[
X_j \quad (j = 1, 2, \ldots, m)
\]

be given nonzero elements of \( KG \). Then there exists an ideal \( C \subseteq KG \) such that the quotient ring \( (KG)/C \) is a finite-dimensional \( K \)-algebra \( K[\bar{G}] \), generated by a finite group \( \bar{G} \) which is the image of \( G \) in \( (KG)/C \). The homomorphism \( \alpha: KG \to \bar{G} \) is extended to a specialization \( \theta: R \to K[\bar{G}] \), whose domain \( R_0 \) contains the elements (1.4). Furthermore the elements \( x_j = \theta(x_j) \ (j = 1, 2, \ldots, m) \) are nonzero elements of \( K[\bar{G}] \).

We will obtain one more result on specializations from \( R \) to algebras finite-dimensional over their central subfields; this is Theorem 1 and its corollary. Let \( H \) be a torsion-free normal subgroup of finite index in \( G \) such that \( H/H_1 \) is free abelian, where \( H_1 \) is the Fitting radical of \( H \). Then Theorem 1 essentially states that there exists a \( G \)-invariant ideal \( A \subseteq KH_1 \) and an ideal \( B = (A)(KG) \) such that the quotient algebra \( (KG)/B \simeq K[\bar{H}] \), where the group \( \bar{H} \) is abelian-by-finite; the images \( \bar{x}_j \ (j = 1, 2, \ldots, m) \) of the elements (1.4) are nonzero in \( K[\bar{G}] \) and a given element \( x_j \) is regular in \( R \) iff its image \( \bar{x}_j \) is regular in \( K[\bar{G}] \). Roseblade’s Theorem 11.2.9 in [4] implies that the ideal \( B \) is localizable in \( KG \).

It is worth remarking that Theorems 1 and 2 provide a method for construction of specializations from \( R \) into finite-dimensional algebras over the same field \( K \); they should be compared with the Reduction Theorem (see [5, Theorem 4.1], [6], or [7, 4.2.1]) which gives specializations into algebras over fields of finite characteristic (see a discussion on this in the book [7, p. 137]).

2

Throughout this section let \( D \) be a field, generated by a polycyclic-by-finite group \( G \). Thus, \( D \) is the field of fractions of its subring generated by the group \( G \); we denote this subring by \( T \). Thus, \( T = Z[G] \) or \( T = Z_p[G] \), depending on the characteristic of \( D \).

Lemma 1. Let (1.4) be given nonzero elements of \( T \). Then there exists an ideal \( A \subseteq T \) such that the quotient ring \( T/A \simeq \prod_{x \in x} \), where \( \prod \) is a finite field and the images of the elements (1.4) are invertible in \( T/A \).

Proof. Wehrtit proved (see [8] or [7, 4.3.12]) that if \( R \) is a finitely generated subring of \( D \), then there exists an ideal \( C \) of \( R \) of finite index with \( \cap_{n=1}^\infty \bigcap_{n=1}^\infty C^n = 0 \); furthermore, every quotient ring \( R/C^n \quad (n = 1, 2, \ldots) \) is finite. We apply this theorem to the subring \( S \) of \( D \), generated by the elements \( x_j, x_j^{-1} \) (j =
1, 2, \ldots, m) and find an ideal \( B \subseteq S \) such that the ring \( \overline{S} = S/B \) is finite. Since the images of the elements \( x_j \in T \ (j = 1, 2, \ldots, m) \) are invertible in the finite ring \( \overline{S} \) they must be invertible in the subring \( T/(T \cap B) \). We see now that an arbitrary maximal ideal \( A \supseteq (T \cap B) \) satisfies the conclusions of the assertion.

**Remark.** The current proof of Lemma 1 is somewhat shorter than the proof given in the original version of the paper, where Lemma 1 was obtained as one of the corollaries of the Reduction Theorem [5].

Now let \( \Pi[G] \) be a domain, generated by a polycyclic-by-finite group \( G \) over a finite field \( \Pi \). We see that \( \Pi[G] \cong Z[G_1], \) where \( G_1 \) is the subgroup of units of \( \Pi[G] \), generated by \( G \) and the multiplicative group of \( \Pi \). We see thus that Lemma 1 is true for this case, when \( T = \Pi[G] \). We will use it in this form in the proof of Proposition 1 below.

**Proposition 1.** Let \( K \) be an arbitrary commutative field, \( G \) be a torsion-free polycyclic group and let \( (1.4) \) be given nonzero elements of \( KG \). Then there exists a maximal ideal \( A \subseteq KG \) such that the quotient algebra \( (KG)/A \) is generated over \( K \) by a finite group \( G \), the image of \( G \) under the natural homomorphism \( (KG) \to (KG)/A \), and the images of the elements \( (1.4) \) in the ring \( KG \) are invertible.

**Proof.** We reduce first the proof to the case when the field \( K \) is finitely generated. Indeed, assume that the theorem is proved for this special case. Let \( K_1 \) be the finitely generated subfield of \( K \), such that \( K_1 G \) contains all the elements \( (1.4) \) and \( A_1 \subseteq K_1 G \) be the ideal, which satisfies all the conclusions of the theorem. Since

\[(KG)/(KA_1) \cong K \otimes ((K_1 G)/A_1),\]

we obtain an ideal \( KA_1 \subseteq KG \) such that the algebra \( (KG)/(KA_1) \) is generated by a finite group and the images of the elements \( (1.4) \) are invertible in it. Since images of the elements \( (1.4) \) are invertible in the algebra \( (KG)/(KA_1) \) they are invertible in every simple homomorphic image of it; this implies easily that an arbitrary maximal ideal \( A \subseteq KG \), which contains \( KA_1 \), satisfies the conclusion of the theorem.

We can assume therefore that the field \( K \) is finitely generated. Let \( K_0 \subseteq K \) be a finitely generated subring such that \( K \) is the field of fractions of \( K_0 \). We have the following representations for the elements \( (1.4) \)

\[(2.1) \quad x_j = \sum_i c_{ij} g_i \quad (c_{ij} \in K; \ j = 1, 2, \ldots, m).\]

An arbitrary coefficient \( c_{ij} \) in \( (2.1) \) has a representation

\[(2.2) \quad c_{ij} = a_{ij} b^{-1}_{ij} \quad (a_{ij}, b_{ij} \in K_0).\]

We can find a maximal ideal \( \mathcal{P} \subseteq K_0 \) which defines a \( p \)-adic valuation in \( K_0 \) and contains no one of the elements \( a_{ij}, b_{ij} \) in \( (2.2) \). If \( K_{\mathcal{P}} \) is the ring of fractions of \( K_0 \) with respect to \( \mathcal{P} \) then all the coefficients \( c_{ij} \) in \( (2.1) \) belong to \( K_{\mathcal{P}} \) and hence

\[x_j \in K_{\mathcal{P}} G \quad (j = 1, 2, \ldots, m).\]

Now consider the natural homomorphism

\[(2.3) \quad \varphi: K_{\mathcal{P}} G \to (K_{\mathcal{P}} G)/(\mathcal{P}),\]
where \((\mathcal{P})\) is the ideal of \(K_{\mathcal{P}}G\), generated by the ideal \(\mathcal{P} \subseteq K_{\mathcal{P}}G\). We observe that the ring \((K_{\mathcal{P}}G)/(\mathcal{P})\) is isomorphic to the group ring \(\Pi G\), where \(\Pi \cong (K_{\mathcal{P}}G)/(\mathcal{P})\) is a finite field and the elements \(\varphi(x_j)\) \((j = 1, 2, \ldots, m)\) are nonzero. Lemma 1 implies that there exists an ideal \(B \subseteq \Pi G\) such that \((\Pi G)/B\) is a simple finite ring and the images \(\bar{x}_j\) of the elements \(\varphi(x_j)\) \((j = 1, 2, \ldots, m)\) are invertible in the ring \((\Pi G)/B\). This together with the homomorphism (2.3) implies that there exists a homomorphism

\[\psi: K_{\mathcal{P}}G \to (\Pi G)/B\]

such that the elements

\[\bar{x}_j = \psi(x_j) \quad (j = 1, 2, \ldots, m)\]

are invertible in the ring \((\Pi G)/B\); clearly, \((\Pi G)/B\) is generated over \(\Pi\) by the finite group \(\bar{G} = \psi(G)\), i.e.,

\[(\Pi G)/B \cong \Pi[\bar{G}].\]

Now take a minimal left ideal \(V\) in the matrix ring \(\Pi[\bar{G}]\); this ideal affords a representation \(\rho\) of the group \(\bar{G}\) and \(\rho(\Pi \bar{G}) \cong \Pi[\bar{G}]\). Let \(\bar{K}_0\) be the \(p\)-adic completion of \(K_0\), \((\pi)\) be the maximal ideal of \(\bar{K}_0\). Since \(G\) is polycyclic, the group \(\bar{G}\) is solvable and Fong-Swan's Theorem implies that there exists a \(\bar{K}_0\bar{G}\)-module \(\bar{V}\), free over \(\bar{K}_0\), such that \(\bar{V}/(\pi)\bar{V} \cong V\). (In fact, this theorem is proven in [9, 22.1] for the case when the group is \(p\)-solvable and \(\bar{K}_0\) contains a primitive root of degree \((\bar{G}:1)\) from 1 but the last condition is unnecessary (see [10]); this can be shown also by a standard argument based on the Galois theory.) If \(\lambda\) is the representation afforded by \(\bar{V}\) and \(\lambda(\bar{K}_0 \bar{G}) \cong \Pi[\bar{G}]\) it is important that the ideal \(\pi R\) is quasiregular in \(R\).

There exists therefore a system of homomorphisms

\[(2.5) \quad \bar{K}_0 G \xrightarrow{\lambda_1} \bar{K}_0 \bar{G} \xrightarrow{\lambda_2} R \xrightarrow{\lambda} \Pi[\bar{G}]\]

where \(\lambda_1\) and \(\lambda_2\) are homomorphisms of \(\bar{K}_0\)-algebras.

The homomorphism

\[(2.6) \quad \lambda_2 \lambda_1: \bar{K}_0 G \to \Pi[\bar{G}]\]

maps the elements (1.4) into invertible elements \(\bar{x}_j\) \((j = 1, 2, \ldots, m)\). Since the kernel of \(\lambda_2\) is a quasiregular ideal we conclude easily that the images of the elements (1.4) under the homomorphism

\[(2.7) \quad \lambda \lambda_1: \bar{K}_0 G \to R\]

are invertible elements of \(R\). Since the field of fractions of \(K_0\) coincides with \(K\) we see that the field of fractions of \(\bar{K}_0\) is isomorphic to the \(p\)-adic completion \(\bar{K}\) of \(K\); homomorphism (2.7) is extended to a homomorphism of \(\bar{K}\)-algebras

\[(2.8) \quad \mu: \bar{K} G \to \bar{K} R.\]

Since the algebra \(\bar{K} R\) is generated over \(\bar{K}\) by the finite group \(\bar{G}\), we see that the \(K\)-algebra \(\mu(KG)\) is also generated over \(K\) by the group \(\bar{G}\), i.e.

\[(2.9) \quad \mu(KG) \cong K[\bar{G}].\]
The homomorphism (2.8) carries out the elements (1.4) into invertible elements of \( \overline{K}R \); we obtain therefore that the images of these elements under the homomorphism (2.9) are invertible elements of \( K[\overline{G}] \). We found thus a homomorphism

\[
KG \rightarrow (KG)/A \simeq K[\overline{G}]
\]

which maps the elements (1.4) into invertible elements of \( K[\overline{G}] \). We can assume, of course, that \( K[\overline{G}] \) is simple, i.e. the ideal \( A \) is maximal. The proof is complete.

3

Let \( G \) be a polycyclic-by-finite group, \( \rho(G) \) be the Fitting radical of \( G \). It is not difficult to verify that \( G \) contains a torsion-free normal subgroup \( H \) of finite index such that the quotient group \( H/\rho(H) \) is free abelian; it is more convenient to denote the subgroup \( \rho(H) \) by \( H_1 \). We observe that if \( A \) is an arbitrary \( G \)-invariant ideal of \( KH_1 \) then \( B = A(KG) \) is an ideal of \( KG \) and \( (KG)/B \simeq K[\overline{G}] \), where the group \( \overline{G} \) is an extension of the normal subgroup \( \overline{H}_1 \) by the group \( \overline{G}/\overline{H}_1 \simeq G/H_1 \). Thus, the algebra \( K[\overline{G}] \) is isomorphic to an appropriate cross product of the algebra \( K[\overline{H}_1] \) and the group \( G/H_1 \) and \( K[\overline{H}] \simeq K[\overline{H}_1] \ast (H/H_1) \).

**Theorem 1.** Let \( K \) be an arbitrary commutative field, \( \text{char} \ K = p > 0 \), and assume that nonzero elements (1.4) of \( KG \) are given. Then there exists a \( G \)-invariant ideal \( A \subseteq KH_1 \) and an ideal \( B = (A)KG \) such that

(i) The image \( \overline{H}_1 \) of the group \( H_1 \) under the natural homomorphism

\[
\varphi : KG \rightarrow (KG)/B \simeq K[\overline{G}]
\]

is a finite \( p' \)-group and hence the group \( \overline{H} \) is finite-by-free abelian. Furthermore, there exists a free abelian normal subgroup \( N \subseteq \overline{G} \) of finite index, which is contained in \( \overline{H} \) and central in it, and whose elements are linearly independent over \( K \); hence \( K[N] \) is isomorphic to the group ring \( KN \).

(ii) The images

\[
\overline{x}_j \quad (j = 1, 2, \ldots, m)
\]

of the elements (1.4) are nonzero elements of \( K[\overline{G}] \). Furthermore, a given element \( x_j \) in (1.4) is regular in \( KG \) if and only if its image \( \overline{x}_j \) is regular in \( K[\overline{G}] \).

(iii) The ideal \( B \) is localizable in \( KG \).

**Proof.** Let \( g_1, g_2, \ldots, g_n \) be a transversal for \( H \) in \( G \). The group ring \( KH \) contains no zero divisors of \( KG \) and we can form the ring \( R \) of fractions of \( KG \) with respect to the set \( (KH) \setminus 0 \). If \( D \) is the field of fractions of \( KH \) then \( R \simeq D \otimes_{KH} KG \) and the transversal \( g_1 = 1, g_2, \ldots, g_n \) gives a basis of the left vector space \( R \) over \( D \).

We can assume without loss of generality that the set (1.4) contains regular elements and these are the first \( m_1 \) elements

\[
(3.1) \quad x_1, x_2, \ldots, x_{m_1}.
\]
These elements must be invertible in $R$; this implies easily that there exist nonzero elements $x_j' \ (j = 1, 2, \ldots, m_1)$ in $KG$ such that

\begin{align*}
y_j = x_j'x_j \in (KH)\setminus 0 \quad (j = 1, 2, \ldots, m_1), \\
x_j'x_j = 0 \quad (j = m_1 + 1, \ldots, m).
\end{align*}

(3.2)

Now let

\begin{align*}
x_j &= \sum_{\alpha=1}^n c_{\alpha j}g_\alpha, \\
x'_j &= \sum_{\alpha=1}^n c'_{\alpha j}g_\alpha
\end{align*}

\begin{align*}
(c_{\alpha j}, c'_{\alpha j} \in KH; \alpha = 1, 2, \ldots, n; j = 1, 2, \ldots, m)
\end{align*}

be the representations of the elements $x_j, x_j'$ $(j = 1, 2, \ldots, m)$. Let

\begin{align*}
c_1, c_2, \ldots, c_r
\end{align*}

be all the nonzero coefficients $c_{ij}$ in (3.3). Let $h_i \ (i \in I)$ be a transversal for $H_1$ in $H$ and

\begin{align*}
c_\beta = \sum_i \lambda_i h_i \quad (\lambda_i, \beta \in KH_1; \beta = 1, 2, \ldots, r).
\end{align*}

(3.5)

Similarly, we have for the elements $y_j$ in (3.2)

\begin{align*}
y_j &= \sum_i \mu_{ij} h_i \quad (\mu_{ij} \in KH_1; j = 1, 2, \ldots, m_1).
\end{align*}

(3.6)

Apply now Proposition 1 and find a maximal ideal $A \subset KH$ such that $(KH)/A \simeq K[\tilde{H}]$, where $\tilde{H}$ is a finite group and for all the elements $\lambda_{i\beta}, \mu_{ij}$ from (3.5) and (3.6) the images of the elements

\begin{align*}
g_\alpha^{-1}\lambda_{i\beta}g_\alpha, g_\alpha^{-1}\mu_{ij}g_\alpha \quad (\alpha = 1, 2, \ldots, n)
\end{align*}

are invertible in $K[\tilde{H}]$. Let

\begin{align*}
A_1 = \bigcap_{\alpha=1}^n g_\alpha^{-1}Ag_\alpha, \quad A_2 = A_1 \cap KH_1, \quad B = (A_2)KG.
\end{align*}

(3.8)

Clearly, $A_1$ is a $G$-invariant ideal of $KH$ and as a result of this $A_2$ is a $G$-invariant ideal of $KH_1$. Hence $B$ is an ideal in $KG$. We have already pointed out that the quotient ring $(KH)/B \simeq K[\tilde{G}]$, where the group $\tilde{G}$ is an extension of the group $H_1$ by the group $G/H_1 \simeq G/H_1$; the group $G/H_1$ is an extension of the free abelian group $H/H_1$ by the finite group $G/H$. On the other hand, we obtain from (3.8),

\begin{align*}
(KH_1)/(KH_1 \cap B) \simeq (KH_1)/A_2 \simeq (KH_1)/(KH_1 \cap A_1).
\end{align*}

(3.9)

The first relation in (3.8) shows that the image of $KH$ under the natural homomorphism $(KH) \to (KH)/A_1$ is a subdirect sum of the rings $(KH)/(g_\alpha^{-1}Ag_\alpha)$ $(\alpha = 1, 2, \ldots, n)$ which are isomorphic to the simple artinian ring $(KH)/A \simeq K[\tilde{H}]$; a routine argument (see [5, Lemma 2.9]) implies that in fact $(KH)/A_1$ is a direct sum of rings isomorphic to $K[\tilde{H}]$. This, together with the relation (3.9), implies first of all that the group $H_1$ which is the image of $H_1$ under the homomorphism $KG \to (KG)/B$, is finite. Furthermore, the images of the
elements (3.7) under the homomorphism $KH \to (KH)/A$ are invertible. This implies that the elements

$$\lambda_{i\beta}, \quad \mu_{ij}$$

become invertible modulo the ideals $g_{a}^{-1}A_{a}$ ($\alpha = 1, 2, \ldots, n$) and hence they are invertible modulo the ideal $A_{1} = \bigcap_{a=1}^{n} g_{a}^{-1}A_{a}$. Since the elements (3.10) belong to $KH_{1}$ the second and the third relations in (3.8) imply that they are invertible modulo the ideal $B$. We have already observed that the image of $KH$ in $(KG)/B$ is isomorphic to

$$K[\tilde{H}] \simeq K[\tilde{H}_{1}] \ast (H/H_{1}).$$

Since the group $H/H_{1}$ is free abelian and all the elements

$$\tilde{\lambda}_{i\beta}, \quad \tilde{\mu}_{ij} \quad (i = 1, 2, \ldots, n)$$

are invertible in $K[\tilde{H}_{1}]$ we conclude easily that the elements

$$\tilde{\varphi}_{\beta} = \sum_{i} \tilde{\lambda}_{i\beta} h_{i} \quad (\beta = 1, 2, \ldots, r)$$

and

$$\tilde{y}_{j} = \sum_{i} \tilde{\mu}_{ij} h_{i} \quad (j = 1, 2, \ldots, n_{1})$$

are regular in $K[\tilde{H}]$. Since $K[\tilde{G}]$ is a free $K[\tilde{H}]$-module a routine argument shows that these elements are also regular in $K[\tilde{G}]$. We obtain from (3.3)

$$\tilde{x}_{j} = \sum_{a=1}^{n} \tilde{c}_{a} g_{a}, \quad \tilde{x}'_{j} = \sum_{a=1}^{n} \tilde{c}'_{a} g_{a}$$

$$(\tilde{c}_{a}, \tilde{c}'_{a} \in K[\tilde{H}], \quad \alpha = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, m).$$

Since the elements (3.5') are nonzero we obtain from (3.3') that $\tilde{x}_{j} \neq 0$ ($j = 1, 2, \ldots, m$). The relations

$$\tilde{y}_{j} = \tilde{x}'_{j} \tilde{x}_{j} \quad (j = 1, 2, \ldots, m_{1})$$

imply, via the regularity of the elements (3.6'), that the elements $\tilde{x}_{j}$ ($j = 1, 2, \ldots, m_{1}$) are regular in $K[\tilde{G}]$. Similarly, the relations $\tilde{x}_{j} \tilde{x}'_{j} = 0$ ($j = m_{1} + 1, \ldots, m$) imply that the elements $\tilde{x}_{j}$ ($j = m_{1} + 1, \ldots, m$) are zero divisors. We completed thus the proof of statement (ii).

To prove statement (iii) we observe that the ideal $B = (A_{2})KG$, where $A_{2}$ is an ideal in the group ring of the nilpotent group $H_{1}$. Since $G$ is polycyclic-by-finite Roseblade's Theorem 11.2.9 in [5] implies that $B$ is localizable and (iii) is proved.

We have already shown that $(KH)/A_{1}$ is a direct sum of rings isomorphic to $(KH)/A_{1} \simeq K[\tilde{H}]$, where $\tilde{H}$ is a finite group and $A$ is a maximal ideal of $KH$. Hence the ring $(KH)/A_{1}$ is semisimple artinian. Furthermore, we have a homomorphism

$$K[\tilde{H}] \to (K[\tilde{H}])/A_{1} \simeq K[H]/A_{1}$$

and the second relation (3.9) implies that

$$A_{1} \cap K[\tilde{H}_{1}] = \emptyset.$$
We have already shown that the group $H_1$ is finite. Assume now that $\text{char } K = p$ and prove that $p^r((H_1 : 1)$. Indeed, we observe first of all that the group $H_1$ is nilpotent since $H_1$ is. Assume now that $p((H_1 : 1)$, let $P$ be the Sylow $p$-subgroup of $H_1$ and let $H_1 \cong P \times Q$. The elements $h - 1 \in P$ generate a nonzero nilpotent ideal in $K[H_1]$ because $P$ is a normal subgroup of $H_1$. Since $K[H]/A_1$ is semisimple we obtain from (3.12) that $(h - 1) \in A_1$ $(h \in P)$ which contradicts (3.13). We proved thus that $H_1$ is a finite $p^r$-group.

To complete the proof we need the following assertion which is part of Lemma 3.2 in [5].

**Lemma 2.** Let $K$ be an arbitrary commutative field and $K[U]$ be a ring, generated by a group $U$, which is an extension of a finite group $V$ by a polycyclic-by-finite group $U/V$. Assume also that $K[U] \cong K[V]*_{(U/V)}$. Then there exists a characteristic poly(infinite cyclic) subgroup $F \subseteq U$ of finite index such that the elements of $F$ are linearly independent over $K$ and, hence, $K[F] \cong KF$.

**Proof.** Let $F$ be a poly-infinite cyclic characteristic subgroup of finite index in $U$. Then $F \cap V = 1$ and it is not difficult to verify that the elements of $F$ are linearly independent over $K[V]$ and hence over $K$.

We complete now the proof of Theorem 1. Since $H_1$ is finite, $H/H_1$ is free abelian, and $H$ is finitely generated we conclude that $H/Z$ is finite, where $Z$ is the center of $H$. The relation (3.11) implies, via Lemma 2, the existence of a characteristic subgroup $F \subseteq H$ of finite index such that $K[F] \cong KF$. Take now $N = F \cap Z$ and statement (iii) follows. The proof is completed.

Let $R$ and $\overline{R}$ be the ring of fractions of $KG$ and $K[\overline{G}]$ correspondingly. The ring $\overline{R}$ is isomorphic to the ring of fractions of $K[\overline{G}]$ with respect to $R$ the subring $KNN$; since $(\overline{G} : N)$ is finite we conclude easily that $\overline{R}$ has a finite left dimension over the subfield $T = (KN)(KN)^{-1}$ and as a result of it is finite-dimensional over a central subfield $Z \subseteq T$. Furthermore, $\overline{R}$ is a homomorphic image of a suitable cross product $T*M/G$; when $\text{char } K = 0$ this cross product is semisimple artinian and so is $\overline{R}$.

If now nonzero elements (1.4) in $R$ are given then

\[(3.14) \quad x_j = a_jb_j^{-1} \quad (a_j \in KG; b_j \in (KG)\setminus 0; j = 1, 2, \ldots, m).\]

We apply Theorem 1 to the set of elements $a_j, b_j \in KG$ ($j = 1, 2, \ldots, n$) and obtain via well-known facts of the localization theory the following corollary.

**Corollary.** Let nonzero elements (3.14) in $R$ be given. Then there exists a localizable ideal $B \subseteq KG$ such that the elements (3.14) belong to the subring $S \subseteq R$, obtained by the localization of the ideal $B$, and $S/BS \cong \overline{R}$, where $\overline{R}$ is the ring of fractions of $K[\overline{G}]$; the ring $\overline{R}$ has a finite dimension over its central subfield $Z$. Clearly, the ideal $BS$ of $S$ is quasiregular.

Let $Q$ be an arbitrary ring. We recall (see Cohn [11] and Passman [12]) that a specialization from $Q$ on ring $\overline{Q}$ is a homomorphism $\alpha : Q_0 \to \overline{Q}$ such that $\ker \alpha$ is a quasiregular ideal of $Q_0$; $Q_0$ is the domain of $\alpha$. Theorem 1 thus gives a method for constructing specializations from the $K$-algebra $R$ to algebras finite-dimensional over their central subfields. Another system of specializations to algebras finite-dimensional over $K$ is obtained from the following theorem.
Theorem 2. Let \( R \) be the ring of fractions of \( KG \) and (3.14) be given nonzero elements of \( R \). Then there exists an ideal \( C \subseteq KG \) such that the quotient ring \((KG)/C\) is a finite-dimensional \( K \)-algebra, generated by a finite group \( \Gamma \), which is the image of \( \Gamma \) in \((KG)/C\). The homomorphism \( \alpha: KG \rightarrow K[\Gamma] \) is extended to a specialization \( \theta: R \rightarrow K[\Gamma] \), whose domain \( R_0 \) contains the elements (3.14). Furthermore, \( x_j = \theta(x_j) \ (j = 1, 2, \ldots, m) \) are nonzero elements of \( K[\Gamma] \).

Proof. Apply first Theorem 1 and its Corollary and obtain a homomorphism
\[
\beta: KG \rightarrow (KG)/B \simeq K[\Gamma]
\]
such that the elements (3.14) belong to the subring \( S \subseteq R \), the domain of the specialization \( \pi: R \rightarrow \overline{R} \) which extend \( \beta \), and
\[
(3.16) \quad \overline{x}_j = \pi(x_j) \neq 0 \quad (j = 1, 2, \ldots, m).
\]

We recall that \( \overline{G} \) contains a free abelian normal subgroup \( N \) of finite index such that \( K[N] \simeq KN \). Let \( T \) be the field of fractions of \( KN \) and \( g_1, g_2, \ldots, g_r \) be a system of elements of \( \overline{G} \) which form a basis of the left vector space \( \overline{R} \) over \( T \). Let
\[
(3.17) \quad x_j = \sum_{i=1}^{r} a_{ij} g_i \quad (a_{ij} \in T; j = 1, 2, \ldots, m).
\]

Let \( a_1, a_2, \ldots, a_s \) be all the elements of \( KN \) which occur in the numerators and denominators of the nonzero elements \( a_{ij} \) in (3.17); clearly, every element \( a_k \ (k = 1, 2, \ldots, s) \) has a finite number of \( \overline{G} \)-conjugates. Then apply Proposition 1 and find an ideal \( A \subseteq KN \) such that the quotient algebra \((KN)/A \simeq K[\tilde{N}] \) where \( \tilde{N} \) is a finite group and
\[
g^{-1}a_k g \notin A \quad (k = 1, 2, \ldots, s; g \in \overline{G}).
\]

Let \( A_1 = \bigcap_{g \in \overline{G}} A \) and \( \overline{C} = A_1(K[\Gamma]) \). The same argument as in the proof of Theorem 1 shows that \( \overline{C} \) contains no one of the elements (3.17) and \( K[\overline{G}] / \overline{C} \simeq K[\tilde{N}] \), where \( \tilde{G} \) is a finite group.

The ideal \( \overline{C} \) is localizable in \( K[\overline{G}] \); this can be verified in a straightforward way or obtained from Roseblade's theorem in [5]. We see therefore that the homomorphism \( \gamma: K[\overline{G}] \rightarrow K[\overline{G}] \) is extended to a specialization \( \tau: \overline{R} \rightarrow K[\overline{G}] \) and
\[
(3.18) \quad \tilde{x}_j = \tau(\overline{x}_j) \neq 0.
\]

Finally, let \( C \) be the inverse image of the ideal \( \overline{C} \) in \( KG \). Clearly,
\[
(KG)/C \simeq (K[\overline{G}])/\overline{C} \simeq K[\tilde{G}].
\]

Furthermore, the natural homomorphism
\[
\alpha: KG \rightarrow (KG)/C \simeq K[\tilde{G}]
\]
is a composition of two homomorphisms \( \beta \) and \( \gamma \), which are extended to specializations \( \pi \) and \( \tau \) correspondingly. We obtain from this (see [8, Chapter 6] or [11]) that \( \alpha \) can be extended to a specialization \( \tau \pi = \theta: \overline{R} \rightarrow K[\tilde{G}] \),
whose domain contains the elements (3.14). The assertion follows now from (3.16) and (3.18).

4

We will need in the proof of Theorem 3 the following fact:

**Lemma 3.** Let $D$ be a field, $x$ be a given matrix from $D_{n \times n}$. Assume that $D$ has a system of subrings $T_i$ $(i \in I)$ such that $x \in (T_i)_{n \times n}$ for all $i \in I$ and

1. given any finite set of elements $M \subseteq D$ there is a $T_i$ with $M \subseteq T_i$,
2. each $T_i$ has an ideal $U_i \not\subseteq T_i$ such that the image of the matrix $x$ in the quotient ring

$$(T_i)_{n \times n}/(U_i)_{n \times n} \simeq (T_i/U_i)_{n \times n}$$

is nilpotent. Then the matrix $x$ is nilpotent.

**Proof.** Assume that $x$ is not nilpotent and hence

$$(4.1) \quad x^n \neq 0.$$  

The powers of $x$ are linearly dependent over $D$; there exists therefore elements

$0 \neq d_j \in D$ $(j = 1, 2, \ldots, r)$ such that

$$
(4.2) \quad \sum_{j=1}^r d_j x^{n_j} = 0 \quad (1 \leq n_1 < n_2 < \cdots < n_r \leq n^2 + 1).
$$

Find in the system of subrings $T_i$ $(i \in I)$ a subring $T$ and its ideal $U \subseteq T$ such that $T$ contains all the elements $d_j, d_j^{-1}$ $(j = 1, 2, \ldots, r)$, all the nonzero entries of the matrix $x$ (and $x^n$) and the inverses of these entries. Since all these elements are invertible in $T$ and $U \neq T$, their images in $T/U$ are nonzero. Let $\overline{X}$ denote the image of a subset $X \subseteq T_{n \times n}$ under the homomorphism $(T)_{n \times n} \to (T/U)_{n \times n}$. We see that the elements $\overline{d_j}$ are invertible in $(T/U)_{n \times n}$, the element $\overline{x}$ is nilpotent but

$$
(4.3) \quad \overline{x}^n \neq 0,
$$

and

$$(4.2') \quad \sum_{j=1}^r \overline{d_j} \overline{x}^{n_j} = 0.$$

Now let $k$ be the smallest natural number such that $\overline{x}^k = 0$. It follows from (4.3) that $k > n$. We multiply (4.2') on the right by $\overline{x}^{k-n_1-1}$ and obtain that $\overline{d}_1 \overline{x}^{k-1} = 0$. Since $\overline{d}_1$ is invertible we see that $\overline{x}^{k-1} = 0$ which contradicts (4.3). Thus assumption (4.1) leads to a contradiction, i.e., $x$ is nilpotent.

The following fact is known (see [13, Lemma II.5.4]).

**Lemma 4.** Let $U$ be a finite-dimensional algebra over a field $K$ of characteristic zero, $Z$ be its center. Then the intersection $[U, U] \cap Z$ is a nilpotent ring.

We can now prove our main result.

**Theorem 3.** Let $G$ be a polycyclic-by-finite group, $K$ be a field of characteristic zero and $R$ be the ring of fractions of $K G$. Let $S$ be a subring of the matrix ring $R_{n \times n}$, $Z$ be its center. Then the intersection $[S, S] \cap Z$ is a nilpotent ring.

**Proof.** In order to prove Theorem 3 it is enough to prove that the ring $[S, S] \cap Z$ is nil because a nil subring of a matrix ring over the artinian ring $R$ must be nilpotent.
Let thus \( z \in ([S, S] \cap \mathbb{Z}) \), where \( S \subseteq R_{m \times m} \). There exist therefore elements \( u_i, v_i \in S \ (i = 1, 2, \ldots, r) \) such that

\[
\sum_{i=1}^{r} [u_i, v_i] = z.
\]

Pick in \( R \) an arbitrary finite subset which has a form

\[
x_1, x_2, \ldots, x_k; \quad x_1^{-1}, x_2^{-1}, \ldots, x_k^{-1}
\]

and contains all the nonzero entries of the matrices \( u_i, v_i \ (i = 1, 2, \ldots, r) \) (and of \( z \)). Apply Theorem 2 and find a subring \( T \subseteq R \) and an ideal \( U \subseteq T \) such that elements (4.5) belong to \( T \) and \( T/\mathbb{Z} \approx K[\tilde{G}] \), where \( K[\tilde{G}] \) is a finite-dimensional algebra over \( K \). Relation (4.4) implies the following relation in \( (T/U)_{m \times m} \) for the images of the elements \( u_i, v_i, z \ (i = 1, 2, \ldots, r) \):

\[
\sum_{i=1}^{r} [\bar{u}_i, \bar{v}_i] = \bar{z}.
\]

Since the element \( \bar{z} \) commutes with all the elements \( \bar{u}_i, \bar{v}_i \ (i = 1, 2, \ldots, r) \), we obtain from Lemma 4 that \( \bar{z} \) is nilpotent. Lemma 3 now implies that \( z \) is nilpotent which completes the proof of Theorem 3.

**Corollary 1.** Let the subring \( S \) in Theorem 3 be semiprime. Then \( [S, S] \cap \mathbb{Z} = 0 \).

Now let \( G \) be a residually torsion-free nilpotent group, \( K \) be a commutative field. Let

\[
G = N_1 \supseteq N_2 \supseteq \cdots
\]

be a series of normal subgroups in \( G \) such that every quotient group \( G/N_i \ (i = 1, 2, \ldots, \) is torsion-free nilpotent and \( \bigcap_{i=1}^{\infty} N_i = 1 \). It is not difficult to define in \( G \) an order such that all the homomorphisms \( G \to G/N_i \) are homomorphisms of ordered groups (see [14]). Let \( K(G) \) be the appropriate Malcev-Neumann power series ring and \( \Delta \) be its subfield, generated by the group ring. We will give now a sketch of proof of the following result.

**Proposition 2.** (i) If \( \text{char } K = 0 \) then the conclusion of Theorem 3 is valid for an arbitrary subring \( S \subseteq \Delta_{n \times n} \).

(ii) If \( K \) has an arbitrary characteristic then

\[
1 \notin [\Delta, \Delta].
\]

**Proof.** Let \( \Delta_i \) be the field of fractions of the group ring \( K(G/N_i) \). The results of [14] imply that for every given \( i \) there exists a specialization \( \theta_i : \Delta \to \Delta_i \), extending the natural homomorphism \( G \to G/N_i \) and that for every given elements of \( D \),

\[
x_1, x_2, \ldots, x_k; \quad x_1^{-1}, x_2^{-1}, \ldots, x_k^{-1}
\]

an index \( i_0 \) can be found such that for every \( i \geq i_0 \) these elements belong to the domain \( T_i \) of the specialization \( \theta_i \). Since Theorem 3 holds for the subrings of \( (\Delta_i)_{n \times n} \) we obtain now easily from Lemma 3 the statement (i).
We prove now (ii). A routine argument reduces the proof to the case when the group $G$ is finitely generated; we can assume also that the field $K$ is algebraically closed. Assume that $1 \in [\Delta, \Delta]$, i.e. there exist nonzero elements $u_j, v_j \in \Delta$ $(j = 1, 2, \ldots, s)$ such that

$$1 = \sum_{j=1}^{s} [u_j, v_j]. \tag{4.7}$$

Apply Proposition 2.8 in [15] and find a specialization $\pi: \Delta \to K[\tilde{G}]$ such that $K[\tilde{G}]$ is a simple algebra generated by a finite $q$-group $\tilde{G}$ where $q$ is an arbitrary prime number unequal to $\text{char } K$ and the domain $T$ of $\pi$ contains all the elements $u_j, v_j$ from (4.7). The relation (4.7) now yields the following relation in $K[\tilde{G}]$,

$$\hat{1} = \sum_{j=1}^{s} [\hat{u}_j, \hat{v}_j]. \tag{4.7'}$$

Since $K[\tilde{G}]$ is a simple algebra over an algebraically closed field $K$ and $q \neq \text{char } K$ we obtain that $K[\tilde{G}]$ is isomorphic to a matrix algebra of degree $q^m$ over $K$. The relation (4.7') however is impossible in the algebra $K_{q^m \times q^m}$ since the trace of the right side is zero whereas $T_r(\hat{1}) = q^m \neq 0$. This completes the proof.

Since free groups and free soluble groups are residually torsion-free nilpotent, we obtain that Proposition 2 is valid for the universal field of fractions of free group rings or for Ore fields of fractions of group rings of free soluble groups.

The truth of (4.6) for a ring of fractions $R$ of a ring $(KG)/P$, where $G$ is a finitely generated nilpotent group, char $K = 0$ and $P$ is a prime ideal of $KG$, was established by M. Lorenz in [16].

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REFERENCES


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