A SIMPLIFIED TRACE FORMULA FOR HECKE OPERATORS FOR $\Gamma_0(N)$

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Abstract. Let $N$ and $n$ be relatively prime positive integers, let $\chi$ be a Dirichlet character modulo $N$, and let $k$ be a positive integer. Denote by $S_k(N, \chi)$ the space of cusp forms on $\Gamma_0(N)$ of weight $k$ and character $\chi$, a space denoted simply $S_k(N)$ when $\chi$ is the trivial character. Beginning with Hijikata's formula for the trace of $T_n$ acting on $S_k(N, \chi)$, we develop a formula which essentially reduces the computation of this trace to looking up values in a table. From this formula we develop very simple formulas for (1) the dimension of $S_k(N, \chi)$ and (2) the trace of $T_n$ acting on $S_k(N)$.

Preliminaries

For each positive integer $N$, let

$$\Gamma_0(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \bigg| a, b, c, d \in \mathbb{Z}, \det(\gamma) = 1 \right\};$$

$\Gamma_0(N)$ is a congruence subgroup of $SL_2(\mathbb{Z})$. Let $\chi$ be a (Dirichlet) character mod $N$. Suppose $N = \prod_{l|N} l^{v_l}$, where each $l$ is a prime and $v_l = \text{ord}_l(N)$. Then $\chi$ can be written as a product $\chi = \prod_{l|N} \chi_l$ of characters, where for each prime $l|N$, $\chi_l$ is a character mod $l^{v_l}$. The exponential conductor $e = e(\chi_l)$ is the smallest value $e$ such that $\chi_l$ is a character mod $l^e$; note that $e = e(\chi_l) \leq v_l$. If $\chi$ is a character and $\gamma = (a \ b \\ c \ d)$, with $a, b, c, d \in \mathbb{Z}$, then by $\chi(\gamma)$ we mean $\chi(a)$. In this paper we will use "|" and "\|" for "divides" and "does not divide," respectively.

Fix a positive integer $k$. For any complex-valued function $f$ and matrix $\gamma = (a \ b \\ c \ d)$ with $a, b, c, d \in \mathbb{R}$ and $\det(\gamma) > 0$ define

$$f \mid \gamma = (\det(\gamma))^{k/2}(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

(where we take the positive root if $k$ is odd).

Let $\mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$ denote the complex upper half plane, and let $f$ be any complex-valued function on $\mathcal{H}$. The cusps of $\Gamma_0(N)$ are the rational

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numbers, along with the point $i\infty$ at infinity. We say that $f$ is a \textit{cusp form on $\Gamma_0(N)$ of weight $k$ and character $\chi$} if $f$ satisfies 

(i) $f$ is holomorphic on $\mathcal{H}$,
(ii) $f$ is 0 at each cusp,
(iii) $f \mid (a \ b) = \chi(a)^{-1}f$ for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

See [Sha, A-L, or Li] for details. The space of cuspforms on $\Gamma_0(N)$ of weight $k$ and character $\chi$ is denoted by $S_k(N, \chi)$, or by $S_k(N)$ if $\chi$ is the trivial character.

For each $n$ with $(n, N) = 1$, let $T_n$ be the standard Hecke operator whose action on $S_k(N, \chi)$ is defined by 

$$f \mid T_n = n^{k-1} \sum_{a, d \equiv 0} \chi(a) f \left( \frac{a \tau + b}{d} \right) d^{-k},$$

where the first sum is over all pairs of integers $a, d$ satisfying $a > 0$, $ad = n$, and $(a, N) = 1$. Note this is the same as the $T'_n$ used by Shimura (see 3.5.7 of [Sha]), and therefore by Hijikata, Pizer, and Shemanske in [H-P-S1, H-P-S2]. If $n = p$, where $p$ is a prime not dividing $N$ and $\chi$ is the trivial character, then our $T_p$ is the same as the $T_p$ operator of Atkin and Lehner in [A-L]. (Note however that our weight $k$ is twice the weight $k$ of Atkin and Lehner.)

\section*{The Simplified Formula; Applications}

We begin by stating the version of the trace formula for the operator $T_n$ acting on $S_k(N, \chi)$ as given in Theorem 2.2 of [H-P-S1] and also in Theorem 2.2 of [H-P-S2]. Denote this trace by $\text{tr}_{N, \chi, k} T_n$.

\textbf{Theorem 1 (Hijikata-Pizer-Shemanske).} Let $k$ be an integer, $k \geq 2$. Let $\chi$ be a character mod $N$ and assume $(-1)^{k+1} \chi(-1) = 1$. Write $\chi = \prod_{l | N} \chi_l$, where for each prime $l$ dividing $N$, $\chi_l$ is a character mod $l^\nu$, where $\nu = \text{ord}_l(N)$. Then for $(n, N) = 1$ we have 

$$\text{tr}_{N, \chi, k} T_n = - \sum_s a(s) \sum f b(s, f) \prod_{l | N} c'_{\chi}(s, f, l) + \delta(\chi) \deg(T_n) + \delta(\sqrt{n}) \frac{k-1}{12} N \prod_{l | N} \left( 1 + \frac{1}{l} \right)$$

$$- \delta(\sqrt{n}) \frac{\sqrt{n}}{2} \prod_{l | N} \text{par}(l),$$

where 

$$\delta(\chi) = \begin{cases} 1 & \text{if } k = 2 \text{ and } \chi \text{ is trivial}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\delta(\sqrt{n}) = \begin{cases} n^{k/2-1} \chi(\sqrt{n}) & \text{if } n \text{ is a perfect square}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\text{par}(l) = \begin{cases} 2l^\nu - \nu & \text{if } e \geq \mu + 1, \\ l^\mu + l^{\mu-1} & \text{if } e \leq \mu \text{ and } \nu \text{ is even}, \\ 2l^\mu & \text{if } e \leq \mu \text{ and } \nu \text{ is odd}. \end{cases}$$
Here for a fixed prime 1 \mid N, \nu = \operatorname{ord}_l(N), \mu = \left[\frac{\nu}{l}\right], \text{ and } e = e(\chi_l).

The meanings of \( s, a(s), b(s, f), \) and \( c'_\chi(s, f, l) \) are given as follows:

Let \( s \) run over all integers such that \( s^2 - 4n \) is a positive square or any negative integer. Hence for some positive integer \( t \) and squarefree negative integer \( m \), \( s^2 - 4n \) has one of the following forms which we classify into the cases (h) or (e) as follows:

\[
s^2 - 4n = \begin{cases} 
  t^2, & (h) \\
  t^2m, & 0 < m \equiv 1 \pmod{4}, \quad (e) \\
  t^2m, & 0 < m \equiv 2, 3 \pmod{4}. \quad (e)
\end{cases}
\]

Let \( \Phi(X) = \Phi_t(X) = X^2 - sX + n \) and let \( x \) and \( y \) be the roots in \( \mathbb{C} \) of \( \Phi(X) = 0 \). Corresponding to the classification of \( s \), put

\[
a(s) = \begin{cases} 
  (\min\{|x|, |y|\})^{k-1} |x - y|^{-1} \operatorname{sgn}(x)^k, & (h) \\
  \frac{1}{2}(x^{k-1} - y^{k-1})/(x - y), & (e)
\end{cases}
\]

For each fixed \( s \), let \( f \) run over all positive divisors of \( t \) and let

\[
b(s, f) = \begin{cases} 
  \frac{1}{2}\phi((s^2 - 4n)^{1/2}/f), & (h) \\
  h((s^2 - 4n)/f^2)/\omega((s^2 - 4n)/f^2), & (e)
\end{cases}
\]

where \( \phi \) is Euler’s function and \( h(d) \) (respectively \( \omega(d) \)) denotes the class number of locally principal ideals (resp. \( \frac{1}{2} \) the cardinality of the unit group) of the order of \( \mathbb{Q}(\sqrt{d}) \) with discriminant \( d \).

Fix a pair \( (s, f) \) and let \( l \) be a prime divisor of \( N \); let \( \nu = \operatorname{ord}_l(N) \) and \( \rho = \operatorname{ord}_l(f) \). Put

\[
\begin{align*}
\tilde{A} &= \{ x \in \mathbb{Z} \mid \Phi(x) \equiv 0 (l^{\nu+2\rho}), 2x \equiv s(l^\rho) \}, \\
\tilde{B} &= \{ x \in \tilde{A} \mid \Phi(x) \equiv 0 (l^{\nu+2\rho+1}) \}.
\end{align*}
\]

Let \( A_p = A(s, f, l) \) (resp. \( B_p = B(s, f, l) \)) be a complete set of representatives of \( \tilde{A} \) (resp. \( \tilde{B} \)) mod \( l^{\nu+\rho} \), and let \( B'_p = B'(s, f, l) = \{ s - z \mid z \in B_p \} \). Then

\[
c'_\chi(s, f, l) = \begin{cases} 
  \sum_x \chi_l(x) & \text{if } (s^2 - 4n)/f^2 \not\equiv 0 (l), \\
  \sum_x \chi_l(x) + \sum_y \chi_l(y) & \text{if } (s^2 - 4n)/f^2 \equiv 0 (l),
\end{cases}
\]

where \( x \) (resp. \( y \)) runs over all elements of \( A_p \) (resp. \( B'_p \)). This ends the statement of the theorem.

Proof. See [Hij, H-P-S1, H-P-S2].

We introduce a classification of prime numbers to be used throughout this paper. Fix integers \( n \) and \( s \), with \( n \geq 1 \), such that \( s^2 - 4n \) is a positive square or any negative integer and write \( s^2 - 4n \) as one of \( t^2, t^2m, \) or \( t^24m \) as in the statement of Theorem 1. Let \( l \) be any prime that divides either \( N \) or \( t \), and classify \( l \) into one of six cases, \( A, B, C, D, E, \) or \( F \), depending on how \( l \)
divides $s^2 - 4n$ and whether or not $l$ is odd, as follows:

$$
\begin{align*}
A & \text{ if } s^2 - 4n = l^2a d^2, \ l \text{ is odd, and } d \text{ is a unit of } \mathbb{Z}_l, \\
B & \text{ if } s^2 - 4n = l^2a u, \ l \text{ is odd, and } u \text{ is a nonsquare unit of } \mathbb{Z}_l, \\
C & \text{ if } s^2 - 4n = l^2a + 1u, \ l \text{ is odd, and } u \text{ is a unit of } \mathbb{Z}_l, \text{ or} \\
D & \text{ if } s^2 - 4n = l^2a w, \ l = 2, \ w \in \mathbb{Z}_2, \ w \equiv 2 \pmod{4}, \\
E & \text{ if } s^2 - 4n = 2a w, \ l = 2, \ u \in \mathbb{Z}_2, \text{ and } u \equiv 5 \pmod{8}, \\
F & \text{ if } s^2 - 4n = 2^a w, \ l = 2, \ w \in \mathbb{Z}_2, \text{ and } w \equiv 3 \pmod{4}.
\end{align*}
$$

We will sometimes denote the case into which $l$ falls by Case$(l)$. Note that $a = \text{ord}_l(t)$, where $a$ is the \\
'$a$' which appears in the expression for $s^2 - 4n$, in whatever case $l$ is actually classified; we will sometimes write $a_l(s^2 - 4n)$ to mean this $a$.

Also let us introduce a convention to be adhered to throughout this paper:

**Convention A.** Let $l$ be a prime and let $n$ be any integer. We agree that any expression of the form $l^n$ or $l^n - 1$ is taken to be 0 if $n < 0$.

We are ready to state the new version of the trace formula.

**Theorem 2.** Let $k$ be an integer, $k \geq 2$. Let $\chi$ be a character mod $N$ and assume $(-1)^k \chi(-1) = 1$. Write $\chi = \prod_{l \mid N} \chi_l$, where for each prime $l$ dividing $N$, $\chi_l$ is a character mod $l^\nu$, where $\nu = \text{ord}_l(N)$. Then for $(n, N) = 1$ we have

$$
\text{tr}_{N, \chi, k} T_n = - \sum_s \left( a(s) b(s) \prod_{l \mid l, l \mid N} \gamma(s, l) \prod_{l \mid N} c(s, l) \right) \\
+ \delta(\chi) \deg(T_n) + \delta(\sqrt{n}) \frac{k - 1}{12} N \prod_{l \mid N} \left( 1 + \frac{1}{l} \right) \\
- \delta(\sqrt{n}) \frac{\sqrt{n}}{2} \prod_{l \mid N} \text{par}(l),
$$

where $\delta(\chi), \delta(\sqrt{n}), \text{ and } \text{par}(l)$ are exactly the same as in Theorem 1.

The meanings of $s$, $a(s)$, $b(s)$, $t$, $\gamma(s, l)$, and $c(s, l)$ are as follows:

Let $s$, $a(s)$, and $t$ be exactly as in Theorem 1. Now fix $s$ and write $s^2 - 4n$ as one of $l^2, t^2 m, t^2 4m$ as in Theorem 1. Let

$$
b(s) = \begin{cases} \\
in \frac{1}{2} & \text{if } s^2 - 4n = l^2, \\
h(m)/\omega(m) & \text{if } s^2 - 4n = l^2 m \text{ or } l^2 4m, \ m < 0,
\end{cases}
$$

where $h(m)$ is the class number of $\mathbb{Q}(\sqrt{m})$ and $\omega(m)$ is one-half the cardinality of the unit group of $\mathbb{Q}(\sqrt{m})$.

Keeping $s$ fixed, now fix a prime $l$ with $l \mid t, l \mid N$, and, according to the classification of $l$, let $a = a_l(s^2 - 4n)$ be the ‘$a$’ which appears in the expression for $s^2 - 4n$ and define

$$
\gamma(s, l) = \begin{cases} \\
l^a & \text{if } l \text{ is a case } A \text{ or } D \text{ prime}, \\
l^{a + 1} - 1 \pmod{l - 1} & \text{if } l \text{ is a case } B \text{ or } E \text{ prime}, \\
l^{a - 1} - 1 \pmod{l - 1} & \text{if } l \text{ is a case } C \text{ or } F \text{ prime}.
\end{cases}
$$
<table>
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<tr>
<th>Case (I)</th>
<th>( \nu )</th>
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<th>( k_2 )</th>
<th>( k_3 )</th>
<th>( k_4 )</th>
<th>( \epsilon )</th>
<th>( k_5 )</th>
<th>( k_6 )</th>
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<td>( l - 1 )</td>
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<td>0</td>
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<td>( l + 1 )</td>
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<td>C</td>
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<td>( l + 1 )</td>
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<td>F</td>
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<td>0</td>
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<td>( l + 1 )</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 1**
Keeping \( s \) fixed, now fix a prime \( l \mid N \). Let \( \nu = \text{ord}_l(N) \), and write \( \nu = 2\mu + \delta \), where \( \delta = 0 \) or \( 1 \). Let \( e = e(\chi_l) \). Classify \( l \) into one of the six cases \( A, \ldots, F \), and, according to the classification, let \( a = a_l(s^2 - 4n) \); if \( l \) is a case \( A \) or \( D \) prime let \( d \) be the \( \prime \) which appears in \( s^2 - 4n \), otherwise let \( d = 1 \). Let \( \chi_l^* = \chi_l(\frac{s^2 + d}{2}) + \chi_l(\frac{s^2 - d}{2}) \); let \( \chi_l = \chi_l(\frac{s^2 + e + d}{2}) \), where \( f = 1 \) if Case(\( l \)) = \( F \), and \( 0 \) otherwise. Then \( c(s, l) \) is given by the expression

\[
(2) \quad \chi_l^* k_l \min(2\mu, \nu - 1, a + \nu - e) + \chi_l k_2 k_3 \nu^{-1} + \chi_l(\frac{s}{2}) k_3 \nu - g(k_4(l^a - \max(\mu, e - \delta)) + 1)/(l - 1) + k_5 l^a - e + e + k_6,
\]

where \( k_1, \ldots, k_6, g, \) and \( \varepsilon \) are determined from Table 1, by knowing the parity of \( \nu \) and the classification of the prime \( l \). In the table, define \( d(x, y) = 1 \) if \( x < y \) and \( 0 \) otherwise, for any integers \( x \) and \( y \). Also, let \( \min_0(x, y) = \max(0, \min(x, y)) \). Note also that Convention A must be followed when evaluating \( (2) \).

We remark that we have used Theorem 2 to write a Turbo Pascal program which finds \( \operatorname{tr}_{N, \chi, k} T_n \) for small values of \( k, N \), and \( n \), and real characters \( \chi \).

The proof of Theorem 2 consists of transforming the first line of the formula given in Theorem 1 into the first line of that given in Theorem 2. We need two lemmas from [H-P-S1] or [H-P-S2]. The first of these is

**Lemma 3.** Let the notation be as Theorem 1. In particular, write \( s^2 - 4n = t^2, t^2m, \) or \( t^24m \) as illustrated there. Let \( l \) be any prime dividing \( N \) or \( t \) and put \( t = l^a t_0 \) where \( (l, t_0) = 1 \). Let \( f | t \) and put \( f = l^p f_0 \) with \( (l, f_0) = 1 \). Then \( b(s, f) = \alpha(s, \rho, 1) \cdot b(s, l^a f_0) \) where

\[
\alpha(s, \rho, 1) = \begin{cases} 
   l^{a-\rho} - l^{a-\rho-1} & \text{if } l \text{ is a case } A \text{ or } D \text{ prime}, \\
   l^{a-\rho} + l^{a-\rho-1} & \text{if } l \text{ is a case } B \text{ or } E \text{ prime}, \\
   l^{a-\rho} & \text{if } l \text{ is a case } C \text{ or } F \text{ prime}.
\end{cases}
\]

Note our Convention A in effect here; if \( \rho = a \) we take \( l^{a-\rho-1} = 0 \).

**Proof.** See Lemma 2.4 of either [H-P-S1] or [H-P-S2].

Fix \( s \) as in Theorem 1 and write \( s^2 - 4n = t^2, t^2m, \) or \( t^24m \) as illustrated there. Let \( l \mid N \) and define \( c'_{\chi}(s, \rho, l) = c'_{\chi}(s, l^p, l) \) for \( \rho = 0, \ldots, \text{ord}_l(t) \). Now let \( f \mid t \), and note that \( c'_{\chi}(s, f, l) \) depends only on \( \text{ord}_l(f) \) once \( s \) and \( l \) are fixed. Write \( f = l^p f_0 \) where \( \rho = \text{ord}_l(f) \); then

\[
c'_{\chi}(s, f, l) = c'_{\chi}(s, l^p f_0, l) = c'_{\chi}(s, l^p, l) = c'_{\chi}(s, \rho, l).
\]

Now, if there are \( v > 0 \) distinct primes \( l \) satisfying \( l \mid t \) and \( l \not\mid N \), let \( \{l_i\}, i = 1, \ldots, v \), be a list of them; if there are no such primes, then for convenience set \( v = 1 \) and define \( l_1 = 1 \) and \( \alpha(s, 0, 1) = 1 \). Next, if \( N \not\mid 1 \) let \( l_{u+1}, \ldots, l_{u+w} \) be a list of the \( w \) distinct primes dividing \( N \), while if \( N = 1 \) then for convenience set \( w = 1 \) and define \( l_{u+1} = 1 \), \( \alpha(s, 0, 1) = 1 \) and \( c'_{\chi}(s, 0, 1) = 1 \). We can write \( t = \prod_{i=1}^{u+w} l_i^{a_i} \) where \( a_i = \text{ord}_l(t) \) if \( l_i \) is a (bona-fide) prime and \( a_i = 0 \) if \( l_i = 1 \). Let \( f \) be any divisor of \( t \); then we
can write $f$ uniquely as $f = \prod_{i=1}^{v+w} l_i^{\rho_i}$, where $0 \leq \rho_i \leq a_i$. We have

$$\sum_{f|t} b(s, f) \prod_{l|N} c'_\chi(s, f, l)$$

(3)

$$= \sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \left( b\left(s, \prod_{i=1}^{v+w} l_i^{\rho_i}\right) \prod_{i=v+1}^{v+w} c'_\chi(s, \rho_i, l_i)\right).$$

The statement of Lemma 3 in our current notation is that, for fixed $\rho_1, \ldots, \rho_{v+w}$, with $0 \leq \rho_i \leq a_i$ for each $i$, $i = 1, \ldots, v + w$, and for some particular $i = i_0$ with $1 \leq i_0 \leq v + w$, we have

$$b\left(s, \prod_{i=1}^{v+w} l_i^{\rho_i}\right) = \alpha(s, \rho_{i_0}, l_{i_0}) b\left(s, \prod_{i\neq i_0}^{v+w} l_i^{\rho_i}\right)$$

(and this is clearly also true if $l_{i_0} = 1$). Repeated applications of this equality transform (3) into

$$\sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \left( \prod_{i=1}^{v+w} \alpha(s, \rho_i, l_i) \cdot \prod_{i=v+1}^{v+w} c'_\chi(s, \rho_i, l_i)\right).$$

Noting that $b(s, t) = b(s)$, this last expression equals

$$b(s) \sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \left( \prod_{i=1}^{v+w} \alpha(s, \rho_i, l_i) \cdot \prod_{i=v+1}^{v+w} c'_\chi(s, \rho_i, l_i)\right).$$

Again for convenience let $c'_\chi(s, \rho_i, l_i) = 1$ for $i = 1, \ldots, v$; the above becomes

$$b(s) \sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \left( \prod_{i=1}^{v+w} \alpha(s, \rho_i, l_i) \cdot \prod_{i=v+1}^{v+w} c'_\chi(s, \rho_i, l_i)\right).$$

All that is needed to prove Theorem 2 is to show two things. First, that $\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) = \gamma(s, l_i)$ for each $i$, $i = 1, \ldots, v$, in the case there are $v > 0$ primes $l$ satisfying $l | t$, $l \nmid N$; if there are no such primes then we arranged things so that $\prod_{l=1}^{v} (\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i)) = \alpha(s, 0, 1) = 1$ which agrees with the "empty" product $\prod_{l|t, l \nmid N} \alpha(l)$. Second, we need to show that

$$\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) c'_\chi(s, \rho, l_i) = c(s, l_i)$$
for each \( i, i = v + 1, \ldots, v + w \), in the case that there are \( w > 0 \) primes \( l \) dividing \( N \); if \( N = 1 \) we arranged for \( \prod_{i=v+1}^{u+w} \sum_{\rho=0}^{\alpha_i} \alpha(s, \rho, l_i) c'(s, \rho, l_i) = \alpha(s, 0, 1) c'(s, 0, 1) = 1 \cdot 1 = 1 \) which agrees with the “empty” product \( \prod_{l \mid N} c(s, l) \).

Suppose then that \( l \) is a prime, \( l \mid t \), \( l \nmid N \). Then \( l = l_i \) for some \( i \) with \( 1 \leq i \leq v \). Let \( a = \text{ord}_l(t) \); we show \( \sum_{\rho=0}^{\alpha_i} \alpha(s, \rho, l) = \gamma(s, l) \).

Suppose \( l \) is a case \( A \) or \( D \) prime. We have

\[
\sum_{\rho=0}^{\alpha} \alpha(s, \rho, l) = \sum_{\rho=0}^{\alpha} (l^{a_\rho} - l^{a_\rho - 1}) = l^a - 1 = l^a = \gamma(s, l).
\]

If Case(\( l \)) = \( B \) or \( E \),

\[
\sum_{\rho=0}^{\alpha} \alpha(s, \rho, l) = \sum_{\rho=0}^{\alpha} l^{a_\rho} + \sum_{\rho=0}^{\alpha} l^{a_\rho - 1} = (l^{a+1} - 1)/(l - 1) + (l^a - 1)/(l - 1) = \gamma(s, l).
\]

If Case(\( l \)) = \( C \) or \( F \),

\[
\sum_{\rho=0}^{\alpha} \alpha(s, \rho, l) = \sum_{\rho=0}^{\alpha} l^{a_\rho} = (l^{a+1} - 1)/(l - 1) = \gamma(s, l).
\]

So all that remains is to explicitly evaluate \( \sum_{\rho=0}^{\alpha_i} \alpha(s, \rho, l_i) c'(s, \rho, l_i) \) for each prime \( l_i \mid N \), and show the result is \( c(s, l_i) \). Fix \( l \mid N \) and write \( \alpha(\rho) \) for \( \alpha(s, \rho, l) \) and \( c(\rho) \) for \( c'(s, \rho, l) \). Now, the task of evaluating \( \sum_{\rho=0}^{\alpha_i} \alpha(\rho)c(\rho) \) and showing that it equals \( c(s, l) \) as given by Table 1 is a long straightforward one, but extremely tedious. We give some details concerning the explicit calculation/evaluation of \( \sum_{\rho=0}^{\alpha_i} \alpha(\rho)c(\rho) \) for Case(\( l \)) = \( A \) and \( \text{ord}_l(N) \) even, leaving all other calculations (i.e., those for Case(\( l \)) = \( B, \ldots, F \), \( \text{ord}_l(N) \) even or odd, and Case(\( l \)) = \( A \) with \( \text{ord}_l(N) \) odd) to the reader. First, let us summarize all the calculations here:

### Explicit Value of \( c(s, l) = \sum_{\rho=0}^{\alpha} \alpha(\rho)c(\rho) \)

<table>
<thead>
<tr>
<th>condition 1</th>
<th>condition 2</th>
<th>for Case(( l )) = ( A ), ( \nu = \text{ord}_l(N) ) even</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a \leq \mu - 1 ) ( e \leq \nu - a ) ( \chi_\nu l^{a_\nu} )</td>
<td>( \chi_\nu l^{a_\nu} )</td>
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</tr>
<tr>
<td>( a &gt; \mu ) ( e &lt; \nu - a ) ( \chi_\nu l^{a_\nu} )</td>
<td>( \chi_\nu l^{a_\nu} + \chi_\nu \left( \frac{\nu}{2} \right) (l - 1) l^{\nu - 1} )</td>
<td>( \chi_\nu l^{a_\nu} + \chi_\nu \left( \frac{\nu}{2} \right) (l - 1) l^{\nu - 1} )</td>
</tr>
<tr>
<td>( a &gt; \mu ) ( e = a ) ( \chi_\nu l^{a_\nu} )</td>
<td>( \chi_\nu l^{a_\nu} + \chi_\nu \left( \frac{\nu}{2} \right) (l - 1) l^{\nu - 1} )</td>
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<th>condition 2</th>
<th>Case(( l )) = ( A ), ( \nu = \text{ord}_l(N) ) odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a \leq \mu ) ( e \leq \nu - a ) ( \chi_\nu l^{a_\nu} )</td>
<td>( \chi_\nu l^{a_\nu} )</td>
<td>( \chi_\nu l^{a_\nu} )</td>
</tr>
<tr>
<td>( a &gt; \mu ) ( e &lt; \nu - a ) ( \chi_\nu l^{a_\nu} )</td>
<td>( \chi_\nu l^{a_\nu} + \chi_\nu \left( \frac{\nu}{2} \right) (l - 1) l^{\nu - 1} )</td>
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</tr>
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</table>
A SIMPLIFIED TRACE FORMULA FOR HECKE OPERATORS FOR $\Gamma_0(N)$

\[ a > \mu \quad e \leq \mu + 1 \quad \chi_i^* l^\nu - 1 + \chi_i((\frac{1}{2}) l^\nu - 1 2(l^\alpha - \mu - 1) \]
\[ \mu + 1 < e \leq a \quad \chi_i^* l^\nu - 1 + \chi_i((\frac{1}{2}) l^\nu - 1 2(l^\alpha - e + 1 - 1) \]
\[ e = a + 1 \quad \chi_i^* l^\nu - 1 (\neq \chi_i^* l^\nu + a - e) \]
\[ e > a + 1 \quad \chi_i^* l^\nu + a - e \]

condition 1 condition 2
Case($l$) = $B$, $\nu = \text{ord}_f(N)$ even
\[ a \leq \mu - 1 \quad (\text{none}) \quad 0 \]
\[ a = \mu \quad e \leq a \quad \chi_i((\frac{1}{2}) (l + 1) l^\nu - 1 \]
\[ e > a \quad 0 \]
\[ a > \mu \quad e \leq \mu \quad \chi_i((\frac{1}{2}) (l + 1) l^\nu - 1 ((l + 1)(l^\alpha - \mu - 1)/(l - 1) + 1) \]
\[ \mu < e \leq a - 1 \quad \chi_i((\frac{1}{2}) (l + 1) l^\nu - 1 ((l + 1)(l^\alpha - e - 1)/(l - 1) + l^\alpha - e + 1) \]
\[ e = a \quad \chi_i((\frac{1}{2}) (l + 1) l^\nu - 1 (l^\alpha - e + 1) \]
\[ e > a \quad 0 \]

condition 1 condition 2
Case($l$) = $B$, $\nu = \text{ord}_f(N)$ odd
\[ a \leq \mu - 1 \quad (\text{none}) \quad 0 \]
\[ a = \mu \quad e \leq a + 1 \quad \chi_i((\frac{1}{2}) (l + 1) l^\nu - 1 2(l^\alpha - \mu - 1)/(l - 1) \]
\[ e > a + 1 \quad 0 \]

condition 1 condition 2
Case($l$) = $C$, $\nu = \text{ord}_f(N)$ even
\[ a \leq \mu - 1 \quad (\text{none}) \quad 0 \]
\[ a = \mu \quad e \leq a + 1 \quad \chi_i((\frac{1}{2}) l^\nu - 1 \]
\[ e > a + 1 \quad 0 \]
\[ a > \mu + 1 \quad e \leq \mu + 1 \quad \chi_i((\frac{1}{2}) l^\nu - 1 (2(l^\alpha - \mu - 1)/(l - 1) + 1) \]
\[ \mu + 1 < e \leq a \quad \chi_i((\frac{1}{2}) l^\nu - 1 (2(l^\alpha - e - 1)/(l - 1) + l^\alpha - e + 1) \]
\[ e = a + 1 \quad \chi_i((\frac{1}{2}) l^\nu - 1 \]
\[ e > a + 1 \quad 0 \]

condition 1 condition 2
Case($l$) = $D$, $\nu = \text{ord}_f(N)$ even
\[ a \leq \mu - 1 \quad e \leq \nu - a \quad \chi_i^* l^2a \]
\[ e > \nu - a \quad \chi_i^* l^\nu + a - e \]
\[ a = \mu \quad e \leq a \quad \chi_i^* l^\nu - 1 + \hat{x}_i l^\nu - 1 \]
\[ e > a \quad \chi_i^* l^\nu + a - e \]
\[ a = \mu + 1 \quad e \leq a - 1 \quad \chi_i^* l^\nu - 1 + \hat{x}_i 2l^\nu - 1 + \chi_i((\frac{1}{2}) l^\nu \]
\[ e = a \quad \chi_i^* l^\nu - 1 + \hat{x}_i 2l^\nu - 1 \]
\[ e > a \quad \chi_i^* l^\nu + a - e \]
\[ a > \mu + 1 \quad e \leq \mu \quad \chi_i^* l^\nu - 1 + 2\hat{x}_i l^\nu - 1 + \chi_i((\frac{1}{2}) l^\nu ((l + 1)(l^\alpha - \mu - 1) - 1) + 1) \]
\[ \mu < e \leq a - 2 \quad \chi_i^* l^\nu - 1 + 2\hat{x}_i l^\nu - 1 \]
\[ + \chi_i((\frac{1}{2}) l^\nu ((l + 1)(l^\alpha - e - 1) - 1) + l^\alpha - e - 1 + 1) \]
\[ e = a - 1 \quad \chi_i^* l^\nu - 1 + 2\hat{x}_i l^\nu - 1 + \chi_i((\frac{1}{2}) l^\nu (l^\alpha - e - 1) + 1) \]
\[ e = a \quad \chi_i^* l^\nu - 1 + 2\hat{x}_i l^\nu - 1 \]
\[ e > a \quad \chi_i^* l^\nu + a - e \]

condition 1 condition 2
Case($l$) = $D$, $\nu = \text{ord}_f(N)$ odd
\[ a \leq \mu \quad e \leq \nu - a \quad \chi_i^* l^2a \]
\[ e > \nu - a \quad \chi_i^* l^\nu + a - e \]
In all of what follows, by the “SUM”, we mean $\sum_{\rho=0}^{a} \alpha(\rho)c(\rho)$. The key to its explicit evaluation is

**Lemma 4.** Let $p$ be a prime and let $\omega$ be a character modulo some power of $p$. Let $e = e(\omega)$ be the exponential conductor of $\omega$. If $\sigma$ and $b$ are nonnegative
integers with $\sigma + b \geq e$ and $2b \geq e$ and if $u$ is a unit mod $p$, then

$$\sum_{z \in \mathbb{Z}/p^n\mathbb{Z}} \omega(u + zp^b) = \begin{cases} p^\sigma \omega(u) & \text{if } e \leq b, \\ 0 & \text{if } e > b. \end{cases}$$

Proof. This is easy. See Lemma 2.1 of [H-P-S1] or [H-P-S2].

Let $A_p$ and $B'_p$ be as in Theorem 1. Specific sets of representatives for $A_p$ and $B'_p$ are calculated by the authors of [H-P-S1, H-P-S2] in their Lemma 2.5 by “easy but tedious calculations”; we copy them here for reference as

**Lemma 5.** Let $A(s, f, l) = A_p$ and $B'(s, f, l) = B'_p$ be the sets appearing in Theorem 1. For fixed $N$, $n$, $s$, and $l$, $A_p$ and $B'_p$ depend only on $p = \text{ord}_l(f)$. Let $\nu = \text{ord}_l(N)$ and set $\nu = 2\mu$ or $\nu = 2\mu + 1$. Classify $l$ as case $A$, $B$, etc., setting $a$ and $d$ according to how $l$ is classified. Then the sets $A_p$ and $B'_p$ are as follows:

**Case** ($l$) \hspace{1cm} $\nu$ \hspace{1cm} condition

**A** odd \hspace{1cm} $a - p \leq \mu$ \hspace{1cm} $A_p = \left\{ \frac{t+b}{2} + zl^{a+2p-a+1} \mid z \in \mathbb{Z}/l^{a-p}\mathbb{Z} \right\}$

$$B'_p = \left\{ \frac{t+b}{2} + zl^{a+2p-a+2} \mid z \in \mathbb{Z}/l^{a-p-1}\mathbb{Z} \right\}$$

$a - p \geq \mu + 1$ \hspace{1cm} $A_p = B'_p = \left\{ \frac{t+b}{2} + zl^{a+1} \mid z \in \mathbb{Z}/l^{a}\mathbb{Z} \right\}$

**B** odd \hspace{1cm} $a - p \leq \mu$ \hspace{1cm} $A_p = B'_p = \emptyset$

$a - p \geq \mu + 1$ \hspace{1cm} $A_p = B'_p = \left\{ \frac{t+b}{2} + zl^{a+1} \mid z \in \mathbb{Z}/l^{a}\mathbb{Z} \right\}$

**C** odd \hspace{1cm} $a - p \leq \mu - 1$ \hspace{1cm} $A_p = B'_p = \emptyset$

$a - p = \mu$ \hspace{1cm} $A_p = \left\{ \frac{t+b}{2} + zl^{a+1} \mid z \in \mathbb{Z}/l^{a}\mathbb{Z} \right\}$

$b'_p = \emptyset$

$a - p \geq \mu + 1$ \hspace{1cm} $A_p = B'_p = \left\{ \frac{t+b}{2} + zl^{a+1} \mid z \in \mathbb{Z}/l^{a}\mathbb{Z} \right\}$

**D** odd \hspace{1cm} $a - p \leq \mu$ \hspace{1cm} $A_p$ and $B'_p$ are the same as for a case $A$ prime

$a - p = \mu + 1$ \hspace{1cm} $A_p = B'_p = \left\{ \frac{t+b}{2} + zl^{a} \mid z \in \mathbb{Z}/l^{a}\mathbb{Z} \right\}$

$a - p \geq \mu + 2$ \hspace{1cm} $A_p$ and $B'_p$ are the same as for a case $A$ prime
even \( a - p \leq \mu - 1 \) \( A_p \) and \( B'_p \) are the same as for a case \( A \) prime

\[
A_p = \left\{ \frac{sl^d}{2} + z^a \mid z \in \mathbb{Z}/l^{a} \mathbb{Z} \right\}
\]

\[
B'_p = \left\{ \frac{sl^{d+1}}{2} + z^{a+1} \mid z \in \mathbb{Z}/l^{a+1} \mathbb{Z} \right\}
\]

\[
a - p = \mu + 1 \quad A_p = \left\{ \frac{sl^d}{2} + z^a \mid z \in \mathbb{Z}/l^{a} \mathbb{Z} \right\}
\]

\[
B'_p = \left\{ \frac{sl^{d+1}}{2} + z^{a+1} \mid z \in \mathbb{Z}/l^{a+1} \mathbb{Z} \right\}
\]

\[
a - p \geq \mu + 2 \quad A_p \) and \( B'_p \) are the same as for a case \( A \) prime
\]

\[
E \quad \text{odd} \quad a - p \leq \mu \quad A_p = B' = \emptyset
\]

\[
a - p \geq \mu + 1 \quad A_p \) and \( B'_p \) are the same as for a case \( D \) prime,
\]

\[
\text{with } d \text{ set to } 1
\]

\[
even \quad a - p \leq \mu - 1 \quad A_p = B'_p = \emptyset
\]

\[
a - p = \mu \quad A_p = \left\{ \frac{sl^d}{2} + z^a \mid z \in \mathbb{Z}/l^{a} \mathbb{Z} \right\}
\]

\[
B'_p = \emptyset
\]

\[
a - p \geq \mu + 1 \quad A_p \) and \( B'_p \) are the same as for a case \( D \) prime,
\]

\[
\text{with } d \text{ set to } 1
\]

\[
F \quad \text{odd} \quad a - p \leq \mu - 1 \quad A_p = B'_p = \emptyset
\]

\[
a - p = \mu \quad A_p = \left\{ \frac{sl^d+1}{2} + z^{a+1} \mid z \in \mathbb{Z}/l^{a+1} \mathbb{Z} \right\}
\]

\[
B'_p = \emptyset
\]

\[
a - p \geq \mu + 1 \quad A_p \) and \( B'_p \) are the same as for a case \( A \) prime
\]

\[
\text{even} \quad a - p \leq \mu - 1 \quad A_p = B'_p = \emptyset
\]

\[
a - p = \mu \quad A_p = \left\{ \frac{sl^d}{2} + z^a \mid z \in \mathbb{Z}/l^{a} \mathbb{Z} \right\}
\]

\[
B'_p = \emptyset
\]

\[
a - p \geq \mu + 1 \quad A_p \) and \( B'_p \) are the same as for a case \( A \) prime
\]

\[
\text{If } \rho = \alpha \text{ then set } B'_p = \emptyset.
\]

We can now make one more observation: With \( s \) fixed and \( l \mid N \), let \( f \mid t \) and \( \rho = \text{ord}_l(f) \) for some \( \rho \geq 0 \). Refer to the definition of \( c''_x(s, f, l) \) as in Theorem 1. We can actually write

\[
c''_x(s, \rho, l) = c''_x(s, f, l) \quad \text{as} \quad \sum_{x \in A_p} \chi_l(x) + \sum_{y \in B'_p} \chi_l(y).
\]

For suppose \( (s^2 - 4n)/(l^p)^2 \neq 0 (l) \). Write \( s^2 - 4n \) as \( t^2 \), \( t^2m \), or \( t^2 4m \) as in Theorem 1 and write \( t = l^a t_0 \), where \( (l, t_0) = 1 \); recall this \( a = a_l(s^2 - 4n) \). Clearly if \( (s^2 - 4n)/(l^p)^2 \neq 0 (l) \) then \( t^2/l^{2a} = l^{2a-2}t_0 \neq 0 (l) \), which implies \( \rho = \alpha \). However, in every case in which \( \rho = \alpha \), Lemma 5 shows that the set \( B'_p \) is empty; consequently \( \sum_{y \in B'_p} \chi_l(y) = 0 \). Let us write “\( A_p\text{sum} \)” and “\( B'_p\text{sum} \)” for \( \sum_{x \in A_p} \chi_l(x) \) and \( \sum_{y \in B'_p} \chi_l(y) \), respectively, so that \( c''_x(s, \rho, l) = A_p\text{sum} + B'_p\text{sum} \). Finally we are ready to begin evaluating \( \sum_{p=0}^{a} \alpha(p)c(p) \). Suppose \( \text{Case}(l) = A \) and \( \nu = \text{ord}_l(N) \) is even; write \( \nu = 2\mu \). Set \( a = a_l(s^2 - 4n) \) and \( e = e(\chi_l) \). If \( a - \rho \leq \mu - 1 \) then by applying Lemma 4 twice on each of the sets \( A_p \) and \( B'_p \) as given in Lemma 5, we have

\[
A_p\text{sum} = \begin{cases}
       l^{a-\rho} \chi_l^* & \text{if } e \leq 2\mu + 2\rho - a, \\
       0 & \text{if } e > 2\mu + 2\rho - a
\end{cases}
\]

and
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$$B'_\rho \text{sum} = \begin{cases} l^{a-\rho-1} \chi^*_l & \text{if } e \leq 2\mu + 2\rho - a + 1, \\
0 & \text{if } e > 2\mu + 2\rho - a + 1. \end{cases}$$

Notice that if $\rho = a$ then $B'_\rho = \emptyset$; in this case the above formula gives $l^{-1} \chi^*_l$ which is 0 by our convention. Adding $A_\rho \text{sum}$ and $B'_\rho \text{sum}$, we have for $a-\rho \leq \mu - 1$

$$c(\rho) = \begin{cases} (l^{a-\rho} + l^{a-\rho-1}) \chi^*_l & \text{if } e \leq 2\mu + 2\rho - a, \\
l^{a-\rho-1} \chi^*_l & \text{if } e = 2\mu + 2\rho - a + 1, \\
0 & \text{if } e > 2\mu + 2\rho - a + 1. \end{cases}$$

If $a-\rho = \mu$, then by Lemmas 4 and 5 we have

$$A_\rho \text{sum} = \begin{cases} l^{\mu} \chi_l(\frac{\sigma}{2}) & \text{if } e \leq a, \\
0 & \text{if } e > a \end{cases}$$

and

$$B'_\rho \text{sum} = \begin{cases} l^{\mu-1} \chi^*_l & \text{if } e \leq a + 1, \\
0 & \text{if } e > a + 1. \end{cases}$$

Adding $A_\rho \text{sum}$ and $B'_\rho \text{sum}$, we have for $a-\rho = \mu$

$$c(\rho) = \begin{cases} l^{\mu} \chi_l(\frac{\sigma}{2}) + l^{\mu} \chi_l(\frac{\sigma}{2}) & \text{if } e \leq a, \\
l^{\mu-1} \chi^*_l & \text{if } e = a + 1, \\
0 & \text{if } e > a + 1. \end{cases}$$

Now suppose $a-\rho \geq \mu + 1$. By Lemmas 4 and 5 we have

$$A_\rho \text{sum} = \begin{cases} l^{\mu} \chi_l(\frac{\sigma}{2}) & \text{if } e \leq \mu + \rho, \\
0 & \text{if } e > \mu + \rho \end{cases}$$

and

$$B'_\rho \text{sum} = \begin{cases} l^{\mu-1} \chi^*_l & \text{if } e \leq \mu + \rho + 1, \\
0 & \text{if } e > \mu + \rho + 1. \end{cases}$$

Adding $A_\rho \text{sum}$ and $B'_\rho \text{sum}$ we have for $a-\rho \geq \mu + 1$

$$c(\rho) = \begin{cases} (l^{\mu} + l^{\mu-1}) \chi_l(\frac{\sigma}{2}) & \text{if } e \leq \mu + \rho, \\
l^{\mu-1} \chi_l(\frac{\sigma}{2}) & \text{if } e = \mu + \rho + 1, \\
0 & \text{if } e > \mu + \rho + 1. \end{cases}$$

Now we are ready to calculate the SUM $= \sum_{\rho=0}^{\mu} \alpha(\rho) c(\rho)$ under the various possibilities.

Suppose $a \leq \mu - 1$. Then for $\rho = 0, \ldots, a$ we have $a - \rho \leq \mu - 1$ and therefore $c(\rho)$ is given by (4).

Suppose $e \leq 2\mu - a (= \nu - a)$. Then for $\rho = 0, \ldots, a$ we have $e \leq 2\mu + 2\rho - a$; by (4) we have

$$\sum_{\rho=0}^{a} \alpha(\rho) c(\rho) = \sum_{\rho=0}^{a} (l^{a-\rho} - l^{a-\rho-1})(l^{a-\rho} + l^{a-\rho-1}) \chi^*_l$$

$$= \chi^*_l \sum_{\rho=0}^{a} (l^{2a-2\rho} - l^{2a-2\rho-2}) = \chi^*_l (l^{2a} - l^{-2}) = \chi^*_l l^{2a}.$$
Suppose \( e > 2\mu - a \). Then either \( e = 2\mu + 2\rho_1 - a \) for some \( \rho_1 > 0 \), or \( e = 2\mu + 2\rho_1 - a + 1 \) for some \( \rho_1 \geq 0 \); we show the SUM is \( X_i^{*} l^\nu + a - e \) in either case. Suppose then that \( e = 2\mu + 2\rho_1 - a \) for some \( \rho_1 > 0 \); note \( \rho_1 \leq a \). If \( \rho \) satisfies \( \rho_1 > \rho \) then \( 2\rho_1 > 2\rho + 1 \) implies \( e = 2\mu + 2\rho_1 - a > 2\mu + 2\rho - a + 1 \); if \( \rho_1 \leq \rho \) then \( e = 2\mu + 2\rho_1 - a \leq 2\mu + 2\rho - a \), so that by (4)

\[
c(\rho) = \begin{cases} 
(l^{a-p} + l^{a-p-1})X_i^{*} & \text{if } \rho \geq \rho_1, \\
0 & \text{if } \rho < \rho_1,
\end{cases}
\]

and the SUM becomes

\[
\sum_{\rho=0}^{\rho_1-1} \alpha(\rho)c(\rho) + \sum_{\rho=\rho_1}^{a} \alpha(\rho)c(\rho)
= 0 + \sum_{\rho=\rho_1}^{a} (l^{a-p} - l^{a-p-1})(l^{a-p} + l^{a-p-1})X_i^{*}
= X_i^{*} \sum_{\rho=\rho_1}^{a} (l^{2a-2\rho} - l^{2a-2\rho-2}) = X_i^{*} l^{2a-2\rho_1} = X_i^{*} l^\nu + a - e.
\]

Suppose \( e = 2\mu + 2\rho_1 - a + 1 \) for some \( \rho_1 > 0 \); note \( \rho_1 + 1 \leq a \). If \( \rho \) satisfies \( \rho_1 > \rho \) then \( e = 2\mu + 2\rho_1 - a + 1 > 2\mu + 2\rho - a + 1 \); if \( \rho_1 \geq \rho \) then \( e < 2\mu + 2\rho - a + 1 \), i.e., \( e \leq 2\mu + 2\rho - a \). We have

\[
c(\rho) = \begin{cases} 
(l^{a-p} + l^{a-p-1})X_i^{*} & \text{if } \rho > \rho_1, \\
l^{a-p_1-1}X_i^{*} & \text{if } \rho = \rho_1, \\
0 & \text{if } \rho < \rho_1,
\end{cases}
\]

so that the SUM equals

\[
\sum_{\rho=0}^{\rho_1-1} \alpha(\rho)c(\rho) + \alpha(\rho_1)l^{a-p_1-1}X_i^{*} + \sum_{\rho=\rho_1+1}^{a} \alpha(\rho)(l^{a-p} + l^{a-p-1})X_i^{*}
= 0 + (l^{a-p_1} - l^{a-p_1-1})l^{a-p_1-1}X_i^{*}
+ \sum_{\rho=\rho_1+1}^{a} (l^{2a-2\rho} - l^{2a-2\rho-2})X_i^{*}
= X_i^{*} (l^{2a-2\rho_1-1} - l^{2a-2\rho_1-2} + l^{2a-2\rho_1-2})
= X_i^{*} l^{2a-2\rho_1-1} = X_i^{*} l^\nu + a - e.
\]

Suppose \( a \geq \mu \); then we can write \( a - \rho_0 = \mu \) for some \( \rho_0 \geq 0 \). Write the SUM as

\[
\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) + \alpha(\rho_0)c(\rho_0) + \sum_{\rho=\rho_0+1}^{a} \alpha(\rho)c(\rho),
\]

where we take \( \sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) = 0 \) in the case \( \rho_0 = 0 \). Suppose \( \rho_0 > 0 \) and consider \( \sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) \). If \( \rho \leq \rho_0 - 1 \) then \( a - \rho \geq a - (\rho_0 - 1) = a - \rho_0 + 1 = \mu + 1 \) so that \( c(\rho) \) is determined using (6).
Suppose $e \leq \mu$; then $e \leq \mu + \rho$ for $\rho = 0, \ldots, \rho_0 - 1$. From (6),

\[
\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) = \sum_{\rho=0}^{\rho_0-1} \left( l^{a-\rho} - l^{a-\rho-1} \right) \left( l^{\mu} + l^{\mu-1} \right) \chi_I(\frac{\xi}{2})
\]

\[
= \chi_I(\frac{\xi}{2}) \left( l^{\mu} + l^{\mu-1} \right) \sum_{\rho=0}^{\rho_0-1} \left( l^{a-\rho} - l^{a-\rho-1} \right)
\]

\[
= \chi_I(\frac{\xi}{2}) l^{\mu-1}(l+1)(l^{a-\rho_0}) = \chi_I(\frac{\xi}{2}) l^{\mu-1}(l+1)l^{\mu}(l^{a-\mu} - 1)
\]

\[
= \chi_I(\frac{\xi}{2}) l^{\mu-1}(l+1)(l^{a-\mu} - 1).
\]

Suppose $e > \mu$; then for some $\rho_1 \geq 0$ we have $e = \mu + \rho_1 + 1$. Suppose $\rho \leq \rho_0 - 1$. Then $e = \mu + \rho_1 + 1 \leq \mu + \rho$ iff $\rho_1 + 1 \leq \rho$; by (6) we have

\[
c(\rho) = \begin{cases} 
(l^{\mu} + l^{\mu-1}) \chi_I(\frac{\xi}{2}) & \text{if } \rho \geq \rho_1 + 1, \\
 l^{\mu-1} \chi_I(\frac{\xi}{2}) & \text{if } \rho = \rho_1, \\
0 & \text{if } \rho < \rho_1.
\end{cases}
\]

Now, $\rho_1 + 1 \leq \rho_0 - 1$ iff $\mu + \rho_1 + 1 \leq \mu + \rho_0 - 1$ iff $e \leq a - 1$. Suppose this is the case. We have

\[
\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) = \sum_{\rho=0}^{\rho_1-1} \alpha(\rho) \cdot 0 + \left( l^{a-\rho_1} - l^{a-\rho_1-1} \right) l^{\mu-1} \chi_I(\frac{\xi}{2})
\]

\[
+ \sum_{\rho=\rho_1+1}^{\rho_0-1} \left( l^{a-\rho} - l^{a-\rho-1} \right) \left( l^{\mu} + l^{\mu-1} \right) \chi_I(\frac{\xi}{2})
\]

\[
= \chi_I(\frac{\xi}{2}) \left( l^{a-e+\mu+1} - l^{a-e+\mu} \right) l^{\mu-1}
\]

\[
+ (l^{\mu} + l^{\mu-1}) (l^{a-\rho_1-1} - l^{a-\rho_0})
\]

\[
= \chi_I(\frac{\xi}{2}) \left( l^{a-e} l^{\mu}(l-1) l^{\mu-1} + l^{\mu-1}(l+1)(l^{a-e+\mu} - l^{\mu}) \right)
\]

\[
= \chi_I(\frac{\xi}{2}) \left( l^{a-e} l^{\mu-1}(l-1) + l^{\mu-1}(l+1)(l^{a-e} - 1) \right)
\]

\[
= \chi_I(\frac{\xi}{2}) (l-1) l^{\mu-1} ((l+1)(l^{a-e} - 1)/(l-1) + l^{a-e}).
\]

Next, $\rho_1 = \rho_0 - 1$ iff $\mu + \rho_1 + 1 = \mu + \rho_0$ iff $e = a$. If this is the case, we have

\[
\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) = \sum_{\rho=0}^{\rho_1-1} \alpha(\rho) \cdot 0 + \left( l^{a-\rho_1} - l^{a-\rho_1-1} \right) l^{\mu-1} \chi_I(\frac{\xi}{2})
\]

\[
= \chi_I(\frac{\xi}{2})(l-1) l^{\mu-1} l^{a-e}.
\]

Finally, if $e > a$ then $\mu + \rho_1 + 1 > \mu + \rho_0$ so $\rho_1 > \rho_0 - 1$. Thus, $\rho \leq \rho_0 - 1$ implies $\rho \leq \rho_1 - 1$ so that by (8)

\[
\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) = \sum_{\rho=0}^{\rho_0-1} \alpha(\rho) \cdot 0 = 0.
\]

Now consider $\alpha(\rho_0)c(\rho_0)$. We have $\alpha(\rho_0) = l^{a-\rho_0} - l^{a-\rho_0-1} = l^{\mu} - l^{\mu-1}$.
$l^{\mu-1}(l - 1)$, so that by (5) directly

if $e \leq a$, \quad \alpha(p_0)c(p_0) = (l^{\mu} - l^{\mu-1})l^{\mu-1} \chi_l^* + l^{\mu-1}(l - 1)l^{\mu} \chi_l\left(\frac{x}{2}\right)

= (l^{\mu-1} - l^{\mu-2})\chi_l^* + l^{\mu-1}(l - 1)\chi_l\left(\frac{x}{2}\right);

if $e = a + 1$, \quad \alpha(p_0)c(p_0) = (l^{\mu} - l^{\mu-1})l^{\mu-1} \chi_l^* = (l^{\mu-1} - l^{\mu-2})\chi_l^*;

if $e > a + 1$, \quad \alpha(p_0)c(p_0) = 0.

Consider next $\sum_{p=p_0+1}^a \alpha(p)c(p)$. Note that $p_0 + 1 \leq a$, for otherwise $1 > a - p_0 = \mu > 0$, a contradiction. If $p \geq p_0 + 1$ then $a - p \leq a - p_0 - 1 = \mu - 1$; hence we use (4) to find $c(p)$.

Suppose $e \leq 2\mu + 2(p_0 + 1) - a (= a + 2)$; then for $p = p_0 + 1, \ldots, a$ we have $e \leq 2\mu + 2p - a$, so that

$$\sum_{p=p_0+1}^a \alpha(p)c(p) = \sum_{p=p_0+1}^a (l^{a-p} - l^{a-p-1})(l^{a-p} + l^{a-p-1})\chi_l^*$$

$$= \chi_l^* \sum_{p=p_0+1}^a (l^{2a-2p} - l^{2a-2p-2}) = \chi_l^* l^{2a-2p_0-2} = \chi_l^* l^{\nu-2}.$$

Suppose $e > 2\mu + 2(p_0 + 1) - a (= a + 2)$. Then either $e = 2\mu + 2p_1 - a$ for some $p_1 > p_0 + 1$ (note $p_1 \leq a$ or a contradiction arises) or $e = 2\mu + 2p_1 - a + 1$ for some $p_1 \geq p_0 + 1$. In either case, $\sum_{p=p_0+1}^a \alpha(p)c(p) = \chi_l^* l^{\nu+a-e}$; the work done to show this is virtually the same as that which showed $\sum_{p=0}^a \alpha(p)c(p) = \chi_l^* l^{\nu+a-e}$ under the conditions $a \leq \mu - 1$ and $e > 2\mu - a$, except that $\sum_{p=0}^a \alpha(p)c(p)$ must be replaced with $\sum_{p=p_0+1}^{p_1-1} \alpha(p)c(p)$; this has no effect on the outcome, as $c(p) = 0$ for each $p$ in either of these two sums.

Now then, to write explicit formulas for (7) let us first add $\alpha(p_0)c(p_0)$ to $\sum_{p=p_0+1}^a \alpha(p)c(p)$ and simplify. We have

if $e \leq a$ : \quad $\chi_l^* (l^{\nu-1} - l^{\nu-2}) + \chi_l\left(\frac{x}{2}\right)l^{\nu-1}(l - 1) + \chi_l^* l^{\nu-2}$

$= \chi_l^* l^{\nu-1} + \chi_l\left(\frac{x}{2}\right)l^{\nu-1}(l - 1)$;

if $e = a + 1$ : \quad $\chi_l^* (l^{\nu-1} - l^{\nu-2}) + \chi_l l^{\nu-2} = \chi_l^* l^{\nu-1} = \chi_l^* l^{\nu+a-e}$;

if $e = a + 2$ : \quad $0 + \chi_l^* l^{\nu-2} = \chi_l^* l^{\nu+a-e}$;

if $e > a + 2$ : \quad $\chi_l^* l^{\nu+a-e}$.

Note two things: First, we can combine the last three lines above into

if $e \geq a + 1$ : \quad $\chi_l^* l^{\nu+a-e}$.

Second, in (7) we take $\sum_{p=0}^{p_0-1} \alpha(p)c(p) = 0$ in case $p_0 = 0$, and this is the case iff $a = \mu$. Therefore, $\sum_{p=0}^a \alpha(p)c(p)$ is given by the above results in the case $a = \mu$. If $p_0 > 0$ (i.e., if $a > \mu$) then we add $\sum_{p=0}^{p_0-1} \alpha(p)c(p)$ to the above
results and simplify to find \( \sum_{\rho=0}^{a} \alpha(\rho)c(\rho) \). We have:

if \( e \leq \mu \):

\[
\chi_i(\frac{\mu}{2}) \nu^{-1}(l+1)(l^{a-\mu} - 1) + \chi_i' \nu^{-1} + \chi_i(\frac{\mu}{2}) \nu^{-1}(l - 1) \\
= \chi_i' \nu^{-1} + \chi_i(\frac{\mu}{2})(l - 1) \nu^{-1}(l+1)(l^{a-\mu} - 1)/(l - 1) + 1;
\]

if \( \mu < e \leq a - 1 \):

\[
\chi_i(\frac{\mu}{2})(l - 1) \nu^{-1}(l+1)(l^{a-e} - 1)/(l - 1) + l^{a-e} \\
+ \chi_i' \nu^{-1} + \chi_i(\frac{\mu}{2}) \nu^{-1}(l - 1) \\
= \chi_i' \nu^{-1} + \chi_i(\frac{\mu}{2})(l - 1) \nu^{-1}(l+1)(l^{a-e} - 1)/(l - 1) + l^{a-e} + 1;
\]

if \( e = a \):

\[
\chi_i(\frac{\mu}{2})(l - 1) \nu^{-1} l^{a-e} + \chi_i' \nu^{-1} + \chi_i(\frac{\mu}{2}) \nu^{-1}(l - 1) \\
= \chi_i' \nu^{-1} + \chi_i(\frac{\mu}{2})(l - 1) \nu^{-1}(l^{a-e} + 1);
\]

if \( e > a + 1 \):

\[
0 + \chi_i' \nu^{a+e-a} = \chi_i' \nu^{a+e-a}.
\]

Refer now to Table 1 for \( c(s, l) \). If Case\((l) = A \) and \( \nu = \text{ord}_{l}(N) = 2\mu \) is even, we have \( c(s, l) \) is equal to

\[
\chi_i \nu^{\min(2a, \nu - 1, \nu + a - e)} \\
= \chi_i(\frac{\nu}{2})(l - 1) \nu^{-1} \{ (l+1)(l^{a-\max(\mu, e)} - 1)/(l - 1) + k_5 l^{a-e} + k_6 \},
\]

where \( k_5 = d(\mu + 1, e)d(\mu + 1, a) \) and \( k_6 = d(e, a)d(\mu, a) \). We show that \( c(s, l) = \sum_{\rho=0}^{a} \alpha(\rho)c(\rho) \) for Case\((l) = A \) and even \( \nu = \text{ord}_{l}(N) = 2\mu \).

Suppose \( a \leq \mu - 1 \). We have \( 2a \leq 2\mu - 2 = \nu - 2 < \nu - 1 \) so that

\[
\min(2a, \nu - 1, \nu + a - e) = \min(2a, \nu + a - e),
\]

so that \( c(s, l) \) gives the \( \chi_i \)-term of the SUM properly. Next, \( a - \max(\mu, e) \leq a - \mu \leq a - 1 - \mu = -1 \); by our convention, then, \( l^{a-\max(\mu, e)} = 1 \). Also, \( a \leq \mu - 1 < \mu < \mu + 1 \) so that \( d(\mu, a) = 0 \) and \( d(\mu + 1, a) = 0 \) so that \( k_6 = k_5 = 0 \). Therefore, since each term in the \( \{ \} \)'s is 0, the \( \chi_i(\frac{\nu}{2}) \)-term is 0.

Suppose \( a = \mu \). Then \( \min(2a, \nu - 1, \nu + a - e) = \min(\nu - 1, \nu + a - e) \), and moreover, \( e > a \) iff \( \nu + a - e = \min(\nu - 1, \nu + a - e) \). Then \( c(s, l) \) gives the \( \chi_i \)-term properly. If \( e = a \), we have \( a - \max(\mu, e) = a - \mu = 0 \) so that \( l^{a-\max(\mu, e)} = 0 \). Clearly \( k_5 = 0 \) and \( k_6 = k_5 = 0 \) so that the \( \chi_i(\frac{\nu}{2}) \)-term is given by \( c(s, l) \) to be \( \chi_i(\frac{\nu}{2})(l - 1) \nu^{-1} \). If \( e > a \), we have \( a - \max(\mu, e) \leq a - e \leq 0 \), so that by convention, \( l^{a-\max(\mu, e)} = 0 \); \( k_5 = k_6 = 0 \) so that each term in the \( \{ \} \)'s is 0, and so no \( \chi_i(\frac{\nu}{2}) \)-term appears.

Suppose \( a > \mu \). Then \( \min(2a, \nu - 1, \nu + a - e) = \min(\nu - 1, \nu + a - e) \); again \( e > a \) iff \( \nu + a - e = \min(\nu - 1, \nu + a - e) \) so that \( c(s, l) \) correctly gives the \( \chi_i \)-term. Now consider the \( \chi_i(\frac{\nu}{2}) \)-term. If \( e \leq \mu \) we have \( l^{a-\max(\mu, e)} = l^{a-\mu} \); \( d(\mu + 1, e) = 0 \) so that \( k_5 = 0 \), while \( k_6 = 1 \). The terms in the \( \{ \} \)'s become \( (l+1)(l^{a-\mu} - 1)/(l - 1) + 1 \). If \( e < a \) we have \( l^{a-\max(\mu, e)} = l^{a-e} \) while \( k_5 = k_6 = 1 \) and the terms in the \( \{ \} \)'s become \( (l+1)(l^{a-e} - 1)/(l - 1) + l^{a-e} + 1 \).

Lastly, if \( e > a \), we have \( a - \max(\mu, e) = a - e < 0 \) so that \( l^{a-\max(\mu, e)} = 0 \); also \( k_5 l^{a-e} = 0 \), and \( k_6 = 0 \) so that each term in the \( \{ \} \)'s and therefore the entire \( \chi_i(\frac{\nu}{2}) \)-term is 0, and we have shown \( c(s, l) \) gives the \( \chi_i(\frac{\nu}{2}) \)-term correctly in each case.
This concludes the proof that \( \sum_{\rho} \alpha(\rho)c(\rho) = c(s, l) \) for Case(l) = A and even \( \nu = \text{ord}_1(N) \); we leave verification in the other cases to the reader.

**Example 6.** Let \( N = 3 \), \( k = 7 \), and \( n = 7 \), and suppose that \( \chi = (\frac{3}{x}) \) is the Legendre symbol. We show how easy Theorem 2 makes the computation of \( \text{tr} F_{N, \chi, k} T_n \). First note that since \( k > 2 \), the \( \delta(\chi) \)-term of the trace formula is 0, while both \( \delta(\sqrt{n}) \)-terms are 0 because \( n \) is not a perfect square. All we need to do is evaluate \( \sum_s \). Now, \( s^2 - 4n < 0 \) for \( 0 \leq s \leq 5 \); \( s^2 - 4n \) is a perfect square for \( \pm s = 8 \).

Suppose \( s = 1 \). We have \( s^2 - 4n = -27 \), so that \( t = 3 \) and \( m = -3 \). The contribution to \( \sum_s \) for \( s = 1 \) is then \( a(1)b(1)c(1, 3) \). To find \( c(1, 3) \), we note that \( \nu = \text{ord}_3(N) = 1 \), so \( \mu = 0 \) and \( \delta = 1 \), that \( e = e_3(\chi) = 1 \), and we determine that \( 3 \) is a case \( C \) prime, with \( a = a_3(s^2 - 4n) = 1 \). By Table 1, \( k_1 = k_2 = k_5 = e = 0 \), \( k_3 = g = 1 \), \( k_4 = 6 \), and \( k_6 = d(1, 2)d(0, 1) = 1 \); also \( a - \max(\mu, e - \delta) + e = 1 \). By (2), \( c(1, 3) = \chi_3(1/2)3^0(6(3 - 1)/2 + 1) = -7 \).

We compute \( a(1) = 60 \), while \( b(1) = h(3)/\omega(3) = 1/3 \). Therefore, the contribution to \( \sum_s \) for \( s = 1 \) is \( 60 \cdot (1/3) \cdot -7 = -140 \).

Similarly, when \( s = 2 \), the contribution to \( \sum_s \) is \( a(2)b(2)c(2, 3) = 51 \cdot (2/1) \cdot 1 = 102 \); if \( s = 4 \) the contribution to \( \sum_s \) is \( a(4)b(4)c(4, 3) = -90 \cdot (1/3) \cdot 4 \cdot -1 = 120 \); for \( s = 5 \) the contribution is \( a(5)b(5)c(5, 3) = 180 \cdot (1/3) \cdot 1 = 60 \); for each of these values of \( s \), \( 3 \) is a case \( C \) prime in the evaluation of \( c(s, 3) \).

Now, if \( s = 0 \) or \( 3 \), then \( c(s, 3) = 0 \). Therefore, there is no need to evaluate \( a(s)b(s) \), nor any products of the \( \gamma(s, l) \)-terms; the contribution to \( \sum_s \) for either of these values of \( s \) is simply 0.

If \( s = 8 \) then \( s^2 - 4n = 36 \), so that \( t = 6 \). The contribution to \( \sum_s \) is then \( a(8)b(8)c(8, 3) \). In finding \( c(8, 3) \), we have \( \nu = \text{ord}_3(N) = 1 \), \( \mu = 0 \) and \( \delta = 1 \), \( e = e_3(\chi) = 1 \), and we determine that \( 3 \) is a case \( A \) prime, with \( a = a_3(s^2 - 4n) = 1 \). By Table 1, \( k_1 = k_2 = k_5 = e = 0 \), \( k_3 = g = 1 \), and \( k_4 = 2 \); also \( \chi^*_1 = \chi_3(1) + \chi_3(7) = 2 \), and \( \chi_3(8/2) = 1 \). By (2), \( c(8, 3) = 2 \cdot 1 \cdot 1 + 0 + 1 \cdot 2 \cdot 1(2(3 - 1)/2 + 0 + 0) = 6 \). We find \( a(8) = 1/6 \), while \( b(8) = 1/2 \). Also, \( \gamma(8, 2) = 2 \), as \( 2 \) is a case \( D \) prime and \( a_2(s^2 - 4n) = 1 \). Therefore, the contribution to \( \sum_s \) for \( s = 8 \) is \( (1/6)(1/2) \cdot 2 \cdot 6 = 1 \).

By Proposition 7 below, the contributions to \( \sum_s \) of \( s \) and \( -s \) are equal, and so finally we obtain \( \text{tr} F_{N, \chi, k} T_n = -\sum_s = -2(-140+102+120+60+1) = -286 \).

The following proposition states that for fixed \( s_0 \) the contributions of the terms corresponding to \( s_0 \) and \( -s_0 \) to the \( \sum_s \) in the trace formula as given in Theorem 2 are the same. Therefore, the formula in Theorem 2 could be modified by taking the \( \sum \) over all the nonnegative integers \( s \) satisfying \( s^2 - 4n \) is a positive square or any negative integer, and replacing (say) \( a(s) \) with \( 2a(s) \), except for \( s = 0 \).

**Proposition 7.** Let the notation be as in Theorem 2. Let \( s \in \mathbb{Z} \) satisfy \( s^2 - 4n \) is a positive square or any negative integer. Then

\[
a(-s)b(-s) \prod_{l \mid t, l \mid N} \gamma(-s, l) \prod_{l \mid N} c(-s, l) = a(s)b(s) \prod_{l \mid t, l \not\mid N} \gamma(s, l) \prod_{l \mid N} c(s, l).
\]

**Proof.** Fix \( s \) satisfying the hypothesis, and write \( s^2 - 4n \) as \( t^2 \), \( t^2m \), or \( t^24m \) as in Theorem 1. Fix a prime \( l \) with \( l \mid t, l \not\mid N \). Note that \( b(s) \) and \( \gamma(s, l) \)
depend only on \( s^2 - 4n = (-s)^2 - 4n \) so that \( b(s) = b(-s) \) and \( \gamma(s, l) = \gamma(-s, l) \). Let \( x \) and \( y \) be the roots in \( \mathbb{C} \) of \( X^2 - sX + n \); then \(-x\) and \(-y\) are the roots of \( X^2 - (-s)X + n \), and it follows that \( a(-s) = (-1)^ka(s) \).

Referring to its definition, note that \( c(s, l) \) is of the form

\[
C_1 \left( \chi_l \left( \frac{s + l^a d}{2} \right) + \chi_l \left( \frac{s - l^a d}{2} \right) \right) + C_2 \left( \chi_l \left( \frac{s + l^{a+f} d}{2} \right) \right) + C_3 \chi_l \left( \frac{s}{2} \right),
\]

where \( C_1, C_2, \) and \( C_3 \) are functions of \( l, e = e(\chi_l), \nu = \text{ord}_l(N) \), and \( a = \text{ord}_l(t) \), where \( s^2 - 4n = (-s)^2 - 4n = t^2, t^2m, \) or \( t^24m \) as the case may be, so that \( C_1, C_2, \) and \( C_3 \) are independent of the sign of \( s \). Therefore, for the same \( C_1, C_2, \) and \( C_3 \), we have \( c(-s, l) \) equals

\[
C_1 \left( \chi_l \left( \frac{-s + l^a d}{2} \right) + \chi_l \left( \frac{-s - l^a d}{2} \right) \right) + C_2 \left( \chi_l \left( \frac{-s + l^{a+f} d}{2} \right) \right) + C_3 \chi_l \left( \frac{-s}{2} \right).
\]

First,

\[
C_1 \left( \chi_l \left( \frac{-s + l^a d}{2} \right) + \chi_l \left( \frac{-s - l^a d}{2} \right) \right) = \chi_l(-1)C_1 \left( \chi_l \left( \frac{s + l^a d}{2} \right) + \chi_l \left( \frac{s - l^a d}{2} \right) \right).
\]

Next, it is clear that \( C_3 \chi_l\left(\frac{s}{2}\right) = \chi_l(-1)C_3 \chi_l\left(\frac{s}{2}\right) \). Furthermore, if \( C_2 \neq 0 \) then we must have \( l = 2 \) and \( d(e, a + f) = 1 \), that is, \( e \leq a + f \). In this case, we have \( 0 \equiv 2a+f \equiv (s + 2a+f d)/2 - (s - 2a+f d)/2 \) (mod \( 2e \)), that is, \( (s + 2a+f d)/2 \equiv (s - 2a+f d)/2 \) (mod \( 2e \)), so that

\[
C_2 \chi_2 \left( \frac{-s + 2a+f d}{2} \right) = C_2 \chi_2(-1) \chi_2 \left( \frac{s - 2a+f d}{2} \right)
\]

\[
= \chi_2(-1)C_2 \chi_2 \left( \frac{s + 2a+f d}{2} \right).
\]

Therefore, \( c(-s, l) = \chi_l(-1)c(s, l) \), and it follows that

\[
\prod_{l|N} c(-s, l) = \chi(-1) \prod_{l|N} c(s, l).
\]

Finally then,

\[
a(-s)b(-s) \prod_{l|t, l|N} \gamma(-s, l) \prod_{l|N} c(-s, l)
\]

\[
= (-1)^k \chi(-1)a(s)b(s) \prod_{l|t, l|N} \gamma(s, l) \prod_{l|N} c(s, l).
\]

This proves the result, because we assume (in both Theorems 1 and 2) that \((-1)^k \chi(-1) = 1\).

The following is easy to show using Theorem 2:
Corollary 8. Let \( k, \chi, \) and \( N \) be as in Theorems 1 and 2. The dimension of the space \( S_k(N, \chi) \) is given by the formula

\[
\dim(S_k(N, \chi)) = -s_0 - s_1 + d + m - p,
\]

where

\[
s_0 = \begin{cases} 
0 & \text{if any one of the following conditions is met: } k \text{ is odd; } 4 \mid N; \chi_l(-1) = -1 \text{ or } \chi_l(1) = -1 \text{ for some odd prime } l \mid N, \\
\frac{1}{4}(-1)^{k/2-1}\chi(r_0)2^n & \text{otherwise, where } r_0 \in \mathbb{Z} \text{ satisfies } r_0^2 \equiv -1 \text{ (mod } N) \text{ and } n \text{ is the number of odd primes which divide } N, 
\end{cases}
\]

\[
s_1 = \begin{cases} 
\frac{3}{2}\chi\left(\frac{1}{2}\right)\prod_{\text{prime } \l \mid N, \l \neq 3} \beta_l & \text{otherwise, where } \alpha = 1 \text{ if } k \equiv 2 \text{ or } 3 \pmod{6} \text{ and } -1 \text{ if } k \equiv 0 \text{ or } 5 \pmod{6} \text{ and } -1, \\
0 & \text{if any one of the following conditions is met: } k \equiv 1 \text{ or } 4 \pmod{6}; 9 \mid N; 2 \mid N; \text{ or } (-3)^{\frac{N}{3}} \equiv -1 \text{ for some odd prime } l \mid N, \ l \neq 3, 
\end{cases}
\]

\[
d = \begin{cases} 
1 & \text{if } k = 2 \text{ and } \chi \text{ is trivial,} \\
0 & \text{otherwise,}
\end{cases}
\]

\[
m = \frac{k-1}{12}N\prod_{\ell \mid N}(1+1/\ell), \\
p = \frac{1}{2}\prod_{\ell \mid N}\text{par}(l),
\]

where \( \text{par}(l) \) is defined as in Theorem 1.

Proof. Since \( T_1 \) is the identity operator, the trace of \( T_1 \) acting on \( S_k(N, \chi) \) gives the dimension of the space, so we need only evaluate Theorem 2 with \( n \) set to 1. Consider the sum over \( s \) in the first part of the trace formula as given in Theorem 2. Now, 0, 1, and \(-1\) are the only values of \( s \) such that \( s^2 - 4n \) is negative, and there are no integral values of \( s \) such that \( s^2 - 4n \) is a positive square.

First, fix \( s = 0 \). We have \( s^2 - 4n = -4 = t^24m \), where \( t = 1 \) and \( m = -1 \equiv 3(4) \). Since \( i \) and \(-i\) are the roots of \( \Phi(X) \), we find \( a(0) = (1/4)(i^{k-2} + (-1)^k) \). If \( k \) is odd, \( a(0) = 0 \); otherwise \( a(0) = (1/2)(-1)^{k/2-1} \). The class number of \( \mathbb{Q}\sqrt{-1} \) is 1, and one-half the cardinality of its unit group is 2 so that \( b(0) = 1/2 \). Since \( t = 1 \), \( \prod_{\ell \mid N, \ell \neq 3} \gamma(0, l) = 1 \). It remains to evaluate \( \prod_{\ell \mid N} c(0, l) \). Let \( l \) be an odd prime dividing \( N \), and set \( \nu = \text{ord}_l(N) \); we have \( s^2 - 4n = -4 = l^2a \cdot -4 \), where \( a = 0 \). Suppose that \( (-4)^{\frac{N}{4}} = 1 \) so that Case(\( l \)) = \( A \). Let \( d_l \in \mathbb{Z} \) satisfy \( d_l^2 = -1 \), so that \( (2d_l)^2 = -4 \). Note that \( 2d_l \) is the ‘\( d \)’ which appears in the classification of \( l \), so that \( (s \pm l^2d)/2 = (0 \pm 1 \cdot 2d)/2 = \pm d_l \). Refer to Table 1 to find \( c(0, l) \): We have \( k_1 = 1 \), \( \min(2a, \nu - 1, a + \nu - e) = 0 \), \( k_2 = 0 \), and \( \chi_l(0/2) = 0 \) so that \( c(0, l) = \chi_l(d_l) + \chi_l(-d_l) = \chi_l(d_l)(1 + \chi_l(-1)) \). If \( \chi_l(-1) = 1 \) then \( c(0, l) = 0 \) and hence the contribution of the \( s = 0 \) term to the trace is 0, while if \( \chi_l(-1) = 1 \) we have \( c(0, l) = 2\chi_l(d_l) \). Now if \( (-4)^{\frac{N}{4}} = -1 \), so that Case(\( l \)) = \( B \), then
referring to Table 1 for \( c(0, l) \) we have \( k_1 = k_2 = 0 \) and \( \chi_l(0/2) = 0 \) so that \( c(0, l) = 0 \) and therefore the contribution of the \( s = 0 \) term to the trace is 0.

Keep \( s = 0 \), and suppose now that \( l = 2 \) and \( l \mid N \). Then Case(\( l \)) = F, \( a = 0, \chi_2(0/2) = 0 \), and Table 1 for \( c(0, l) \) gives \( k_1 = 0 \). Let \( \nu = \text{ord}_2(N) \); write \( \nu = 2\mu \) or \( 2\mu + 1 \) as the case may be. If \( 4 \mid N \) then \( \mu \geq 1 \) so that \( d(\mu, a) = d(\mu, 0) = 0 \), thus \( k_2 = 0 \) and so \( c(0, 2) = 0 \). If \( 2 \mid N \) then \( \nu = 1 \) and \( \mu = 0 \), and \( \chi_2 \) is the trivial character, so that \( e = e(\chi_2) = 0 \). We have \( k_2 = d(e, a + 1)d(\mu, a) = d(0, 1)d(0, 0) = 1 \); in this case \( c(0, 2) = k_2k_3\chi_2 = 1 \cdot 1 \cdot \chi_2((0 + 2^{0+1} \cdot 1)/2) = \chi_2(1) = 1 \).

Therefore the contribution of the \( s = 0 \) term to the trace is 0 unless \( k \) is even, \( 4 \nmid N \), \( \chi_l(-1) = 1 \) for all odd primes \( l \mid N \), and \( (\frac{-1}{l}) = 1 \) for all odd \( l \mid N \). Suppose that all these conditions are met. In particular, since \( 4 \nmid N \) and \( (\frac{-1}{l}) = 1 \) for all odd \( l \mid N \), there is some \( r \in \mathbb{Z} \) with \( r^2 \equiv -1(N) \); note \( r \) is odd if \( 2 \mid N \) so that \( \chi_2(r) = 1 \). If \( l \) is an odd prime dividing \( N \) and \( d_l \) is a unit in \( \mathbb{Z}_l \) satisfying \( d_l^2 = -1 \), then \( \chi_l(d_l) = \chi_l(r) \), because \( d_l \equiv \pm r \pmod{\text{ord}_l(N)} \) and \( \chi_l(-1) = 1 \). Finally then,

\[
\prod_{l \mid N, l \text{ odd}} 2\chi_l(d_l) = \prod_{l \mid N, l \text{ odd}} 2\chi_l(r) = 2^n\chi(r),
\]

where \( n \) is the number of odd primes dividing \( N \).

Now fix \( s = 1 \). We have \( s^2 - 4n = -3 = t^2m \), where \( t = 1 \) and \( m = -3 \). The roots \( x \) and \( y \) of \( \Phi(X) = X^2 - X + 1 \) are \((1 \pm \sqrt{-3})/2 \); de Moivre's formula gives \((x^n - y^n)/(x - y) = 2\sin((k - 1)\pi/3)/(i\sqrt{3}) \) so that

\[
a(1) = \frac{1}{2} \cdot \begin{cases} 
1 & \text{if } k \equiv 2, 3 \pmod{6}, \\
0 & \text{if } k \equiv 1, 4 \pmod{6}, \\
-1 & \text{if } k \equiv 0, 5 \pmod{6}.
\end{cases}
\]

The class number of \( \mathbb{Q}_\sqrt{-3} \) is 1, and one-half the cardinality of the unit group is 3, so that \( b(1) = 1/3 \). Since \( t = 1 \), \( \prod_{l \mid t, l \neq 3} \gamma(1, l) = 1 \); it remains to evaluate \( \prod_{l \mid N} c(1, l) \). Let \( l \mid N \) be an odd prime, \( l \neq 3 \). Then \( s^2 - 4n = -3 = l^{2a(-3)} \) with \( a = 0 \). Let \( \nu = \text{ord}_l(N) \) and set \( \nu = 2\mu \) or \( 2\mu + 1 \) as appropriate. Suppose \((\frac{-3}{l}) = 1 \), so that Case(\( l \)) = A. Let \( d_l \) be a unit in \( \mathbb{Z}_l \) satisfying \( d_l^2 = -3 \). Referring to Table 1 for \( c(1, l) \) we see that \( k_2 = 0, k_1 = 1, \text{ and } l^{\min(2a,\nu-1,\nu+a-e)} = l^0 = 1 \). If \( \nu \) is even, then \( \mu > a \) so that \( d(\mu + 1, a) \) and \( d(\mu, a) \) are both 0, hence \( k_5 = k_6 = 0 \), while if \( \nu \) is odd, \( k_5 \) and \( k_6 \) are 0 automatically. Since \( a - \max(\mu, e - \delta) \leq 0 \), the \( k_4 \)-term is 0, and hence for any \( \nu \), the contribution to \( c(1, l) \) from the \( \chi_l(\frac{\xi}{l}) \)-term is 0, and so \( c(1, l) = \chi_l((1 + d_l)/2) + \chi_l((1 - d_l)/2) \). If \((\frac{-3}{l}) = -1 \) then Case(\( l \)) = B; here \( k_1 = 0 \) while the other \( k_l \)-terms are the same as for Case(\( l \)) = A. Thus if \((\frac{-3}{l}) = -1 \), then \( c(1, l) = 0 \) and therefore the contribution of the \( s = 1 \) term to the trace is 0.

Suppose now that \( l \mid N \) and \( l = 3 \); we have \( s^2 - 4n = -3 = 3^{2a+1}(-1) \), where \( a = 0 \) and \( -1 \) is a unit in \( \mathbb{Z}_3 \), so that Case(\( l \)) = C. Let \( \nu = \text{ord}_3(N) \) and set \( \nu = 2\mu \) or \( 2\mu + 1 \). Suppose first that \( \nu \) is even; refer to Table 1 for \( c(1, 3) \). We have \( k_1 = k_2 = k_6 = 0 \); also \( d(\mu, a) = 0 \) so \( k_5 = 0 \). Furthermore, \( a - \max(\mu, e + 1) \leq 0 \) so the \( k_4 \)-term is 0. Therefore the \( \chi_l(\frac{\xi}{l}) \)-term is 0, so that if \( \nu \) is even, \( c(1, 3) = 0 \). Now, if \( \nu \) is odd, we have \( k_1 = k_2 = k_5 = 0 \),
and \( a - \max(\mu, e - 1) \leq 0 \) so the \( k_4 \)-term is 0. Consider \( k_6 \). If \( \mu \geq 1 \) then \( d(\mu, a) = 0 \), so \( k_6 = 0 \), and therefore \( c(1, 3) = 0 \); combined with the fact that \( c(1, 3) = 0 \) if \( \nu \) is even we have that the contribution of the \( s = 1 \) term to the trace is 0 if \( 3^2 \mid N \). Suppose \( 3 \mid N \); then \( \mu \) = 0, and it follows that \( k_6 = 1 \). Therefore \( c(1, 3) = \chi_3(\frac{1}{2}) \cdot 3^{1-1}(0 + 0 + 1) = \chi_3(\frac{1}{2}) \); since \( \chi_3 \) is now either the trivial character or \( (\frac{3}{3}) \), we have \( c(1, 3) = 1 \) or \(-1\), respectively.

Suppose \( l = 2 \) and \( 2 \mid N \); then \( s^2 - 4n = -3 = 2^{2a}(-3) \) with \( a = 0 \) and \(-3 \equiv 5 \pmod{8} \) is a unit in \( \mathbb{Z}_2 \), so that \( \text{Case}(l) = E \). Refer to Table 1 for \( c(1, 2) \). We have \( k_1 = 0 \). Let \( \nu = \text{ord}_2(N) \) and set \( \nu = 2\mu \) or \( 2\mu + 1 \). Suppose first that \( \nu \) is even. Then \( \mu > a \) so that \( k_2 = k_5 = k_6 = 0 \) and the \( k_4 \)-term is 0 and therefore \( c(1, 2) = 0 \). If \( \nu \) is odd, then \( k_2 \) and the \( k_4 \)-term are again 0, while \( k_5 \) and \( k_6 \) are automatically 0, so that \( c(1, 2) = 0 \). Hence, if \( 2 \mid N \), the contribution of the \( s = 1 \) term to the trace is 0.

We have shown that \( \prod_{|l|N} c(1, l) = 0 \) unless \( 9 \not| N \), \( 2 \not| N \), and \( (-3) = 1 \) for each odd prime \( l \mid N \), \( l \neq 3 \). Suppose in fact that all these conditions are satisfied. It is then possible to find \( r \in \mathbb{Z} \) such that \( r^2 \equiv -3 \pmod{N} \) if \( (3, N) = 1 \), and \( r^2 \equiv -3 \pmod{(\frac{N}{2})} \) if \( 3 \mid N \), and satisfying the following: for each odd prime \( l \mid N \), \( l \neq 3 \), we have \( r \equiv \pm d_l \pmod{\text{ord}(N)} \), where \( d_l \in \mathbb{Z}_l \) is a unit with \( d_l^2 = -3 \). For each such \( l \) we have \( \chi_l(\frac{1}{2})(\chi_l(1 + r) + \chi_l(1 - r)) = \chi_l((1 + d_l)/2) + \chi_l((1 - d_l)/2) \). Taking \( \{\chi_3(\frac{1}{2})\} \) to mean 1 if \( 3 \not| N \) and \( \chi_3(\frac{1}{2}) \) if \( 3 \mid N \), we have \( \prod_{|l|N} c(1, l) = \{\chi_3(\frac{1}{2})\} \cdot \prod_{|l|N, l \neq 3} \chi_l((1 + d_l)/2) + \chi_l((1 - d_l)/2) = \chi(\frac{1}{2}) \prod_{|l|N, l \neq 3} \chi_l(1 + r) + \chi_l(1 - r) \).

By Proposition 7, the contribution of the \( s = -1 \) term to the trace equals that of the \( s = 1 \) term. The remaining terms in the dimension formula come immediately from the corresponding terms in either Theorem 1 or 2.

Consider the trace formula as given in Theorem 2. If \( \chi \) is the trivial character, we can make additional simplifications to the formula, the most important being that \( c(s, l) \) can be given by a very simple table; this is the result of our next corollary.

**Corollary 9.** Let \( k, \chi, N, \) and \( n \) be as in Theorem 2, and suppose furthermore that \( \chi \) is the trivial character. Then for \( (n, N) = 1 \) we have

\[
\text{tr}_{N, \chi, k}(T_n) = -\sum_s \left( a(s)b(s) \prod_{l | (s, l) \not| N} \gamma_0(s, l) \prod_{l | N} c_0(s, l) \right) \\
+ \left\lfloor \frac{2}{k} \right\rfloor \deg(T_n) + \delta_0(\sqrt{n}) \frac{k - 1}{12} N \prod_{l | N} (1 + 1/l) \\
- \delta_0(\sqrt{n}) \frac{\sqrt{n}}{2} \prod_{l | N} \text{par}_0(l),
\]

where \( n, a(s), b(s), t, \) and \( \gamma(s, l) \) are exactly the same as in Theorem 2, and

\[
\delta_0(\sqrt{n}) = \begin{cases} n^{k/2} & \text{if } n \text{ is a perfect square}, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\text{par}_0(l) = \begin{cases} l^\mu + l^{\mu - 1} & \text{if } \nu = \text{ord}_l(N) = 2\mu, \\
2l^\mu & \text{if } \nu = \text{ord}_l(N) = 2\mu + 1,
\end{cases}
\]
and $c_0(s, l)$ is defined as follows. Fix $s$ and a prime $l | N$. Let $\nu = \text{ord}_l(N)$, write $\nu = 2\mu + \delta$, where $\delta = 0$ or $1$. Classify $l$ into one of the six cases $A$, $\ldots$, $F$, and referring to how Case($l$) is determined, let $a = a_l(s^2 - 4n)$. Let $d(x, y) = 1$ if $x \leq y$ and $0$ otherwise, for any $x$, $y \in \mathbb{Z}$. Then $c_0(s, l)$ is given by the expression

$$k_12l^{\min(2\nu, \nu - 1)} + k_3l^{\nu - 1}(c_4(l^{2\nu + \epsilon} - 1)/(l - 1) + c_6),$$

where the values of $k_1$, $k_3$, $c_4$, $c_6$, and $\epsilon$ are determined from Table 2.

<table>
<thead>
<tr>
<th>Case($l$)</th>
<th>$\nu$</th>
<th>$k_1$</th>
<th>$k_3$</th>
<th>$c_4$</th>
<th>$\epsilon$</th>
<th>$c_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ or $D$</td>
<td>odd</td>
<td>1</td>
<td>$l - 1$</td>
<td>$l + 1$</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$B$ or $E$</td>
<td>odd</td>
<td>0</td>
<td>$l + 1$</td>
<td>$l + 1$</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$C$ or $F$</td>
<td>odd</td>
<td>0</td>
<td>1</td>
<td>$l + 1$</td>
<td>$2l$</td>
<td>1</td>
</tr>
</tbody>
</table>

Remarks. Note that $k_1$ and $k_3$ are the same $k_1$ and $k_3$ as appear in Theorem 2, while $c_4$ and $c_6$ are similar to the $k_4$ and $k_6$ (respectively) of the same theorem. Also, one must heed Convention A in evaluating (9). In Theorems 1 and 2 we assume $(-1)^k\chi(-1) = 1$; the corollary’s additional hypothesis that $\chi$ is trivial implies that $k$ is even.

Proof. Let $\chi$ be the trivial character mod $N$, so that $\chi = \prod_{l|N} \chi_l$, where for each prime $l | N$, $\chi_l$ is the trivial character mod $l^{\text{ord}_l(N)}$; note $e = e(\chi_l) = 0$ for each prime $l | N$. Let the trace $\text{tr}_N, \chi, k T_n$ be as given by Theorem 2. Clearly the last three lines of the formula in the statement of Corollary 9 follow directly from the corresponding lines of Theorem 2. All one has to do is show how Table 1 “collapses” into Table 2 by showing that $c_0(s, l) = c(s, l)$ for each fixed $s$ and fixed $l | N$, for any classification of the prime $l$, and any relationship between $a = a_l(s^2 - 4n)$ and $\mu$ where $\text{ord}_l(N) = 2\mu$ or $2\mu + 1$. We leave the details to the reader.

References


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