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# ON THE GENUS OF SMOOTH 4-MANIFOLDS

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ABSTRACT. The projective complex plane and the "twisted" $S^3$ bundle over $S^1$ are proved to be the unique closed prime connected (smooth or PL) 4-manifolds of genus two. Then the classification of the nonorientable 4-manifolds of genus 4 is given. Finally the genus of a manifold $M$ is shown to be related with the 2nd Betti number of $M$ and some applications are proved in the general (resp. simply-connected) case.

## 1. INTRODUCTION

Throughout the paper, we consider smooth or combinatorial (PL) closed 4-manifolds indifferently. In fact, a well-known result states that if $M$ is a PL 4-manifold, then $M$ possesses a $C^\infty$-differentiable structure compatible with the triangulation (see [20]).

For the topology of 4-manifolds we refer to [9, 15].

We recall the definition of (regular) genus for a closed (PL) $n$-manifold as introduced in [10].

An $(n+1)$-*colored graph* is a pair $(G, c)$, where $G = (V, E)$ is a finite multigraph, regular of degree $n+1$, and $c: E \to \Delta_n = \{0, 1, \dots, n\}$ is an edge-coloration on $G$ with $n+1$ colors (i.e., $c(e) \neq c(f)$ for any pair of adjacent edges $e, f \in E$). The graph $(G, c)$ is said to be *contracted* if the partial subgraph $G_{\hat{i}} = (V, c^{-1}(\Delta_n - \{i\}))$ is connected for each color $i \in \Delta_n$. An $n$-pseudocomplex (see [14]) $K = K(G)$ can be uniquely associated with $(G, c)$ so that $|G|$ becomes its dual 1-skeleton (see [5]). A *crystallization* of a closed (PL) $n$-manifold $M$ is a contracted $(n+1)$-colored graph $(G, c)$ which *represents* $M$, i.e. $|K(G)| \simeq_{\mathrm{PL}} M$.

It is well known that each closed connected (PL) $n$-manifold admits a crystallization as proved in [17].

Given a crystallization $(G, c)$, the minimum genus of a closed (connected) surface into which $(G, c)$ regularly imbeds (see [19]) is denoted by $g(G)$. The *regular genus* (or simply called the *genus*) of a closed (PL) $n$-manifold $M$ is

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defined as the nonnegative integer (see [10])

$$g(M) = \min\{g(G)/(G, c) \text{ is a crystallization of } M\}.$$

Given a closed smooth connected 4-manifold $M$, the following results are well known. If $g(M) = 0$, then $M \simeq_{\mathrm{PL}} S^4$ (4-sphere) (see [6]). If $g(M) = 1$, then $M \simeq_{\mathrm{PL}} S^1 \times S^3$ (see [2]). If $\Pi_1(M)$ is the fundamental group of $M$, then $g(M) \geq \operatorname{rank} \Pi_1(M)$ (see [1]). If $M$ is nonorientable, then $g(M)$ is even (see [10]). In [7] bounds were determined for the genus of any 4-manifold which is a product of $S^1$ by a closed 3-manifold or a product of two closed surfaces.

*Remark.* Let $(G_i, c_i)$ be a crystallization of a closed connected $n$-manifold $M_i$ $(i = 1, 2)$, and let $f_i \colon |G_i| \to S_i$ be a regular imbedding of $G_i$ into the closed connected surface $S_i$ or genus $g(M_i)$. By direct construction it is very easy to obtain a regular imbedding of $G_1 \# G_2$ into the surface $S_1 \# S_2$. If $M_1$ and $M_2$ are both orientable (resp. nonorientable), then the genus is subadditive, i.e. $g(M_1 \# M_2) \leq g(M_1) + g(M_2)$, since the genus of $S_1 \# S_2$ is exactly $g(M_1) + g(M_2)$. If $M_1$ is orientable and $M_2$ is nonorientable, then we have $g(M_1 \# M_2) \leq 2g(M_1) + g(M_2)$ since the genus of $S_1 \# S_2$ is just $2g(M_1) + g(M_2)$.

Now we state the main results of the present paper:

**Proposition 1.** *Let $M^4$ be a smooth (or PL) closed connected 4-manifold of genus $g$.*

(a) *If $g = 2$ and $M$ is orientable, then $M$ is (PL) homeomorphic to either the projective complex plane $CP^2$ or the connected sum $\#_2(S^1 \times S^3)$.*

(b) *If $g = 2$ and $M$ is nonorientable, then $M$ is (PL) homeomorphic to $S^1 \underset{\sim}{\times} S^3$ (the "twisted" $S^3$ bundle over $S^1$).*

(c) *If $g = 4$ and $M$ is nonorientable, then $M$ is (PL) homeomorphic to either $\#_2(S^1 \underset{\sim}{\times} S^3)$ or $(S^1 \times S^3) \# (S^1 \underset{\sim}{\times} S^3)$.*

**Corollary 1.**

$$g(RP^4) = g\left(\underset{2}{\#}(S^1 \times S^3) \# (S^1 \underset{\sim}{\times} S^3)\right) = g(\pm CP^2 \# S^1 \underset{\sim}{\times} S^3) = 6,$$

*where $RP^4$ denotes the real projective 4-dimensional space.*

The formulae, used in the proof of Proposition 1, imply the following

**Proposition 2.** *Let $M^4$ be a closed connected smooth (or PL) orientable 4-manifold of genus $g$. Then $b_2(M) \leq [(5/2)g]$. If $M$ is simply-connected, then $b_2(M) \leq [g/2]$. Here $[x]$ and $b_k(M)$ denote the integer part of $x$ and the $k$th Betti number of $M$ respectively.*

**Corollary 2.**

$$g\left(\underset{k}{\#}(S^1 \times S^3)\right) = k, \tag{1}$$

$$g\left(\underset{k}{\#} \pm CP^2\right) = 2k, \tag{2}$$

$$g\left(\underset{k}{\#}(S^2 \times S^2)\right) = 4k. \tag{3}$$

Now we can apply the Freedman classification of simply-connected smooth 4-manifolds (see [9]) to obtain the following consequence of Proposition 2:

**Proposition 3.** *If $M^4$ is a closed simply-connected smooth 4-manifold of genus $g \leq 31$, then we have either*

$$M \underset{\mathrm{TOP}}{\cong} \#_r(\pm CP^2) \quad or \quad M \underset{\mathrm{TOP}}{\cong} \#_r(S^2 \times S^2)$$

*where* $r = b_2(M)$.

Other applications of these results and some open questions related with the 4-dimensional Poincaré conjecture complete the paper.

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## 2. Proof of Proposition 1: The orientable case

In order to prove Proposition 1.a, we recall some constructions and results given in [2].

Let $M$ be a closed connected orientable smooth (or PL) 4-manifold of genus $g$. Let $(G, c)$ be a crystallization of $M$ and $\{v_i | i \in \Delta_4\}$ the vertex-set of $K = K(G)$. If $\{i, j\} = \Delta_4 - \{r, s, t\}$, then $K(i, j)$ (resp. $K(r, s, t)$) represents the subcomplex of $K$ generated by the vertices $v_i$ and $v_j$ (resp. $v_r$, $v_s$ and $v_t$). By $g_{rst}$ (resp. $g_{ij}$) we denote the number of edges (resp. triangles) of $K(i, j)$ (resp. $K(r, s, t)$). It is very easy to see that $g_{rst}$ and $g_{ij}$ also represent the numbers of components of the subgraphs $G_{\{r,s,t\}}$ and $G_{\{i,j\}}$ respectively.

Here $G_B$ denotes the subgraph $(V, c^{-1}(B))$ for any subset $B$ of $\Delta_4$.

If $\mathrm{Sd}\,K$ is the first barycentric subdivision of $K$, let $H(i, j)$ be the largest subcomplex of $\mathrm{Sd}\,K$ disjoint from $\mathrm{Sd}\,K(i, j) \cup \mathrm{Sd}\,K(r, s, t)$. Then the polyhedron $|H(i, j)|$ is a closed orientable 3-manifold which splits $M$ into two complementary 4-manifolds $N(i, j)$ and $N(r, s, t)$ with common boundary. Further $N(i, j)$ and $N(r, s, t)$ are regular neighborhoods in $M$ of $|\mathrm{Sd}\,K(i, j)|$ and $|\mathrm{Sd}\,K(r, s, t)|$ respectively.

Following [2], we can always assume that $(G, c)$ regularly imbeds into the closed orientable surface of genus $g$ and of Euler characteristic $2 - 2g = g_{01} + g_{12} + g_{23} + g_{34} + g_{40} - 3p$, where $p$ is the order of $G$ divided by 2.

As proved in [2], we have

$$\begin{array}{llll}
(1) & g_{013} = 1 + g - g_{\hat{2}} - g_{\hat{4}}, & (6) & g_{14} = g_{014} + g - g_{\hat{0}}, \\
(2) & g_{023} = 1 + g - g_{\hat{1}} - g_{\hat{4}}, & (7) & g_{02} = g_{012} + g - g_{\hat{1}}, \\
(3) & g_{024} = 1 + g - g_{\hat{1}} - g_{\hat{3}}, & (8) & g_{13} = g_{123} + g - g_{\hat{2}}, \\
(4) & g_{124} = 1 + g - g_{\hat{0}} - g_{\hat{3}}, & (9) & g_{24} = g_{234} + g - g_{\hat{3}}, \\
(5) & g_{134} = 1 + g - g_{\hat{0}} - g_{\hat{2}}, & (10) & g_{03} = g_{034} + g - g_{\hat{4}},
\end{array}$$

$$(11) \quad \chi(M) = 2 - 2g + \sum_i g_{\hat{i}}$$

where $g_{\hat{i}}$ $(0 \leq g_{\hat{i}} \leq g)$ is the genus of an orientable closed surface into which the subgraph $G_{\hat{i}}$ $(i \in \Delta_4)$ regularly imbeds and $\chi(M)$ is the Euler characteristic of $M$.

If $g = 2$, then the sum

$$R = g_{013} + g_{023} + g_{024} + g_{124} + g_{134} = 5 + 5g - 2\sum_i g_{\hat{i}}$$

belongs to the set $\{5, 7, 9, 11, 13, 15\}$ since $R$ $(R \geq 5)$ is odd.

Now we show that the cases $R \in \{9, 13, 7, 11\}$ give a contradiction, while the cases $R = 5$ and $R = 15$ imply that $M$ is (PL) homeomorphic to the complex projective plane and the connected sum $\#_2(S^1 \times S^3)$ respectively

*Case* $R = 9$. If $R = 9$, then $\sum_i g_{\hat{i}} = 3$ and the manifold $M$ is simply-connected because at least one of the above $g_{ijk}$'s equals 1 (see [11] for the generators of the fundamental group of $M$). This implies that $\chi(M) = 2 + b_2(M) \geq 2$, i.e. a contradiction since $\chi(M) = 1$ by (11).

*Case* $R = 13$. If $R = 13$, then $\sum_i g_{\hat{i}} = 1$ and $\chi(M) = -1$. The fundamental group of $M$ is trivial or cyclic since at least one of the above $g_{ijk}$'s must be $\leq 2$ (see [11]). The inequality $b_1(M) \leq 1$ implies that $\chi(M) = 2 - 2b_1(M) + b_2(M) \geq b_2(M) \geq 0$, i.e. a contradiction.

*Case* $R = 7$. If $R = 7$, then $\sum_i g_{\hat{i}} = 4$ and $\chi(M) = 2$. Because at least one of the above $g_{ikj}$'s equals 1, the manifold $M$ is simply-connected, whence $\chi(M) = 2$ implies that $b_2(M) = 0$.

Now the addendum of the sum $R$ may assume the values listed in Table 1.

The fifteen cases to verify can be reduced to three by cyclic permutations in the color set. In fact, doing this type of change of names in the color set $\Delta_4$ the permutation of $\Delta_4$ giving regular imbedding of $G$ does not change. Thus we can only examine the cases 7.1, 7.6 and 7.7.

(n. 7.1) If $g_{023} = g_{024} = g_{124} = g_{134} = 1$ and $g_{013} = 3$, then the relations (1), ..., (5) imply that $g_{\hat{0}} = g_{\hat{1}} = 2$ and $g_{\hat{2}} = g_{\hat{3}} = g_{\hat{4}} = 0$. By (6), ..., (10) we obtain $g_{14} = g_{014}$, $g_{02} = g_{012}$, $g_{13} = g_{123} + 2$, $g_{24} = g_{234} + 2$ and $g_{03} = g_{034} + 2$. Since $g_{134} = 1$, then $K(0, 2)$ consists of exactly one edge, hence $N(0, 2)$ is a 4-ball. Furthermore $K(1, 3)$ and $K(1, 4)$ are also formed by one edge, each one as $g_{024} = g_{023} = 1$. Thus all triangles of $K(1, 3, 4)$ have two edges of $K(1, 3)$ and $K(1, 4)$ in common. The other edge of each triangle of $K(1, 3, 4)$ is free since $g_{02} = g_{012}$. Therefore $K(1, 3, 4)$ is a cone which collapses to a point, i.e. $N(1, 3, 4)$ is a 4-ball. This implies that $M$ is (PL) homeomorphic to the 4-sphere $S^4$ which is a contradiction since $g(S^4) = 0$.

(n. 7.6) If $g_{013} = g_{023} = 2$ and $g_{024} = g_{124} = g_{134} = 1$, then $g_{\hat{0}} = g_{\hat{1}} = g_{\hat{2}} = g_{\hat{3}} = 1$, $g_{\hat{4}} = 0$, $g_{14} = g_{014} + 1$, $g_{02} = g_{012} + 1$, $g_{13} = g_{123} + 1$, $g_{24} = g_{234} + 1$ and $g_{03} = g_{034} + 2$. The manifold $N(1, 3)$ is a 4-ball since $g_{024} = 1$. Further $K(0, 2)$ (resp. $K(2, 4)$) consists of exactly one (resp. two) edge since $g_{134} = 1$ (resp. $g_{013} = 2$). Because $g_{13} = g_{123} + 1$, the pseudocomplex $K(0, 2, 4)$ contains many triangles but one as there are edges in $K(0, 4)$. The two triangles of $K(0, 2, 4)$, which have two edges $\langle v_2, v_4\rangle$ and $\langle v_0, v_4\rangle$ in common, cannot have the same boundary since $FH_2(M) \simeq 0$ and $H_2(N(1, 3)) \simeq 0$ imply that $H_2(N(0, 2, 4)) \simeq 0$ (use the Mayer-Vietoris sequence of the pair $(N(1, 3)$, $N(0, 2, 4))$. Thus $K(0, 2, 4)$ collapses to a point, i.e. $N(0, 2, 4)$ is (PL) homeomorphic to a 4-ball. This gives a contradiction since $M \simeq_{\mathrm{PL}} S^4$ (4-sphere) and $g(M) = 2$.

(n. 7.7) If $g_{013} = g_{024} = 2$ and $g_{023} = g_{124} = g_{134} = 1$, then we have $g_{\hat{0}} = 2$, $g_{\hat{2}} = g_{\hat{3}} = 0$, $g_{\hat{1}} = g_{\hat{4}} = 1$, $g_{14} = g_{014}$, $g_{02} = g_{012} = 1$, $g_{03} = g_{034} = 1$,

TABLE 1

| n. | $g_{013}$ | $g_{023}$ | $g_{024}$ | $g_{124}$ | $g_{134}$ |
|---|---|---|---|---|---|
| 7.1 | 3 | 1 | 1 | 1 | 1 |
| 7.2 | 1 | 3 | 1 | 1 | 1 |
| 7.3 | 1 | 1 | 3 | 1 | 1 |
| 7.4 | 1 | 1 | 1 | 3 | 1 |
| 7.5 | 1 | 1 | 1 | 1 | 3 |
| 7.6 | 2 | 2 | 1 | 1 | 1 |
| 7.7 | 2 | 1 | 2 | 1 | 1 |
| 7.8 | 2 | 1 | 1 | 2 | 1 |
| 7.9 | 2 | 1 | 1 | 1 | 2 |
| 7.10 | 1 | 2 | 2 | 1 | 1 |
| 7.11 | 1 | 2 | 1 | 2 | 1 |
| 7.12 | 1 | 2 | 1 | 1 | 2 |
| 7.13 | 1 | 1 | 2 | 2 | 1 |
| 7.14 | 1 | 1 | 2 | 1 | 2 |
| 7.15 | 1 | 1 | 1 | 2 | 2 |

$g_{13} = g_{123} + 2$ and $g_{24} = g_{234} + 2$. Because $g_{023} + 1$ and $(g_{124} = g_{134} + 1$, $g_{14} = g_{014})$, the contradiction follows as in the case n. 7.1 by using the pair $(N(1, 4),\ N(0, 2, 3))$.

*Case* $R = 11$. If $R = 11$, then $\sum_i g_{\hat{i}} = 2$ and $\chi(M) = 0$. The manifold $M$ cannot be simply-connected since $\chi(M) = 0$. Thus we have $g_{ijk} \geq 2$ for any $i, j, k$, whence the addendum of $R$ may assume the values listed in Table 2.

TABLE 2

| n. | $g_{013}$ | $g_{023}$ | $g_{024}$ | $g_{124}$ | $g_{134}$ |
|---|---|---|---|---|---|
| 11.1 | 2 | 2 | 2 | 2 | 3 |
| 11.2 | 2 | 2 | 2 | 3 | 2 |
| 11.3 | 2 | 2 | 3 | 2 | 2 |
| 11.4 | 2 | 3 | 2 | 2 | 2 |
| 11.5 | 3 | 2 | 2 | 2 | 2 |

All these cases are equivalent to one by the same reason that $R = 7$.

The fundamental group of $M$ is cyclic and nontrivial since $b_1(M) \neq 0$ and at least one of the above $g_{ijk}$'s equals 2. This implies that $H_1(M) \simeq H_3(M) \simeq Z$ and $FH_2(M) \simeq 0$ (use $\chi(M) = 0$).

(n. 11.1) If $g_{013} = g_{023} = g_{024} = g_{124} = 2$ and $g_{134} = 3$, then we have $g_{\hat{0}} = g_{\hat{1}} = g_{\hat{2}} = 0$, $g_{\hat{3}} = g_{\hat{4}} = 1$, $g_{03} = g_{034} + 1$, $g_{24} = g_{234} + 1$, $g_{13} = g_{123} + 2$, $g_{02} = g_{012} + 2$ and $g_{14} = g_{014} + 2$ (use (1), ..., (10)). Since $g_{013} = 2$, then $K(2, 4)$ is formed by two vertices joined by exactly two edges, whence $N(2, 4)$ is (PL) homeomorphic to $S^1 \times B^3$. Here $B^3$ represents a (closed) 3-ball. Furthermore $K(1, 3)$ and $K(0, 3)$ consist of exactly two edges each one as $g_{124} = g_{024} = 2$. Because $g_{24} = g_{234} + 1$, the pseudocomplex $K(0, 1, 3)$ contains many triangles but one as there are edges in $K(0, 1)$. However any two triangles of $K(0, 1, 3)$ cannot have the same boundary since $FH_2(M) \simeq 0$ and $H_2(N(2, 4)) \simeq 0$ imply that $H_2(N(0, 1, 3)) \simeq 0$ (use the Mayer-Vietoris sequence of the pair $(N(2, 4), N(0, 1, 3))$. Thus $K(0, 1, 3)$ collapses to a circle, whence $N(0, 1, 3)$ is (PL) homeomorphic to $S^1 \times B^3$. By [16, Theorem 2], we have that $M \simeq_{\mathrm{PL}} S^1 \times S^3$ which is a contradiction since $g(S^1 \times S^3) = 1$ by [7].

*Case* $r = 15$: *The connected sum.* $\#_2(S^1 \times S^3)$. If $R = 15$, then it follows that $g_{013} = g_{023} = g_{024} = g_{124} = g_{134} = 3$ and $g_{\hat{i}} = 0$ for each $i \in \Delta_4$. The rank of $\Pi_1(M)$ is $\leq 2$ as each one of the above $g_{ijk}$'s equals 3. This implies that $b_1(M) = b_3(M) = 2$ and $b_2(M) = 0$ since $\chi(M) = -2$. Because $g_{024} = 3$, the pseudocomplex $K(1, 3)$ consists of exactly three edges, hence $N(1, 3)$ is (PL) homeomorphic to the connected sum $\#_2(S^1 \times B^3)$. Furthermore $K(0, 2)$ and $K(2, 4)$ are also formed by three edges each one as $g_{134} = g_{013} = 3$. Because $g_{13} = g_{123} + 2$ (use (8)), the complex $K(0, 4)$ has many edges but two as there are triangles in $K(0, 2, 4)$. Since $H_1(M) \simeq H_3(M) \simeq Z \oplus Z$, $FH_2(M) \simeq 0$ and $H_2(N(1, 3)) \simeq 0$, the Mayer-Vietoris sequence of the pair $(N(1, 3), N(0, 2, 4))$ gives

$$0 \longrightarrow Z \oplus Z \longrightarrow Z \oplus Z \longrightarrow H_2(N(0, 2, 4)) \longrightarrow 0,$$

whence $H_2(N(0, 2, 4)) \simeq 0$, i.e. there are no two triangles of $K(0, 2, 4)$ with common boundary. Now it is very easy to see that the pseudocomplex $K(0, 2, 4)$ collapses to a one-dimensional subcomplex formed by two vertices joined by three edges. Therefore $N(0, 2, 4)$ is (PL) homeomorphic to the connected sum $\#_2(S^1 \times B^3)$.

Thus $M \simeq_{\mathrm{PL}} \#_2(S^1 \times S^3)$ by Theorem 2 of [16]. Now the result in Proposition 1 follows as $g(\#_2(S^1 \times S^3)) = 2$ by [7] and the subadditivity of the genus.

*Case* $R = 5$: *The projective complex plane.* If $R = 5$, then $g_{013} = g_{023} = g_{024} = g_{124} = g_{134} = 1$ and $\chi(M) = 3$. By (1), ..., (5) it follows that $g_{\hat{1}} = 1$ for each $i \in \Delta_4$. In fact, suppose on the contrary that $g_{\hat{1}} = 2$ (recall that $g_{\hat{i}} \leq 2$). Then we have $g_{\hat{0}} = g_{\hat{2}} = 2$ and $g_{\hat{3}} = g_{\hat{4}} = 0$. This is a contradiction since $\sum_i g_{\hat{i}} = 5$.

Furthermore, if $g_{\hat{1}} = 0$, then $g_{\hat{0}} = g_{\hat{2}} = 0$ and $g_{\hat{3}} = g_{\hat{4}} = 2$ which give a contradiction as shown above.

Therefore we must have $g_{\hat{1}} = 1$, whence $g_{\hat{0}} = g_{\hat{2}} = g_{\hat{3}} = g_{\hat{4}} = 1$. By

(6), ... , (10) the relations $g_{14} = g_{014} + 1$, $g_{02} = g_{012} + 1$, $g_{13} = g_{123} + 1$, $g_{24} = g_{234} + 1$, and $g_{03} = g_{034} + 1$ hold. Since $g_{024} = 1$, then $K(1, 3)$ consists of exactly one edge, hence $N(1, 3)$ is a 4-ball. Further $K(0, 2)$ and $K(2, 4)$ are also formed by one edge each one since $g_{134} = g_{013} = 1$. Thus all triangles of $K(0, 2, 4)$ have two edges in common. Because $g_{13} = g_{123} + 1$, the complex $K(0, 2, 4)$ has many triangles but one as there are edges in $K(0, 4)$.

Therefore $K(0, 2, 4)$ collapses to a combinatorial 2-sphere, formed by exactly two triangles $T_1$, $T_2$ of $K(0, 2, 4)$, with common boundary. Then we have that $H_0(N(0, 2, 4)) \simeq H_2(N(0, 2, 4)) \simeq Z$ and $H_1(N(0, 2, 4)) \simeq H_3(N(0, 2, 4)) \simeq 0$. By isotopy we can always suppose that $T_1$ is the standard 2-simplex in $M$. Let $\widehat{T}_1$ be the barycenter of $T_1$ and $\mathrm{Sd}^2 K$ be the second barycentric subdivision of $K = K(G)$. Then $N(0, 2, 4)$ is the orientable bordered 4-manifold obtained by adding a 2-handle (a regular neighborhood of $\widehat{T}_1$ in $\mathrm{Sd}^2 K$) onto the boundary of a 4-ball (a small regular neighborhood of $T_2$ in $M$) along a knot $N$. Since the surgery is given by attaching 2-handles in dimension 4, the surgery coefficient associated to $N$ must be an integer and by homological reasons equal to $\pm 1$. Therefore $\partial N(0, 2, 4) = S^3$ so by Gordon and Luecke[1], Theorem 2, $N$ is the trivial knot and the manifold $N(0, 2, 4)$ is (PL) homeomorphic to $\pm CP^2$–(4-ball), whence $M$ is the projective complex plane as required. Now the proof is completed because a crystallization of $CP^2$ with genus 2 is really constructed in [12].

## 3. Proof of Proposition 1: The nonorientable case

Let $M$ be a closed smooth (or PL) nonorientable connected 4-manifold of genus $h$. As proved in the orientable case, we can obtain the following relations:

$$
\begin{aligned}
g_{013} &= 1 + h/2 - g_{\hat{2}} - g_{\hat{4}}, & g_{14} &= g_{014} + h/2 - g_{\hat{0}},\\
g_{023} &= 1 + h/2 - g_{\hat{1}} - g_{\hat{4}}, & g_{02} &= g_{012} + h/2 - g_{\hat{1}},\\
g_{024} &= 1 + h/2 - g_{\hat{1}} - g_{\hat{3}}, & g_{13} &= g_{123} + h/2 - g_{\hat{2}},\\
g_{124} &= 1 + h/2 - g_{\hat{0}} - g_{\hat{3}}, & g_{24} &= g_{234} + h/2 - g_{\hat{3}},\\
g_{134} &= 1 + h/2 - g_{\hat{0}} - g_{\hat{2}}, & g_{03} &= g_{034} + h/2 - g_{\hat{4}},
\end{aligned}
$$

$$\chi(M) = 2 - h + \sum_i g_{\hat{i}}.$$

(Statement 1.b). If $h = 2$, then the sum

$$R = g_{013} + g_{023} + g_{024} + g_{124} + g_{134} = 10 - 2\sum_i g_{\hat{i}}$$

is even, hence $R$ belongs to the set $\{6, 8, 10\}$ as $R \geq 6$ and $g_{\hat{i}} \geq 0$. Now we show that the cases $R = 6$ and $R = 8$ give a contradiction, while the case $R = 10$ implies that the manifold $M$ is (PL) homeomorphic to the "twisted" $S^3$ bundle over $S^1$, i.e. $M \simeq_{\mathrm{PL}} S^1 \underset{\sim}{\times} S^3$. If $R = 6$ or $R = 8$, then at least one of the above $g_{ijk}$'s equals 1. Thus the fundamental group of $M$ is trivial so that $M$ is orientable, i.e. a contradiction.

If $R = 10$, then we have $\chi(M) = 0$ and $g_{\hat{i}} = 0$ for each $i \in \Delta_4$. Furthermore the relations $g_{124} = g_{013} = g_{023} = g_{024} = g_{134} = 2$ imply that

[1] C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, Bull. Amer. Math. Soc. (N.S.) **20** (1989), 83–89.

$\Pi_1(M) \simeq H_1(M) \simeq Z$. Since $g_{024} = 2$, then $K(1,3)$ consists of two vertices joined by two edges, hence $N(1,3) \simeq_{\mathrm{PL}} (S^1 \times B^3)$ or $S^1 \underset{\sim}{\times} B^3$ (the "twisted" $B^3$ bundle over $S^1$) and $\partial N(1,3) \simeq_{\mathrm{PL}} S^1 \times S^2$ or $S^1 \underset{\sim}{\times} S^2$ respectively. The Mayer-Vietoris sequence of the pair $(N(1,3), N(0,2,4))$ gives

$$\begin{aligned} 0 &\longrightarrow FH_3(M) \longrightarrow FH_2(\partial N(1,3)) \simeq Z \longrightarrow H_2(N(0,2,4)) \\ &\longrightarrow H_2(M) \longrightarrow Z \longrightarrow Z \oplus H_1(N(0,2,4)) \longrightarrow Z \longrightarrow 0, \end{aligned}$$

whence $H_2(N(0,2,4)) \simeq 0$ since $FH_3(M) \simeq Z$ and $FH_2(M) \simeq H_2(M) \simeq 0$ (note that $H_1(N(0,2,4)) \simeq Z$). In this case the relations $g_{14} = g_{014}+1$, $g_{02} = g_{012}+1$, $g_{13} = g_{123}+1$, $g_{24} = g_{234}+1$ and $g_{03} = g_{034}+1$ hold too. Now the formulae $g_{013} = g_{134} = 2$ and $g_{13} = g_{123}+1$ imply that $K(0,2,4)$ collapses to a circle since $H_2(K(0,2,4)) \simeq 0$. Therefore the manifold $N(0,2,4)$ is (PL) homeomorphic to either $S^1 \times B^3$ or $S^1 \underset{\sim}{\times} B^3$. Since $M$ is nonorientable, it follows that $M \simeq_{\mathrm{PL}} S^1 \underset{\sim}{\times} S^3$ by Theorem 2 of [16]. Now the proof is completed because a crystallization of $S^1 \underset{\sim}{\times} S^3$ with genus 2 is shown in [13, Figure 1, p. 155].

(Statement 1.c) By the "subadditivity" of the genus we have that

$$g\left(\underset{2}{\#}(S^1 \underset{\sim}{\times} S^3)\right) \leq 2g(S^1 \underset{\sim}{\times} S^3) = 4$$

and $g(S^1 \times S^3 \# S^1 \underset{\sim}{\times} S^3) \leq 2g(S^1 \times S^3) + g(S^1 \underset{\sim}{\times} S^3) = 4$. If $h = 4$, then the sum $R = 15 - 2\sum_i g_{\hat{i}}$ is odd, whence $R \in \{5, 7, 9, 11, 13, 15\}$. We show that the cases $R \in \{5, 7, 9, 11, 13\}$ give a contradiction while the case $R = 15$ implies the statement (c) in Proposition 1.

If $R \in \{5, 7, 9\}$, then at least one of the $g_{ijk}$'s in $R$ must be one, so that $\Pi_1(M) = 0$ and $M$ is orientable, i.e. a contradiction.

If $R = 11$, then $\sum_i g_{\hat{i}} = 2$ and $\chi(M) = 0$. Thus the addendum of $R$ may assume the values listed in the table n. 11 (§2). Since these cases are equivalent to one, we can suppose that $g_{013} = g_{023} = g_{024} = g_{124} = 2$ and $g_{134} = 3$. As in the case n. 11.1 (§2), the formulae $g_{\hat{0}} = g_{\hat{1}} = g_{\hat{2}} = 0$, $g_{\hat{3}} = g_{\hat{4}} = 1$, $g_{03} = g_{034}+1$, $g_{24} = g_{234}+1$, $g_{13} = g_{123}+2$, $g_{02} = g_{012}+2$ and $g_{14} = g_{014}+2$ hold too. Since $g_{013} = 2$, then $\operatorname{rk} \Pi_1(M) \leq 1$, whence $b_1(M; Z_2) \leq 1$ and $\chi(M) = 0 = 2 - 2b_1(M; Z_2) + b_2(M; Z_2)$ implies that $H_2(M; Z_2) = 0$. Then by the same arguments used in the proof of n. 11.1 (§2), we have that $N(1,4)$ and $N(0,2,3)$ are both (PL) homeomorphic to either $S^1 \times B^3$ or $S^1 \underset{\sim}{\times} B^3$. Since $M$ is nonorientable, it follows that $M \simeq_{\mathrm{PL}} S^1 \underset{\sim}{\times} S^3$ by Theorem 2 of [16]. This is a contradiction because $g(S^1 \underset{\sim}{\times} S^3) = 2$.

If $R = 13$, then $\sum_i g_{\hat{i}} = 1$ and $\chi(M) = -1$. Since the manifold $M$ cannot be simply-connected, at least one of $g_{ijk}$'s in $R$ equals 2. Thus we have $\operatorname{rk}\Pi_1(M) \leq 1$, $b_1(M; Z_2) \leq 1$ and $\chi(M) = -1 = 2 - 2b_1(M; Z_2) + b_2(M; Z_2) \geq 0$, which is a contradiction.

If $R = 15$, then $g_{013} = g_{023} = g_{024} = g_{124} = g_{134} = 3$ and $g_{\hat{i}} = 0$ for each $i \in \Delta_4$. The rank of $\Pi_1(M)$ is $\leq 2$ as each one of the above $g_{ijk}$'s equals 3. This implies that $H_1(M; Z_2) = H_3(M; Z_2) = Z_2 \oplus Z_2$ and $H_2(M; Z_2) = 0$ since $\chi(M) = -2$. Then by the same arguments used in the case $R = 15$ (§2), it follows that $N(1,3)$ and $N(0,2,4)$ are both (PL) homeomorphic to either $\#_2(S^1 \underset{\sim}{\times} B^3)$ or $S^1 \times B^3 \# S^1 \underset{\sim}{\times} B^3$. Note that the case $\#_2(S^1 \times B^3)$ can be avoided since $M$ is nonorientable. Thus we have $M \simeq_{\mathrm{PL}} \#_2(S^1 \underset{\sim}{\times} S^3)$ or $M \simeq_{\mathrm{PL}} S^1 \times S^3 \# S^1 \underset{\sim}{\times} S^3$ as required.

Now Corollary 1 directly follows from Proposition 1 and the "subadditivity" of the genus (see §1). Furthermore there exists a crystallization of $RP^4$ with genus six as shown in [11].

## 4. The general case

*Proof of Proposition* 2. The sum

$$R = g_{134} + g_{124} + g_{024} + g_{023} + g_{013} = 5 + 5g - 2\sum_i g_{\hat{i}}$$

is odd (resp. even) whenever $g$ is even (resp. odd). Therefore it follows that

$$0 \le \sum_i g_{\hat{i}} \le [(5/2)g],$$

whence

$$\begin{aligned} 2 - 2g + b_2(M) &\le 2 - 2b_1(M) + b_2(M) \\ &= \chi(M) = 2 - 2g + \sum_i g_{\hat{i}} \le 2 - 2g + [(5/2)g] \end{aligned}$$

since $b_1(M) \le g$ (see [1]). If $M$ is simply-connected, then the inequality

$$2 + b_2(M) = \chi(M) \le 2 - 2g + [(5/2)g]$$

implies that $b_2(M) \le [g/2]$ as required. □

*Proof of Corollary* 2. (1) Use $g(S^1 \times S^3) = 1$ (see [2]), the subadditivity of the genus and

$$g\left(\underset{k}{\#}(S^1 \times S^3)\right) \ge \operatorname{rank} \Pi_1 \left(\underset{k}{\#}(S^1 \times S^3)\right) = k$$

(see [1]).

(2) Since $CP^2$ is simply-connected, we have

$$k = b_2\left(\underset{k}{\#} \pm CP^2\right) \le [g/2],$$

whence $g \ge 2k$. Now the relation $g(CP^2) = 2$ (see [2]) and the subadditivity of the genus prove the statement.

(3) Since $\#_k(S^1 \times S^2)$ is simply-connected, we have

$$2k = b_2\left(\underset{k}{\#}(S^2 \times S^2)\right) \le [g/2],$$

whence $g \ge 4k$. Then the statement follows by $g(S^2 \times S^2) \le 4$ (use the crystallization shown in [7]) and the subadditivity of the genus. □

*Proof of Proposition* 3. Since $r \le [g/2] \le [31/2] = 15$, the intersection form $\omega_M$, induced on $M$ by the cup product, has rank $\le 15$. Recall that $\omega_M$ may only be either odd or indefinite even whenever $M$ is smooth. In the last case the signature $\sigma(M)$ of $M$ is divisible by 16 (see [3, 4, 9] and the Rohlin theorem). Furthermore $\omega_M$ is equivalent (over the integers) to either $(\pm 1) \oplus \cdots \oplus (\pm 1)$ ($r$ times) or a $E_8 \oplus b\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ where $a$ is even, $b \neq 0$ and $a, b \in Z$ (see [3, 4]). Since the rank of a $E_8 \oplus b\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is $8|a| + 2|b|$, it follows that $a = 0$, (use $\operatorname{rank} \omega_M \leq 15$ and $a$ even). Now the Freedman classification of simply-connected smooth 4-manifolds implies the statement (see [9]). □

**Corollary 3.** (1) *Let $M^4$ be a smooth homotopy 4-sphere. Then there exists an integer $k$ such that*

$$g\left(M \# \underset{k}{\#}(S^2 \times S^2)\right) = 4k.$$

(2) *If $V_n$ is an algebraic nonsingular hypersurface of degree $n$ in $CP^3$, then $n^3 - 4n^2 + 6n - 2 \leq [g/2]$.*

(3) *If $V$ is a simply-connected complex surface, then $12p_g + 10 - c_1^2[V] \leq [g/2]$.*

*If $V$ is minimal elliptic, then $12p_g + 10 \leq [g/2]$.*
*Here $p_g$ and $c_1[V]$ denote the geometric genus and the first Chern class of $V$ respectively.*

*Proof.* (1) If $M$ is a homotopy 4-sphere, then there exists an integer $k$ such that $M \# \#_k (S^2 \times S^2) \simeq_{\mathrm{DIFF}} \#_k (S^2 \times S^2)$. Now use Corollary 2.

(2) Recall that $b_2(V_n) = n^3 - 4n^2 + 6n - 2$ (see [15]). Now use Proposition 2 as $V_n$ is simply-connected.

(3) It is well known that $b_2(V) = 12p_g + 10 - c_1^2[V]$ (see [15]). Furthermore we have that $c_1^2[V] = 0$ for the minimal elliptic case. Now we use Proposition 2. □

## 5. Relations with the Poincaré conjecture

Let $P(4)$ be the Poincaré conjecture of 4-dimension. Now we state some conjectures which are related with $P(4)$.

**Conjecture** $C(1)$**.** If $M$ is a closed smooth simply-connected 4-manifold, then $g(M) = 2b_2(M)$.

**Conjecture** $C(2)$**.** The genus is additive with respect to the connected sum of simply-connected smooth (or PL) 4-manifolds.

The conjecture $C(2)$ and its relation with $P(4)$ are also established in [8]. We prove that $C(1)$ and $C(2)$ are equivalent:

$C(1) \Rightarrow C(2)$. Let $M_1$, $M_2$ be closed smooth (or PL) simply-connected 4-manifolds. Then we have $\frac{1}{2}g(M_1 \# M_2) = b_2(M_1 \# M_2) = b_2(M_1) + b_2(M_2) = \frac{1}{2}g(M_1) + \frac{1}{2}g(M_2)$, whence $g(M_1 \# M_2) = g(M_1) + g(M_2)$.

$C(2) \Rightarrow C(1)$. If $M$ is a closed smooth (or PL) simply-connected 4-manifold, then there exist two integers $p, q \in Z$ such that

$$M \# \underset{p}{\#} CP^2 \# \underset{q}{\#}(-CP^2) \underset{\mathrm{DIFF}}{\simeq} \underset{a}{\#} CP^2 \# \underset{b}{\#}(-CP^2)$$

where

$$a = p + \tfrac{1}{2}(b_2(M) + \sigma(M)),$$

$$b = q + \tfrac{1}{2}(b_2(M) - \sigma(M)),$$

$$\sigma(M) = \text{signature of } M,$$

(see [15]). Thus we have

$$\begin{aligned} g\left(M\#\#_p CP^2\#\#_q(-CP^2)\right) &= g\left(\#_a CP^2\#_b(-CP^2)\right) = 2(a+b) \\ &= 2(p+q)+2b_2(M) = g(M)+g\left(\#_p CP^2\#\#_q(-CP^2)\right) \\ &= g(M)+2(p+q), \end{aligned}$$

whence $g(M) = 2b_2(M)$ as required. □

Obviously $C(1)$ (or $C(2)$) implies $P(4)$ as follows. Let $M$ be a closed smooth (or PL) homotopy 4-sphere. Since $b_2(M) = 0$, we have $g(M) = 2b_2(M) = 0$, whence $M \simeq_{\mathrm{PL}} S^4$ (4-sphere).

We also have the following

**Conjecture** $C(3)$**.** If $M$ is a smooth (or PL) homotopy 4-sphere, then

$$g((M)\#\#_k(S^2\times S^2)) \geq g(M)+4k$$

for any integer $k$.

We prove that $P(4)$ and $C(3)$ are equivalent.

$C(3) \Rightarrow P(4)$. If $M$ is a homotopy 4-sphere, then there exists an integer $k$ (see Corollary 3) such that $4k = g(M\#\#_k(S^2\times S^2)) \geq g(M)+4k$, whence $g(M) = 0$ and $M \simeq_{\mathrm{PL}} S^4$.

$P(4) \Rightarrow C(3)$. If $M$ is a smooth homotopy 4-sphere, then $M \simeq_{\mathrm{PL}} S^4$, whence

$$g\left(M\#\#_k(S^2\times S^2)\right) = g\left(\#_k(S^2\times S^2)\right) = 4k \geq g(M)+4k$$

since $g(M) = 0$.

We conclude the paper by noting that it is an open question whether the number of closed smooth (or PL) connected 4-manifolds of a fixed genus $g$ is finite or not.

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