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A CHERN CHARACTER IN CYCLIC HOMOLOGY

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ABSTRACT. We show that inner derivations act trivially on the cyclic cohomology of the normalized cyclic complex $\mathscr{C}(\Omega)/\mathscr{D}(\Omega)$ where Ω is a differential graded algebra. This is then used to establish the fact that the map introduced in [GJ] defines a Chern character in K theory.

J.-M. Bismut recently extended the classical theory of Chern characters to the equivariant theory of the free loop space LX; he explicitly constructed an S^1-invariant equivariantly closed form $\omega(E, \nabla)$ on LX whose restriction to X, the space of constant loops, is $\mathrm{Ch}(E) = \mathrm{tr}(\exp \nabla^2)$. Closely related to the equivariant theory of the loop space of X is A. Connes' cyclic theory of the differential graded algebra Ω of differential forms on X. More precisely, Chen's theory of iterated integrals allows us to view cyclic cycles of forms on X as equivariantly closed forms on the loop space LX. This is a generalization of the point of view that a one-form ω on X can be viewed as a function on LX by $\gamma \to \int_\gamma \omega$.

This suggests the possibility of interpreting the classical theory of Chern characters in the context of cyclic homology i.e., of extending $\mathrm{Ch}(E)$ off the fixed point set X of the circle action on LX by way of a cyclic cycle over Ω. In [GJP], Getzler, Jones, and Petrack, gave an explicit formula for a cyclic cycle $\psi(E, e)$ which via iterated integrals maps to the Bismut Chern character $\Omega(E)$ over LX. A priori, their construction of $\psi(E, e)$ appears to depend on the choice of idempotent e describing the vector bundle E, but in fact, it is shown that $\psi(E, e)$ depends only on the isomorphism class of E in $K_0(X)$, and hence defines the desired Chern character. This is essentially proved by establishing the triviality of certain group actions on the cyclic homology of a differential graded algebra Ω. These actions are natural generalizations of the usual adjoint action $\mathrm{Ad}(g)$, given by conjugation by an invertible element g, and its infinitesimal analogue ad, given by inner derivation, to Connes' cyclic bar complex of Ω.

1. Cyclic homology

In this section we review the definition of the cyclic homology of a DGA.

According to C. Kassel, by a *mixed complex* we mean a triplet $(\mathscr{C}^*; d; B)$ where $\mathscr{C}^*$ is a graded algebra and where $d: \mathscr{C}^* \to \mathscr{C}^{*+1}$, and $B: \mathscr{C}^* \to \mathscr{C}^{*-1}$, are derivations of degrees $+1$ and -1 respectively, and subject to the rela-

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tion $[d, B] = 0$. The associated chain complex $(\mathscr{C}^{\lambda}_{*}; d + B)$, called the *cyclic complex* is defined as follows:

$$\mathscr{C}^{\lambda}_{n} = \mathscr{C}^{n} \oplus \mathscr{C}^{n+2} \oplus \mathscr{C}^{n+4} \oplus \cdots ,$$

and

$$d_n : \mathscr{C}^{\lambda}_{n} \to \mathscr{C}^{\lambda}_{n-1},$$

by

$$d_n(\omega^n, \omega^{n+2}, \omega^{n+4}, \dots) = (B\omega^n, d\omega^n + B\omega^{n+2}, d\omega^{n+2} + B\omega^{n+4}, \dots),$$

in short $d_n = d + B$. We define the *cyclic homology* $HC_*(\mathscr{C}^*)$ of the mixed complex $(\mathscr{C}^*; d; B)$ to be the homology of the associated chain complex. Similarly, we define the *even* and *odd cyclic homology groups* $HC^{\text{even}}(\mathscr{C}^*)$ and $HC^{\text{odd}}(\mathscr{C}^*)$.

We now apply this construction to define the *cyclic homology theories of a* DGA. Let $(\Omega; d)$ be a DGA over $\mathbb{C}$ with unit 1. Then we set

$$\mathscr{C}(\Omega) = \sum_p \Omega \otimes (I\Omega)^{\otimes p},$$

where $I\Omega = \Omega/\mathbb{C}$. We make $\mathscr{C}(\Omega)$ into a graded algebra by defining

$$\deg(\omega_0 \otimes \cdots \otimes \omega_k) = \sum_{i=0}^{k} |\omega_i| - k.$$

(The product structure is of course given by the shuffle product.) Next, we define operators d and b

$$d, b : \mathscr{C}^p(\Omega) \to \mathscr{C}^{p+1}(\Omega),$$

by

$$d(\omega_0 \otimes \cdots \otimes \omega_k) = -\sum_{i=0}^{k} (-1)^{\varepsilon_{i-1}} \omega_0 \otimes \cdots \otimes \omega_{i-1} \otimes d\omega_i \otimes \omega_{i+1} \otimes \cdots \otimes \omega_k,$$

and

$$\begin{aligned} b(\omega_0 \otimes \cdots \otimes \omega_k) = & -\sum_{i=0}^{k} (-1)^{\varepsilon_i} \omega_0 \otimes \cdots \otimes \omega_i\omega_{i+1} \otimes \cdots \otimes \omega_k \\ & + (-1)^{(|\omega_k|-1)\varepsilon_{k-1}} \omega_k\omega_0 \otimes \omega_1 \otimes \cdots \otimes \omega_{k-1}, \end{aligned}$$

where

$$\varepsilon_i = |\omega_0| + \cdots + |\omega_i| - i.$$

Then, it is easy to show that $(d + b)^2 = 0$, in fact that $d^2 = b^2 = db + bd = 0$. We also define the A. Connes operator B_0 $B : \mathbb{C}^p(\Omega) \to \mathscr{C}^{p-1}(\Omega)$ by the formula

$$B(\omega_0 \otimes \cdots \otimes \omega_k) = \sum_{i=0}^{k} (-1)^{(\varepsilon_{i-1}+1)(\varepsilon_k - \varepsilon_{i-1})} 1 \otimes \omega_i \otimes \cdots \otimes \omega_k \otimes \omega_0 \otimes \cdots \otimes \omega_{i-1}.$$

It is easily seen that $[B; d + b] = 0$ so that $(\mathscr{C}(\Omega); d + b; B)$ constitutes a mixed complex. Thus we define the *cyclic homology theory* of Ω as $HC(\Omega) = HC(\mathscr{C}(\Omega))$.

For a DGA Ω we define $\mathscr{D}(\Omega)$ to be the subspace of $\mathscr{C}(\Omega)$ generated by the images of the operators $S_i(f)$ and $R_i(f)$, where f is of degree 0 and

$$S_i(f)(\omega_0 \otimes \cdots \otimes \omega_k) = \omega_0 \otimes \cdots \otimes \omega_i \otimes f \otimes \omega_{i+1} \otimes \cdots \otimes \omega_k,$$

and

$$R_i(f) = [(b+d), S_i(f)].$$

Then, it is easy to show that the differentials $(b+d)$ and B map $\mathscr{D}(\Omega)$ into itself and hence $(\mathscr{D}(\Omega); (b+d); B)$ is a sub mixed complex of $(\mathscr{C}(\Omega); (b+d); B)$. Chen's normalized cyclic complex is then defined to be the quotient complex $\mathscr{C}(\Omega)/\mathscr{D}(\Omega)$.

2. The construction of a cyclic cycle

Let X be a smooth manifold, E a complex vector bundle over X, and let T denote the unit circle S^1. In this section, we construct an element $\psi(E, e)$ in the complex $\mathscr{C}(\Omega)$ where

$$\Omega = \Omega(X) \otimes \Omega_T(T) = \Omega_T(X \times T),$$

where T acts trivially on X and by multiplication on T. We shall show that $\psi(E, e)$ is $(d+b)+B$ closed in the reduced complex $\overline{\mathscr{C}(\Omega)} = \mathscr{C}(\Omega)/\mathscr{D}(\Omega)$ and hence defines a class in HC^{even} of the normalized complex. (Unfortunately, $\psi(E, e)$ is not closed in $\mathscr{C}(\Omega)$.) Moreover, $\psi(E, e)$ is an extension of the classical Chern character $\mathrm{Ch}(E)$; that is, if we view $\psi(E, e)$ as an equivariantly closed form on the free loop space LX (via the mapping σ) and restrict it to X, the space of constant loops, we obtain $\mathrm{Ch}(E)$. The above construction is due to Getzler, Jones, and Petrack [GJP].

A general element of Ω is of the form $\omega = \alpha + \beta\, dt$, where α, $\beta \in \Omega(X)$ and where dt is the standard 1-form on the unit circle T. Then we define the differential d_T on Ω by the following rule:

$$d_T(\alpha + \beta\, dt) = d\alpha + (-1)^{\deg \beta} \beta + d\beta\, dt.$$

Let E be a vector bundle over X given by an idempotent e, and let ∇_e denote the Levi-Civita connection on E. So, $\nabla_E = ed$. Let $E^\perp$ denote the vector bundle determined by the idempotent $e^\perp = 1 - e$. Similarly, let $\nabla_{E^\perp}$ denote the Levi-Civita connection on $E^\perp$. Let $\mathscr{A}$ be the matrix of 1-forms on X determined by the connection $\nabla_E \oplus \nabla_{E^\perp}$ on the trivial bundle $X \times \mathbb{C}^n = E \oplus E^\perp$, that is,

$$\mathscr{A} = ed + e^\perp d - d.$$

So if $s \in C^\infty(X)^n$, then we find

$$\begin{aligned}
\mathscr{A} s &= ed(es) + e^\perp d(e^\perp s) - ds \\
&= edes + eds + e^\perp de^\perp s + e^\perp ds - ds \\
&= (ede + e^\perp de^\perp)s + eds + (1-e)ds - ds \\
&= (ede + e^\perp de^\perp)s.
\end{aligned}$$

So we find

$$\mathscr{A} = ede + e^\perp de^\perp.$$

Let $\mathscr{R}$ denote the curvature associated to the connection $\nabla_E \oplus \nabla_{E^\perp} = d + \mathscr{A}$ on the trivial bundle $X \times \mathbb{C}^n$. So

$$\mathscr{R} = d\mathscr{A} + \mathscr{A} \wedge \mathscr{A} = dede.$$

Next we set $\mathscr{N} = \mathscr{A} - \mathscr{R}\,dt$.

Lemma 2.1. (1) $d_T e = [e, \mathscr{N}]$, (2) $d_T \mathscr{N} = -\mathscr{N} \wedge \mathscr{N}$.

Next we define $\mathscr{N}_k \in \mathscr{C}(\Omega)$ by

$$\mathscr{N}_k = e \otimes \underbrace{\mathscr{N} \otimes \mathscr{N} \otimes \cdots \otimes \mathscr{N}}_{k\text{-times}}.$$

Lemma 2.2. $d_T \mathscr{N}_{k-1} = -b\mathscr{N}_k$.

Finally, we define

$$\psi(E, e) = \sum_{k=0}^{\infty} \operatorname{Tr}(\mathscr{N}_k),$$

where Tr denotes the generalized trace map. It follows from the previous Lemma that $(d+b)\psi(E, e) = 0$. However, we note that $B\psi(E, e)$ is not equal to 0 in $\mathscr{C}(\Omega)$. On the other hand, since $B\psi(E, e) \in \mathscr{D}(\Omega)$ it follows that $\psi(E, e)$ is a closed form in the normalized complex $\mathscr{C}(\Omega)/\mathscr{D}(\Omega)$. In what follows, we shall prove that the class of $\psi(E, e)$ is independent of the choice of idempotent e and hence defines a Chern character on the Grothendieck group of vector bundles.

3. Triviality of ad on the cyclic homology of a DGA

In this section, we extend the definition of the adjoint action to a DGA Ω and prove that the induced action on the cyclic homology of Ω is trivial.

Define for all $X \in \Omega^0$,

$$\begin{aligned}\operatorname{ad}(X)(\omega_0 \otimes \cdots \otimes \omega_k) = &\sum_{i=0}^{k} \omega_0 \otimes \cdots \otimes [\omega_i, X] \otimes \cdots \otimes \omega_k \\ &+ \sum_{i=0}^{k} \omega \otimes \cdots \otimes \omega_i \otimes dX \otimes \omega_{i+1} \otimes \cdots \otimes \omega_k.\end{aligned}$$

We shall write

$$\operatorname{ad}(X)(\omega_0 \otimes \cdots \otimes \omega_k) = \alpha(X)(\omega_0 \otimes \cdots \otimes \omega_k) + \beta(X)(\omega_0 \otimes \cdots \otimes \omega_k).$$

Then clearly $\operatorname{ad}(X)$ is degree preserving, i.e., $\operatorname{ad}(X) : \mathscr{C}^p(\Omega) \to \mathscr{C}^p(\Omega)$.

Proposition 3.1. $\operatorname{ad}(X)$ *commutes with both the operators* B *and* $d + b$, *and hence defines an action of* Ω^0 *on the cyclic homology of* Ω.

Proposition 3.2. $\operatorname{ad}(X)$ *acts on the normalized cyclic complex* $\mathscr{C}(\Omega)/\mathscr{D}(\Omega)$. *In fact it maps* $\mathscr{D}(\Omega)$ *into itself.*

Proof. The proof is clear from the definition of $\operatorname{ad}(X)$, and the fact that $\mathscr{D}(\Omega)$ is generated by the images of the operators $S_i(f)$ and $R_i(f)$ defined in §1.

Proposition 3.3. *Define* h_X *by the following formula*:

$$h_X(\omega_0 \otimes \cdots \otimes \omega_k) = -\sum_{i=0}^{k} (-1)^{\varepsilon_i} \omega_0 \otimes \cdots \otimes \omega_i \otimes X \otimes \cdots \otimes \omega_k.$$

Then

$$(d+b)h_X + h_X(d+b) = \mathrm{ad}(X),$$

and

$$Bh_X + h_X B = 0.$$

Hence $\mathrm{ad}(X)$ *acts trivially on the cyclic homology of the DGA* Ω.

4. Triviality of Ad on the cyclic homology of a DGA

For each $g \in \Omega^{0^*}$ we define the mapping $\mathrm{Ad}(g) : \mathscr{C}^p(\Omega) \to \mathscr{C}^p(\Omega)$, by the formula

$$\mathrm{Ad}(g)(\omega_0 \otimes \cdots \otimes \omega_k) = \sum g^{-1}\omega_0 \underbrace{g^{-1}\,dg \otimes \cdots \otimes g^{-1}\,dg}_{i_0} \otimes \cdots$$
$$\otimes\, g^{-1}\omega_k g \otimes \underbrace{g^{-1}\,dg \otimes \cdots \otimes g^{-1}\,dg}_{i_k},$$

where the sum runs over all $(i_0, i_1, \dots, i_k) \in \mathbb{N}^k$. Then clearly $\mathrm{Ad}(g)$ is degree preserving.

Proposition 4.1. $\mathrm{Ad}(g)$ *commutes both with* B *and* $d+b$ *and hence defines an action of* Ω^{0^*} *on the cyclic homology* $HC(\Omega)$ *of the DGA* Ω.

Proof. Clearly $\mathrm{Ad}(g)$ commutes with B. That $\mathrm{Ad}(g)$ also commutes with $d+b$ follows from a straightforward calculation using the following two relations:

(1) $d(g^{-1}dg) = -g^{-1}\,dgg^{-1}\,dg$,

(2) $d(g^{-1}\omega_i g) = -g^{-1}\,dgg^{-1}\omega_i g + g^{-1}\,d\omega_i g + (-1)^{|\omega_i|} g^{-1}\omega_i gg^{-1}\,dg$.

Proposition 4.2. $\mathrm{Ad}(g)$ *acts on the normalized cyclic complex* $\mathscr{C}(\Omega)/\mathscr{D}(\Omega)$; *in fact, it maps* $\mathscr{D}(\Omega)$ *into itself.*

Proof. Clearly $\mathrm{Ad}(g)S_i(f)$ is contained in $\bigcup_k S_k(g^{-1}fg)$.

Theorem 4.3. $\mathrm{Ad}(g)$ *acts trivially on the cyclic homology* $HC(\Omega)$ *of the* DGA Ω.

Proof. Let τ be a $(b+d)+B$ cycle. Then τ will consist of a sum of terms of the form $\omega_0 \otimes \cdots \otimes \omega_k$. We shall begin by replacing $\omega_0 \otimes \cdots \otimes \omega_k$ by

$$\begin{pmatrix} \omega_0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \omega_k & 0 \\ 0 & 0 \end{pmatrix},$$

and g by $\left(\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix}\right)$ and show that $\mathrm{Ad}(\left(\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix}\right))$ acts trivially on the cyclic cohomology of the DGA M

$$M = M_{2\times 2}(\Omega^0) \oplus M_{2\times 2}(\Omega^1) \oplus M_{2\times 2}(\Omega^2) \oplus \cdots.$$

This will be enough since the generalized trace map $\mathrm{Tr} : M^{\otimes k} \to \Omega^{\otimes k}$ defined by

$$\mathrm{Tr}(M^1 \otimes \cdots \otimes M^k) = \sum M^1_{i_1, i_2} \otimes M^2_{i_2, i_3} \otimes \cdots \otimes M^k_{i_k, i_1},$$

induces a homomorphism in homology $HC(M; W) \to HC(\Omega; W)$.

Let $\mathscr{B}$ denote the algebra $\mathbb{Q}[x, y]/(x^2 + y^2 - 1)$. Then we define

$$m(x, y) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ y & x \end{pmatrix},$$

and observe that $m(0, 1) = I_{2\times 2}$ while $m(1, 0) = \left(\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix}\right)$.

Let τ now denote a $(b+d)+B$ cycle in $\mathscr{C}(M)$. Then $\omega = \mathrm{Ad}(m(x, y))\tau - \tau$ is a $(b+d)+B$ cycle over the DGA $M \otimes_{\mathbb{Q}} \mathscr{B}$. Then, if we define the derivation D of $\mathscr{B}$ by $D = yd/dx$; then $D(x) = y$ and $D(y) = -x$ and moreover the following relations are easily verified:

$$D\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

$$D\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} = -\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$D\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

$$D\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} = -\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

therefore, it follows that

$$\begin{aligned} D\omega = {} & \mathrm{Ad}\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right)\mathrm{ad}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\mathrm{Ad}\left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\right)\tau \\ & + \mathrm{Ad}\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix}\right) \\ & \times \mathrm{ad}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\mathrm{Ad}\left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\right)\tau. \end{aligned}$$

Therefore, since ad acts trivially on $HC(M \otimes \mathscr{B})$, it follows that $[D\omega] = 0$ in $HC(M \otimes \mathscr{B})$, where $[D\omega]$ denotes the equivalence class of $D\omega$ in $HC(M \otimes \mathscr{B})$.

But, in view of the identification

$$HC(M \otimes \mathscr{B}) \xrightarrow{\sim} HC(M) \otimes \mathscr{B},$$

where we view $M \otimes \mathscr{B}$ as a DGA over $\mathscr{B}$, we can write

$$[\omega] = \sum_i \alpha_i \otimes \beta_i,$$

where $\alpha_i \in HC(M)$ are linearly independent and $\beta_i \in \mathscr{B}$. Then,

$$[D\omega] = \sum_i \alpha_i \otimes D\beta_i = 0,$$

and so we can conclude that $D\beta_i = 0$ for all i. A straightforward calculation shows that if $\beta \in \mathscr{B} = \mathbb{Q}[x, y]/(x^2+y^2-1)$ and $D\beta = 0$, then β is a constant. This means that the evaluation map is constant on $[\omega] = [\mathrm{Ad}(m(x, y))\tau] - [\tau]$. Since, we say that $m(0, 1) = I_{2\times 2}$, it follows that $\mathrm{Ad}(m(1, 0)) = \mathrm{Ad}(\left(\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix}\right))$ acts trivially on $HC(M)$.

Theorem 4.4. *The $(d+b)+uB$ cycle $\psi(E, e)$ of the reduced cyclic complex $\mathscr{C}(\Omega)/\mathscr{D}(\Omega)$ defines a Chern character from the Grothendieck group $K_0(X)$, of all isomorphism classes of vector bundles over X, into $HC^{\text{even}}(\Omega)$ where $\Omega = \Omega_t(X \otimes T)$; that is, $\psi(e, E)$ depends only on the conjugacy class of the idempotent e.*

Proof. We recall, that

$$\psi(E, e) = \sum_{k=0}^{\infty} \mathrm{Tr}\mathscr{N}_k,$$

where

$$\mathscr{N}_k = e \otimes \underbrace{\mathscr{N} \otimes \mathscr{N} \otimes \otimes \cdots \otimes \mathscr{N}}_{k},$$

and where $\mathscr{N} = \mathscr{A} - \mathscr{R}\,dt$. $\mathscr{A} = ede + e^{\perp}de^{\perp}$ and $\mathscr{R} = de\,de$, are the connection and curvature form respectively of the connection $\nabla_E \oplus \nabla_{E^{\perp}}$ on the trivial bundle $X \times \mathbb{C}^n$. The transformation $e \to g^{-1}eg$ leads to the transformations $\mathscr{A} \to g^{-1}\mathscr{A}g + g^{-1}dg$ and $\mathscr{R} \to g^{-1}\mathscr{R}g$, and hence $\psi(E, g^{-1}eg) = \mathrm{Ad}(g)\psi(E, e)$. Since $\mathrm{Ad}(g)$ acts trivially on $HC(\Omega)$, it follows that the class of $\psi(E, e)$ is independent of the choice of idempotent e describing the vector bundle E. Moreover, it is easy to see from the definition of $\psi(E, e)$, that

$$\psi\left(E, \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}\right) = \psi(E, e),$$

and that

$$\psi\left(E \oplus F, \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right) = \psi(E, e) + \psi(F, f).$$

Therefore, since $K_0(X)$ is defined as the $\bigcup_n$ idempotents in $M_n(C^\infty(X))$, it follows that $\psi(E, e)$ defines a Chern character on $K_0(X)$.

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