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A SIMPLIFED TRACE FORMULA FOR HECKE OPERATORS FOR $\Gamma_0(N)$

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ABSTRACT. Let N and n be relatively prime positive integers, let χ be a Dirichlet character modulo N, and let k be a positive integer. Denote by $S_k(N, \chi)$ the space of cusp forms on $\Gamma_0(N)$ of weight k and character χ, a space denoted simply $S_k(N)$ when χ is the trivial character. Beginning with Hijikata's formula for the trace of T_n acting on $S_k(N, \chi)$, we develop a formula which essentially reduces the computation of this trace to looking up values in a table. From this formula we develop very simple formulas for (1) the dimension of $S_k(N, \chi)$ and (2) the trace of T_n acting on $S_k(N)$.

PRELIMINARIES

For each positive integer N, let

$$\Gamma_0(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \middle| a, b, c, d \in \mathbf{Z}, \det(\gamma) = 1 \right\};$$

$\Gamma_0(N)$ is a congruence subgroup of $SL_2(\mathbf{Z})$. Let χ be a (Dirichlet) character mod N. Suppose $N = \prod_{l|N} l^{\nu_l}$, where each l is a prime and $\nu_l = \mathrm{ord}_l(N)$. Then χ can be written as a product $\chi = \prod_{l|N} \chi_l$ of characters, where for each prime $l \mid N$, χ_l is a character mod l^{ν_l}. The *exponential conductor* $e = e(\chi_l)$ is the smallest value e such that χ_l is a character mod l^e; note that $e = e(\chi_l) \leq \nu_l$. If χ is a character and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $a, b, c, d \in \mathbf{Z}$, then by $\chi(\gamma)$ we mean $\chi(a)$. In this paper we will use "$\mid$" and "$\nmid$" for "divides" and "does not divide," respectively.

Fix a positive integer k. For any complex-valued function f and matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbf{R}$ and $\det(\gamma) > 0$ define

$$f \mid \gamma = (\det(\gamma))^{k/2}(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

(where we take the positive root if k is odd).

Let $\mathscr{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$ denote the complex upper half plane, and let f be any complex-valued function on $\mathscr{H}$. The *cusps* of $\Gamma_0(N)$ are the rational

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numbers, along with the point $i\infty$ at infinity. We say that f is a *cusp form on* $\Gamma_0(N)$ *of weight* k *and character* χ if f satisfies

(i) f is holomorphic on $\mathscr{H}$,
(ii) f is 0 at each cusp,
(iii) $f \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \chi(a)^{-1} f$ for each $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.

See [Sha, A-L, or Li] for details. The space of cuspforms on $\Gamma_0(N)$ of weight k and character χ is denoted by $S_k(N, \chi)$, or by $S_k(N)$ if χ is the trivial character.

For each n with $(n, N) = 1$, let T_n be the standard Hecke operator whose action on $S_k(N, \chi)$ is defined by

$$f \mid T_n = n^{k-1} \sum_{a,d} \sum_{b=0}^{d-1} \chi(a) f\left(\frac{a\tau + b}{d}\right) d^{-k}, \tag{1}$$

where the first sum is over all pairs of integers a, d satisfying $a > 0$, $ad = n$, and $(a, N) = 1$. Note this is the same as the T'_n used by Shimura (see 3.5.7 of [Sha]), and therefore by Hijikata, Pizer, and Shemanske in [H-P-S_1, H-P-S_2]. If $n = p$, where p is a prime not dividing N and χ is the trivial character, then our T_p is the same as the T_p operator of Atkin and Lehner in [A-L]. (Note however that our weight k is *twice* the weight k of Atkin and Lehner.)

The simplified formula; Applications

We begin by stating the version of the trace formula for the operator T_n acting on $S_k(N, \chi)$ as given in Theorem 2.2 of [H-P-S_1] and also in Theorem 2.2 of [H-P-S_2]. Denote this trace by $\mathrm{tr}_{N,\chi,k}\, T_n$.

Theorem 1 (Hijikata-Pizer-Shemanske). *Let* k *be an integer,* $k \geq 2$. *Let* χ *be a character* mod N *and assume* $(-1)^k \chi(-1) = 1$. *Write* $\chi = \prod_{l|N} \chi_l$, *where for each prime* l *dividing* N, χ_l *is a character* mod l^ν, *where* $\nu = \mathrm{ord}_l(N)$. *Then for* $(n, N) = 1$ *we have*

$$\begin{aligned} \mathrm{tr}_{N,\chi,k}\, T_n = & -\sum_s a(s) \sum_f b(s, f) \prod_{l|N} c'_\chi(s, f, l) \\ & + \delta(\chi) \deg(T_n) + \delta(\sqrt{n}) \frac{k-1}{12} N \prod_{l|N} \left(1 + \frac{1}{l}\right) \\ & - \delta(\sqrt{n}) \frac{\sqrt{n}}{2} \prod_{l|N} \mathrm{par}(l), \end{aligned}$$

where

$$\delta(\chi) = \begin{cases} 1 & \textit{if } k = 2 \textit{ and } \chi \textit{ is trivial}, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\delta(\sqrt{n}) = \begin{cases} n^{k/2-1} \chi(\sqrt{n}) & \textit{if } n \textit{ is a perfect square}, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\mathrm{par}(l) = \begin{cases} 2l^{\nu-e} & \textit{if } e \geq \mu + 1, \\ l^\mu + l^{\mu-1} & \textit{if } e \leq \mu \textit{ and } \nu \textit{ is even}, \\ 2l^\mu & \textit{if } e \leq \mu \textit{ and } \nu \textit{ is odd}. \end{cases}$$

Here for a fixed prime $l \mid N$, $\nu = \operatorname{ord}_l(N)$, $\mu = \left[\frac{\nu}{2}\right]$, *and* $e = e(\chi_l)$.

The meanings of s, $a(s)$, $b(s, f)$, *and* $c'_\chi(s, f, l)$ *are given as follows*:

Let s run over all integers such that $s^2 - 4n$ *is a positive square or any negative integer. Hence for some positive integer* t *and squarefree negative integer* m, $s^2 - 4n$ *has one of the following forms which we classify into the cases* (h) *or* (e) *as follows*:

$$s^2 - 4n = \begin{cases} t^2, & & \text{(h)} \\ t^2 m, & 0 > m \equiv 1 \pmod 4, & \text{(e)} \\ t^2 4m, & 0 > m \equiv 2, 3 \pmod 4. & \text{(e)} \end{cases}$$

Let $\Phi(X) = \Phi_s(X) = X^2 - sX + n$ *and let* x *and* y *be the roots in* $\mathbf{C}$ *of* $\Phi(X) = 0$. *Corresponding to the classification of* s, *put*

$$a(s) = \begin{cases} (\min\{|x|, |y|\})^{k-1} |x - y|^{-1} \operatorname{sgn}(x)^k, & \text{(h)} \\ \frac{1}{2}(x^{k-1} - y^{k-1})/(x - y). & \text{(e)} \end{cases}$$

For each fixed s, *let* f *run over all positive divisors of t and let*

$$b(s, f) = \begin{cases} \frac{1}{2}\phi((s^2 - 4n)^{1/2}/f), & \text{(h)} \\ h((s^2 - 4n)/f^2)/\omega((s^2 - 4n)/f^2), & \text{(e)} \end{cases}$$

where ϕ *is Euler's function and* $h(d)$ (*respectively* $\omega(d)$) *denotes the class number of locally principal ideals* (*resp.* $\frac{1}{2}$ *the cardinality of the unit group*) *of the order of* $\mathbf{Q}(\sqrt{d})$ *with discriminant* d.

Fix a pair (s, f) *and let* l *be a prime divisor of* N; *let* $\nu = \operatorname{ord}_l(N)$ *and* $\rho = \operatorname{ord}_l(f)$. *Put*

$$\widetilde{A} = \{x \in \mathbf{Z} \mid \Phi(x) \equiv 0\,(l^{\nu+2\rho}),\, 2x \equiv s(l^\rho)\},$$
$$\widetilde{B} = \{x \in \widetilde{A} \mid \Phi(x) \equiv 0\,(l^{\nu+2\rho+1})\}.$$

Let $A_\rho = A(s, f, l)$ (*resp.* $B_\rho = B(s, f, l)$) *be a complete set of representatives of* $\widetilde{A}$ (*resp.* $\widetilde{B}$) mod $l^{\nu+\rho}$, *and let* $B'_\rho = B'(s, f, l) = \{s - z \mid z \in B_\rho\}$. *Then*

$$c'_\chi(s, f, l) = \begin{cases} \sum_x \chi_l(x) & \text{if } (s^2 - 4n)/f^2 \not\equiv 0\,(l), \\ \sum_x \chi_l(x) + \sum_y \chi_l(y) & \text{if } (s^2 - 4n)/f^2 \equiv 0\,(l), \end{cases}$$

where x (*resp.* y) *runs over all elements of* A_ρ (*resp.* B'_ρ). *This ends the statement of the theorem.*

Proof. See [Hij, H-P-S_1, H-P-S_2].

We introduce a classification of prime numbers to be used throughout this paper. Fix integers n and s, with $n \geq 1$, such that $s^2 - 4n$ is a positive square or any negative integer and write $s^2 - 4n$ as one of t^2, $t^2 m$, or $t^2 4m$ as in the statement of Theorem 1. Let l be any prime that divides either N or t, and classify l into one of six cases, A, B, C, D, E, or F, depending on how l

divides s^2-4n and whether or not l is odd, as follows:

$$l \text{ is case} \begin{cases} A & \text{if } s^2-4n=l^{2a}d^2,\ l \text{ is odd, and } d \text{ is a unit of } \mathbf{Z}_l, \\ B & \text{if } s^2-4n=l^{2a}u,\ l \text{ is odd, and } u \text{ is a nonsquare unit of } \mathbf{Z}_l, \\ C & \text{if } s^2-4n=l^{2a+1}u,\ l \text{ is odd, and } u \text{ is a unit of } \mathbf{Z}_l, \text{ or} \\ & \quad s^2-4n=l^{2a}4w,\ l=2,\ w\in\mathbf{Z}_2,\ w\equiv 2 \pmod 4, \\ D & \text{if } s^2-4n=2^{2a}d^2,\ l=2, \text{ and } d \text{ is a unit of } \mathbf{Z}_2, \\ E & \text{if } s^2-4n=2^{2a}u,\ l=2,\ u\in\mathbf{Z}_2, \text{ and } u\equiv 5 \pmod 8), \\ F & \text{if } s^2-4n=2^{2a}4w,\ l=2,\ w\in\mathbf{Z}_2, \text{ and } w\equiv 3 \pmod 4). \end{cases}$$

We will sometimes denote the case into which l falls by Case(l). Note that $a=\operatorname{ord}_l(t)$, where a is the 'a' which appears in the expression for s^2-4n, in whatever case l is actually classified; we will sometimes write $a_l(s^2-4n)$ to mean this a.

Also let us introduce a convention to be adhered to throughout this paper:

Convention A. Let l be a prime and let n be any integer. We agree that any expression of the form l^n or l^n-1 is taken to be 0 if $n<0$.

We are ready to state the new version of the trace formula.

Theorem 2. *Let k be an integer, $k\geq 2$. Let χ be a character* mod N *and assume $(-1)^k\chi(-1)=1$. Write $\chi=\prod_{l|N}\chi_l$, where for each prime l dividing N, χ_l is a character* mod l^ν, *where $\nu=\operatorname{ord}_l(N)$. Then for $(n, N)=1$ we have*

$$\begin{aligned} \operatorname{tr}_{N,\chi,k} T_n = & -\sum_s \left(a(s)b(s) \prod_{l|t,\, l\nmid N} \gamma(s,l) \prod_{l|N} c(s,l) \right) \\ & + \delta(\chi)\deg(T_n) + \delta(\sqrt{n})\frac{k-1}{12} N \prod_{l|N}\left(1+\frac{1}{l}\right) \\ & - \delta(\sqrt{n})\frac{\sqrt{n}}{2}\prod_{l|N}\operatorname{par}(l), \end{aligned}$$

where $\delta(\chi)$, $\delta(\sqrt{n})$, and par(l) *are exactly the same as in Theorem* 1.

The meanings of s, $a(s)$, $b(s)$, t, $\gamma(s, l)$, and $c(s, l)$ are as follows:

Let s, $a(s)$, and t be exactly as in Theorem 1. *Now fix s and write s^2-4n as one of t^2, t^2m, t^24m as in Theorem* 1. *Let*

$$b(s)=\begin{cases} \frac{1}{2} & \text{if } s^2-4n=t^2, \\ h(m)/\omega(m) & \text{if } s^2-4n=t^2m \text{ or } t^24m,\quad m<0, \end{cases}$$

where $h(m)$ is the class number of $\mathbf{Q}(\sqrt{m})$ and $\omega(m)$ is one-half the cardinality of the unit group of $\mathbf{Q}(\sqrt{m})$.

Keeping s fixed, now fix a prime l with $l\mid t$, $l\nmid N$, and, according to the classification of l, let $a=a_l(s^2-4n)$ be the 'a' which appears in the expression for s^2-4n and define

$$\gamma(s,l)=\begin{cases} l^a & \text{if } l \text{ is a case } A \text{ or } D \text{ prime}, \\ (l^a(l+1)-2)/(l-1) & \text{if } l \text{ is a case } B \text{ or } E \text{ prime}, \\ (l^{a+1}-1)/(l-1) & \text{if } l \text{ is a case } C \text{ or } F \text{ prime.} \end{cases}$$

Case(l)	ν	k_1	k_2	k_3	g	k_4	ε	k_5	k_6
A	*even*	1	0	$l-1$	1	$l+1$	0	$d(\mu+1, e)d(\mu+1, a)$	$d(e, a)d(\mu, a)$
	odd					2		0	0
B	*even*	0	0	$l+1$	1	$l+1$	0	$d(\mu+1, e)d(\mu+1, a)$	$d(e, a)d(\mu, a)$
	odd					2		0	0
C	*even*	0	0	1	1	$l+1$	1	$d(\mu+1, e)d(\mu, a)$	0
	odd					$2l$	0	0	$d(e, a+1)d(\mu, a)$
D	*even*	1	$d(e, a)\min_0(a-\mu+1, 2)$	1	0	$l+1$	-1	$d(\mu+1, e)d(\mu+2, a)$	$d(e, a-1)d(\mu, a-1)$
	odd		$2d(e, a)d(\mu+1, a)$			2		0	0
E	*even*	0	$d(e, a)\min_0(a-\mu+1, 2)$	$l+1$	0	$l+1$	-1	$d(\mu+1, e)d(\mu+2, a)$	$d(e, a-1)d(\mu, a-1)$
	odd		$2d(e, a)d(\mu+1, a)$			2		0	0
F	*even*	0	$d(e, a+1)d(\mu, a)$	1	0	$l+1$	0	$d(\mu+1, e)d(\mu+1, a)$	$d(e, a)d(\mu, a)$
	odd					2		0	0

TABLE 1

Keeping s fixed, now fix a prime $l \mid N$. Let $\nu = \mathrm{ord}_l(N)$, and write $\nu = 2\mu + \delta$, where $\delta = 0$ or 1. Let $e = e(\chi_l)$. Classify l into one of the six cases $A, \ldots, F$, and, according to the classification, let $a = a_l(s^2 - 4n)$; if l is a case A or D prime let d be the 'd' which appears in $s^2 - 4n$, otherwise let $d = 1$. Let $\chi_l^* = \chi_l(\frac{s+l^a d}{2}) + \chi_l(\frac{s-l^a d}{2})$; let $\hat{\chi}_l = \chi_l(\frac{s+l^{a+f}d}{2})$, where $f = 1$ if Case$(l) = F$, *and* 0 *otherwise. Then $c(s, l)$ is given by the expression*

$$(2) \quad \chi_l^* k_1 l^{\min(2a, \nu-1, a+\nu-e)} + \hat{\chi}_l k_2 k_3 l^{\nu-1} + \chi_l(\tfrac{s}{2}) k_3 l^{\nu-g}(k_4(l^{a-\max(\mu, e-\delta)+\varepsilon} - 1)/(l-1) + k_5 l^{a-e+\varepsilon} + k_6),$$

where $k_1, \ldots, k_6$, g, and ε are determined from Table 1, *by knowing the parity of ν and the classification of the prime l. In the table, define $d(x, y) = 1$ if $x \le y$ and* 0 *otherwise, for any integers x and y. Also, let* $\min_0(x, y) = \max(0, \min(x, y))$. *Note also that Convention* A *must be followed when evaluating* (2).

We remark that we have used Theorem 2 to write a Turbo Pascal program which finds $\mathrm{tr}_{N,\chi,k} T_n$ for small values of k, N, and n, and real characters χ.

The proof of Theorem 2 consists of transforming the first line of the formula given in Theorem 1 into the first line of that given in Theorem 2. We need two lemmas from [H-P-S$_1$] or [H-P-S$_2$]. The first of these is

Lemma 3. *Let the notation be as Theorem* 1. *In particular, write $s^2 - 4n = t^2$, $t^2 m$, or $t^2 4m$ as illustrated there. Let l be any prime dividing N or t and put $t = l^a t_0$ where $(l, t_0) = 1$. Let $f \mid t$ and put $f = l^\rho f_0$ with $(l, f_0) = 1$. Then $b(s, f) = \alpha(s, \rho, l) \cdot b(s, l^a f_0)$ where*

$$\alpha(s, \rho, l) = \begin{cases} l^{a-\rho} - l^{a-\rho-1} & \text{if } l \text{ is a case } A \text{ or } D \text{ prime}, \\ l^{a-\rho} + l^{a-\rho-1} & \text{if } l \text{ is a case } B \text{ or } E \text{ prime}, \\ l^{a-\rho} & \text{if } l \text{ is a case } C \text{ or } F \text{ prime}. \end{cases}$$

Note our Convention A *in effect here; if $\rho = a$ we take $l^{a-\rho-1} = 0$.*

Proof. See Lemma 2.4 of either [H-P-S$_1$] or [H-P-S$_2$].

Fix s as in Theorem 1 and write $s^2 - 4n = t^2$, $t^2 m$, or $t^2 4m$ as illustrated there. Let $l \mid N$ and define $c''_\chi(s, \rho, l) = c'_\chi(s, l^\rho, l)$ for $\rho = 0, \ldots, \mathrm{ord}_l(t)$. Now let $f \mid t$, and note that $c'_\chi(s, f, l)$ depends *only* on $\mathrm{ord}_l(f)$ once s and l are fixed. Write $f = l^\rho f_0$ where $\rho = \mathrm{ord}_l(f)$; then

$$c'_\chi(s, f, l) = c'_\chi(s, l^\rho f_0, l) = c'_\chi(s, l^\rho, l) = c''_\chi(s, \rho, l).$$

Now, if there are $v > 0$ distinct primes l satisfying $l \mid t$ and $l \nmid N$, let $\{l_i\}$, $i = 1, \ldots, v$, be a list of them; if there are no such primes, then for convenience set $v = 1$ and define $l_1 = 1$ and $\alpha(s, 0, 1) = 1$. Next, if $N \ne 1$ let $l_{v+1}, \ldots, l_{v+w}$ be a list of the w distinct primes dividing N, while if $N = 1$ then for convenience set $w = 1$ and define $l_{v+1} = 1$, $\alpha(s, 0, 1) = 1$ and $c''_\chi(s, 0, 1) = 1$. We can write $t = \prod_{i=1}^{v+w} l_i^{a_i}$ where $a_i = \mathrm{ord}_{l_i}(t)$ if l_i is a (bona-fide) prime and $a_i = 0$ if $l_i = 1$. Let f be any divisor of t; then we can write f uniquely as $f = \prod_{i=1}^{v+w} l_i^{\rho_i}$, where $0 \le \rho_i \le a_i$. We have

$$\begin{aligned}(3)\qquad &\sum_{f|t} b(s, f) \prod_{l|N} c'_\chi(s, f, l)\\ &= \sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \left(b\left(s, \prod_{i=1}^{v+w} l_i^{\rho_i} \right) \prod_{i=v+1}^{v+w} c''_\chi(s, \rho_i, l_i) \right).\end{aligned}$$

The statement of Lemma 3 in our current notation is that, for fixed $\rho_1, \dots, \rho_{v+w}$, with $0 \le \rho_i \le a_i$ for each i, $i = 1, \dots, v+w$, and for some particular $i = i_0$ with $1 \le i_0 \le v+w$, we have

$$b\left(s, \prod_{i=1}^{v+w} l_i^{\rho_i} \right) = \alpha(s, \rho_{i_0}, l_{i_0}) b \left(s, l_{i_0}^{a_{i_0}} \cdot \prod_{\substack{i \ne i_0 \\ i=1}}^{v+w} l_i^{\rho_i} \right)$$

(and this is clearly also true if $l_{i_0} = 1$). *Repeated* applications of this equality transform (3) into

$$\sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \Bigg(\alpha(s, \rho_1, l_1) \cdots \alpha(s, \rho_{v+w}, l_{v+w}) \cdot b(s, \underbrace{l_1^{a_1} \cdots l_{v+w}^{a_{v+w}}}_{=t}) \prod_{i=v+1}^{v+w} c''_\chi(s, \rho_i, l_i) \Bigg).$$

Noting that $b(s, t) = b(s)$, this last expression equals

$$b(s) \sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \left(\prod_{i=1}^{v+w} \alpha(s, \rho_i, l_i) \cdot \prod_{i=v+1}^{v+w} c''_\chi(s, \rho_i, l_i) \right).$$

Again for convenience let $c''_\chi(s, \rho_i, l_i) = 1$ for $i = 1, \dots, v$; the above becomes

$$\begin{aligned} b(s) &\sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \left(\prod_{i=1}^{v+w} \alpha(s, \rho_i, l_i) c''_\chi(s, \rho_i, l_i) \right) \\ &= b(s) \prod_{i=1}^{v+w} \left(\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) c''_\chi(s, \rho, l_i) \right) \\ &= b(s) \prod_{i=1}^{v} \left(\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) \right) \cdot \prod_{i=v+1}^{v+w} \left(\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) c''_\chi(s, \rho, l_i) \right). \end{aligned}$$

All that is needed to prove Theorem 2 is to show two things. First, that $\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) = \gamma(s, l_i)$ for each i, $i = 1, \dots, v$, in the case there *are* $v > 0$ primes l satisfying $l \mid t$, $l \nmid N$; if there are no such primes then we arranged things so that $\prod_{i=1}^{v} (\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i)) = \alpha(s, 0, 1) = 1$ which agrees with the "empty" product $\prod_{l|t, l \nmid N} \alpha(l)$. Second, we need to show that

$$\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) c''_\chi(s, \rho, l_i) = c(s, l_i)$$

for each i, $i = v+1, \ldots, v+w$, in the case that there *are* $w > 0$ primes l dividing N; if $N = 1$ we arranged for $\prod_{i=v+1}^{v+w} \sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) c''_\chi(s, \rho, l_i) = \alpha(s, 0, 1) c''_\chi(s, 0, 1) = 1 \cdot 1 = 1$ which agrees with the "empty" product $\prod_{l|N} c(s, l)$.

Suppose then that l is a prime, $l \mid t$, $l \nmid N$. Then $l = l_i$ for some i with $1 \le i \le v$. Let $a = \mathrm{ord}_l(t)$; we show $\sum_{\rho=0}^{a} \alpha(s, \rho, l) = \gamma(s, l)$.

Suppose l is a case A or D prime. We have

$$\sum_{\rho=0}^{a} \alpha(s, \rho, l) = \sum_{\rho=0}^{a} (l^{a-\rho} - l^{a-\rho-1}) = l^a - l^{-1} = l^a = \gamma(s, l).$$

If $\mathrm{Case}(l) = B$ or E,

$$\sum_{\rho=0}^{a} \alpha(s, \rho, l) = \sum_{\rho=0}^{a} l^{a-\rho} + \sum_{\rho=0}^{a} l^{a-\rho-1}$$
$$= (l^{a+1} - 1)/(l-1) + (l^a - 1)/(l-1) = \gamma(s, l).$$

If $\mathrm{Case}(l) = C$ or F,

$$\sum_{\rho=0}^{a} \alpha(s, \rho, l) = \sum_{\rho=0}^{a} l^{a-\rho} = (l^{a+1} - 1)/(l-1) = \gamma(s, l).$$

So all that remains is to explicitly evaluate $\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) c''_\chi(s, \rho, l_i)$ for each prime $l_i \mid N$, and show the result is $c(s, l_i)$. Fix $l \mid N$ and write $\alpha(\rho)$ for $\alpha(s, \rho, l)$ and $c(\rho)$ for $c''_\chi(s, \rho, l)$. Now, the task of evaluating $\sum_{\rho=0}^{a} \alpha(\rho) c(\rho)$ and showing that it equals $c(s, l)$ as given by Table 1 is a long straightforward one, but extremely tedious. We give some details concerning the explicit calculation/evaluation of $\sum_{\rho=0}^{a} \alpha(\rho) c(\rho)$ for $\mathrm{Case}(l) = A$ and $\mathrm{ord}_l(N)$ even, leaving all other calculations (i.e., those for $\mathrm{Case}(l) = B, \ldots, F$, $\mathrm{ord}_l(N)$ even or odd, and $\mathrm{Case}(l) = A$ with $\mathrm{ord}_l(N)$ odd) to the reader. First, let us *summarize* all the calculations here:

$$\text{Explicit Value of } c(s, l) = \sum_{\rho=0}^{a} \alpha(\rho) c(\rho)$$

condition 1	condition 2	$\sum_{\rho=0}^{a} \alpha(\rho)c(\rho)$ for $\mathrm{Case}(l) = A$, $\nu = \mathrm{ord}_l(N)$ even
$a \le \mu - 1$	$e \le \nu - a$	$\chi_l^* l^{2a}$
	$e > \nu - a$	$\chi_l^* l^{\nu+a-e}$
$a = \mu$	$e \le a$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{s}{2})(l-1) l^{\nu-1}$
	$e > a$	$\chi_l^* l^{\nu+a-e}$
$a > \mu$	$e \le \mu$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{s}{2})(l-1) l^{\nu-1}((l+1)(l^{a-\mu} - 1)/(l-1) + 1)$
	$\mu < e \le a-1$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{s}{2})(l-1) l^{\nu-1}$ $\cdot((l+1)(l^{a-e} - 1)/(l-1) + l^{a-e} + 1)$
	$e = a$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{s}{2})(l-1) l^{\nu-1}(l^{a-e} + 1)$
	$e > a$	$\chi_l^* l^{\nu+a-e}$

condition 1	condition 2	$\mathrm{Case}(l) = A$, $\nu = \mathrm{ord}_l(N)$ odd
$a \le \mu$	$e \le \nu - a$	$\chi_l^* l^{2a}$
	$e > \nu - a$	$\chi_l^* l^{\nu+a-e}$

$a > \mu$	$e \le \mu + 1$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{s}{2}) l^{\nu-1} 2(l^{a-\mu} - 1)$
	$\mu + 1 < e \le a$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{s}{2}) l^{\nu-1} 2(l^{a-e+1} - 1)$
	$e = a + 1$	$\chi_l^* l^{\nu-1} \quad (= \chi_l^* l^{\nu+a-e})$
	$e > a + 1$	$\chi_l^* l^{\nu+a-e}$

condition 1	condition 2	Case$(l) = B$, $\nu = \mathrm{ord}_l(N)$ even
$a \le \mu - 1$	(none)	0
$a = \mu$	$e \le a$	$\chi_l(\frac{s}{2})(l+1) l^{\nu-1}$
	$e > a$	0
$a > \mu$	$e \le \mu$	$\chi_l(\frac{s}{2})(l+1) l^{\nu-1}((l+1)(l^{a-\mu} - 1)/(l-1) + 1)$
	$\mu < e \le a - 1$	$\chi_l(\frac{s}{2})(l+1) l^{\nu-1}((l+1)(l^{a-e} - 1)/(l-1) + l^{a-e} + 1)$
	$e = a$	$\chi_l(\frac{s}{2})(l+1) l^{\nu-1}(l^{a-e} + 1)$
	$e > a$	0

condition 1	condition 2	Case$(l) = B$, $\nu = \mathrm{ord}_l(N)$ odd
$a \le \mu$	(none)	0
$a > \mu$	$e \le \mu + 1$	$\chi_l(\frac{s}{2})(l+1) l^{\nu-1} 2(l^{a-\mu} - 1)/(l-1)$
	$\mu + 1 < e \le a$	$\chi_l(\frac{s}{2})(l+1) l^{\nu-1} 2(l^{a-e+1} - 1)/(l-1)$
	$e > a$	0

condition 1	condition 2	Case$(l) = C$, $\nu = \mathrm{ord}_l(N)$ even
$a \le \mu - 1$	(none)	0
$a \ge \mu$	$e \le \mu$	$\chi_l(\frac{s}{2}) l^{\nu-1}(l+1)(l^{a-\mu+1} - 1)/(l-1)$
	$\mu < e \le a$	$\chi_l(\frac{s}{2}) l^{\nu-1}((l+1)(l^{a-e+1} - 1)/(l-1) + l^{a-e+1})$
	$e = a + 1$	$\chi_l(\frac{s}{2}) l^{\nu-1} l^{a-e+1}$
	$e > a + 1$	0

condition 1	condition 2	Case$(l) = C$, $\nu = \mathrm{ord}_l(N)$ odd
$a \le \mu - 1$	(none)	0
$a = \mu$	$e \le a + 1$	$\chi_l(\frac{s}{2}) l^{\nu-1}$
	$e > a + 1$	0
$a \ge \mu + 1$	$e \le \mu + 1$	$\chi_l(\frac{s}{2}) l^{\nu-1}(2l(l^{a-\mu} - 1)/(l-1) + 1)$
	$\mu + 1 < e \le a$	$\chi_l(\frac{s}{2}) l^{\nu-1}(2l(l^{a-e+1} - 1)/(l-1) + 1)$
	$e = a + 1$	$\chi_l(\frac{s}{2}) l^{\nu-1}$
	$e > a + 1$	0

condition 1	condition 2	Case$(l) = D$, $\nu = \mathrm{ord}_l(N)$ even
$a \le \mu - 1$	$e \le \nu - a$	$\chi_l^* l^{2a}$
	$e > \nu - a$	$\chi_l^* l^{\nu+a-e}$
$a = \mu$	$e \le a$	$\chi_l^* l^{\nu-1} + \hat{\chi}_l l^{\nu-1}$
	$e > a$	$\chi_l^* l^{\nu+a-e}$
$a = \mu + 1$	$e \le a - 1$	$\chi_l^* l^{\nu-1} + \hat{\chi}_l 2 l^{\nu-1} + \chi_l(\frac{s}{2}) l^{\nu}$
	$e = a$	$\chi_l^* l^{\nu-1} + \hat{\chi}_l 2 l^{\nu-1}$
	$e > a$	$\chi_l^* l^{\nu+a-e}$
$a > \mu + 1$	$e \le \mu$	$\chi_l^* l^{\nu-1} + 2\hat{\chi}_l l^{\nu-1} + \chi_l(\frac{s}{2}) l^{\nu}((l+1)(l^{a-\mu-1} - 1) + 1)$
	$\mu < e \le a - 2$	$\chi_l^* l^{\nu-1} + 2\hat{\chi}_l l^{\nu-1}$ $+\chi_l(\frac{s}{2}) l^{\nu}((l+1)(l^{a-e-1} - 1) + l^{a-e-1} + 1)$
	$e = a - 1$	$\chi_l^* l^{\nu-1} + 2\hat{\chi}_l l^{\nu-1} + \chi_l(\frac{s}{2}) l^{\nu}(l^{a-e-1} + 1)$
	$e = a$	$\chi_l^* l^{\nu-1} + 2\hat{\chi}_l l^{\nu-1}$
	$e > a$	$\chi_l^* l^{\nu+a-e}$

condition 1	condition 2	Case$(l) = D$, $\nu = \mathrm{ord}_l(N)$ odd
$a \le \mu$	$e \le \nu - a$	$\chi_l^* l^{2a}$
	$e > \nu - a$	$\chi_l^* l^{\nu+a-e}$

$a=\mu+1$	$e\le a$	$\chi_l^*\, l^{\nu-1}+\hat\chi_l 2 l^{\nu-1}$
	$e>a$	$\chi_l^*\, l^{\nu+a-e}$
$a>\mu+1$	$e\le\mu+1$	$\chi_l^*\, l^{\nu-1}+\hat\chi_l 2l^{\nu-1}+\chi_l(\frac{s}{2})l^\nu 2(l^{a-\mu-1}-1)$
	$\mu+1<e<a$	$\chi_l^*\, l^{\nu-1}+\hat\chi_l 2l^{\nu-1}+\chi_l(\frac{s}{2})l^\nu 2(l^{a-(e-1)-1}-1)$
	$e=a$	$\chi_l^*\, l^{\nu-1}+\hat\chi_l 2l^{\nu-1}$
	$e>a$	$\chi_l^*\, l^{\nu+a-e}$

condition 1	condition 2	Case$(l)=E$, $\nu=\mathrm{ord}_l(N)$ even
$a\le\mu-1$	(none)	0
$a=\mu$	$e\le a$	$\hat\chi_l(l+1)l^{\nu-1}$
	$e>a$	0
$a=\mu+1$	$e\le a-1$	$\hat\chi_l 2(l+1)l^{\nu-1}+\chi_l(\frac{s}{2})(l+1)l^\nu$
	$e=a$	$\hat\chi_l 2(l+1)l^{\nu-1}$
	$e>a$	0
$a>\mu+1$	$e\le\mu$	$\hat\chi_l 2(l+1)l^{\nu-1}+\chi_l(\frac{s}{2})(l+1)l^\nu((l+1)(l^{a-\mu-1}-1)+1)$
	$\mu<e\le a-2$	$\hat\chi_l 2(l+1)l^{\nu-1}+\chi_l(\frac{s}{2})(l+1)l^\nu$ $\cdot((l+1)(l^{a-e-1}-1)+l^{a-e-1}+1)$
	$e=a-1$	$\hat\chi_l 2(l+1)l^{\nu-1}+\chi_l(\frac{s}{2})(l+1)l^\nu(l^{a-e-1}+1)$
	$e=a$	$\hat\chi_l 2(l+1)l^{\nu-1}$
	$e>a$	0

condition 1	condition 2	Case$(l)=E$, $\nu=\mathrm{ord}_l(N)$ odd
$a\le\mu$	(none)	0
$a=\mu+1$	$e\le a$	$\hat\chi_l 2(l+1)l^{\nu-1}$
	$e>a$	0
$a>\mu+1$	$e\le\mu+1$	$\hat\chi_l 2(l+1)l^{\nu-1}+\chi_l(\frac{s}{2})(l+1)l^\nu 2(l^{a-\mu-1}-1)$
	$\mu+1<e<a$	$\hat\chi_l 2(l+1)l^{\nu-1}+\chi_l(\frac{s}{2})(l+1)l^\nu 2(l^{a-(e-1)-1}-1)$
	$e=a$	$\hat\chi_l 2(l+1)l^{\nu-1}$
	$e>a$	0

condition 1	condition 2	Case$(l)=F$, $\nu=\mathrm{ord}_l(N)$ even
$a\le\mu-1$	(none)	0
$a=\mu$	$e\le a$	$\hat\chi_l l^{\nu-1}+\chi_l(\frac{s}{2})l^\nu$
	$e=a+1$	$\hat\chi_l l^{\nu-1}$
	$e>a+1$	0
$a\ge\mu+1$	$e\le\mu$	$\hat\chi_l l^{\nu-1}+\chi_l(\frac{s}{2})l^\nu((l+1)(l^{a-\mu}-1)+1)$
	$\mu<e\le a-1$	$\hat\chi_l l^{\nu-1}+\chi_l(\frac{s}{2})l^\nu((l+1)(l^{a-e}-1)+l^{a-e}+1)$
	$e=a$	$\hat\chi_l l^{\nu-1}+\chi_l(\frac{s}{2})l^\nu(l^{a-e}+1)$
	$e=a+1$	$\hat\chi_l l^{\nu-1}$
	$e>a+1$	0

condition 1	condition 2	Case$(l)=F$, $\nu=\mathrm{ord}_l(N)$ odd
$a\le\mu-1$	(none)	0
$a=\mu$	$e\le a+1$	$\hat\chi_l l^{\nu-1}$
	$e>a+1$	0
$a\ge\mu+1$	$e\le\mu+1$	$\hat\chi_l l^{\nu-1}+\chi_l(\frac{s}{2})l^\nu 2(l^{a-\mu}-1)$
	$\mu+1<e\le a$	$\hat\chi_l l^{\nu-1}+\chi_l(\frac{s}{2})l^\nu 2(l^{a-e+1}-1)$
	$e=a+1$	$\hat\chi_l l^{\nu-1}$
	$e>a+1$	0

In all of what follows, by the "**SUM**", we mean $\sum_{\rho=0}^{a}\alpha(\rho)c(\rho)$. The key to its explicit evaluation is

Lemma 4. *Let p be a prime and let ω be a character modulo some power of p. Let $e=e(\omega)$ be the exponential conductor of ω. If σ and b are nonnegative* *integers with* $\sigma + b \geq e$ *and* $2b \geq e$ *and if* u *is a unit* mod p, *then*

$$\sum_{z \in \mathbf{Z}/p^\sigma \mathbf{Z}} \omega(u + zp^b) = \begin{cases} p^\sigma \omega(u) & \text{if } e \leq b, \\ 0 & \text{if } e > b. \end{cases}$$

Proof. This is easy. See Lemma 2.1 of [H-P-S$_1$] or [H-P-S$_2$].

Let A_ρ and B'_ρ be as in Theorem 1. Specific sets of representatives for A_ρ and B'_ρ are calculated by the authors of [H-P-S$_1$, H-P-S$_2$] in their Lemma 2.5 by "easy but tedious calculations"; we copy them here for reference as

Lemma 5. *Let* $A(s, f, l) = A_\rho$ *and* $B'(s, f, l) = B'_\rho$ *be the sets appearing in Theorem* 1. *For fixed* N, n, s, *and* l, A_ρ *and* B'_ρ *depend only on* $\rho = \mathrm{ord}_l(f)$. *Let* $\nu = \mathrm{ord}_l(N)$ *and set* $\nu = 2\mu$ *or* $\nu = 2\mu + 1$. *Classify* l *as case* A, B, *etc., setting* a *and* d *according to how* l *is classified. Then the sets* A_ρ *and* B'_ρ *are as follows*:

Case(l)	ν	*condition*	
A	*odd*	$a - \rho \leq \mu$	$A_\rho = \left\{\frac{s \pm l^a d}{2} + z l^{2\mu+2\rho-a+1} \mid z \in \mathbf{Z}/l^{a-\rho}\mathbf{Z}\right\}$
			$B'_\rho = \left\{\frac{s \pm l^a d}{2} + z l^{2\mu+2\rho-a+2} \mid z \in \mathbf{Z}/l^{a-\rho-1}\mathbf{Z}\right\}$ †
		$a - \rho \geq \mu + 1$	$A_\rho = B'_\rho = \left\{\frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^{\mu}\mathbf{Z}\right\}$
	even	$a - \rho \leq \mu - 1$	$A_\rho = \left\{\frac{s \pm l^a d}{2} + z l^{2\mu+2\rho-a} \mid z \in \mathbf{Z}/l^{a-\rho}\mathbf{Z}\right\}$
			$B'_\rho = \left\{\frac{s \pm l^a d}{2} + z l^{2\mu+2\rho-a+1} \mid z \in \mathbf{Z}/l^{a-\rho-1}\mathbf{Z}\right\}$ †
		$a - \rho = \mu$	$A_\rho = \left\{\frac{s}{2} + z l^{a} \mid z \in \mathbf{Z}/l^{\mu}\mathbf{Z}\right\}$
			$B'_\rho = \left\{\frac{s \pm l^a d}{2} + z l^{a+1} \mid z \in \mathbf{Z}/l^{\mu-1}\mathbf{Z}\right\}$
		$a - \rho \geq \mu + 1$	$A_\rho = \left\{\frac{s}{2} + z l^{\mu+\rho} \mid z \in \mathbf{Z}/l^{\mu}\mathbf{Z}\right\}$
			$B'_\rho = \left\{\frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^{\mu-1}\mathbf{Z}\right\}$
B	*odd*	$a - \rho \leq \mu$	$A_\rho = B'_\rho = \varnothing$
		$a - \rho \geq \mu + 1$	$A_\rho = B'_\rho = \left\{\frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^{\mu}\mathbf{Z}\right\}$
	even	$a - \rho \leq \mu - 1$	$A_\rho = B'_\rho = \varnothing$
		$a - \rho = \mu$	$A_\rho = \left\{\frac{s}{2} + z l^{a} \mid z \in \mathbf{Z}/l^{\mu}\mathbf{Z}\right\}$
			$B'_\rho = \varnothing$
		$a - \rho \geq \mu + 1$	$A_\rho = \left\{\frac{s}{2} + z l^{\mu+\rho} \mid z \in \mathbf{Z}/l^{\mu}\mathbf{Z}\right\}$
			$B'_\rho = \left\{\frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^{\mu-1}\mathbf{Z}\right\}$
C	*odd*	$a - \rho \leq \mu - 1$	$A_\rho = B'_\rho = \varnothing$
		$a - \rho = \mu$	$A_\rho = \left\{\frac{s}{2} + z l^{a+1} \mid z \in \mathbf{Z}/l^{\mu}\mathbf{Z}\right\}$
			$B'_\rho = \varnothing$
		$a - \rho \geq \mu + 1$	$A_\rho = B'_\rho = \left\{\frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^{\mu}\mathbf{Z}\right\}$
	even	$a - \rho \leq \mu - 1$	$A_\rho = B'_\rho = \varnothing$
		$a - \rho \geq \mu$	$A_\rho = \left\{\frac{s}{2} + z l^{\mu+\rho} \mid z \in \mathbf{Z}/l^{\mu}\mathbf{Z}\right\}$
			$B'_\rho = \left\{\frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^{\mu-1}\mathbf{Z}\right\}$
D	*odd*	$a - \rho \leq \mu$	A_ρ *and* B'_ρ *are the same as for a case* A *prime*
		$a - \rho = \mu + 1$	$A_\rho = B'_\rho = \left\{\frac{s + l^a d}{2} + z l^{a} \mid z \in \mathbf{Z}/l^{\mu}\mathbf{Z}\right\}$
		$a - \rho \geq \mu + 2$	A_ρ *and* B'_ρ *are the same as for a case* A *prime*

	even	$a-\rho \le \mu-1$	A_ρ *and* B'_ρ *are the same as for a case* A *prime*
		$a-\rho=\mu$	$A_\rho = \left\{\frac{s+l^a d}{2} + zl^a \mid z \in \mathbf{Z}/l^\mu\mathbf{Z}\right\}$
			$B'_\rho = \left\{\frac{s\pm l^a d}{2} + zl^{a+1} \mid z \in \mathbf{Z}/l^{\mu-1}\mathbf{Z}\right\}$
		$a-\rho=\mu+1$	$A_\rho = \left\{\frac{s}{2} + zl^{a-1} \mid z \in \mathbf{Z}/l^\mu\mathbf{Z}\right\}$
			$B'_\rho = \left\{\frac{s+l^a d}{2} + zl^{a} \mid z \in \mathbf{Z}/l^{\mu-1}\mathbf{Z}\right\}$
		$a-\rho \ge \mu+2$	A_ρ *and* B'_ρ *are the same as for a case* A *prime*
E	*odd*	$a-\rho \le \mu$	$A_\rho = B'_\rho = \varnothing$
		$a-\rho \ge \mu+1$	A_ρ *and* B'_ρ *are the same as for a case* D *prime*, with d set to 1
	even	$a-\rho \le \mu-1$	$A_\rho = B'_\rho = \varnothing$
		$a-\rho=\mu$	$A_\rho = \left\{\frac{s+l^a}{2} + zl^a \mid z \in \mathbf{Z}/l^\mu\mathbf{Z}\right\}$
			$B'_\rho = \varnothing$
		$a-\rho \ge \mu+1$	A_ρ *and* B'_ρ *are the same as for a case* D *prime*, with d set to 1
F	*odd*	$a-\rho \le \mu-1$	$A_\rho = B'_\rho = \varnothing$
		$a-\rho=\mu$	$A_\rho = \left\{\frac{s+l^{a+1}}{2} + zl^{a+1} \mid z \in \mathbf{Z}/l^\mu\mathbf{Z}\right\}$
			$B'_\rho = \varnothing$
		$a-\rho \ge \mu+1$	A_ρ *and* B'_ρ *are the same as for a case* A *prime*
	even	$a-\rho \le \mu-1$	$A_\rho = B'_\rho = \varnothing$
		$a-\rho=\mu$	$A_\rho = \left\{\frac{s}{2} + zl^{a} \mid z \in \mathbf{Z}/l^\mu\mathbf{Z}\right\}$
			$B'_\rho = \left\{\frac{s+l^{a+1}}{2} + zl^{a+1} \mid z \in \mathbf{Z}/l^{\mu-1}\mathbf{Z}\right\}$
		$a-\rho \ge \mu+1$	A_ρ *and* B'_ρ *are the same as for a case* A *prime*

†*If* $\rho = a$ *then set* $B'_\rho = \varnothing$.

We can now make one more observation: With s fixed and $l \mid N$, let $f \mid t$ and $\rho = \operatorname{ord}_l(f)$ for some $\rho \ge 0$. Refer to the definition of $c'_\chi(s, f, l)$ as in Theorem 1. We can actually write

$$c''_\chi(s, \rho, l) = c'_\chi(s, f, l) \quad \text{as} \quad \sum_{x\in A_\rho} \chi_l(x) + \sum_{y \in B'_\rho} \chi_l(y).$$

For suppose $(s^2-4n)/(l^\rho)^2 \not\equiv 0\ (l)$. Write s^2-4n as t^2, t^2m, or $t^2 4m$ as in Theorem 1 and write $t = l^a t_0$, where $(l, t_0) = 1$; recall this $a = a_l(s^2-4n)$. Clearly if $(s^2-4n)/(l^\rho)^2 \not\equiv 0\ (l)$ then $t^2/l^{2\rho} = l^{2a-2\rho} t_0 \not\equiv 0\,(l)$, which implies $\rho = a$. However, in every case in which $\rho = a$, Lemma 5 shows that the set B'_ρ is empty; consequently $\sum_{y\in B'_\rho} \chi_l(y) = 0$. Let us write "$A_\rho$sum" and "$B'_\rho$sum" for $\sum_{x\in A_\rho}\chi_l(x)$ and $\sum_{y\in B'_\rho}\chi_l(y)$, respectively, so that $c''_\chi(s, \rho, l) = A_\rho\text{sum} + B'_\rho\text{sum}$. Finally we are ready to begin evaluating $\sum_{\rho=0}^{a} \alpha(\rho)c(\rho)$. Suppose Case$(l) = A$ and $\nu = \operatorname{ord}_l(N)$ is even; write $\nu = 2\mu$. Set $a = a_l(s^2-4n)$ and $e = e(\chi_l)$. If $a-\rho \le \mu-1$ then by applying Lemma 4 twice on each of the sets A_ρ and B'_ρ as given in Lemma 5, we have

$$A_\rho\text{sum} = \begin{cases} l^{a-\rho}\chi_l^* & \text{if } e \le 2\mu+2\rho-a, \\ 0 & \text{if } e > 2\mu+2\rho-a \end{cases}$$

and

$$B'_\rho\text{sum} = \begin{cases} l^{a-\rho-1}\chi_l^* & \text{if } e \le 2\mu+2\rho-a+1, \\ 0 & \text{if } e > 2\mu+2\rho-a+1. \end{cases}$$

Notice that if $\rho = a$ then $B'_\rho = \varnothing$; in this case the above formula gives $l^{-1}\chi_l^*$ which is 0 by our convention. Adding $A_\rho$sum and B'_ρsum, we have for $a-\rho \le \mu-1$

$$c(\rho) = \begin{cases} (l^{a-\rho}+l^{a-\rho-1})\chi_l^* & \text{if } e \le 2\mu+2\rho-a, \\ l^{a-\rho-1}\chi_l^* & \text{if } e = 2\mu+2\rho-a+1, \\ 0 & \text{if } e > 2\mu+2\rho-a+1. \end{cases} \tag{4}$$

If $a-\rho=\mu$, then by Lemmas 4 and 5 we have

$$A_\rho\text{sum} = \begin{cases} l^\mu\chi_l(\frac{s}{2}) & \text{if } e \le a, \\ 0 & \text{if } e > a \end{cases}$$

and

$$B'_\rho\text{sum} = \begin{cases} l^{\mu-1}\chi_l^* & \text{if } e \le a+1, \\ 0 & \text{if } e > a+1. \end{cases}$$

Adding $A_\rho$sum and B'_ρsum, we have for $a-\rho=\mu$

$$c(\rho) = \begin{cases} l^{\mu-1}\chi_l^* + l^\mu\chi_l(\frac{s}{2}) & \text{if } e \le a, \\ l^{\mu-1}\chi_l^* & \text{if } e = a+1, \\ 0 & \text{if } e > a+1. \end{cases} \tag{5}$$

Now suppose $a-\rho \ge \mu+1$. By Lemmas 4 and 5 we have

$$A_\rho\text{sum} = \begin{cases} l^\mu\chi_l(\frac{s}{2}) & \text{if } e \le \mu+\rho, \\ 0 & \text{if } e > \mu+\rho \end{cases}$$

and

$$B'_\rho\text{sum} = \begin{cases} l^{\mu-1}\chi_l(\frac{s}{2}) & \text{if } e \le \mu+\rho+1, \\ 0 & \text{if } e > \mu+\rho+1. \end{cases}$$

Adding $A_\rho$sum and B'_ρsum we have for $a-\rho \ge \mu+1$

$$c(\rho) = \begin{cases} (l^\mu+l^{\mu-1})\chi_l(\frac{s}{2}) & \text{if } e \le \mu+\rho, \\ l^{\mu-1}\chi_l(\frac{s}{2}) & \text{if } e = \mu+\rho+1, \\ 0 & \text{if } e > \mu+\rho+1. \end{cases} \tag{6}$$

Now we are ready to calculate the **SUM**$=\sum_{\rho=0}^a \alpha(\rho)c(\rho)$ under the various possibilities.

Suppose $a \le \mu-1$. Then for $\rho = 0, \dots, a$ we have $a-\rho \le \mu-1$ and therefore $c(\rho)$ is given by (4).

Suppose $e \le 2\mu-a$ $(=\nu-a)$. Then for $\rho=0,\dots,a$ we have $e \le 2\mu+2\rho-a$; by (4) we have

$$\sum_{\rho=0}^a \alpha(\rho)c(\rho) = \sum_{\rho=0}^a (l^{a-\rho}-l^{a-\rho-1})(l^{a-\rho}+l^{a-\rho-1})\chi_l^*$$
$$= \chi_l^*\sum_{\rho=0}^a (l^{2a-2\rho}-l^{2a-2\rho-2}) = \chi_l^*(l^{2a}-l^{-2}) = \chi_l^*\, l^{2a}.$$

Suppose $e > 2\mu - a$. Then either $e = 2\mu + 2\rho_1 - a$ for some $\rho_1 > 0$, or $e = 2\mu + 2\rho_1 - a + 1$ for some $\rho_1 \geq 0$; we show the **SUM** is $\chi_l^* \, l^{\nu+a-e}$ in either case. Suppose then that $e = 2\mu + 2\rho_1 - a$ for some $\rho_1 > 0$; note $\rho_1 \leq a$. If ρ satisfies $\rho_1 > \rho$ then $2\rho_1 > 2\rho + 1$ implies $e = 2\mu + 2\rho_1 - a > 2\mu + 2\rho - a + 1$; if $\rho_1 \leq \rho$ then $e = 2\mu + 2\rho_1 - a \leq 2\mu + 2\rho - a$, so that by (4)

$$c(\rho) = \begin{cases} (l^{a-\rho} + l^{a-\rho-1})\chi_l^* & \text{if } \rho \geq \rho_1, \\ 0 & \text{if } \rho < \rho_1, \end{cases}$$

and the **SUM** becomes

$$\begin{aligned} &\sum_{\rho=0}^{\rho_1-1} \alpha(\rho)c(\rho) + \sum_{\rho=\rho_1}^{a} \alpha(\rho)c(\rho) \\ &\quad = 0 + \sum_{\rho=\rho_1}^{a} (l^{a-\rho} - l^{a-\rho-1})(l^{a-\rho} + l^{a-\rho-1})\chi_l^* \\ &\quad = \chi_l^* \sum_{\rho=\rho_1}^{a} (l^{2a-2\rho} - l^{2a-2\rho-2}) = \chi_l^* \, l^{2a-2\rho_1} = \chi_l^* \, l^{\nu+a-e}. \end{aligned}$$

Suppose $e = 2\mu + 2\rho_1 - a + 1$ for some $\rho_1 \geq 0$; note $\rho_1 + 1 \leq a$. If ρ satisfies $\rho_1 > \rho$ then $e = 2\mu + 2\rho_1 - a + 1 > 2\mu + 2\rho - a + 1$; if $\rho_1 < \rho$ then $e < 2\mu + 2\rho - a + 1$, i.e., $e \leq 2\mu + 2\rho - a$. We have

$$c(\rho) = \begin{cases} (l^{a-\rho} + l^{a-\rho-1})\chi_l^* & \text{if } \rho > \rho_1, \\ l^{a-\rho_1-1}\chi_l^* & \text{if } \rho = \rho_1, \\ 0 & \text{if } \rho < \rho_1, \end{cases}$$

so that the **SUM** equals

$$\begin{aligned} &\sum_{\rho=0}^{\rho_1-1} \alpha(\rho)0 + \alpha(\rho_1)l^{a-\rho_1-1}\chi_l^* + \sum_{\rho=\rho_1+1}^{a} \alpha(\rho)(l^{a-\rho} + l^{a-\rho-1})\chi_l^* \\ &\quad = 0 + (l^{a-\rho_1} - l^{a-\rho_1-1})l^{a-\rho_1-1}\chi_l^* \\ &\qquad + \sum_{\rho=\rho_1+1}^{a} (l^{2a-2\rho} - l^{2a-2\rho-2})\chi_l^* \\ &\quad = \chi_l^* (l^{2a-2\rho_1-1} - l^{2a-2\rho_1-2} + l^{2a-2\rho_1-2}) \\ &\quad = \chi_l^* \, l^{2a-2\rho_1-1} = \chi_l^* \, l^{\nu+a-e}. \end{aligned}$$

Suppose $a \geq \mu$; then we can write $a - \rho_0 = \mu$ for some $\rho_0 \geq 0$. Write the **SUM** as

$$\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) + \alpha(\rho_0)c(\rho_0) + \sum_{\rho=\rho_0+1}^{a} \alpha(\rho)c(\rho), \tag{7}$$

where we take $\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) = 0$ in the case $\rho_0 = 0$. Suppose $\rho_0 > 0$ and consider $\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho)$. If $\rho \leq \rho_0 - 1$ then $a - \rho \geq a - (\rho_0 - 1) = a - \rho_0 + 1 = \mu + 1$ so that $c(\rho)$ is determined using (6).

Suppose $e \le \mu$; then $e \le \mu + \rho$ for $\rho = 0, \dots, \rho_0 - 1$. From (6),

$$\begin{aligned}\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) &= \sum_{\rho=0}^{\rho_0-1}(l^{a-\rho} - l^{a-\rho-1})(l^\mu + l^{\mu-1})\chi_l(\tfrac{s}{2}) \\ &= \chi_l(\tfrac{s}{2})(l^\mu + l^{\mu-1})\sum_{\rho=0}^{\rho_0-1}(l^{a-\rho} - l^{a-\rho-1}) \\ &= \chi_l(\tfrac{s}{2})l^{\mu-1}(l+1)(l^a - l^{a-\rho_0}) = \chi_l(\tfrac{s}{2})l^{\mu-1}(l+1)l^\mu(l^{a-\mu} - 1) \\ &= \chi_l(\tfrac{s}{2})l^{\nu-1}(l+1)(l^{a-\mu} - 1).\end{aligned}$$

Suppose $e > \mu$; then for some $\rho_1 \ge 0$ we have $e = \mu + \rho_1 + 1$. Suppose $\rho \le \rho_0 - 1$. Then $e = \mu + \rho_1 + 1 \le \mu + \rho$ iff $\rho_1 + 1 \le \rho$; by (6) we have

$$c(\rho) = \begin{cases} (l^\mu + l^{\mu-1})\chi_l(\tfrac{s}{2}) & \text{if } \rho \ge \rho_1 + 1, \\ l^{\mu-1}\chi_l(\tfrac{s}{2}) & \text{if } \rho = \rho_1, \\ 0 & \text{if } \rho < \rho_1. \end{cases} \tag{8}$$

Now, $\rho_1 + 1 \le \rho_0 - 1$ iff $\mu + \rho_1 + 1 \le \mu + \rho_0 - 1$ iff $e \le a - 1$. Suppose this is the case. We have

$$\begin{aligned}\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) &= \sum_{\rho=0}^{\rho_1-1} \alpha(\rho)\cdot 0 \; + (l^{a-\rho_1} - l^{a-\rho_1-1})l^{\mu-1}\chi_l(\tfrac{s}{2}) \\ &\quad + \sum_{\rho=\rho_1+1}^{\rho_0-1} (l^{a-\rho} - l^{a-\rho-1})(l^\mu + l^{\mu-1})\chi_l(\tfrac{s}{2}) \\ &= \chi_l(\tfrac{s}{2})((l^{a-e+\mu+1} - l^{a-e+\mu})l^{\mu-1} \\ &\qquad + (l^\mu + l^{\mu-1})(l^{a-\rho_1-1} - l^{a-\rho_0})) \\ &= \chi_l(\tfrac{s}{2})(l^{a-e}l^\mu(l-1)l^{\mu-1} + l^{\mu-1}(l+1)(l^{a-e+\mu} - l^\mu)) \\ &= \chi_l(\tfrac{s}{2})(l^{a-e}l^{\nu-1}(l-1) + l^{\nu-1}(l+1)(l^{a-e} - 1)) \\ &= \chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}\;((l+1)(l^{a-e} - 1)/(l-1) \; + l^{a-e}).\end{aligned}$$

Next, $\rho_1 = \rho_0 - 1$ iff $\mu + \rho_1 + 1 = \mu + \rho_0$ iff $e = a$. If this is the case, we have

$$\begin{aligned}\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) &= \sum_{\rho=0}^{\rho_1-1} \alpha(\rho)\cdot 0 \; + (l^{a-\rho_1} - l^{a-\rho_1-1})l^{\mu-1}\chi_l(\tfrac{s}{2}) \\ &= \chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}l^{a-e}.\end{aligned}$$

Finally, if $e > a$ then $\mu + \rho_1 + 1 > \mu + \rho_0$ so $\rho_1 > \rho_0 - 1$. Thus, $\rho \le \rho_0 - 1$ implies $\rho \le \rho_1 - 1$ so that by (8)

$$\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) = \sum_{\rho=0}^{\rho_0-1} \alpha(\rho)\cdot 0 = 0.$$

Now consider $\alpha(\rho_0)c(\rho_0)$. We have $\alpha(\rho_0) = l^{a-\rho_0} - l^{a-\rho_0-1} = l^\mu - l^{\mu-1} =$

$l^{\mu-1}(l-1)$, so that by (5) directly

$$\begin{aligned}
\text{if } e \le a, \quad & \alpha(\rho_0)c(\rho_0) = (l^{\mu} - l^{\mu-1})l^{\mu-1}\chi_l^* + l^{\mu-1}(l-1)l^{\mu}\chi_l(\tfrac{s}{2}) \\
& \qquad = (l^{\nu-1} - l^{\nu-2})\chi_l^* + l^{\nu-1}(l-1)\chi_l(\tfrac{s}{2}); \\
\text{if } e = a+1, \quad & \alpha(\rho_0)c(\rho_0) = (l^{\mu} - l^{\mu-1})l^{\mu-1}\chi_l^* = (l^{\nu-1} - l^{\nu-2})\chi_l^*; \\
\text{if } e > a+1, \quad & \alpha(\rho_0)c(\rho_0) = 0.
\end{aligned}$$

Consider next $\sum_{\rho=\rho_0+1}^{a} \alpha(\rho)c(\rho)$. Note that $\rho_0 + 1 \le a$, for otherwise $1 > a - \rho_0 = \mu > 0$, a contradiction. If $\rho \ge \rho_0 + 1$ then $a - \rho \le a - \rho_0 - 1 = \mu - 1$; hence we use (4) to find $c(\rho)$.

Suppose $e \le 2\mu + 2(\rho_0 + 1) - a$ $(= a + 2)$; then for $\rho = \rho_0 + 1, \dots, a$ we have $e \le 2\mu + 2\rho - a$, so that

$$\begin{aligned}
\sum_{\rho=\rho_0+1}^{a} \alpha(\rho)c(\rho) &= \sum_{\rho=\rho_0+1}^{a} (l^{a-\rho} - l^{a-\rho-1})(l^{a-\rho} + l^{a-\rho-1})\chi_l^* \\
&= \chi_l^* \sum_{\rho=\rho_0+1} (l^{2a-2\rho} - l^{2a-2\rho-2}) = \chi_l^* l^{2a-2\rho_0-2} = \chi_l^* l^{\nu-2}.
\end{aligned}$$

Suppose $e > 2\mu + 2(\rho_0 + 1) - a$ $(= a + 2)$. Then either $e = 2\mu + 2\rho_1 - a$ for some $\rho_1 > \rho_0 + 1$ (note $\rho_1 \le a$ or a contradiction arises) or $e = 2\mu + 2\rho_1 - a + 1$ for some $\rho_1 \ge \rho_0 + 1$. In either case, $\sum_{\rho=\rho_0+1}^{a} \alpha(\rho)c(\rho) = \chi_l^* l^{\nu+a-e}$; the work done to show this is virtually the same as that which showed $\sum_{\rho=0}^{a} \alpha(\rho)c(\rho) = \chi_l^* l^{\nu+a-e}$ under the conditions $a \le \mu - 1$ and $e > 2\mu - a$, except that $\sum_{\rho=0}^{\rho_1} \alpha(\rho)c(\rho)$ must be replaced with $\sum_{\rho=\rho_0+1}^{\rho_1-1} \alpha(\rho)c(\rho)$; this has no effect on the outcome, as $c(\rho) = 0$ for each ρ in either of these two sums.

Now then, to write explicit formulas for (7) let us first add $\alpha(\rho_0)c(\rho_0)$ to $\sum_{\rho=\rho_0+1}^{a} \alpha(\rho)c(\rho)$ and simplify. We have

$$\begin{aligned}
\text{if } e \le a: \quad & \chi_l^*(l^{\nu-1} - l^{\nu-2}) + \chi_l(\tfrac{s}{2})l^{\nu-1}(l-1) + \chi_l^* l^{\nu-2} \\
& \quad = \chi_l^* l^{\nu-1} + \chi_l(\tfrac{s}{2})l^{\nu-1}(l-1); \\
\text{if } e = a+1: \quad & \chi_l^*(l^{\nu-1} - l^{\nu-2}) + \chi_l^* l^{\nu-2} = \chi_l^* l^{\nu-1} = \chi_l^* l^{\nu+a-e}; \\
\text{if } e = a+2: \quad & 0 + \chi_l^* l^{\nu-2} = \chi_l^* l^{\nu+a-e}; \\
\text{if } e > a+2: \quad & \chi_l^* l^{\nu+a-e}.
\end{aligned}$$

Note two things: First, we can combine the last three lines above into

$$\text{if } e \ge a+1: \quad \chi_l^* l^{\nu+a+e}.$$

Second, in (7) we take $\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) = 0$ in case $\rho_0 = 0$, and this is the case iff $a = \mu$. Therefore, $\sum_{\rho=0}^{a} \alpha(\rho)c(\rho)$ is given by the above results in the case $a = \mu$. If $\rho_0 > 0$ (i.e., if $a > \mu$) then we add $\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho)$ to the above results and simplify to find $\sum_{\rho=0}^{a}\alpha(\rho)c(\rho)$. We have:

if $e\le\mu$:

$$\chi_l(\tfrac{s}{2})l^{\nu-1}(l+1)(l^{a-\mu}-1)+\chi_l^*\,l^{\nu-1}+\chi_l(\tfrac{s}{2})l^{\nu-1}(l-1)$$
$$=\chi_l^*\,l^{\nu-1}+\chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}((l+1)(l^{a-\mu}-1)/(l-1)\ +1);$$

if $\mu<e\le a-1$:

$$\chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}((l+1)(l^{a-e}-1)/(l-1)+l^{a-e})$$
$$+\chi_l^*\,l^{\nu-1}+\chi_l(\tfrac{s}{2})l^{\nu-1}(l-1)$$
$$=\chi_l^*\,l^{\nu-1}+\chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}((l+1)(l^{a-e}-1)/(l-1)+l^{a-e}+1);$$

if $e=a$:

$$\chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}l^{a-e}+\chi_l^*\,l^{\nu-1}+\chi_l(\tfrac{s}{2})l^{\nu-1}(l-1)$$
$$=\chi_l^*\,l^{\nu-1}+\chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}(l^{a-e}+1);$$

if $e\ge a+1$:

$$0+\chi_l^*\,l^{\nu+a-e}=\chi_l^*\,l^{\nu+a-e}.$$

Refer now to Table 1 for $c(s,l)$. If $\mathrm{Case}(l)=A$ and $\nu=\mathrm{ord}_l(N)=2\mu$ is even, we have $c(s,l)$ is equal to

$$\chi_l^*\,l^{\min(2a,\nu-1,\nu+a-e)}$$
$$+\chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}\{(l+1)(l^{a-\max(\mu,e)}-1)/(l-1)+k_5l^{a-e}+k_6\},$$

where $k_5=d(\mu+1,e)d(\mu+1,a)$ and $k_6=d(e,a)d(\mu,a)$. We show that $c(s,l)=\sum_{\rho=0}^{a}\alpha(\rho)c(\rho)$ for $\mathrm{Case}(l)=A$ and even $\nu=\mathrm{ord}_l(N)=2\mu$.

Suppose $a\le\mu-1$. We have $2a\le 2\mu-2=\nu-2<\nu-1$ so that $\min(2a,\nu-1,\nu+a-e)=\min(2a,\nu+a-e)$. Now, $e\le\nu-a$ iff $2a\le\nu+a-e$, so that $c(s,l)$ gives the χ_l^*-term of the **SUM** properly. Next, $a-\max(\mu,e)\le a-\mu\le\mu-1-\mu=-1$; by our convention, then, $l^{a-\max(\mu,e)}-1=0$. Also, $a\le\mu-1<\mu<\mu+1$ so that $d(\mu,a)=0$ and $d(\mu+1,a)=0$ so that $k_6=k_5=0$. Therefore, since each term in the { }'s is 0, the $\chi_l(\frac{s}{2})$-term is 0.

Suppose $a=\mu$. Then $\min(2a,\nu-1,\nu+a-e)=\min(\nu-1,\nu+a-e)$, and moreover, $e>a$ iff $\nu+a-e=\min(\nu-1,\nu+a-e)$ so that $c(s,l)$ gives the χ_l^*-term properly. If $e\le a$, we have $a-\max(\mu,e)=a-\mu=0$ so that $l^{a-\max(\mu,e)}-1=0$. Clearly $k_5=0$ and k_6 is 1 so the $\chi_l(\frac{s}{2})$-term is given by $c(s,l)$ to be $\chi_l(\frac{s}{2})(l-1)l^{\nu-1}$. If $e>a$, we have $a-\max(\mu,e)\le a-e<0$, so that by convention, $l^{a-\max(\mu,e)}-1=0$; $k_5=k_6=0$ so that each term in the { }'s is 0, and so no $\chi_l(\frac{s}{2})$-term appears.

Suppose $a>\mu$. Then $\min(2a,\nu-1,\nu+a-e)=\min(\nu-1,\nu+a-e)$; again $e>a$ iff $\nu+a-e=\min(\nu-1,\nu+a-e)$ so that $c(s,l)$ correctly gives the χ_l^*-term. Now consider the $\chi_l(\frac{s}{2})$-term. If $e\le\mu$ we have $l^{a-\max(\mu,e)}=l^{a-\mu}$; $d(\mu+1,e)=0$ so that $k_5=0$, while $k_6=1$. The terms in { }'s become $(l+1)(l^{a-\mu}-1)/(l-1)+1$. If $\mu<e\le a-1$ we have $l^{a-\max(\mu,e)}=l^{a-e}$ while $k_5=k_6=1$ and the terms in { }'s become $(l+1)(l^{a-e}-1)/(l-1)+l^{a-e}+1$. If $e=a$ we get $l^{a-\max(\mu,e)}-1=l^{a-e}-1=0$; $k_5=k_6=1$ so the terms in { }'s become $l^{a-e}+1$. Lastly, if $e>a$, we have $a-\max(\mu,e)=a-e<0$ so that $l^{a-\max(\mu,e)}-1=0$; also $k_5l^{a-e}=0$, and $k_6=0$ so that each term in the { }'s and therefore the entire $\chi_l(\frac{s}{2})$-term is 0, and we have shown $c(s,l)$ gives the $\chi_l(\frac{s}{2})$-term correctly in each case.

This concludes the proof that $\sum_{\rho=0}^{a} \alpha(\rho)c(\rho) = c(s, l)$ for $Case(l) = A$ and even $\nu = \text{ord}_l(N)$; we leave verification in the other cases to the reader.

Example 6. Let $N = 3$, $k = 7$, and $n = 7$, and suppose that $\chi = (\frac{*}{3})$ is the Legendre symbol. We show how easy Theorem 2 makes the computation of $\text{tr}_{N,\chi,k} T_n$. First note that since $k > 2$, the $\delta(\chi)$-term of the trace formula is 0, while both $\delta(\sqrt{n})$-terms are 0 because n is not a perfect square. All we need to do is evaluate $\sum_s$. Now, $s^2 - 4n < 0$ for $0 \leq \pm s \leq 5$; $s^2 - 4n$ is a perfect square for $\pm s = 8$.

Suppose $s = 1$. We have $s^2 - 4n = -27$, so that $t = 3$ and $m = -3$. The contribution to $\sum_s$ for $s = 1$ is then $a(1)b(1)c(1, 3)$. To find $c(1, 3)$, we note that $\nu = \text{ord}_3(N) = 1$, so $\mu = 0$ and $\delta = 1$, that $e = e_3(\chi) = 1$, and we determine that 3 is a case C prime, with $a = a_3(s^2 - 4n) = 1$. By Table 1, $k_1 = k_2 = k_5 = \varepsilon = 0$, $k_3 = g = 1$, $k_4 = 6$, and $k_6 = d(1, 2)d(0, 1) = 1$; also $a - \max(\mu, e - \delta) + \varepsilon = 1$. By (2), $c(1, 3) = \chi_3(1/2)3^0(6(3-1)/2 + 1) = -7$. We compute $a(1) = 60$, while $b(1) = h(-3)/\omega(-3) = 1/3$. Therefore, the contribution to $\sum_s$ for $s = 1$ is $60 \cdot (1/3) \cdot -7 = -140$.

Similarly, when $s = 2$, the contribution to $\sum_s$ is $a(2)b(2)c(2, 3) = 51 \cdot (2/1) \cdot 1 = 102$; if $s = 4$ the contribution to $\sum_s$ is $a(4)b(4)\gamma(4, 2)c(4, 3) = -90 \cdot (1/3) \cdot 4 \cdot -1 = 120$; for $s = 5$ the contribution is $a(5)b(5)c(5, 3) = 180 \cdot (1/3) \cdot 1 = 60$; for each of these values of s, 3 is a case C prime in the evaluation of $c(s, 3)$.

Now, if $s = 0$ or 3, then $c(s, 3) = 0$. Therefore, there is no need to evaluate $a(s)b(s)$, nor any products of the $\gamma(s, l)$-terms; the contribution to $\sum_s$ for either of these values of s is simply 0.

If $s = 8$ then $s^2 - 4n = 36$, so that $t = 6$. The contribution to $\sum_s$ is then $a(8)b(8)\gamma(8, 2)c(8, 3)$. In finding $c(8, 3)$, we have $\nu = \text{ord}_3(N) = 1$, $\mu = 0$ and $\delta = 1$, $e = e_3(\chi) = 1$, and we determine that 3 is a case A prime, with $a = a_3(s^2 - 4n) = 1$. By Table 1, $k_2 = k_5 = k_6 = \varepsilon = 0$, $k_1 = g = 1$, and $k_3 = k_4 = 2$; also $\min(2a, \nu - 1, a + \nu - e) = 0$, $a - \max(\mu, e - \delta) + \varepsilon = 1$, $\chi_l^* = \chi_3(1) + \chi_3(7) = 2$, and $\chi_3(8/2) = 1$. By (2), $c(8, 3) = 2 \cdot 1 \cdot 1 + 0 + 1 \cdot 2 \cdot 1(2(3-1)/2 + 0 + 0) = 6$. We find $a(8) = 1/6$, while $b(8) = 1/2$. Also, $\gamma(8, 2) = 2$, as 2 is a case D prime and $a_2(s^2 - 4n) = 1$. Therefore, the contribution to $\sum_s$ for $s = 8$ is $(1/6)(1/2) \cdot 2 \cdot 6 = 1$.

By Proposition 7 below, the contributions to $\sum_s$ of s and $-s$ are equal, and so finally we obtain $\text{tr}_{N,\chi,k} T_n = -\sum_s = -2(-140+102+120+60+1) = -286$.

The following proposition states that for fixed s_0 the contributions of the terms corresponding to s_0 and $-s_0$ to the $\sum_s$ in the trace formula as given in Theorem 2 are the same. Therefore, the formula in Theorem 2 could be modified by taking the $\sum$ over all the *nonnegative* integers s satisfying $s^2 - 4n$ is a positive square or any negative integer, and replacing (say) $a(s)$ with $2a(s)$, except for $s = 0$.

Proposition 7. *Let the notation be as in Theorem 2. Let $s \in \mathbf{Z}$ satisfy $s^2 - 4n$ is a positive square or any negative integer. Then*

$$a(-s)b(-s) \prod_{l|t,\, l\nmid N} \gamma(-s, l) \prod_{l|N} c(-s, l) = a(s)b(s) \prod_{l|t,\, l\nmid N} \gamma(s, l) \prod_{l|N} c(s, l).$$

Proof. Fix s satisfying the hypothesis, and write $s^2 - 4n$ as t^2, $t^2 m$, or $t^2 4m$ as in Theorem 1. Fix a prime l with $l \mid t$, $l \nmid N$. Note that $b(s)$ and $\gamma(s, l)$

depend only on $s^2 - 4n = (-s)^2 - 4n$ so that $b(s) = b(-s)$ and $\gamma(s, l) = \gamma(-s, l)$. Let x and y be the roots in $\mathbf{C}$ of $X^2 - sX + n$; then $-x$ and $-y$ are the roots of $X^2 - (-s)X + n$, and it follows that $a(-s) = (-1)^k a(s)$. Referring to its definition, note that $c(s, l)$ is of the form

$$C_1\left(\chi_l\left(\frac{s + l^a d}{2}\right) + \chi_l\left(\frac{s - l^a d}{2}\right)\right) + C_2\left(\chi_l\left(\frac{s + l^{a+f} d}{2}\right)\right) + C_3\chi_l\left(\frac{s}{2}\right),$$

where C_1, C_2, and C_3 are functions of l, $e = e(\chi_l)$, $\nu = \operatorname{ord}_l(N)$, and $a = \operatorname{ord}_l(t)$, where $s^2 - 4n = (-s)^2 - 4n = t^2$, $t^2 m$, or $t^2 4m$ as the case may be, so that C_1, C_2, and C_3 are independent of the sign of s. Therefore, for the same C_1, C_2, and C_3, we have $c(-s, l)$ equals

$$C_1\left(\chi_l\left(\frac{-s + l^a d}{2}\right) + \chi_l\left(\frac{-s - l^a d}{2}\right)\right) + C_2\left(\chi_l\left(\frac{-s + l^{a+f} d}{2}\right)\right) + C_3\chi_l\left(\frac{-s}{2}\right).$$

First,

$$C_1\left(\chi_l\left(\frac{-s + l^a d}{2}\right) + \chi_l\left(\frac{-s - l^a d}{2}\right)\right) = \chi_l(-1)C_1\left(\chi_l\left(\frac{s + l^a d}{2}\right) + \chi_l\left(\frac{s - l^a d}{2}\right)\right).$$

Next, it is clear that $C_3\chi_l(\frac{-s}{2}) = \chi_l(-1)C_3\chi_l(\frac{s}{2})$. Furthermore, if $C_2 \neq 0$ then we must have $l = 2$ and $d(e, a + f) = 1$, that is, $e \leq a + f$. In this case, we have $0 \equiv 2^{a+f}d \equiv (s + 2^{a+f}d)/2 - (s - 2^{a+f}d)/2 \pmod{2^e}$, that is, $(s + 2^{a+f}d)/2 \equiv (s - 2^{a+f}d)/2 \pmod{2^e}$, so that

$$C_2\chi_2\left(\frac{-s + 2^{a+f}d}{2}\right) = C_2\chi_2(-1)\chi_2\left(\frac{s - 2^{a+f}d}{2}\right) = \chi_2(-1)C_2\chi_2\left(\frac{s + 2^{a+f}d}{2}\right).$$

Therefore, $c(-s, l) = \chi_l(-1)c(s, l)$, and it follows that

$$\prod_{l|N} c(-s, l) = \chi(-1)\prod_{l|N} c(s, l).$$

Finally then,

$$a(-s)b(-s)\prod_{l|t,\, l\nmid N} \gamma(-s, l)\prod_{l|N} c(-s, l) = (-1)^k\chi(-1)a(s)b(s)\prod_{l|t,\, l\nmid N} \gamma(s, l)\prod_{l|N} c(s, l).$$

This proves the result, because we assume (in both Theorems 1 and 2) that $(-1)^k\chi(-1) = 1$.

The following is easy to show using Theorem 2:

Corollary 8. *Let* k, χ, *and* N *be as in Theorems* 1 *and* 2. *The dimension of the space* $S_k(N, \chi)$ *is given by the formula*

$$\dim(S_k(N, \chi)) = -s_0 - s_1 + d + m - p,$$

where

$$s_0 = \begin{cases} 0 & \text{if any one of the following conditions is met: } k \text{ is odd; } 4 \mid N;\ \chi_l(-1) = -1 \text{ or } (\frac{-1}{l}) = -1 \text{ for some odd prime } l \mid N, \\ \frac{1}{4}(-1)^{k/2-1}\chi(r_0)2^n & \text{otherwise, where } r_0 \in \mathbf{Z} \text{ satisfies } r_0^2 \equiv -1 \pmod{N} \text{ and } n \text{ is the number of odd primes which divide } N, \end{cases}$$

$$s_1 = \begin{cases} 0 & \text{if any one of the following conditions is met: } k \equiv 1 \text{ or } 4 \pmod 6;\ 9 \mid N;\ 2 \mid N; \text{ or } (\frac{-3}{l}) = -1 \text{ for some odd prime } l \mid N,\ l \neq 3, \\ \frac{\alpha}{3}\chi(\frac{1}{2})\prod_{l \mid N, l \neq 3} \beta_l & \text{otherwise, where } \alpha = 1 \text{ if } k \equiv 2 \text{ or } 3 \pmod 6 \text{ and } -1 \text{ if } k \equiv 0 \text{ or } 5 \pmod 6;\ \beta_l = \chi_l(1 + r_1) + \chi_l(1 - r_1) \text{ where } r_1 \in \mathbf{Z} \text{ satisfies } r_1^2 \equiv -3 \pmod{N} \text{ if } (N, 3) = 1, \text{ and } r_1^2 \equiv -3 \pmod{\frac{N}{3}} \text{ if } 3 \| N, \end{cases}$$

$$d = \begin{cases} 1 & \text{if } k = 2 \text{ and } \chi \text{ is trivial,} \\ 0 & \text{otherwise,} \end{cases}$$

$$m = \frac{k-1}{12} N \prod_{l \mid N} (1 + 1/l), \qquad p = \frac{1}{2} \prod_{l \mid N} \operatorname{par}(l),$$

where $\operatorname{par}(l)$ *is defined as in Theorem* 1.

Proof. Since T_1 is the identity operator, the trace of T_1 acting on $S_k(N, \chi)$ gives the dimension of the space, so we need only evaluate Theorem 2 with n set to 1. Consider the sum over s in the first part of the trace formula as given in Theorem 2. Now, 0, 1, and -1 are the only values of s such that $s^2 - 4n$ is negative, and there are no integral values of s such that $s^2 - 4n$ is a positive square.

First, fix $s = 0$. We have $s^2 - 4n = -4 = t^2 4m$, where $t = 1$ and $m = -1 \equiv 3\,(4)$. Since i and $-i$ are the roots of $\Phi(X)$, we find $a(0) = (1/4)i^{k-2}(1 + (-1)^k)$. If k is odd, $a(0) = 0$; otherwise $a(0) = (1/2)(-1)^{k/2-1}$. The class number of $\mathbf{Q}\sqrt{-1}$ is 1, and one-half the cardinality of its unit group is 2 so that $b(0) = 1/2$. Since $t = 1$, $\prod_{l \mid t, l \nmid N} \gamma(0, l) = 1$. It remains to evaluate $\prod_{l \mid N} c(0, l)$. Let l be an odd prime dividing N, and set $\nu = \operatorname{ord}_l(N)$; we have $s^2 - 4n = -4 = l^{2a} \cdot -4$, where $a = 0$. Suppose that $(\frac{-4}{l}) = 1$ so that $\operatorname{Case}(l) = A$. Let $d_l \in \mathbf{Z}_l$ satisfy $d_l^2 = -1$, so that $(2d_l)^2 = -4$. Note that $2d_l$ is the 'd' which appears in the classification of l, so that $(s \pm l^a d)/2 = (0 \pm 1 \cdot 2d_l)/2 = \pm d_l$. Refer to Table 1 to find $c(0, l)$: We have $k_1 = 1$, $\min(2a, \nu - 1, a + \nu - e) = 0$, $k_2 = 0$, and $\chi_l(0/2) = 0$ so that $c(0, l) = \chi_l(d_l) + \chi_l(-d_l) = \chi_l(d_l)(1 + \chi_l(-1))$. If $\chi_l(-1) = -1$ then $c(0, l) = 0$ and hence the contribution of the $s = 0$ term to the trace is 0, while if $\chi_l(-1) = 1$ we have $c(0, l) = 2\chi_l(d_l)$. Now if $(\frac{-4}{l}) = -1$, so that $\operatorname{Case}(l) = B$, then referring to Table 1 for $c(0, l)$ we have $k_1 = k_2 = 0$ and $\chi_l(0/2) = 0$ so that $c(0, l) = 0$ and therefore the contribution of the $s = 0$ term to the trace is 0.

Keep $s = 0$, and suppose now that $l = 2$ and $l \mid N$. Then Case$(l) = F$, $a = 0$, $\chi_2(0/2) = 0$, and Table 1 for $c(0, l)$ gives $k_1 = 0$. Let $\nu = \mathrm{ord}_2(N)$; write $\nu = 2\mu$ or $2\mu + 1$ as the case may be. If $4 \mid N$ then $\mu \geq 1$ so that $d(\mu, a) = d(\mu, 0) = 0$, thus $k_2 = 0$ and so $c(0, 2) = 0$. If $2 \parallel N$ then $\nu = 1$ and $\mu = 0$, and χ_2 is the trivial character, so that $e = e(\chi_2) = 0$. We have $k_2 = d(e, a+1)d(\mu, a) = d(0, 1)d(0, 0) = 1$; in this case $c(0, 2) = k_2 k_3 \hat{\chi}_2 = 1 \cdot 1 \cdot \chi_2((0 + 2^{0+1} \cdot 1)/2) = \chi_2(1) = 1$.

Therefore the contribution of the $s = 0$ term to the trace is 0 unless k is even, $4 \nmid N$, $\chi_l(-1) = 1$ for all odd primes $l \mid N$, and $(\frac{-1}{l}) = 1$ for all odd $l \mid N$. Suppose that all these conditions are met. In particular, since $4 \nmid N$ and $(\frac{-1}{l}) = 1$ for all odd $l \mid N$, there is some $r \in \mathbf{Z}$ with $r^2 \equiv -1 (N)$; note r is odd if $2 \mid N$ so that $\chi_2(r) = 1$. If l is an odd prime dividing N and d_l is a unit in $\mathbf{Z}_l$ satisfying $d_l^2 = -1$, then $\chi_l(d_l) = \chi_l(r)$, because $d_l \equiv \pm r$ (mod $l^{\mathrm{ord}_l(N)}$) and $\chi_l(-1) = 1$. Finally then,

$$\prod_{l \mid N,\, l \text{ odd}} 2\chi_l(d_l) = \prod_{l \mid N,\, l \text{ odd}} 2\chi_l(r) = 2^n \chi(r),$$

where n is the number of odd primes dividing N.

Now fix $s = 1$. We have $s^2 - 4n = -3 = t^2 m$, where $t = 1$ and $m = -3$. The roots x and y of $\Phi(X) = X^2 - X + 1$ are $(1 \pm \sqrt{-3})/2$; deMoivre's formula gives $(x^{k-1} - y^{k-1})/(x - y) = 2i \sin((k-1)\pi/3)/(i\sqrt{3})$ so that

$$a(1) = \frac{1}{2} \cdot \begin{cases} 1 & \text{if } k \equiv 2, 3\,(6), \\ 0 & \text{if } k \equiv 1, 4\,(6), \\ -1 & \text{if } k \equiv 0, 5\,(6). \end{cases}$$

The class number of $\mathbf{Q}\sqrt{-3}$ is 1, and one-half the cardinality of the unit group is 3, so that $b(1) = 1/3$. Since $t = 1$, $\prod_{l \mid t,\, l \nmid N} \gamma(1, l) = 1$; it remains to evaluate $\prod_{l \mid N} c(1, l)$. Let $l \mid N$ be an odd prime, $l \neq 3$. Then $s^2 - 4n = -3 = l^{2a}(-3)$ with $a = 0$. Let $\nu = \mathrm{ord}_l(N)$ and set $\nu = 2\mu$ or $2\mu + 1$ as appropriate. Suppose $(\frac{-3}{l}) = 1$, so that Case$(l) = A$. Let d_l be a unit in $\mathbf{Z}_l$ satisfying $d_l^2 = -3$. Referring to Table 1 for $c(1, l)$ we see that $k_2 = 0$, $k_1 = 1$, and $l^{\min(2a, \nu-1, \nu+a-e)} = l^0 = 1$. If ν is even, then $\mu > a$ so that $d(\mu + 1, a)$ and $d(\mu, a)$ are both 0, hence $k_5 = k_6 = 0$, while if ν is odd, k_5 and k_6 are 0 automatically. Since $a - \max(\mu, e - \delta) \leq 0$, the k_4-term is 0, and hence for any ν, the contribution to $c(1, l)$ from the $\chi_l(\frac{s}{2})$-term is 0, and so $c(1, l) = \chi_l((1 + d_l)/2) + \chi_l((1 - d_l/2)$. If $(\frac{-3}{l}) = -1$ then Case$(l) = B$; here $k_1 = 0$ while the other k_i-terms are the same as for Case$(l) = A$. Thus if $(\frac{-3}{l}) = -1$, then $c(1, l) = 0$ and therefore the contribution of the $s = 1$ term to the trace is 0.

Suppose now that $l \mid N$ and $l = 3$; we have $s^2 - 4n = -3 = 3^{2a+1}(-1)$, where $a = 0$ and -1 is a unit in $\mathbf{Z}_3$, so that Case$(l) = C$. Let $\nu = \mathrm{ord}_3(N)$ and set $\nu = 2\mu$ or $2\mu + 1$. Suppose first that ν is even; refer to Table 1 for $c(1, 3)$. We have $k_1 = k_2 = k_6 = 0$; also $d(\mu, a) = 0$ so $k_5 = 0$. Furthermore, $a - \max(\mu, e) + 1 \leq 0$ so the k_4-term is 0. Therefore the $\chi_l(\frac{s}{2})$-term is 0, so that if ν is even, $c(1, 3) = 0$. Now, if ν is odd, we have $k_1 = k_2 = k_5 = 0$, and $a-\max(\mu, e-1)\leq 0$ so the k_4-term is 0. Consider k_6. If $\mu \geq 1$ then $d(\mu, a)=0$, so $k_6=0$, and therefore $c(1,3)=0$; combined with the fact that $c(1,3)=0$ if ν is even we have that the contribution of the $s=1$ term to the trace is 0 if $3^2 \mid N$. Suppose $3 \parallel N$; then $\mu=0$, and it follows that $k_6=1$. Therefore $c(1,3)=\chi_3(\frac{1}{2})\cdot 3^{1-1}(0+0+1)=\chi_3(\frac{1}{2})$; since χ_3 is now either the trivial character or $(\frac{*}{3})$, we have $c(1,3)=1$ or -1, respectively.

Suppose $l=2$ and $2 \mid N$; then $s^2-4n=-3=2^{2a}(-3)$ with $a=0$ and $-3\equiv 5 \pmod 8$ is a unit in $\mathbf{Z}_2$, so that $\mathrm{Case}(l)=E$. Refer to Table 1 for $c(1,2)$. We have $k_1=0$. Let $\nu=\mathrm{ord}_2(N)$ and set $\nu=2\mu$ or $2\mu+1$. Suppose first that ν is even. Then $\mu>a$ so that $k_2=k_5=k_6=0$ and the k_4-term is 0 and therefore $c(1,2)=0$. If ν is odd, then k_2 and the k_4-term are again 0, while k_5 and k_6 are automatically 0, so that $c(1,2)=0$. Hence, if $2\mid N$, the contribution of the $s=1$ term to the trace is 0.

We have shown that $\prod_{l\mid N} c(1,l)=0$ unless $9\nmid N$, $2\nmid N$, and $(\frac{-3}{l})=1$ for each odd prime $l\mid N$, $l\neq 3$. Suppose in fact that all these conditions are satisfied. It is then possible to find $r\in\mathbf{Z}$ such that $r^2\equiv -3 \pmod N$ if $(3,N)=1$, and $r^2\equiv -3 \pmod{(\frac{N}{3})}$ if $3\parallel N$, and satisfying the following: for each odd prime $l\mid N$, $l\neq 3$, we have $r\equiv \pm d_l \pmod{l^{\mathrm{ord}_l(N)}}$, where $d_l\in\mathbf{Z}_l$ is a unit with $d_l^2=-3$. For each such l we have $\chi_l(\frac{1}{2})(\chi_l(1+r)+\chi_l(1-r))=\chi_l(\frac{1}{2})(\chi_l(1\pm d_l)+\chi_l(1\mp d_l))=\chi_l((1+d_l)/2)+\chi_l((1-d_l)/2)$. Taking $\{\chi_3(\frac{1}{2})\}$ to mean 1 if $3\nmid N$ and $\chi_3(\frac{1}{2})$ if $3\parallel N$, we have $\prod_{l\mid N}c(1,l)=\{\chi_3(\frac{1}{2})\}\cdot\prod_{l\mid N,\, l\neq 3}(\chi_l((1+d_l)/2)+\chi_l((1-d_l)/2))=\chi(\frac{1}{2})\prod_{l\mid N,\, l\neq 3}(\chi_l(1+r)+\chi_l(1-r))$.

By Proposition 7, the contribution of the $s=-1$ term to the trace equals that of the $s=1$ term. The remaining terms in the dimension formula come immediately from the corresponding terms in either Theorem 1 or 2.

Consider the trace formula as given in Theorem 2. If χ is the trivial character, we can make additional simplifications to the formula, the most important being that $c(s,l)$ can be given by a very simple table; this is the result of our next corollary.

Corollary 9. *Let k, χ, N, and n be as in Theorem 2, and suppose furthermore that χ is the trivial character. Then for $(n,N)=1$ we have*

$$\mathrm{tr}_{N,\chi,k}T_n = -\sum_s\left(a(s)b(s)\prod_{l\mid t,\, l\nmid N}\gamma(s,l)\prod_{l\mid N}c_0(s,l)\right) + \left[\frac{2}{k}\right]\deg(T_n)+\delta_0(\sqrt{n})\frac{k-1}{12}N\prod_{l\mid N}(1+1/l) - \delta_0(\sqrt{n})\frac{\sqrt{n}}{2}\prod_{l\mid N}\mathrm{par}_0(l),$$

where s, $a(s)$, $b(s)$, t, and $\gamma(s,l)$ are exactly the same as in Theorem 2, and

$$\delta_0(\sqrt{n})=\begin{cases} n^{k/2-1} & \text{if } n \text{ is a perfect square},\\ 0 & \text{otherwise},\end{cases}$$

$$\mathrm{par}_0(l)=\begin{cases} l^{\mu}+l^{\mu-1} & \text{if } \nu=\mathrm{ord}_l(N)=2\mu,\\ 2l^{\mu} & \text{if } \nu=\mathrm{ord}_l(N)=2\mu+1,\end{cases}$$

and $c_0(s, l)$ is defined as follows. Fix s and a prime $l \mid N$. Let $\nu = \operatorname{ord}_l(N)$, write $\nu = 2\mu + \delta$, where $\delta = 0$ or 1. Classify l into one of the six cases A, ..., F, and referring to how Case(l) *is determined, let $a = a_l(s^2 - 4n)$. Let $d(x, y) = 1$ if $x \le y$ and 0 otherwise, for any x, $y \in \mathbf{Z}$. Then $c_0(s, l)$ is given by the expression*

(9) $$k_1 2l^{\min(2a, \nu-1)} + k_3 l^{\nu-1}(c_4(l^{a-\mu+\varepsilon} - 1)/(l-1) + c_6),$$

where the values of k_1, k_3, c_4, c_6, and ε are determined from Table 2.

TABLE 2

Case(l)	ν	k_1	k_3	c_4	ε	c_6
A or D	*even* *odd*	1	$l-1$	$l+1$ 2	0	$d(\mu, a)$ 0
B or E	*even* *odd*	0	$l+1$	$l+1$ 2	0	$d(\mu, a)$ 0
C or F	*even* *odd*	0	1	$l+1$ $2l$	1 0	0 $d(\mu, a)$

Remarks. Note that k_1 and k_3 are the same k_1 and k_3 as appear in Theorem 2, while c_4 and c_6 are similar to the k_4 and k_6 (respectively) of the same theorem. Also, one must heed Convention A in evaluating (9). In Theorems 1 and 2 we assume $(-1)^k\chi(-1) = 1$; the corollary's additional hypothesis that χ is trivial implies that k is even.

Proof. Let χ be the trivial character mod N, so that $\chi = \prod_{l \mid N} \chi_l$, where for each prime $l \mid N$, χ_l is the trivial character mod $l^{\operatorname{ord}_l(N)}$; note $e = e(\chi_l) = 0$ for each prime $l \mid N$. Let the trace $\operatorname{tr}_{N,\chi,k} T_n$ be as given by Theorem 2. Clearly the last three lines of the formula in the statement of Corollary 9 follow directly from the corresponding lines of Theorem 2. All one has to do is show how Table 1 "collapses" into Table 2 by showing that $c_0(s, l) = c(s, l)$ for each fixed s and fixed $l \mid N$, for any classification of the prime l, and any relationship between $a = a_l(s^2 - 4n)$ and μ where $\operatorname{ord}_l(N) = 2\mu$ or $2\mu + 1$. We leave the details to the reader.

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