TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 331, Number 1, May 1992

TWO-DIMENSIONAL CREMONA GROUPS ACTING ON SIMPLICIAL COMPLEXES

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ABSTRACT. We show that the 2-dimensional Cremona group

$$\mathrm{Cr}_2 = \mathrm{Aut}_k\, k(X, Y)$$

acts on a 2-dimensional simplicial complex C, which has as vertices certain models in the function field $k(X, Y)$. The fundamental domain consists of one face F. This yields a structural description of Cr_2 as an amalgamation of three subgroups along pairwise intersections. The subgroup $\mathrm{GA}_2 = \mathrm{Aut}_k\, k[X, Y]$ (integral Cremona group) acts on C by restriction. The face F has an edge E such that the GA_2 translates of E form a tree T. The action of GA_2 on T yields the well-known structure theory for GA_2 as an amalgamated free product, using Serre's theory of groups acting on trees.

1. INTRODUCTION

1.1. This discussion sheds light on the relationship between two well-understood automorphism groups. One is the group of k-automorphisms of the polynomial ring $k[X, Y]$, for k is a field. This is often viewed anti-isomorphically as the group of algebraic automorphisms of the affine plane $\mathbb{A}_k^2$. The other group consists of the k-automorphisms of the rational function field $k(X, Y)$; elements of this group corresponds to birational automorphisms of $\mathbb{A}_k^2$, or equivalently, of the projective plane $\mathbb{P}_k^2$.

1.2. Sections 2 and 3 summarize existing knowledge of these groups and present them as free products with amalgamation (Theorems 2.4 and 3.11). For $\mathrm{Aut}_k\, k[X, Y]$ this draws from the classical theorem of Jung and Van der Kulk, which asserts that this group is generated by the set of elements which are of linear or elementary type, and the theorem of Nagata which describes the group as an amalgamated free product of two groups. For $\mathrm{Aut}_k\, k(X, Y)$, the classical Noether's Theorem asserts that, for k algebraically closed, this group is generated by the linear fractional transformations together with the standard quadratic transformation. However, our conclusions are based on recent results of Iskovskikh which give a set of defining relations of $\mathrm{Aut}_k\, k(X, Y)$ in terms of generators slightly different from those of Noether's Theorem. From Iskovskikh's generators and relations, we deduce that this group is the free product of three subgroups amalgamated along pairwise intersections.

Received by the editors August 25, 1989 and, in revised form, February 16, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14J50, 13B10, 13B35, 05C05, 20F05.

1.3. Although $\mathrm{Aut}_k\, k(X, Y)$ contains $\mathrm{Aut}_k\, k[X, Y]$ as a subgroup, the relationship between the structures of these two groups has long been a mystery. For although the proofs which yield generators and relations for the two groups seem related in that they both involve the technique of blowing up points of indeterminacy, it seems that neither theorem can be deduce from the other. The purpose of this paper, then, is to present a framework which unifies the two structures. The method is suggested by the fact that both groups admit a description as amalgamated free products. Topologists have long recognized that such a structure is tantamount to an action of the group on a simply connected simplicial complex. We will exhibit these complexes in such a way that their vertices correspond to certain "multiprojective spaces" having function field $k(X, Y)$. The complex on which $\mathrm{Aut}_k\, k(X, Y)$ acts contains the one for $\mathrm{Aut}_k\, k[X, Y]$ in a way compatible with the containment $\mathrm{Aut}_k\, k(X, Y) \supset \mathrm{Aut}_k\, k[X, Y]$; moreover there is containment between suitably chosen fundamental domains for the respective actions.

We begin by introducing some notation and stating the theorems. As above, k will be a field; we denote by k^* the set of nonzero elements of k.

2. The automorphism group of $k[X, Y]$

2.1. **Integral Cremona group.** Let $\mathrm{GA}_2(k)$, or just GA_2, denote the group of k-automorphisms of the polynomial ring $k[X, Y]$. This is called the *integral Cremona group.*

Letting $W = \mathrm{Spec}\, k[X, Y]$, we see that elements of GA_2 correspond anti-isomorphically to automorphisms of the variety W. In §4 our dicussion will involve different models in $k(X, Y)$ which contain a fixed $\mathbb{A}^2_k$ as a Zariski open set, and we will let this fixed $\mathbb{A}^2_k$ be W. We will refer to W as the *standard* $\mathbb{A}^2$) (in $k(X, Y)$).

2.2. **Vector representation.** An element φ of GA_2 can be represented as a pair of polynomials (F, G), where $F = \varphi(X)$, $G = \varphi(Y)$.

2.3. **Linear and triangular elements.** We denote by Af the subgroup of GA_2 consisting of those elements $\varphi = (F, G)$ for which F and G have total degree one in X and Y (but are not necessarily homogeneous).

We let BA be the subgroup of GA_2 consisting of all $\varphi = (F, G)$ of the form

$$(1) \qquad F = aX + b, \qquad G = cY + g(X),$$

where $a, c \in k^*$, $b \in k$, and $g(X) \in k[X]$. Elements of this subgroup are called *triangular*, since these are precisely the automorphisms which preserve the containment $k[X] \subset k[X, Y]$. It is clear that the intersection of Af and BA, which we denote by B, consists of those $\varphi = (F, G)$ which are of the form (1) where $g(X)$ has degree ≤ 1.

The well-known structure theorem for GA_2 is

2.4. **Theorem.** GA_2 *has the amalgamated free product structure*

$$\mathrm{GA}_2 = Af *_B BA.$$

(A proof will be given in §4.)

2.5. **Remarks on the origin of this theorem.** That GA_2 is generated by Af and BA was first proved by Jung [10] for k of characteristic zero. Van der Kulk [19] generalized this to arbitrary characteristic and proved a factorization theorem which essentially gives the amalgamated free product structure, although he did not state it in this language. Nagata [12] seems to be the first to have stated and proved the assertion as it appears above. The techniques in these proofs require that k be algebraically closed. However, it is not hard to deduce the general case from this (see [20]).

Some fairly recent proofs have been given which use purely algebraic techniques, and for which it is not necessary to assume k is algebraically closed [3], [11].

3. The automorphism group of $k(X, Y)$

3.1. **Cremona group.** Let Cr_2 denote the group of k-automorphisms of the field $k(X, Y)$. It will be called the (full) *Cremona group.* This group is anti-isomorphic to the group of birational automorphisms of the projective (or affine) plane. (It should be noted that most sources use the term "Cremona group" and the symbol Cr to refer to the group of birational automorphisms.) Note that Cr_2 contains the integral Cremona group GA_2 as a subgroup.

3.2. **Homogeneous and nonhomogeneous vector representations.** An element $\varphi \in Cr_2$ can be represented by the pair of rational functions (F, G), where $F = \varphi(X)$, $G = \varphi(Y)$.

Another way to realize elements of Cr_2 is as follows: Letting $X = x/z$ and $Y = y/z$, $k(X, Y)$ becomes the field of homogeneous rational functions of degree zero in $k(x, y, z)$. Given $\varphi = (F, G) \in Cr_2$, write $F = F_0/H_0$, $G = G_0/H_0$, where F_0, G_0, and H_0 are polynomials in X and Y. Now replace X, Y by x/z, y/z and homogenize to get $F = f(x, y, z)/h(x, y, z)$, $G = g(x, y, z)/h(x, y, z)$, where f, g and h are forms of the same degree. Then φ is represented by the triple $(f\colon g\colon h)$, uniquely up to common factors of f, g, and h. We may take f, g, and h to have no common factors. They serve as the coordinate functions for the birational automorphism of $\mathbb{P}^2$ determined by φ.

We will be shifting back and forth between these representation of φ. The representation $\varphi = (f\colon g\colon h)$ will be called the *homogeneous* representation; the representation $\varphi = (F, G)$ will be called *nonhomogeneous.*

3.3. **Linear fractional transformations.** The group $PGL_3(k)$ is contained as a subgroup of Cr_2, whereby the class of the matrix

$$\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix},$$

is identified with the Cremona transformation having homogeneous representation

$$(ax + by + cz\colon a'x + b'y + c'z\colon a''x + b''y + c''z),$$

and nonhomogeneous representation

$$\left(\frac{aX + bY + c}{a''X + b''Y + c''}, \frac{a'X + b'Y + c'}{a''X + b''Y + c''}\right).$$

(Note: it may seem to the reader that we should have taken the transpose of the above matrix. But recall that we are viewing Cremona transformations as field automorphisms, not as transformations of $\mathbb{P}^2$.) Such elements are sometimes called *linear fractional transformations.*

Since $k(X, Y)$ can be identified as the function field of $\mathbb{P}^2_k$ in many ways, let Proj $k[x, y, z]$, where $X = x/z$, $Y = y/z$, be called the *standard* $\mathbb{P}^2$ (in $k(X, Y)$). This surface contains the standard $\mathbb{A}^2$, W (see 2.1), the complement of W being the line $z = 0$, which we will call the *line at infinity* and denote l_∞. Note that the linear fractional transformations correspond to the actual automorphisms (not just birational automorphisms) of the standard $\mathbb{P}^2$.

3.4. **Standard quadratic transformation.** Another familiar element of Cr_2 is the standard *quadratic transformation* σ, given by $(yz\colon xz\colon xy) = (1/X, 1/Y)$. Clearly σ has order 2. The birational automorphism of $\mathbb{P}^2_k$ corresponding to σ can be explained as the blow-up of the points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ followed by the blow-down of the proper transforms of the x, y, and z axes.

3.5. **Noether's Theorem.** *For* k *algebraically closed,* Cr_2 *is generated by* $\mathrm{PGL}_3(k)$ *together with the standard quadratic transformation* σ.

3.6. *Remarks.* This result bears the name "Noether's Theorem" because it was first claimed by Max Noether [14], although his proof was flawed. The first correct proof seems to have been given by Castelnuovo [2]; another early proof appears in [6]. More recent proofs, with modern terminology, can be found in [16] and [13]. These proofs all assume k is algebraically closed, and unlike the Jung-Van der Kulk-Nagata Theorem (Theorem 2.3), this hypothesis is essential, according to some new results of Iskovskikh.

3.7. **The standard $\mathbb{P}^1_k \times \mathbb{P}^1_k$ and its automorphism group.** There is a unique model in $k(X, Y)$ isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ whose projection maps correspond to the inclusions $k(X) \to k(X, Y)$ and $k(Y) \to k(X, Y)$. We will call this model the *standard* $\mathbb{P}^1 \times \mathbb{P}^1$ (in $k(X, Y)$).

The subgroup of Cr_2 which corresponds to the automorphisms of the standard $\mathbb{P}^1 \times \mathbb{P}^1$ is $(\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)) \rtimes \langle\tau\rangle$, where

$$\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}\right) \in \mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)$$

is identified with the element

$$\left(\frac{aX+b}{cX+d}, \frac{a'Y+b'}{c'Y+d'}\right) \in \mathrm{Cr}_2,$$

and $\tau = (Y, X)$. This group will play a key role in our structure theorem for Cr_2.

3.8. **Triangular subgroups.** We view $k(X, Y)$ as $k(X)(Y)$ and consider elements of Cr_2 which fix $k(X)$. This subgroup is identified with $\mathrm{PGL}_2(k(X))$, where the matrix class

$$\begin{pmatrix} a(X) & c(X) \\ b(X) & d(X) \end{pmatrix} \in \mathrm{PGL}_2(k(X)),$$

is identified with

$$\left(X, \frac{a(X)Y + b(X)}{c(X)Y + d(X)}\right) \in \mathrm{Cr}_2.$$

Note that this subgroup contains the element $\varepsilon = (X, X/Y) = (xy\colon xz\colon yz)$. This automorphism appears in Iskovskikh's Theorem, stated below.

Let us note that a k-automorphism of $k(X)$, which can be identified with an element of $\mathrm{PGL}_2(k)$, gives rise to a unique automorphism of $k(X, Y)$ which fixes Y. In this way, $\mathrm{PGL}_2(k)$ is identified with the subgroup of elements of Cr_2 of the form

$$\left(\frac{aX+b}{cX+d}, Y\right),$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(k)$.

Letting $\mathrm{PGL}_2(k)$ act on $k(X)$ gives rise to a semidirect product

$$\mathrm{PGL}_2(k) \ltimes \mathrm{PGL}_2(k(X)).$$

An element

$$\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} a'(X) & c'(X) \\ b'(X) & d'(X) \end{pmatrix}\right),$$

in this group is identified with the element

$$\left(\frac{aX+b}{cX+d}, \frac{a'(X)Y+b'(X)}{c'(X)Y+d'(X)}\right) \in \mathrm{Cr}_2,$$

and this is the subgroup of Cr_2 which preserves the containment $k(X) \subset k(X, Y)$. Its elements are called triangular automorphisms. Note that this group contains the direct product $\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)$, which contains the standard quadratic automorphism $\sigma = (1/X, 1/Y)$.

Note also that this subgroup corresponds to the automorphism group of $\mathbb{P}^1_{k(X)}$. We will refer to the k-scheme $\mathbb{P}^1_{k(X)}$ as the *standard* $\mathbb{P}^1$ (in $k(X, Y)$).

3.9. **The group** $G^{(n)}$. Crucial to our dicussion in §§4 and 5 are the groups $G^{(n)}$, $n \geq 1$, defined by

$$G^{(n)} = \left\{\left(\frac{aX+b}{cX+d}, \frac{tY+f(X)}{(cX+d)^n}\right) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k),\ t \in k^*,\ \deg f \leq n\right\}$$

Note that $G^{(n)}$ is contained in the triangular subgroup of Cr_2. Elements of $G^{(n)}$ correspond to automorphisms of the classical surface F_n, as described below.

3.10. **The surfaces** F_n. The reader is referred to [13] or [16] for a full discussion of these surfaces. Briefly, F_n is a smooth rational surface for $n \geq 0$, minimal if $n \neq 1$, with $F_0 \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$. There is a map $\pi\colon F_n \to \mathbb{P}^1_k$ by which F_n is a $\mathbb{P}^1_k$-bundle over $\mathbb{P}^1_k$. For $n \geq 1$, π is unique and F_n contains a special section Δ_n having the property that $(\Delta_n^2) = -n$. This curve Δ_n is the only prime divisor in F_n having negative self-intersection. All fibers of π are linearly equivalent, and if f is such a fiber we have $(f^2) = 0$, $(f, \Delta_n) = 1$, $\mathrm{Pic}\, F_n = \mathbb{Z}\cdot[f] \oplus \mathbb{Z}\cdot[\Delta_n]$ (the free abelian group on the divisor classes of f and Δ_n). Given a point $p \in F_n$, there is an elementary transformation elm_p centered at p which transforms F_n to an F_{n+1} or an F_{n-1}. This consists of first blowing up at p, then blowing down the proper transform of the fiber containing p. If $p \in \Delta_n$, the resulting surface is an F_{n+1}; if not (and $n \geq 1$) it is an F_{n-1}.

3.11. **The standard** F_n. F_n can are realized as follows. Let

$$W = \operatorname{Spec} k[X, Y] \quad \text{(the standard } \mathbb{A}^2 \text{ (see 2.1))},$$
$$V = \operatorname{Spec} k[X, Y^{-1}], \quad W' = \operatorname{Spec} k[X^{-1}, Y'], \quad V' = \operatorname{Spec} k[X^{-1}, Y'^{-1}],$$

where $Y' = X^{-n}Y$. then $W \cup V \cup W' \cup V' \cong F_n$. The complement of $W \cup W'$ is Δ_n, and the map π arises from the containment $k(X) \to k(X, Y)$. We call this model the *standard* F_n (in $k(X, Y)$). Note that for $n = 0$ this is the standard $\mathbb{P}^1 \times \mathbb{P}^1$ (see 3.8). The complement of W in the standard F_n is the union of Δ_n and one fiber of π, the latter of which we call the *fiber at infinity* (with respect to W) and denote by $f_{n,\infty}$. Thus we have

$$F_n = W \cup \Delta_n \cup f_{n,\infty}.$$

For $n = 1$, this surface is obtained from the standard $\mathbb{P}^2$ (see 3.3) by blowing up the point $(0 : 1 : 0)$. The resulting exceptional curve becomes Δ_1 and the proper transform of l_∞ (see 3.3) becomes $f_{1,\infty}$. If we perform the elementary transformation elm_p on the standard F_n, where $p = \Delta_n \cap f_{n,\infty}$, we obtain the standard F_{n+1}.

For $n \geq 1$, any automorphism of F_n must preserve Δ_n, since this curve is characterized by the property $\Delta_n^2 = -n$. An easy argument using intersection numbers shows that any such automorphism must carry fibers (of π) to fibers. From these facts one can show without too much difficulty that, for the standard F_n, such automorphisms correspond precisely to elements of $G^{(n)}$, so that $G^{(n)}$ is anti-isomorhic to $\operatorname{Aut}(F_n)$.

An important breakthough was recently made by V. A. Iskovskikh, who proved the following:

3.12. **Theorem** (Iskovskikh). Cr_2 *is generated by the triangular group* $\mathrm{PGL}_2(k) \ltimes \mathrm{PGL}_2(k(X))$ *together with the element* $\tau = (Y, X)$. *Moreover, a complete set of relations is given by the group law of the triangular group,* $\tau^2 = 1$, *and the relations*

(*) $\tau \cdot (F, G) \cdot \tau = (G, F)$, *for all* $(F, G) \in \mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)$;

(**) $(\tau\varepsilon)^3 = \sigma$ (*where, as above,* $\varepsilon = (X, \frac{X}{Y})$ *and* $\sigma = (\frac{1}{X}, \frac{1}{Y})$).

(*See* [8] *and* [9].)

(We will comment in 3.14 on the relationship between this theorem and Noether's Theorem.)

We now state a structure theorem, based on Iskovskikh's result, which presents Cr_2 as a product with amalgamations along pairwise intersections.

3.13. **Theorem.** *Let*

$$A_1 = \mathrm{PGL}_3(k),$$
$$A_2 = (\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)) \rtimes \langle\tau\rangle,$$
$$A_3 = \mathrm{PGL}_2(k) \ltimes \mathrm{PGL}_2(k(X)),$$

(*each identified as a subgroup of* Cr_2 *as described in* 3.3, 3.7, *and* 3.8). *Then* Cr_2 *is the free product of* A_1, A_2, *and* A_3 *amalgamated along their pairwise intersections in* Cr_2.

Proof. The proof appeals to Iskovskikh's Theorem (3.12). Let G be the group obtained by amalgamating A_1, A_2, and A_3 along their pairwise intersections in Cr_2. We clearly have a group homomorphism $\alpha: G \to \mathrm{Cr}_2$ restricting to the identity on $A_1 \cup A_2 \cup A_3$. The map α is surjective, since the image contains A_3 and $\tau(\in A_2)$, which generate Cr_2 according to Iskovskikh's Theorem.

We now wish to define a homomorphism $\beta: \mathrm{Cr}_2 \to G$. According to Iskovskikh's Theorem, Cr_2 is generated by A_3 and $\{1, \tau\}$, the latter lying inside A_2, so we have a map $\tilde{\beta}$ from the free product $\{1, \tau\} * A_3$ to G. We must show that the relations $(*)$ and $(**)$ of Theorem 3.12 hold in G.

Note that $(*)$ obviously holds, since it takes place in A_2.

As for $(**)$, note that the equality $\varepsilon = \rho\sigma$, where $\rho = (1/X, Y/X)$, holds in A_3, hence in G. Since σ and ρ commute in A_3, they commute in G. Since σ and τ commute in A_2, they commute in G. Thus we have the following equations in G:

$$(1) \qquad (\tau\varepsilon)^3 = (\tau\rho\sigma)^3 = (\tau\rho)^3\sigma^3.$$

Observe that τ and ρ lie in A_1, and that $(\tau\rho)^3 = 1$ in A_1, hence in G. Since $\sigma^3 = \sigma$ (in A_2, hence in G), it follows from (1) that $(\tau\varepsilon)^3 = \sigma$ in G, as desired.

Hence $\tilde{\beta}$ induces a map $\beta: \mathrm{Cr}_2 \to G$ which restricts to the identity on A_3 and $\{1, \tau\}$. Since Cr_2 is generated by A_3 and $\{1, \tau\}$, it is clear that $\alpha\circ\beta = 1$. It will follow that α is an isomorphism (with $\alpha^{-1} = \beta$) once we show that β is surjective.

To see that β is surjective, note that its image contains $A_3 \subset G$ and $\tau \in A_1 \cap A_2 \subset G$. It is an easy exercise to see that both A_1 and A_2 are generated by their intersection with A_3 (in Cr_2) together with τ. Therefore A_1 and A_2 are in the image of β, and since G is generated by $A_1 \cup A_2 \cup A_3$, β is surjective. The theorem is proved.

3.14. *Remark.* One easily proves that A_2 is generated by its intersection with A_1 together with σ. It can be shown (though not so easily) that A_3 is also generated by its intersection with A_1 together with σ. (This is done by Iskovskikh in [7].) Thus, from Iskovskikh's results one can deduce Noether's Theorem (3.5), which says that Cr_2 is generated by A_1 together with σ.

4. Tree actions which yield the structure theorem for GA_2

4.1. **Tree theory.** The fact that GA_2 has a decomposition as an amalgamated free product (Theorem 2.4) says that it acts *without inversion* on a tree (see [15]). Such a tree can be constructed abstractly, but it is more useful to realize it in a natural context. This was done by Roger Alperin [1] (see also [3]), where the vertices of the tree correspond to certain subvector spaces of $k[X, Y]$. This section will culminate in a realization of what is essentially Alperin's tree, but our approach is closer in spirit to that of Gizatullin and Danilov [4] and [5], which is to consider the GA_2 action on a tree whose vertices correspond to certain smooth compactifications of $\mathbb{A}^2$.

4.2. **Let k be an algebraically closed field.** Any separated reduced, irreducible k-scheme whose function field is $k(X, Y)$ can be identified with collection of local rings in $k(X, Y)$ corresponding to its points. Thus Cr_2 (and GA_2) acts on the set of such k-schemes.

4.3. **W-admissible models.** Let W be the standard $\mathbb{A}^2$ (see 2.1). As above, we identify W with the set of local rings in $k(X; Y)$ corresponding to its points; these will be localizations of $k[X, Y]$. Let S be a complete, nonsingular surface containing W as a Zariski open subset. We will say that S is a *W-admissible model* if S satisfies one of these (mutually exclusive) conditions:

(1) $S \cong \mathbb{P}^2_k$ (*in* which case it is automatic that $S = W \cup l$, where l is the "line at infinity" with respect to W), or

(2) $S \cong F_n$ for some integer $n \geq 1$, *and* $S = W \cup \Delta_n \cup f_n$, where F_n is the classical surface described in 3.11, Δ_n is its special section, and f_n is a fiber of the canonical map $\pi: F_n \to \mathbb{P}^1_k$.

If the first condition holds, we say that S *is a* $\mathbb{P}^2$ (in $k(X, Y)$; if the second condition holds for some n, we say S *is an* F_n (in $k(X, Y)$). Note that any W-admissible model S is a relatively minimal model (in the sense of [16]) unless S is an F_1.

4.4. **The graph $\widetilde{T}$ (of W-admissible models).** For $W \cong \mathbb{A}^1$ as in 4.3, we construct a graph $\widetilde{T}$ whose vertices consist of all W-admissible models. An F_n and an F_{n+1} are connected by an edge if the F_{n+1} is obtained from the F_n by means of the elementary transformation elm_p (see 3.11) with center $p = f_n \cap \Delta_n$ (f_n and Δ_n are as in condition (2) of 4.3). This transformation blows up p, then blows down the proper transform of f_n (see [16, Chapter V, §1] or [13, §2]). A $\mathbb{P}^2$ and an F_1 are connected by an edge if the F_1 is obtained from the $\mathbb{P}^2$ by blowing up a point on l (l is as in condition (1) of 4.3).

4.5. **Adjacency in $\widetilde{T}$.** Two vertices are called *adjacent* if they are connected by an edge. Note that there is precisely one F_{n+1} adjacent in $\widetilde{T}$ to a given F_n. However, for $n \geq 2$, there are many F_{n+1}'s adjacent to a given F_n, since one such is obtained by performing elm_p for any p on f_n but not on Δ_n. Also note that each F_1 is adjacent to precisely one $\mathbb{P}^2$; but a given $\mathbb{P}^2$ is adjacent to many F_1's—one for each point on l_∞.

4.6. **Type of a vertex.** It will be convenient for us to say that a vertex S in $\widetilde{T}$ has *type* n if S is an F_n ($n \geq 1$), or has *type zero* if S is a $\mathbb{P}^2$. Clearly two adjacent vertices in $\widetilde{T}$ have types which differ by one. According to 4.5, each vertex of type $n \geq 1$ is adjacent to precisely one vertex of type $n + 1$.

4.7. **Paths.** A sequence of vertices $V_1, \dots, V_r$ in $\widetilde{T}$ with V_{i-1} adjacent to V_i for $i = 1, \dots, r$ determines a path in $\widetilde{T}$. We say the path has *no backtracks* if $V_{i-2} \neq V_i$, whenever $2 \leq i \leq r$.

4.8. **Lemma.** *Let $V_0, \dots, V_r$, $r \geq 1$, be a path in $\widetilde{T}$ with no backtracks. Suppose V_0 is an F_1 and V_1 is a $\mathbb{P}^2$. Let O be the discrete valuation ring which is the local ring of the special section Δ in V_0. Then O dominates the local ring of a closed point of V_r.*

Proof. Note that V_1 is obtained by blowing down Δ to a point q on V_1, and hence O dominates the local ring of q. If $r \geq 2$, V_2 is an F_1 obtained by blowing up a point p on V_1, and since $V_2 \neq V_0$, we have $p \neq q$. Thus O dominates a point on V_1 which is not the center of the blow-up that yields V_2.

Let $U_1, \dots, U_n$ be the subsequence of $V_0, \dots, V_r$ consisting of all its $\mathbb{P}^2$'s. Note that $n \geq 1$, since $U_1 = V_1$. If $n > 1$, then, for $i = 1, \dots, n-1$, the path from U_i to U_{i+1} has the form

(1) $$U_i, S_1, S_2, \dots, S_{m_i-1}, S_{m_i} = S'_{m_i}, S'_{m_i-1}, \dots, S'_2, S'_1, U_{i+1},$$

with $m_i \geq 2$ (since there are no backtracks) and S_j, $S'_j \cong F_j$, for $j = 1, \ldots, m_i$. If $i = n$, the path from U_n to V_r may be extended past V_r to such a path. Hence it suffices to show O dominates the local ring of a closed point on each surface in (1).

We may assume by induction that O dominates a closed point q on U_i which is not the center of the blow-up which yields S_1. (We have observed that this is the case when $i = 1$.) Thus q is a point on S_1 lying on the fiber at infinity (with respect to W), but *not* on its special section. One easily sees that the local ring of q, and hence O, dominates a point of S_2 ($\cong F_2$) satisfying the same conditions, and so on up to S_{m_i}. Moreover, the elementary transformation leading from S_{m_i} to S'_{m_i-1} does not blow up the center of O, otherwise we would have a backtrack in the path. Therefore the center of O on S'_{m_i-1} is the intersection of its special section and its fiber at infinity, and this holds for S'_{m_i-2} down to S'_1. Thus on U_{i+1}, O dominates a point at infinity which is not the center of the blow-up which yields the next surface along the path. This completes the induction, and the proof of the lemma.

4.9. **Proposition.** (*Each connected component of*) $\widetilde{T}$ *is simply connected.*

(That $\widetilde{T}$ is connected is asserted in Theorem 4.11.)

Proof. This is equivalent to the nonexistence of paths $V_0, \ldots, V_r$ with $r \geq 2$, no backtracks, and $V_0 = V_r$. Suppose such a path exists. By possibly extending the path on both ends, one can easily arrange that $V_0, \ldots, V_r$ satisfies the conditions of Lemma 4.8. Therefore V_0 is an F_1 and the local ring O of its special section dominates a closed point of V_r. But this is absurd, since $V_r = V_0$.

4.10. **Action of** GA_2 **on** $\widetilde{T}$. Let S be a W-admissible model (hence a vertex in $\widetilde{T}$), and φ an element of GA_2. Extending φ to an automorphism of $k(X, Y)$, we note that φ carries the set of local rings of W onto itself, and hence it carries the local rings of S onto those of another W-admissible model S', with S' being the same type as S (in the sense of 4.6). Thus GA_2 acts on the set of W-admissible models. Clearly this action preserves adjacency in $\widetilde{T}$, so we have an action of GA_2 on $\widetilde{T}$, since edges in $\widetilde{T}$ are determined by their end vertices

4.11. **Theorem.**

(1) $\widetilde{T}$ *is a tree, and* GA_2 *acts on* $\widetilde{T}$ *without version.*

(2) *A fundamental domain for the action is any geodesic*

$$\underset{S_0}{\bullet}\overset{e_0}{\text{———}}\underset{S_1}{\bullet}\overset{e_1}{\text{———}}\underset{S_2}{\bullet}\overset{e_2}{\text{———}}\underset{S_3}{\bullet}\text{———}\cdots,$$

where S_0 *is a* $\mathbb{P}^2$ *and* S_i *is an* F_i *for* $i \geq 1$.

(3) *The fundamental domain of* (2) *can be chosen so that* S_0 *is the standard* $\mathbb{P}^2$ (*see* 3.3), *and,* S_i *is the standard* F_i (*see* 3.11). *Therefore the stabilizer of* S_0 *is* Af, *and the stabilizer of* F_j *is the subgroup*

$$BA^{(i)} = \{(aX + b, cY + g(X))\,|\,a, c \in k^*; b \in k; \; g(X) \in K[X] \text{ and } \deg g(X) \leq i)\}.$$

Proof. It is clear that any $\mathbb{P}^2$ or F_i is a translate, via some Cremona transformation, of the standard $\mathbb{P}^2$ or F_i. Since $Af = \mathrm{PGL}_3 \cap \mathrm{GA}_2$ and $BA^{(i)} = G^{(i)} \cap GA_2$ (see 3.9, 3.11), assertions (2) and (3) hold. As for (1), the simple connectivity of $\widetilde{T}$ was established in Proposition 4.9. It remains to show that $\widetilde{T}$ is connected.

The proof of the connectivity is relegated to the appendix, since it uses essentially the same techniques as several of the published proofs that GA_2 is generated by linear and elementary automorphisms. We note, however, the elegance with which this framework leads to the proof of the complete structure theorem for GA_2 (not just the fact that GA_2 is generated by Af and BA) as explained below in 4.12.

4.12. **Consequence.** It follows from Theorem 4.11, and from the theory of groups acting on trees [15] that GA_2 is the free product of the stabilizers of S_i, $i \geq 0$, amalgamated in pairwise fashion along the stablilizer of e_i, $i \geq 0$. The stabilizer of e_i is the intersection of the stabilizers of S_i and S_{i+1}, so one sees easily that the stabilizer of e_0 is B (as defined in 2.3), and for $i \geq 1$ the stabilizer of e_i is $BA^{(i)}$, since $BA^{(i)} \subset BA^{(i+1)}$. We thus have GA_2 presented as the amalgamated free product

$$Af *_B BA^{(1)} *_{BA^{(1)}} BA^{(2)} *_{BA^{(2)}} * BA^{(3)} * \cdots .$$

Noting that $\bigcup_{i \geq 1} BA^{(i)} = BA$ (defined in 2.3), we readily obtain Theorem 2.4:

$$\mathrm{GA}_2 = Af *_B BA, \qquad (2)$$

as a consequence of the GA_2-action on $\widetilde{T}$.

4.13. **The simplified tree** T. Our next step is to obtain from $\widetilde{T}$ a tree on which GA_2 acts with a fundamental domain consisting of only one edge, yielding the decomposition (2). Of course, we know such a tree can be constructed abstractly, using Serre's theory [15]. Moreover, the tree Alperin produces in [1] has this property. But since $\widetilde{T}$ contains the apparatus which so naturally shows GA_2 to be *generated* by Af and BA, our task seems a worthy goal.

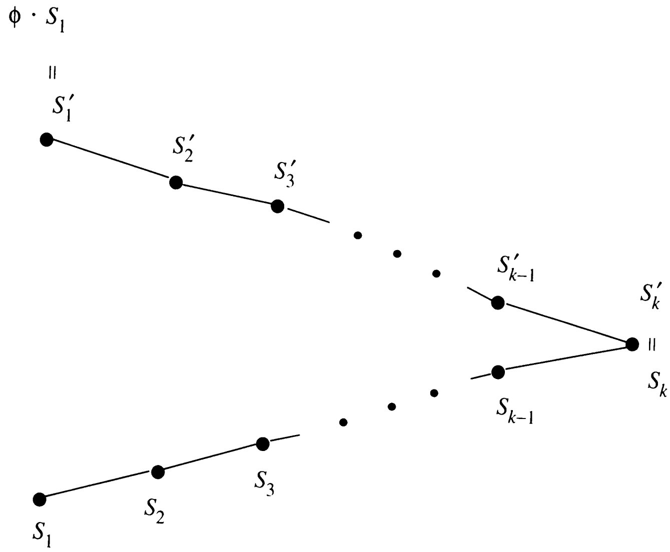
We proceed as follows: Consider the graph Z obtained by removing from $\widetilde{T}$ all vertices of type zero (see 4.6) and all edges which have a type zero vertex as an end point. Form a new graph T whose vertices consist of the type zero vertices of T (i.e. the $\mathbb{P}^2$'s) together with the connected components of Z. Connect a $\mathbb{P}^2$ with a component of Z by an edge if they are connected by an edge in $\widetilde{T}$. T is clearly a tree, since $\widetilde{T}$ is. This construction just amounts to shrinking the connected components of Z, which are subtrees of $\widetilde{T}$, down to points. Since the GA_2 action on $\widetilde{T}$ preserves the types of the vertices, it is clear that elements of GA_2 carry components of Z to components of Z, and therefore GA_2 acts on T. Moreover, it follows from (2) of Theorem 4.11 that a fundamental domain for the action of GA_2 on T consists of an edge

$$\underset{S_0}{\bullet} \overset{e_0}{\text{———}} \underset{Z_0}{\bullet}$$

where S_0 is the $\mathbb{P}^2$ of (2) and Z_0 is the connected component of Z containing the geodesic

$$\underset{S_1}{\bullet} \overset{e_1}{\text{———}} \underset{S_2}{\bullet} \overset{e_2}{\text{———}} \underset{S_3}{\bullet} \overset{e_3}{\text{———}} \cdots$$

of (2). Taking S_0 to be the standard of $\mathbb{P}^2$ and S_i, $i \geq 1$, to be the standard F_i, the stabilizer of S_0 is Af and the stabilizer of of S_i, $i \geq 1$, is $BA^{(i)}$. We claim that the stabilizer of Z_0 is BA. It clearly contains the stabilizers of $S_1, S_2, S_3, \ldots,$, hence contains their union, which is BA. Conversely, suppose $\phi \in \mathrm{GA}_2$ is in the stabilizer of Z_0. Consider the geodesic in $\widetilde{T}$ from S_1 to $\phi \cdot S_1$, which necessarily lies within Z_0. Since a vertex of type $i \geq 1$ in $\widetilde{T}$ is adjacent to precisely one vertex of type $i+1$ (see 4.6), this geodesic is of the form



Moreover, it follows inductively that for $i = 2, \ldots, k$ we must have $\phi \cdot S_i = S_i'$, since S_i' and $\phi \cdot S_i$ are both the unique type i vertex adjacent to S_{i-1}'. Hence ϕ fixes S_k, so $\phi \in BA^{(k)} \subset BA$, and the claim is proved. Thus the tree T has the desired properties.

4.14. T **as a tree of models.** In order for us to establish the connection between the structure of GA_2 and that of Cr_2, it is useful to realize the tree T as a tree whose vertices, like those of $\widetilde{T}$, are certain models in $k(X, Y)$. To do so, it will be necessary to allow as "models" certain k-schemes which are not varieties over k.

4.15. $\mathbb{P}^1$**'s in** $k(X, Y)$. We consider k-schemes R such that (1) $R \cong \mathbb{P}^1_K$ where K is a field containing k and (2) the function field of R is $k(X, Y)$. Thus $k \subset K \subset k(X, Y)$ and K is of transcendence degree 1 over k. It follows from Luroth's Theorem that $K = k(t)$, with t transcendental over k. We can view such an R as a certain collection of local rings in $k(X, Y)$, and it becomes clear that Cr_2 acts transitively on the set of such k-schemes. We will call such a scheme a $\mathbb{P}^1$ in $k(X, Y)$. The standard $\mathbb{P}^1$, introduced in 3.8, is the one for which $K = k(X)$.

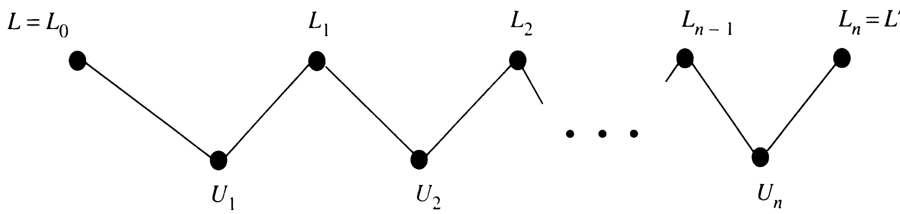
4.16. **The generic** $\mathbb{P}^1$ **associated to an** F_n $(n \geq 1)$. If S is an F_n in $k(X, Y)$, the generic fiber R of the map (unique for $n \geq 1$) $\pi : S \to \mathbb{P}^1_k$ is isomorphic to $\mathbb{P}^1_K$ where K is the function field of $\mathbb{P}^1_k$. Thus we have subscheme R (not open or closed) of S, canonical if $n \neq 0$, which is a $\mathbb{P}^1$ in

$k(X, Y)$ (as defined in 4.15). For $n \geq 1$, we call R the *generic* $\mathbb{P}^1$ *associated to* S.

4.17. **The $\mathbb{P}^1$ associated to certain vertices of T.** Let S be an F_n, with $n \geq 1$, and let p be any point on the section Δ_n of S. The elementary transformation elm_p, which blows up p, then blows down the proper transform of its fiber, does not disturb the generic $\mathbb{P}^1$, call it R, of S. Therefore R is also the generic $\mathbb{P}^1$ of the resulting F_{n+1}. It follows that if S is an F_n and S' is an F_m, both corresponding to vertices of T and lying in the same connected component of Z, then S and S' have the same generic $\mathbb{P}^1$, which we can therefore associate to this component. Since the components of Z are vertices of T, all such vertices (i.e. those in the GA_2 orbit of Z_0) have an associated $\mathbb{P}^1$ in $k(X, Y)$.

4.18. **Lemma.** *Distinct vertices in T yield distinct $\mathbb{P}^1$'s.*

Proof. Let L and L' be distinct vertices in the GA_2-orbit of Z_0. There is a unique geodesic in T connecting L and L', which is of the form



with $L_0, \ldots, L_n$ in the GA_2-orbit of Z_0 and $U_1, \ldots, U_n$ in the GA_2-orbit of S_0. This means, of course, that $U_1, \ldots, U_n$ are $\mathbb{P}^2$'s. There exist unique representative models V of $L = L_0$ and V' of $L' = L_n$ which are F_1's such that V is adjacent to U_1 ($\cong \mathbb{P}^2$) in $\widetilde{T}$ and V' is adjacent to U_n in $\widetilde{T}$. The geodesic in $\widetilde{T}$ from V to V' contains the vertices $U_1, \ldots, U_n$ of $\widetilde{T}$, and goes through the components $L_1, \ldots, L_{n-1}$ of Z. For $i = 1, \ldots, n-1$, there exists a unique path with no backtracks through L_i in $\widetilde{T}$ from U_i to U_{i+1}. By juxtaposing these we obtain a path in $\widetilde{T}$ from V to V' satisfying the conditions of Lemma 4.8. The lemma tells us that the local ring O of the special section Δ on V dominates a closed point on V'. It follows that O is the local ring of a point on R, the generic $\mathbb{P}^1$ of V (and of L), but is not the local ring of a point on R', the generic $\mathbb{P}^1$ of V' (and of L'). Therefore $R \neq R'$.

4.19. **W-admissible $\mathbb{P}^1$'s.** It is clear that a $\mathbb{P}^1$, call it R, in $k(X, Y)$ corresponds to a vertex of T if and only if there exist, $F, G \in k[X, Y]$ such that $k[F, G] = k[X, Y]$ and such that $R = \operatorname{Spec} k(F)[G] \cup \operatorname{Spec} k(F)[G^{-1}] \cong \mathbb{P}^1_{k(F)}$. If this is the case we say R is a *W-admissible* $\mathbb{P}^1$.

4.20. **T as a tree of admissible models.** We conclude from the above discussion that T is a tree whose vertices consist of all W-admissible $\mathbb{P}^2$'s and $\mathbb{P}^1$'s in $k(X, Y)$. Given S an admissible $\mathbb{P}^2$ and R an admissible $\mathbb{P}^1$, S and R are adjacent vertices in T if and only if there is a point p at infinity in S (with respect to W) such that R is the generic $\mathbb{P}^1$ of the F_1 obtained by blowing up S at p. The fundamental domain for the GA_2-action on T becomes

$$E = \underset{S}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\underset{R}{\bullet} \tag{3}$$

where S and R may be chosen to be $S = \operatorname{Proj} k[x, y, z]$, $R = \operatorname{Spec} k(X)[Y] \cup \operatorname{Spec} k(X)[Y^{-1}] = \mathbb{P}^1_{k(X)}$, where $X = x/z$, $Y = y/z$. One can reaffirm that the stabilizer in GA_2 of S is Af, and the stabilizer of R consists of those k-automorphisms of $k[X, Y]$ which preserve the containment $k[X] \subset k[X, Y]$—precisely BA.

We can summarize 4.14–4.20 as follows:

4.21. **Theorem.** *Let T be the graph whose vertices consist of all W-admissible $\mathbb{P}^2$'s and $\mathbb{P}^1$'s, where S, a $\mathbb{P}^2$, is connected by an edge to R, a $\mathbb{P}^1$, precisely when S has a point p at infinity (with respect to W) such that R is the generic $\mathbb{P}^1$ of the F_1 obtained by blowing up S at p. Then*

(1) *T is a tree, on which GA_2 acts without inversion.*
(2) *A fundamental domain for the action is*

$$\overset{S}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\overset{R}{\bullet}$$

where S is the standard $\mathbb{P}^2$, R is the standard $\mathbb{P}^1$ (see 3.3, 3.8*).*
(3) *The stabilizer of S in GA_2 is Af; the stabilizer of R is BA.*

From this theorem, the decomposition $GA_2 = Af *_B BA$ is immediate.

5. The simplicial complex which yields the structure theorem for Cr_2

5.1. **General theory.** The amalgamated product group structure of Cr_2 laid out in §3 reflects the fact that it acts on a simply connected 2-dimensional simplicial complex. This follows from a higher dimensional analogue of Serre's tree theory. (This is folklore amongst topologists, but see [18] or [17].) We wish to realize this space in such a way that its vertices again correspond to *models* in $k(X, Y)$, and such that it contains the tree T of Theorem 4.21 as a subcomplex, compatibly with the containment $GA_2 \subset Cr_2$.

5.2. **Admissible models.** Consider the set of models S (*model* now means *reduced, irreducible, separated k-scheme having function field $k(X, Y)$) satisfying one of these three properties*:

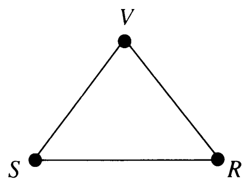
(1) $S \cong \mathbb{P}^2_k$,
(2) $S \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$, or
(3) $S \cong \mathbb{P}^1_K$ for some subfield K of $k(X, Y)$ (necessarily of pure transcendence degree 1 over k).

Such a k-scheme S will be called an *admissible model*. We say S is a $\mathbb{P}^2$, S is a $\mathbb{P}^1 \times \mathbb{P}^1$, or S is a $\mathbb{P}^1$ according to whether (1), (2), or (3), respectively, is satisfied.

5.3. **The complex C.** We construct a two-dimensional simplicial complex C using as vertices the set of admissible models. We declare that three models S, a $\mathbb{P}^2$, V a $\mathbb{P}^1 \times \mathbb{P}^1$, and R a $\mathbb{P}^1$, determine a face when there exist two distinct points p and q on S such that (a) V is the $\mathbb{P}^1 \times \mathbb{P}^1$ ($\cong F_0$) obtained by blowing up S at p and q, then blowing down the proper transform of the line in S containing p and q, and (b) R is the generic $\mathbb{P}^1$ of the F_i obtained by blowing up S at p.

Taking S to be the standard $\mathbb{P}^2$ (see 3.3), $p = (0:1:0)$, and $q = (1:0:0)$, the resulting V and R are the standard $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1$, respectively (see 3.7, 3.8). Therefore these standard models form a face, which we will call the *standard face* in C.

5.4. **Fundamental domain.** It is clear from the construction of C that Cr_2 acts on C without inverting any edge or rotating any face. Moreover, a fundamental domain for the action is given by any one face



We may choose (1) to be the standard face (see 5.3). For this choice, the stabilizers of S, V, and R are, respectively, the groups

$$\begin{aligned} A_1 &\cong \mathrm{PGL}_3(k), \\ A_2 &\cong (\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)) \rtimes \langle \tau \rangle, \qquad \text{and} \\ A_3 &\cong \mathrm{PGL}_2(k) \ltimes \mathrm{PGL}_2(k(X)), \end{aligned}$$

identified as subgroups of Cr_2 as in 3.3, 3.8, and 3.9.

5.5. **Theorem.** *The simplicial complex C is 1-connected.*

Proof. This follows, using standard arguments, from the fact that (1) is a fundamental domain for the action of Cr_2 on C, and the fact that Cr_2 is the amalgamated free product of the stabilizers A_1, A_2, and A_3 along their pairwise intersections (Theorem 3.13). We sketch the proof.

To see that C is connected, consider a face F', and let F denote the standard face. We will show that there a path from F to F'. If E is a face in C, then E shares a vertex of F if and only if $E = hF$ for some $h \in A_1 \cup A_2 \cup A_3$. Since F is a fundamental domain, there exists $g \in \mathrm{Cr}_2$ such that $gF = F'$. Let $g = g_1 \cdots g_r$ be a factorization of g such that $g_1, \dots, g_r \in A_1 \cup A_2 \cup A_3$. Consider the sequence of faces $F = F_0, F_1, \dots, F_r = F'$, where $F_i = g_1 \cdots g_i F$, $i = 0, \dots, r$. For $i = 0, \dots, r-1$, $g_{i+1}F$ touches F, as previously observed, hence F_{i+1} touches F_i (translating by $g_1 \cdots g_i$). Therefore the union of these faces contains a path from F to F'. This shows C is connected.

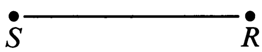
To show simple connectivity, we associate a loop to a sequence of faces $F_0, \dots, F_r$ such that F_i and F_{i+1} have a common vertex, for $i = 0, \dots, r-1$, and such that $F_0 = F_r$. We may assume that F_0 and F_r are the standard face. Thus we have a sequence $g_1, \dots, g_r \in A_1 \cup A_2 \cup A_3$ such that $F_i = g_1 \cdots g_i F_0$, for $i = 1, \dots, r$. Since $F_r = F_0$, we see that $g_1 \cdots g_r F_0$ is in the stabilizer of F_0, which is $A_1 \cap A_2 \cap A_3$. The fact that Cr_2 is the pairwise amalgamated product of A_1, A_2, and A_3 (see Theorem 3.13, and §5.4) implies that the sequence $g_1, \dots, g_r$ can be transformed into the sequence consisting only of the element $g = g_1 \cdots g_r$ by a series of the following types of changes:

(a) replace a sequence entry h by $h'h''$, where h, h', $h'' \in A_i$ and $h = h'h''$.

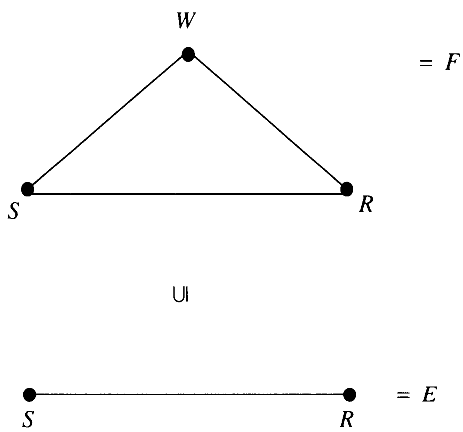
(b) replace two successive entries h', h'' by h, where h', $h'' \in A_i$ and $h = h'h''$.

An alteration of either of these types replaces the *path* $F_0, \dots, F_r$ by a homotopic path. Thus the path contracts into the face F_0. This concludes the proof.

5.6. **The** Cr_2 **complex** C **contains the** GA_2 **tree** T. If we choose for the fundamental domain F of C the standard face, (1) and let W be the standard $\mathbb{A}^2$, which is the complement in S of the line containing $p = (0:1:0)$ and $q = (1:0:0)$, then the tree T of §4 is the union of the GA_2 translates of the edge



of F. Thus the complex C contains the tree T as a subcomplex, and the face F contains the fundamental domain E of T, as follows:



6. Appendix

6.1. Here we sketch a proof of the connectivity of the graph $\widetilde{T}$, which completes the proof of Theorem 4.11. The reader is assumed to be familiar with the basic facts about birational maps between smooth surfaces, and with intersection numbers and their behavior under the blowing up of points.

As before, all varieties discussed are assumed to be k-varieties, where k is a fixed algebraically closed field.

6.2. **Lemma.** *Let V be a nonsingular complete surface and $\omega\colon V \to \mathbb{P}^1_k$ a morphism making V a ruled surface (i.e., for some open set $U \subseteq \mathbb{P}^1_k$, $\omega^{-1}(U) \cong \mathbb{P}^1_k \times U$). Then each fiber of ω contains a component having self-intersection ≥ -1.*

Proof. Let $W = \mathbb{P}^1_k \times \mathbb{P}^1_k$. Taking $\pi\colon W \to \mathbb{P}^1_k$ to be one of the projections, there exists a birational map $\gamma\colon V \dashrightarrow W$ such that

$$\begin{array}{ccc} V & \overset{\gamma}{\dashrightarrow} & W \\ & \omega \searrow \quad \swarrow \pi & \\ & \mathbb{P}^1_k & \end{array} \tag{1}$$

commutes.

Suppose D is a fiber of ω all of whose components have self-intersection ≤ -2. We will arrive at a contradiction by resolving γ to a morphism. Note that the commutativity of (1) implies that any curve of V which maps to a point in W must be contained in a fiber of ω, and the same must hold if we blow up a point of V and replace V by the surface thus obtained. By successively blowing up points of indeterminacy for γ which do not lie on D, we may assume γ is defined at all points not lying on D. (These blow-ups do not alter the self-intersection of the components of D.)

First we argue that γ is not everywhere defined. Since $(D^2) = 0$, D has more than one component, and the image of D under γ lies within a fiber C of π. If γ were a morphism it would factor into a sequence of blowing downs, and D would contain a component E with $(E^2) = -1$, in violation of our assumption.

So let x be a point on D at which γ is not defined. We proceed to resolve γ by blowing up x, creating a new surface V_1 with exceptional curve E_1, and induced rational map $\gamma_1\colon V_1 \dashrightarrow W$. V_1 is a ruled surface and the proper transform D_1 of D is a fiber whose components are the proper transforms of the components of D, together with E_1. Note that all the components of D_1, except E_1, have self-intersection ≤ -2, and $(E_1^2) = -1$.

Case 1. γ_1 is defined at all points of E_1. In this case we must have $\gamma_1(E_1) = C$; for if $\gamma_1(E_1)$ is a point, then γ_1 factors through the blowing down of E_1, contradicting the fact that x was a point of indeterminacy for γ. It easily follows that γ_1 is defined everywhere. For if not, we could resolve γ_1 to a morphism by blowing up more points, and the last exceptional curve must necessarily map into C (otherwise the last blow-up would have been unnecessary). However a birational morphism of surfaces cannot carry two distinct curves onto the same curves. So γ_1 is a morphism, with D_1 mapping into C and E_1 mapping onto C. Since γ_1 is the product of blowing downs, and all components of D_1 other than E_1 collapse to a point, one of them must have self-intersection -1. This is a contradiction.

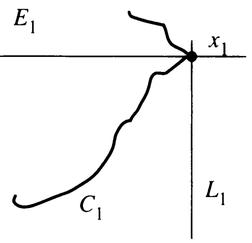
Case 2. There exists a point $x_1 \in E_1$ which is a point of indeterminacy for γ_1. We blow up x_1 to obtain a surface V_2 with exceptional curve E_2 and the induced birational map $\gamma_2\colon V_2 \dashrightarrow W$. V_2 has fiber F_2 over $\mathbb{P}^1_k$ which is the total transform of D_1; all of its components, other than E_2, have self-intersection ≤ -2. If γ_2 is defined along E_2 we reach the same contradiction as in Case 1. Otherwise we again blow up a point of indeterminacy on E_2. This process must end (in a contradiction) when γ is finally resolved to a morphism.

6.3. To show that $\widetilde{T}$ is connected, it suffices to show that, given any two W-admissible $\mathbb{P}^2$'s, S_0 and S, there is a path in $\widetilde{T}$ from S_0 to S. (Recall that W is the standard $\mathbb{A}^2$ (see 2.1).) We may assume that S_0 is the standard $\mathbb{P}^2$ (see 3.3). Let P, $Q \in k[X, Y]$ be a system of variables which define straight lines in W relative to the embedding $W \subset S \cong \mathbb{P}^2$. If P and Q are both linear polynomials in X and Y, then $S = S_0$ and there is nothing to prove. In the general case, let $d = \deg P$; we will reduce to the case $d = 1$.

6.4. Let L be the line at infinity (relative to W) in S_0, and let C be the closure in S_0 of the curve in W defined by P. Since $C \cap W \cong \mathbb{A}_k^1$, C has one point x on L, and it is a *one place point*, i.e., the closure of $C \cap W$ in any smooth complete surface containing W will always have one point at infinity (relative to W), and that point will have only one tangential direction. Note that $(L \cdot C) = d$, by Bezout's Theorem. Let $d_1 = \text{mult}_x C$, the multiplicity of x on C. Since $d > 1$, L must be tangent to C at x; for otherwise $d = d_1$, and letting H be the line on S_0 tangent to C at x, we have $(H \cdot C) > d$, violating Bezout's Theorem. Let $m = d - d_1$ and write $d = nm + r$ with $0 \le r < m$. We will eventually show that $r = 0$, i.e., $m|d$. (This is the crux of the proof.)

6.5. We blow up S_0 at x to obtain a W-admissible surface $S_1 \cong F_1$. Note that S_1 is a vertex in $\widetilde{T}$ adjacent to S_0. Let E_1 be the resulting exceptional curve on S_1, and let L_1 and C_1 be, respectively, the proper transforms of L and C. The complement of W in S_1 is $E_1 \cup L_1$, and C_1 intersects this complement at the point where E_1 and L_1 intersect; call this point x_1. We have $(E_1 \cdot C_1) = d_1 = d - m$, and an easy argument shows $(L_1 \cdot C_1) = m$.

6.6. If $d_1 > m$, then E_1 is tangent to C_1 at x_1,

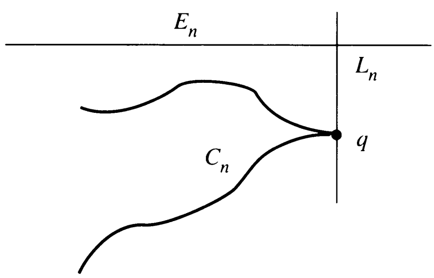


and $m = \text{mult}_{x_1} C_1$. In this case, we blow up x_1, separating E_1 and L_1, and contract the proper transform of L_1 to obtain a W-admissible surface $S_2 \cong F_2$, adjacent to S_1 in $\widetilde{T}$. Let E_2 and C_2 be the proper transform of E_1 and C_1 in S_2, and let L_2 be the proper transforms of the exceptional curve obtained from blowing up x_1. The complement of W in S_2 is $E_2 \cup L_2$, and C_2 intersects this complement at the point x_2 where E_2 and L_2 intersect. We have $(E_2 \cdot C_2) = d_1 - m = d - 2m$ and $(L_2 \cdot C_2) = m$.

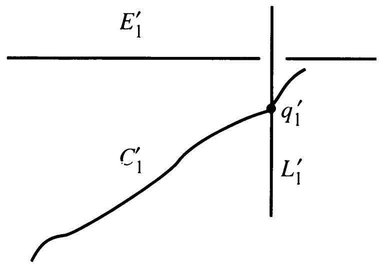
6.7. If $d - 2m > m$, then E_2 is tangent to C_2 at x_2, and $m = \text{mult}_{x_2} C_2$. Repeating this process n times, where $d = nm+r$ (possibly proceeding through the situation $d-(n-1)m = m$, in case $m|d$), we obtain a W-admissible surface $S_n \cong F_n$, connected to S_0 in $\widetilde{T}$ by the path $S_0, S_1, \ldots, S_n$. The complement of W in S_n is $E_n \cup L_n$, and C_n is the proper transform of C. Letting x_n be the point where E_n and L_n intersect, we have $(E_n \cdot C_n) = d - mn = r$ and $(L_n \cdot C_n) = m$. We have $0 \le r < m$ and x_n lies on C_n if and only if $r \ne 0$. In either case, the point at infinity (with respect to W) of C_n lies on L_n. Call this point q. Then $q = x_n$ if and only if $r \ne 0$. Note that these conditions must hold if $n = 1$ (a situation we later deem impossible).

6.8. We claim that L_n is not tangent to C_n at q. To prove this, suppose L_n and C_n *do* meet tangentially, and consider the rational map $\omega: S_n \dashrightarrow \mathbb{P}^1_k$ induced by the containment $k(F) \subset k(X, Y)$. The divisor of F on S_n is easily seen to be $(F) = C_n - dL_n - mE_n$; it follows that q is the only point of indeterminacy for ω, as it is the only point where neither F nor F^{-1} is defined. We proceed to resolve ω to a morphism $\tilde{\omega}: V \to \mathbb{P}^1_k$ by blowing up points of indeterminacy, beginning with q. In this process, each time we blow-up we get an exceptional curve E such that either (1) E contains at least one point of indeterminacy, or (2) $\tilde{\omega}$ is defined at all points of E. In case (1), the proper transform of E on V has self-intersection ≤ -2 (since $(E^2) = -1$ and blowing up the point(s) of indeterminacy on E causes its self-intersection to drop). In case (2) the image of E under ω is all of $\mathbb{P}^1_k$ (for if its image were a point, the last blow-up would have been unnecessary). It follows that all components of the complement of W in V, excepting certain curves which map onto $\mathbb{P}^1$, have self-intersection ≤ -2. This is due to the fact that C_n is tangent to L_n at q, and $(L_n^2) = 0$. Thus at least two blow-ups occur on L_n or its proper transforms, so that its proper transform on V also has self-intersection ≤ -2. As for E_n, we have $(E_n^2) = -n$, and the only problem could arise when $n = 1$. But in this case we have noted that $q \in E_n$, so that the proper transform of E_n has self-intersection ≤ 2 as well. Let $p \in \mathbb{P}^1_k$ be the point at which F has its pole, and note that $\omega^{-1}(p)$ lies in the compliment of W in V, but obviously does not contain the curve(s) which map onto $\mathbb{P}^1_k$. Hence this fiber consists of components having self-intersection ≤ -2, in violation of Lemma 6.2. The claim is proved.

6.9. We conclude that $r = 0$, i.e. $m|d$. For if not then $q = x_n$, as was seen in 6.7, and the fact that $(L_n \cdot C_n) = m > r = (E_n \cdot C_n)$ would say that L_n is tangent to C_n at q, which we know from 6.8 is not the case. Moreover, it follows from this and from 6.7 that $n > 1$. The situation is as depicted below:

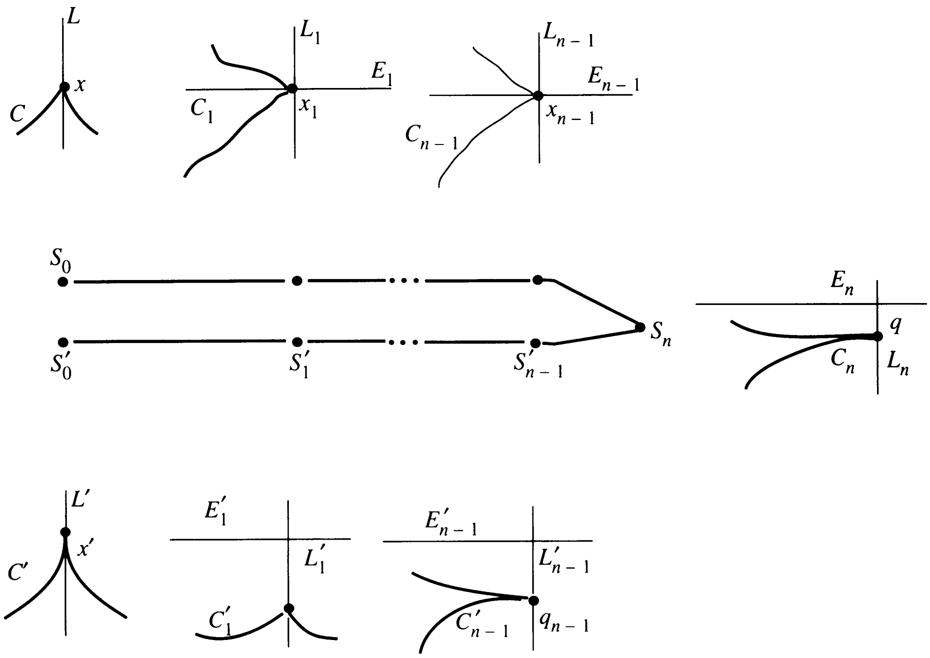


Note that $m = \text{mult}_q C_n$. Now we blow up q, separating C_n and L_n, and blow down the proper transform of L_n to obtain a W-admissible surface $S'_{n-1} \cong F_{n-1}$, adjacent to S_n in $\widetilde{T}$. Letting E'_{n-1} and C'_{n-1} be the proper transforms of E_n and C_n, and letting L'_{n-1} be the new fiber at infinity, we have $(L'_{n-1} \cdot C'_{n-1}) = m$, and $(E'_{n-1} \cdot C'_{n-1}) = 0$. The argument of 6.8 shows that L'_{n-1} is not tangent to C'_{n-1} at their point of intersection, call it q_{n-1}, provided $n-1 \geq 2$. Thus $\text{mult}_{q_{n-1}} C'_{n-1} = m$. We repeat this process and thereby continue along a path in $\widetilde{T}$ to arrive at a surface $S'_1 \cong F_1$ and the situation depicted below:



with $(C_1' \cdot L_1') = m$ (but we no longer know, at this step, that L_1' is not tangent to C_1' at q_1'). Now we blow down E_1' to obtain a W-admissible surface $S_0' \cong \mathbb{P}_k^2$, with C' the proper transform of C_1' and L' the proper transform of L_1'. L' is the line at infinity with respect to W, and $(L' \cdot C') = m$. Therefore C' is a curve of degree m on S_0', and if we write F as a polynomial in variables X', Y' which define straight lines in S_0', its degree is m. Since $m < d$ the degree of F has been lowered.

The path in T which has been traversed is illustrated below:



6.10. We can repeat this procedure until $d = 1$. Now consider $e = \deg G$, and assume $e > 1$. We will perform the same operation as above, with respect to G instead of F. So now let C be the curve defined by G, and let D be the curve defined by F, which is a straight line in S_0, since $d = 1$. We again let L be the line at infinity. Since C and D meet once on W, they intersect $e-1$ times at their common point at infinity. We now trace the proper transform of D through the path illustrated in 6.9. Note that the proper transform of D on S_1 is a fiber of F_1, and this holds at each vertex in the path, through S_1'. Thus the proper transform D' of D in S_0' is again a straight line. Hence as a polynomial in X' and Y', F is still linear, and the degree of G has been lowered. When we finally achieve $d = e = 1$, we have $S_0 = S$, and the proof is complete.

BIBLIOGRAPHY

1. R. C. Alperin, *Homology of the group of automorphisms of* $k[x, y]$, J. Pure Appl. Algebra **15** (1979), 109–115.
2. G. Castelnuovo, *La trasformazioni generatrici del gruppo Cremoniano nel piano*, Atti Acad. Sci. Torino **36** (1901), 861–874.
3. W. Dicks, *Automorphisms of the polynomial ring in two variables*, Publ. Mat. **27** (1983).
4. M. H. Gizatullin and V. I. Danilov, *Automorphisms of affine surfaces.* I, Math. USSR-Izv. **9** (1975), no. 3, 493–534.
5. ——, *Automorphisms of affine surfaces.* II, Math. USSR-Izv. **11** (1977), no. 1, 51–98.
6. H. P. Hudson, *Cremona transformations*, Cambridge University Press, Cambridge, 1927.
7. V. A. Iskovskikh, *Birational automorphisms of three-dimensional algebraic varieties*, J. Soviet Math. **13** (1980), 815–868.
8. ——, *Generators and relations in a two-dimensional Cremona group*, Vestnik Moskov. Univ. Ser. I. Mat. Mekh. **38** (1983), no. 5, 43–48.
9. ——, *Proof of a theorem on relations in the two dimensional Cremona group*, Uspekhi Mat. Nauk **40** (1985), no. 5 (245), 255–256.
10. H. W. E. Jung, *Über ganze birationale Transformationen der Ebene*, J. Reine Angew. Math. **184** (1942), 161–174.
11. J. H. McKay and S. Wang, *An elementary proof of the automorphism theorem of a polynomial ring in two variables*, J. Pure Appl. Algebra **52** (1988), 91–102.
12. M. Nagata, *On automorphism groups of* $k[X, Y]$, Kyoto Univ. Lectures in Math. No. 5, Kinokuniya-Tokyo, 1972.
13. ——, *Our rational surfaces.* I, Mem. College Sci. Kyoto Univ. **32** (1960).
14. M. Noether, *Zur Theorie der eindeutigen Ebenentransformationen*, Math. Ann. **5** (1972), 635–639.
15. J.-P. Serre, *Arbes, amalgames, et* SL_2, Astérisque **46** (1977).
16. I. R. Shafarevich, *Algebraic surfaces*, Proc. Steklov Inst. Math. **75** (1965).
17. C. Soulé, *Groupes opérant sur un complexe simplicial avec domaine fondamental*, C. R. Acad. Sci. Paris Sér. A **276** (1973), 607–609.
18. R. G. Swan, *Generators and relations for certain special linear groups*, Adv. in Math. **6** (1971), 1–77.
19. W. Van der Kulk, *On polynomial rings in two variables*, Nieuw Arch. Wisk. **1** (1953), 33–41.
20. D. Wright, *Algebras which resemble symmetric algebras*, Thesis, Columbia University, 1975.

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