

## A CHERN CHARACTER IN CYCLIC HOMOLOGY

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**ABSTRACT.** We show that inner derivations act trivially on the cyclic cohomology of the normalized cyclic complex  $\mathcal{E}(\Omega)/\mathcal{D}(\Omega)$  where  $\Omega$  is a differential graded algebra. This is then used to establish the fact that the map introduced in [GJ] defines a Chern character in  $K$  theory.

J.-M. Bismut recently extended the classical theory of Chern characters to the equivariant theory of the free loop space  $LX$ ; he explicitly constructed an  $S^1$ -invariant equivariantly closed form  $\omega(E, \nabla)$  on  $LX$  whose restriction to  $X$ , the space of constant loops, is  $\text{Ch}(E) = \text{tr}(\exp \nabla^2)$ . Closely related to the equivariant theory of the loop space of  $X$  is A. Connes' cyclic theory of the differential graded algebra  $\Omega$  of differential forms on  $X$ . More precisely, Chen's theory of iterated integrals allows us to view cyclic cycles of forms on  $X$  as equivariantly closed forms on the loop space  $LX$ . This is a generalization of the point of view that a one-form  $\omega$  on  $X$  can be viewed as a function on  $LX$  by  $\gamma \rightarrow \int_\gamma \omega$ .

This suggests the possibility of interpreting the classical theory of Chern characters in the context of cyclic homology i.e., of extending  $\text{Ch}(E)$  off the fixed point set  $X$  of the circle action on  $LX$  by way of a cyclic cycle over  $\Omega$ . In [GJP], Getzler, Jones, and Petrack, gave an explicit formula for a cyclic cycle  $\psi(E, e)$  which via iterated integrals maps to the Bismut Chern character  $\Omega(E)$  over  $LX$ . A priori, their construction of  $\psi(E, e)$  appears to depend on the choice of idempotent  $e$  describing the vector bundle  $E$ , but in fact, it is shown that  $\psi(E, e)$  depends only on the isomorphism class of  $E$  in  $K_0(X)$ , and hence defines the desired Chern character. This is essentially proved by establishing the triviality of certain group actions on the cyclic homology of a differential graded algebra  $\Omega$ . These actions are natural generalizations of the usual adjoint action  $\text{Ad}(g)$ , given by conjugation by an invertible element  $g$ , and its infinitesimal analogue  $\text{ad}$ , given by inner derivation, to Connes' cyclic bar complex of  $\Omega$ .

### 1. CYCLIC HOMOLOGY

In this section we review the definition of the cyclic homology of a DGA.

According to C. Kassel, by a *mixed complex* we mean a triplet  $(\mathcal{E}^*; d; B)$  where  $\mathcal{E}^*$  is a graded algebra and where  $d: \mathcal{E}^* \rightarrow \mathcal{E}^{*+1}$ , and  $B: \mathcal{E}^* \rightarrow \mathcal{E}^{*-1}$ , are derivations of degrees  $+1$  and  $-1$  respectively, and subject to the rela-

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tion  $[d, B] = 0$ . The associated chain complex  $(\mathcal{E}_*^\lambda; d + B)$ , called the *cyclic complex* is defined as follows:

$$\mathcal{E}_n^\lambda = \mathcal{E}^n \oplus \mathcal{E}^{n+2} \oplus \mathcal{E}^{n+4} \oplus \dots,$$

and

$$d_n : \mathcal{E}_n^\lambda \rightarrow \mathcal{E}_{n-1}^\lambda,$$

by

$$d_n(\omega^n, \omega^{n+2}, \omega^{n+4}, \dots) = (B\omega^n, d\omega^n + B\omega^{n+2}, d\omega^{n+2} + B\omega^{n+4}, \dots),$$

in short  $d_n = d + B$ . We define the *cyclic homology*  $HC_*(\mathcal{E}^*)$  of the mixed complex  $(\mathcal{E}^*; d; B)$  to be the homology of the associated chain complex. Similarly, we define the *even* and *odd cyclic homology groups*  $HC^{\text{even}}(\mathcal{E}^*)$  and  $HC^{\text{odd}}(\mathcal{E}^*)$ .

We now apply this construction to define the *cyclic homology theories of a DGA*. Let  $(\Omega; d)$  be a DGA over  $\mathbb{C}$  with unit 1. Then we set

$$\mathcal{E}(\Omega) = \sum_p \Omega \otimes (I\Omega)^{\otimes p},$$

where  $I\Omega = \Omega/\mathbb{C}$ . We make  $\mathcal{E}(\Omega)$  into a graded algebra by defining

$$\text{deg}(\omega_0 \otimes \dots \otimes \omega_k) = \sum_{i=0}^k |\omega_i| - k.$$

(The product structure is of course given by the shuffle product.) Next, we define operators  $d$  and  $b$

$$d, b : \mathcal{E}^p(\Omega) \rightarrow \mathcal{E}^{p+1}(\Omega),$$

by

$$d(\omega_0 \otimes \dots \otimes \omega_k) = - \sum_{i=0}^k (-1)^{\varepsilon_{i-1}} \omega_0 \otimes \dots \otimes \omega_{i-1} \otimes d\omega_i \otimes \omega_{i+1} \otimes \dots \otimes \omega_k,$$

and

$$\begin{aligned} b(\omega_0 \otimes \dots \otimes \omega_k) = & - \sum_{i=0}^k (-1)^{\varepsilon_i} \omega_0 \otimes \dots \otimes \omega_i \omega_{i+1} \otimes \dots \otimes \omega_k \\ & + (-1)^{(|\omega_k|-1)\varepsilon_{k-1}} \omega_k \omega_0 \otimes \omega_1 \otimes \dots \otimes \omega_{k-1}, \end{aligned}$$

where

$$\varepsilon_i = |\omega_0| + \dots + |\omega_i| - i.$$

Then, it is easy to show that  $(d + b)^2 = 0$ , in fact that  $d^2 = b^2 = db + bd = 0$ . We also define the *A. Connes operator*  $B_0 B : \mathcal{C}^p(\Omega) \rightarrow \mathcal{E}^{p-1}(\Omega)$  by the formula

$$B(\omega_0 \otimes \dots \otimes \omega_k) = \sum_{i=0}^k (-1)^{(\varepsilon_{i-1}+1)(\varepsilon_k-\varepsilon_{i-1})} 1 \otimes \omega_i \otimes \dots \otimes \omega_k \otimes \omega_0 \otimes \dots \otimes \omega_{i-1}.$$

It is easily seen that  $[B; d + b] = 0$  so that  $(\mathcal{E}(\Omega); d + b; B)$  constitutes a mixed complex. Thus we define the *cyclic homology theory of  $\Omega$*  as  $HC(\Omega) = HC(\mathcal{E}(\Omega))$ .

For a DGA  $\Omega$  we define  $\mathcal{D}(\Omega)$  to be the subspace of  $\mathcal{E}(\Omega)$  generated by the images of the operators  $S_i(f)$  and  $R_i(f)$ , where  $f$  is of degree 0 and

$$S_i(f)(\omega_0 \otimes \cdots \otimes \omega_k) = \omega_0 \otimes \cdots \otimes \omega_i \otimes f \otimes \omega_{i+1} \otimes \cdots \otimes \omega_k,$$

and

$$R_i(f) = [(b + d), S_i(f)].$$

Then, it is easy to show that the differentials  $(b + d)$  and  $B$  map  $\mathcal{D}(\Omega)$  into itself and hence  $(\mathcal{D}(\Omega); (b + d); B)$  is a sub mixed complex of  $(\mathcal{E}(\Omega); (b + d); B)$ . Chen's normalized cyclic complex is then defined to be the quotient complex  $\mathcal{E}(\Omega)/\mathcal{D}(\Omega)$ .

### 2. THE CONSTRUCTION OF A CYCLIC CYCLE

Let  $X$  be a smooth manifold,  $E$  a complex vector bundle over  $X$ , and let  $T$  denote the unit circle  $S^1$ . In this section, we construct an element  $\psi(E, e)$  in the complex  $\mathcal{E}(\Omega)$  where

$$\Omega = \Omega(X) \otimes \Omega_T(T) = \Omega_T(X \times T),$$

where  $T$  acts trivially on  $X$  and by multiplication on  $T$ . We shall show that  $\psi(E, e)$  is  $(d+b)+B$  closed in the reduced complex  $\overline{\mathcal{E}(\Omega)} = \mathcal{E}(\Omega)/\mathcal{D}(\Omega)$  and hence defines a class in  $HC^{\text{even}}$  of the normalized complex. (Unfortunately,  $\psi(E, e)$  is not closed in  $\mathcal{E}(\Omega)$ .) Moreover,  $\psi(E, e)$  is an extension of the classical Chern character  $\text{Ch}(E)$ ; that is, if we view  $\psi(E, e)$  as an equivariantly closed form on the free loop space  $LX$  (via the mapping  $\sigma$ ) and restrict it to  $X$ , the space of constant loops, we obtain  $\text{Ch}(E)$ . The above construction is due to Getzler, Jones, and Petrack [GJP].

A general element of  $\Omega$  is of the form  $\omega = \alpha + \beta dt$ , where  $\alpha, \beta \in \Omega(X)$  and where  $dt$  is the standard 1-form on the unit circle  $T$ . Then we define the differential  $d_T$  on  $\Omega$  by the following rule:

$$d_T(\alpha + \beta dt) = d\alpha + (-1)^{\text{deg } \beta} \beta + d\beta dt.$$

Let  $E$  be a vector bundle over  $X$  given by an idempotent  $e$ , and let  $\nabla_e$  denote the Levi-Civita connection on  $E$ . So,  $\nabla_E = ed$ . Let  $E^\perp$  denote the vector bundle determined by the idempotent  $e^\perp = 1 - e$ . Similarly, let  $\nabla_{E^\perp}$  denote the Levi-Civita connection on  $E^\perp$ . Let  $\mathcal{A}$  be the matrix of 1-forms on  $X$  determined by the connection  $\nabla_E \oplus \nabla_{E^\perp}$  on the trivial bundle  $X \times \mathbb{C}^n = E \oplus E^\perp$ , that is,

$$\mathcal{A} = ed + e^\perp d - d.$$

So if  $s \in C^\infty(X)^n$ , then we find

$$\begin{aligned} \mathcal{A}s &= ed(es) + e^\perp d(e^\perp s) - ds \\ &= edes + eds + e^\perp de^\perp s + e^\perp ds - ds \\ &= (ede + e^\perp de^\perp)s + eds + (1 - e)ds - ds \\ &= (ede + e^\perp de^\perp)s. \end{aligned}$$

So we find

$$\mathcal{A} = ede + e^\perp de^\perp.$$

Let  $\mathcal{R}$  denote the curvature associated to the connection  $\nabla_E \oplus \nabla_{E^\perp} = d + \mathcal{A}$  on the trivial bundle  $X \times \mathbb{C}^n$ . So

$$\mathcal{R} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = dede.$$

Next we set  $\mathcal{N} = \mathcal{A} - \mathcal{R} dt$ .

**Lemma 2.1.** (1)  $d_T e = [e, \mathcal{N}]$ , (2)  $d_T \mathcal{N} = -\mathcal{N} \wedge \mathcal{N}$ .

Next we define  $\mathcal{N}_k \in \mathcal{E}(\Omega)$  by

$$\mathcal{N}_k = e \otimes \underbrace{\mathcal{N} \otimes \mathcal{N} \otimes \cdots \otimes \mathcal{N}}_{k\text{-times}}.$$

**Lemma 2.2.**  $d_T \mathcal{N}_{k-1} = -b\mathcal{N}_k$ .

Finally, we define

$$\psi(E, e) = \sum_{k=0}^{\infty} \text{Tr}(\mathcal{N}_k),$$

where  $\text{Tr}$  denotes the generalized trace map. It follows from the previous Lemma that  $(d + b)\psi(E, e) = 0$ . However, we note that  $B\psi(E, e)$  is not equal to 0 in  $\mathcal{E}(\Omega)$ . On the other hand, since  $B\psi(E, e) \in \mathcal{D}(\Omega)$  it follows that  $\psi(E, e)$  is a closed form in the normalized complex  $\mathcal{E}(\Omega)/\mathcal{D}(\Omega)$ . In what follows, we shall prove that the class of  $\psi(E, e)$  is independent of the choice of idempotent  $e$  and hence defines a Chern character on the Grothendieck group of vector bundles.

### 3. TRIVIALITY OF $\text{ad}$ ON THE CYCLIC HOMOLOGY OF A DGA

In this section, we extend the definition of the adjoint action to a DGA  $\Omega$  and prove that the induced action on the cyclic homology of  $\Omega$  is trivial.

Define for all  $X \in \Omega^0$ ,

$$\begin{aligned} \text{ad}(X)(\omega_0 \otimes \cdots \otimes \omega_k) &= \sum_{i=0}^k \omega_0 \otimes \cdots \otimes [\omega_i, X] \otimes \cdots \otimes \omega_k \\ &+ \sum_{i=0}^k \omega \otimes \cdots \otimes \omega_i \otimes dX \otimes \omega_{i+1} \otimes \cdots \otimes \omega_k. \end{aligned}$$

We shall write

$$\text{ad}(X)(\omega_0 \otimes \cdots \otimes \omega_k) = \alpha(X)(\omega_0 \otimes \cdots \otimes \omega_k) + \beta(X)(\omega_0 \otimes \cdots \otimes \omega_k).$$

Then clearly  $\text{ad}(X)$  is degree preserving, i.e.,  $\text{ad}(X) : \mathcal{E}^p(\Omega) \rightarrow \mathcal{E}^p(\Omega)$ .

**Proposition 3.1.**  $\text{ad}(X)$  commutes with both the operators  $B$  and  $d + b$ , and hence defines an action of  $\Omega^0$  on the cyclic homology of  $\Omega$ .

**Proposition 3.2.**  $\text{ad}(X)$  acts on the normalized cyclic complex  $\mathcal{E}(\Omega)/\mathcal{D}(\Omega)$ . In fact it maps  $\mathcal{D}(\Omega)$  into itself.

*Proof.* The proof is clear from the definition of  $\text{ad}(X)$ , and the fact that  $\mathcal{D}(\Omega)$  is generated by the images of the operators  $S_i(f)$  and  $R_i(f)$  defined in §1.

**Proposition 3.3.** Define  $h_X$  by the following formula:

$$h_X(\omega_0 \otimes \cdots \otimes \omega_k) = - \sum_{i=0}^k (-1)^{e_i} \omega_0 \otimes \cdots \otimes \omega_i \otimes X \otimes \cdots \otimes \omega_k.$$

Then

$$(d + b)h_X + h_X(d + b) = \text{ad}(X),$$

and

$$Bh_X + h_X B = 0.$$

Hence  $\text{ad}(X)$  acts trivially on the cyclic homology of the DGA  $\Omega$ .

4. TRIVIALITY OF Ad ON THE CYCLIC HOMOLOGY OF A DGA

For each  $g \in \Omega^{0^*}$  we define the mapping  $\text{Ad}(g) : \mathcal{C}^p(\Omega) \rightarrow \mathcal{C}^p(\Omega)$ , by the formula

$$\begin{aligned} \text{Ad}(g)(\omega_0 \otimes \cdots \otimes \omega_k) = & \sum g^{-1} \omega_0 \underbrace{g^{-1} dg \otimes \cdots \otimes g^{-1} dg}_{i_0} \otimes \cdots \\ & \otimes g^{-1} \omega_k g \otimes \underbrace{g^{-1} dg \otimes \cdots \otimes g^{-1} dg}_{i_k}, \end{aligned}$$

where the sum runs over all  $(i_0, i_1, \dots, i_k) \in \mathbb{N}^k$ . Then clearly  $\text{Ad}(g)$  is degree preserving.

**Proposition 4.1.**  $\text{Ad}(g)$  commutes both with  $B$  and  $d + b$  and hence defines an action of  $\Omega^{0^*}$  on the cyclic homology  $HC(\Omega)$  of the DGA  $\Omega$ .

*Proof.* Clearly  $\text{Ad}(g)$  commutes with  $B$ . That  $\text{Ad}(g)$  also commutes with  $d + b$  follows from a straightforward calculation using the following two relations:

- (1)  $d(g^{-1}dg) = -g^{-1}dg g^{-1}dg$ ,
- (2)  $d(g^{-1}\omega_i g) = -g^{-1}dg g^{-1}\omega_i g + g^{-1}d\omega_i g + (-1)^{|\omega_i|}g^{-1}\omega_i g g^{-1}dg$ .

**Proposition 4.2.**  $\text{Ad}(g)$  acts on the normalized cyclic complex  $\mathcal{C}(\Omega)/\mathcal{D}(\Omega)$ ; in fact, it maps  $\mathcal{D}(\Omega)$  into itself.

*Proof.* Clearly  $\text{Ad}(g)S_i(f)$  is contained in  $\bigcup_k S_k(g^{-1}fg)$ .

**Theorem 4.3.**  $\text{Ad}(g)$  acts trivially on the cyclic homology  $HC(\Omega)$  of the DGA  $\Omega$ .

*Proof.* Let  $\tau$  be a  $(b + d) + B$  cycle. Then  $\tau$  will consist of a sum of terms of the form  $\omega_0 \otimes \cdots \otimes \omega_k$ . We shall begin by replacing  $\omega_0 \otimes \cdots \otimes \omega_k$  by

$$\begin{pmatrix} \omega_0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \omega_k & 0 \\ 0 & 0 \end{pmatrix},$$

and  $g$  by  $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$  and show that  $\text{Ad}(\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix})$  acts trivially on the cyclic cohomology of the DGA  $M$

$$M = M_{2 \times 2}(\Omega^0) \oplus M_{2 \times 2}(\Omega^1) \oplus M_{2 \times 2}(\Omega^2) \oplus \cdots.$$

This will be enough since the generalized trace map  $\text{Tr} : M^{\otimes k} \rightarrow \Omega^{\otimes k}$  defined by

$$\text{Tr}(M^1 \otimes \cdots \otimes M^k) = \sum M_{i_1, i_2}^1 \otimes M_{i_2, i_3}^2 \otimes \cdots \otimes M_{i_k, i_1}^k,$$

induces a homomorphism in homology  $HC(M; W) \rightarrow HC(\Omega; W)$ .

Let  $\mathcal{B}$  denote the algebra  $\mathbb{Q}[x, y]/(x^2 + y^2 - 1)$ . Then we define

$$m(x, y) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ y & x \end{pmatrix},$$

and observe that  $m(0, 1) = I_{2 \times 2}$  while  $m(1, 0) = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ .

Let  $\tau$  now denote a  $(b+d)+B$  cycle in  $\mathcal{E}(M)$ . Then  $\omega = \text{Ad}(m(x, y))\tau - \tau$  is a  $(b+d)+B$  cycle over the DGA  $M \otimes_{\mathbb{Q}} \mathcal{B}$ . Then, if we define the derivation  $D$  of  $\mathcal{B}$  by  $D = yd/dx$ ; then  $D(x) = y$  and  $D(y) = -x$  and moreover the following relations are easily verified:

$$\begin{aligned} D \begin{pmatrix} x & -y \\ y & x \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \\ D \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} &= - \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ D \begin{pmatrix} x & y \\ -y & x \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \\ D \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} &= - \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

therefore, it follows that

$$\begin{aligned} D\omega &= \text{Ad} \left( \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) \text{ad} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \text{Ad} \left( \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \right) \tau \\ &+ \text{Ad} \left( \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \right) \\ &\times \text{ad} \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \text{Ad} \left( \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \right) \tau. \end{aligned}$$

Therefore, since  $\text{ad}$  acts trivially on  $HC(M \otimes \mathcal{B})$ , it follows that  $[D\omega] = 0$  in  $HC(M \otimes \mathcal{B})$ , where  $[D\omega]$  denotes the equivalence class of  $D\omega$  in  $HC(M \otimes \mathcal{B})$ .

But, in view of the identification

$$HC(M \otimes \mathcal{B}) \simeq HC(M) \otimes \mathcal{B},$$

where we view  $M \otimes \mathcal{B}$  as a DGA over  $\mathcal{B}$ , we can write

$$[\omega] = \sum_i \alpha_i \otimes \beta_i,$$

where  $\alpha_i \in HC(M)$  are linearly independent and  $\beta_i \in \mathcal{B}$ . Then,

$$[D\omega] = \sum_i \alpha_i \otimes D\beta_i = 0,$$

and so we can conclude that  $D\beta_i = 0$  for all  $i$ . A straightforward calculation shows that if  $\beta \in \mathcal{B} = \mathbb{Q}[x, y]/(x^2 + y^2 - 1)$  and  $D\beta = 0$ , then  $\beta$  is a constant. This means that the evaluation map is constant on  $[\omega] = [\text{Ad}(m(x, y))\tau] - [\tau]$ . Since, we say that  $m(0, 1) = I_{2 \times 2}$ , it follows that  $\text{Ad}(m(1, 0)) = \text{Ad}(\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix})$  acts trivially on  $HC(M)$ .

**Theorem 4.4.** *The  $(d + b) + uB$  cycle  $\psi(E, e)$  of the reduced cyclic complex  $\mathcal{E}(\Omega)/\mathcal{D}(\Omega)$  defines a Chern character from the Grothendieck group  $K_0(X)$ , of all isomorphism classes of vector bundles over  $X$ , into  $HC^{\text{even}}(\Omega)$  where  $\Omega = \Omega_i(X \otimes T)$ ; that is,  $\psi(e, E)$  depends only on the conjugacy class of the idempotent  $e$ .*

*Proof.* We recall, that

$$\psi(E, e) = \sum_{k=0}^{\infty} \text{Tr } \mathcal{N}_k,$$

where

$$\mathcal{N}_k = e \otimes \underbrace{\mathcal{N} \otimes \mathcal{N} \otimes \cdots \otimes \mathcal{N}}_k,$$

and where  $\mathcal{N} = \mathcal{A} - \mathcal{R} dt$ .  $\mathcal{A} = ede + e^\perp de^\perp$  and  $\mathcal{R} = dede$ , are the connection and curvature form respectively of the connection  $\nabla_E \oplus \nabla_{E^\perp}$  on the trivial bundle  $X \times \mathbb{C}^n$ . The transformation  $e \rightarrow g^{-1}eg$  leads to the transformations  $\mathcal{A} \rightarrow g^{-1}\mathcal{A}g + g^{-1}dg$  and  $\mathcal{R} \rightarrow g^{-1}\mathcal{R}g$ , and hence  $\psi(E, g^{-1}eg) = \text{Ad}(g)\psi(E, e)$ . Since  $\text{Ad}(g)$  acts trivially on  $HC(\Omega)$ , it follows that the class of  $\psi(E, e)$  is independent of the choice of idempotent  $e$  describing the vector bundle  $E$ . Moreover, it is easy to see from the definition of  $\psi(E, e)$ , that

$$\psi\left(E, \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}\right) = \psi(E, e),$$

and that

$$\psi\left(E \oplus F, \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right) = \psi(E, e) + \psi(F, f).$$

Therefore, since  $K_0(X)$  is defined as the  $\bigcup_n$  idempotents in  $M_n(C^\infty(X))$ , it follows that  $\psi(E, e)$  defines a Chern character on  $K_0(X)$ .

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