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A HAAR-TYPE THEORY OF BEST L_1-APPROXIMATION WITH CONSTRAINTS

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ABSTRACT. A general setting for constrained L^1-approximation is presented. Let U_n be a finite dimensional subspace of $C[a, b]$ and L be a linear operator from U_n to $C^r(K)$ $(r = 0, 1)$ where K is a finite union of disjoint, closed, bounded intervals. For $v, u \in C^r(K)$ with $v < u$, the approximating set is $\widetilde{U}_n(v, u) = \{p \in U_n : v \leq Lp \leq u \text{ on } K\}$ and the norm is $\|f\|_w = \int_a^b |f| w\,dx$ where w a positive continuous function on $[a, b]$. We obtain necessary and sufficient conditions for $\widetilde{U}_n(v, u)$ to admit unique best $\|\cdot\|_w$-approximations to all $f \in C[a, b]$ for all positive continuous w and all $v, u \in C^r(K)$ $(r = 0, 1)$ satisfying a nonempty interior condition. These results are applied to several L^1-approximation problems including polynomial and spline approximation with restricted derivatives, lacunary polynomial approximation with restricted derivatives, and others.

1. INTRODUCTION

In this paper we shall study uniqueness of best L_1-approximation of continuous functions by elements of certain convex sets, resulting from imposing constraints on finite-dimensional spaces. Problems of this type were investigated in the literature of approximation theory mainly for the L_∞-norm (see, e.g., the papers by Chalmers [1] and Chalmers and Taylor [2] and references therein).

Recently much progress has been made in the study of uniqueness of best L_1-approximation of continuous functions from finite-dimensional spaces. A Haar-type theory was developed for this setting with the so-called A-spaces being analogs of Haar spaces for the L_1-norm [4, 5, 9, 10, 13]. In this paper we are concerned with providing a similar Haar-type theory for constrained L_1-approximation, i.e., giving necessary and sufficient conditions for uniqueness. In a recent paper by Pinkus and Strauss [11] the problem of uniqueness of constrained L_1-approximation was studied for the special cases of restricted range and restricted coefficient approximation with constraints imposed by fixed boundary functions. Our goal is to develop a general theory of constrained L_1-

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approximation imposing constraints by linear operators and thus characterizing uniqueness in terms of the given operator. Furthermore, our approach also differs from that in [11] in another respect. Instead of studying uniqueness for constraints with *fixed* boundaries we shall give *boundary independent* characterizations of uniqueness which leads to simpler and more utile descriptions. On the other hand our results depend on whether we work with continuous or smooth (C^1)-boundaries.

We first fix some notations. $C_w[a, b]$ denotes the set of real continuous functions on $[a, b]$ endowed with the norm $\|f\| = \int_a^b |f(x)|w(x)\,dx$ where $w \in W$—the set of positive continuous functions on $[a, b]$. Let U_n be an n-dimensional subspace of $C[a, b]$. Let K be a finite union of disjoint closed bounded intervals in $\mathbf{R}$, and for $r = 0$ or 1, denote by $C^r(K)$ the set of real continuous $(r = 0)$ or continuously differentiable $(r = 1)$ functions on K. Consider functions v, $u \in C^r(K)$ $(r = 0, 1)$ satisfying $v < u$ on K and let $L\colon U_n \to C^r(K)$ be a linear operator mapping U_n into $C^r(K)$. Then set

$$\widetilde{U}_n(v, u) = \{p \in U_n : v \le Lp \le u \text{ on } K\}.$$

We shall say that $\operatorname{Int}\widetilde{U}_n(v, u) \neq \phi$ if for some $\tilde{p} \in U_n$ we have $v < L\tilde{p} < u$ on K. Recall that $\widetilde{U}_n(v, u)$ is said to be a *uniqueness set* in $C_w[a, b]$ if every $f \in C_w[a, b]$ has a unique best approximant in $\widetilde{U}_n(v, u)$. (Note that we approximate in the L_1-norm on $[a, b]$, while the constraints are imposed on K.)

Recently it has been noted that various necessary and sufficient conditions for uniqueness of L_1-approximation depend on the weight w which defines the L_1-norm. The study of *weight independent* uniqueness involves the so-called A-spaces. A subspace U_n is called an A-space if for every $g \in U_n\setminus\{0\}$ and continuous function $\sigma : \operatorname{supp} g \to \{-1, 1\}$ there exists $g_1 \in U_n\setminus\{0\}$ such that $\sigma g_1 \ge 0$ on $\operatorname{supp} g$ and $g_1 = 0$ a.e. on $Z_0(g)$. (There and in what follows $Z_0(g) = \{x \in [a, b] : g(x) = 0\}$ and $\operatorname{supp} g = [a, b]\setminus Z_0(g)$.) It is known [5, 10, 13] that in order for U_n to be a uniqueness subspace in $C_w[a, b]$ for every $w \in W$ it is necessary and sufficient that U_n be an A-space. The main goal of the present paper consists in obtaining similar results for constrained approximation. Thus we shall study the following problem.

Problem. Given $U_n \subset C_w[a, b]$, $r = 0$ or 1, and $L\colon U_n \to C^r(K)$, find a necessary and sufficient condition so that for every $w \in W$ and every v, $u \in C^r(K)$ with $\operatorname{Int}\widetilde{U}_n(v, u) \neq \phi$, $\widetilde{U}_n(v, u)$ is a uniqueness set in $C_w[a, b]$.

Thus we shall consider separately the case of continuous boundaries $(r = 0)$ and C^1-boundaries $(r = 1)$. It turns out that considering C- or C^1-boundaries leads to essentially different solutions and correspondingly distinct applications.

Our paper is organized as follows. §2 provides a complete solution to the problem outlined above. §3 consists of applications for the case of C-boundaries while §4 gives various applications for C^1-boundaries. It turns out that our theory can be widely applied for different operators and spaces of polynomial and spline functions. Finally, let us mention that a similar study of constrained approximation in the L_∞-norm is given in our recent paper [6].

2. General theory of best L_1-approximation with constraints

In this section we give a complete weight and boundary independent characterization of uniqueness for approximating from $\widetilde{U}_n(v, u)$ in $C_w[a, b]$. In order to accomplish this we shall need the following characterization of best constrained L_1-approximants.

Theorem 2.1. *Let $L : U_n \to C^0(K)$, v, $u \in C^0(K)$ be such that $\operatorname{Int} \widetilde{U}_n(v, u) \neq \phi$, $w \in W$ and $f \in C_w[a, b]$. Then $p_0 \in \widetilde{U}_n(v, u)$ is a best approximant of f if and only if there exist $h \in L^\infty[a, b]$ with $|h| \leq 1$, points $y_1, \dots, y_m \in Z_0(v - Lp_0)$, $y_{m+1}, \dots, y_s \in Z_0(u - Lp_0)$ satisfying $\dim LU_n|_{\{y_i\}_{i=1}^s} = s$ $(0 \leq s \leq n)$, and positive numbers $\alpha_1, \dots, \alpha_s$ such that*

$$\int_a^b h(f - p_0)w\,dx = \int_a^b |f - p_0|w\,dx \tag{2.1}$$

and for every $p \in U_n$

$$\int_a^b hpw\,dx + \sum_{i=1}^{m} \alpha_i (Lp)(y_i) - \sum_{i=m+1}^{s} \alpha_i (Lp)(y_i) = 0. \tag{2.2}$$

For a function $g \in C^0(K)$ as above $Z_0(g)$ is the set of its zeros. If $g \in C^1(K)$ we denote $Z_1(g) = \{x \in Z_0(g) : g'(x) = 0 \text{ if } x \in \operatorname{Int} K\}$. Our next theorem is the main result of this paper.

Theorem 2.2. *Let $L : U_n \to C^r(K)$ $(r = 0, 1)$. Then the following are equivalent:*

(i) *For all v, $u \in C^r(K)$ satisfying $\operatorname{Int} \widetilde{U}_n(v, u) \neq \phi$ and $w \in W$, $\widetilde{U}_n(v, u)$ is a uniqueness set in $C_w[a, b]$;*

(ii) *for every $g \in U_n \backslash \{0\}$, continuous mapping $\sigma : \operatorname{supp} g \to \{-1, 1\}$ and points $\{y_i\}_{i=1}^s \subseteq Z_r(L(g))$ such that $\dim LU_n|_{\{y_i\}_{i=1}^s} = s$ $(0 \leq s \leq n-1)$ there exists a $g_1 \in U_n \backslash \{0\}$ such that*

$$g_1 = 0 \quad \text{a.e. on } Z_0(g), \tag{2.3}$$

$$\sigma g_1 \geq 0 \quad \text{on } \operatorname{supp} g, \tag{2.4}$$

$$(Lg_1)(y_i) = 0 \qquad (1 \leq i \leq s). \tag{2.5}$$

One can notice a simple connection between the above result and the A-property. Indeed, properties (2.3) and (2.4) of g_1 are the ones needed for the A-property, while (2.5) is the extra requirement resulting from the constraints imposed by operator L. Thus (2.3)–(2.5) combine the A-property and the features of operator L. This combination is reflected in the next definition.

Definition. Let L: $U_n \to C^r(K)$ $(r = 0, 1)$. Then we call U_n an L^r-A-space if property (ii) in the above theorem holds.

Since $Z_1(g) \subseteq Z_0(g)$ for every $g \in C^1(K)$ condition (ii) for $r = 1$ is less restrictive than for $r = 0$. This indicates that the set of L^1-A-spaces might be wider than that of L^0-A-spaces. Our various applications will show that this is the case. Evidently, L^0-A-spaces and L^1-A-spaces are, in particular, A-spaces.

We now prove the two theorems stated above.

Proof of Theorem 2.1. *Sufficiency.* Let $p \in \widetilde{U}_n(v, u)$, i.e., $v \le Lp \le u$. Then by (2.2)

$$\begin{aligned}\int_a^b hp_0 w\,dx &= -\sum_{i=1}^m \alpha_i (Lp_0)(y_i) + \sum_{i=m+1}^s \alpha_i (Lp_0)(y_i) \\ &= -\sum_{i=1}^m \alpha_i v(y_i) + \sum_{i=m+1}^s \alpha_i u(y_i) \\ &\ge -\sum_{i=1}^m \alpha_i (Lp)(y_i) + \sum_{i=m+1}^s \alpha_i (Lp)(y_i) = \int_a^b hpw\,dx.\end{aligned}$$

Using (2.1) we obtain

$$\int_a^b |f - p_0| w\,dx = \int_a^b h(f - p_0) w\,dx \le \int_a^b h(f-p)w\,dx \le \int_a^b |f-p|w\,dx$$

and p_0 is a best approximant of f.

Necessity. The proof is essentially that of Theorem 5.1 in [11]. Assume now that p_0 is a best approximant of f. Since $\widetilde{U}_n(v, u)$ is a convex set, a well-known characterization of best approximants yields the existence of $h \in L^\infty[a, b]$ such that $|h| \le 1$, $\int_a^b |f - p_0| w\,dx = \int_a^b h(f-p_0)w\,dx$ and

$$\text{(2.6)} \qquad \int_a^b hp_0 w\,dx \ge \int_a^b hpw\,dx, \qquad p \in \widetilde{U}_n(v, u).$$

Let $\{p_1, \dots, p_n\}$ be a basis for U_n,

$$\begin{aligned}\mathscr{P} &= \{-((Lp_k)(y))_{k=1}^n : y \in Z_0(v - Lp_0)\} \\ &\quad \cup \{((Lp_k)(y))_{k=1}^n : y \in Z_0(u - Lp_0)\} \subseteq \mathbf{R}^n\end{aligned}$$

and denote by Q the smallest convex cone containing $\mathscr{P}$. Consider the vector $\bar{c} = (\int_a^b hp_k w\,dx)_{k=1}^n$ in $\mathbf{R}^n$ and suppose that $\bar{c} \notin Q$. Then there is a hyperplane supporting Q at the origin which strictly separates $\bar{c}$ from Q, i.e., for some $\bar{a} = (a_k)_{k=1}^n \in \mathbf{R}^n \backslash \{0\}$ we have

$$\text{(2.7)} \qquad \sum_{k=1}^n a_k \int_a^b hp_k w\,dx > 0 \ge \sum_{k=1}^n a_k q_k, \qquad \bar{q} = (q_k)_{k=1}^n \in Q.$$

Set $p^* = \sum_{k=1}^n a_k p_k \in U_n \backslash \{0\}$. Then (2.7) yields that $\int_a^b hp^* w\,dx > 0$, $Lp^* \ge 0$ on $Z_0(v - Lp_0)$ and $Lp^* \le 0$ on $Z(u - Lp_0)$. Since $\operatorname{Int} \widetilde{U}_n(v, u) \ne \phi$, for some $\tilde{p} \in U_n$ we have $v < L\tilde{p} < u$ on K. Set $p_t = p^* + t(\tilde{p} - p_0)$ $(t > 0)$. Then $Lp_t > 0$ on $Z_0(v - Lp_0)$ and $Lp_t < 0$ on $Z_0(u - Lp_0)$ for every $t > 0$. In addition, $\int_a^b hp_t w\,dx > 0$ if $t > 0$ is small enough. Choosing $\varepsilon > 0$ to be sufficiently small we have $p_0 + \varepsilon p_t \in \widetilde{U}_n(v, u)$ and

$$\int_a^b h(p_0 + \varepsilon p_t) w\,dx > \int_a^b hp_0 w\,dx.$$

This contradicts (2.6). Thus $\bar{c} \in Q$. If $\bar{c} = 0$, then (2.2) holds with $s = 0$. Otherwise, $t\bar{c} \in \operatorname{co}\mathscr{P}$ (the convex hull of $\mathscr{P}$) for some $t > 0$. Since $\operatorname{co}\mathscr{P}$ is closed and bounded, $\rho\bar{c} \in \operatorname{co}\mathscr{P}$ where $\rho := \sup\{t > 0 : t\bar{c} \in \operatorname{co}\mathscr{P}\}$, $0 < \rho < \infty$. Choose $\bar{q}_1, \dots, \bar{q}_s \in \mathscr{P}$ such that $\rho\bar{c} \in \operatorname{co}\{\bar{q}_1, \dots, \bar{q}_s\}$ and s is minimal. If $\{\bar{q}_1, \dots, \bar{q}_s\}$ were linearly dependent, then $l := \dim\operatorname{span}\{\bar{q}_1, \dots, \bar{q}_s\} < s$. But $\rho\bar{c}$ is on the boundary of $\operatorname{co}\{\bar{q}_1, \dots, \bar{q}_s\}$ relative to $\operatorname{span}\{\bar{q}_1, \dots, \bar{q}_s\}$ and therefore by the Carathéodory Theorem $\rho\bar{c}$ is expressible as a convex combination of l or fewer vectors in $\{\bar{q}_1, \dots, \bar{q}_s\}$ contradicting the minimality of s. Thus $\{\bar{q}_1, \dots, \bar{q}_s\}$ is linearly independent. We thus have $y_1, \dots, y_m \in Z_0(v - Lp_0)$, $y_{m+1}, \dots, y_s \in Z_0(u - Lp_0)$ where $\dim LU_n|_{\{y_i\}_{i=1}^s} = s$ $(0 \le s \le n)$ and positive numbers $\beta_1, \dots, \beta_s$ so that

$$\rho \int_a^b h p_k w\, dx = -\sum_{i=1}^{m} \beta_i (Lp_k)(y_i) + \sum_{i=m+1}^{s} \beta_i (Lp_k)(y_i) \qquad (1 \le k \le n).$$

Hence (2.2) holds with $\alpha_i = \beta_i/\rho$ $(1 \le i \le s)$.

Proof of Theorem 2.2. (ii) $\Rightarrow$ (i).

Assume that (ii) holds, but for some $w \in W$, v, $u \in C^r(K)$ with $\operatorname{Int} \widetilde{U}_n(v, u) \ne \phi$, and $f \in C_w[a, b]$ there are two distinct best approximants p_1, $p_2 \in \widetilde{U}_n(v, u)$ for f. Then $(p_1 + p_2)/2 \in \widetilde{U}_n(v, u)$ is also a best approximant and setting $g = p_1 - p_2$, we have

$$\begin{gathered} Z_0(f - (p_1 + p_2)/2) \subseteq Z_0(f - p_1) \cap Z_0(f - p_2) \subseteq Z_0(g), \\ Z_0(v - L((p_1 + p_2)/2)) \subseteq Z_r(v - Lp_1) \cap Z_r(v - Lp_2) \subseteq Z_r(Lg), \\ Z_0(u - L((p_1 + p_2)/2)) \subseteq Z_r(u - Lp_1) \cap Z_r(u - Lp_2) \subseteq Z_r(Lg). \end{gathered}$$

Now choose h; $y_1, \dots, y_s$; $\alpha_1, \dots, \alpha_s$ as in Theorem 2.1 for the best approximant $(p_1 + p_2)/2$ of f. Let $\sigma = \operatorname{sgn}(f - (p_1 + p_2)/2)$. Since $\operatorname{supp} g \subseteq \operatorname{supp}(f - (p_1 + p_2)/2)$, σ maps $\operatorname{supp} g$ continuously to $\{-1, 1\}$. Moreover, $\sigma = h$ on $\operatorname{supp} g$, and $\{y_i\}_{i=1}^s \subseteq Z_r(Lg)$. By (ii) there exists a $g_1 \in U_n \backslash \{0\}$ satisfying (2.3)–(2.5). Therefore

$$\int_a^b h g_1 w\, dx = \int_{\operatorname{supp} g} h g_1 w\, dx = \int_{\operatorname{supp} g} \sigma g_1 w\, dx = \int_a^b |g_1| w\, dx. \tag{2.8}$$

On the other hand (2.2) should hold for $p = g_1$ which (in view of (2.5)) contradicts (2.8).

(i) $\Rightarrow$ (ii). Assume that (i) holds. We show now that for given g, σ and $\{y_i\}_{i=1}^s$ as in (ii) there exists $g_1 \in U_n \backslash \{0\}$ satisfying (2.3)–(2.5). Let us replace (2.5) by a seemingly weaker condition (2.5$'$) : $\gamma_i (Lg_1)(y_i) \ge 0$, where $\gamma_i = 1$ or -1 are chosen arbitrarily $(1 \le i \le s)$. First we show that the existence of $g_1 \in U_n \backslash \{0\}$ satisfying (2.3), (2.4) and (2.5$'$) (with arbitrary $\gamma_i = \pm 1$) implies existence of $\tilde{g}_1 \in U_n \backslash \{0\}$ satisfying (2.3)–(2.5). For $\bar{\gamma} = (\gamma_i)_{i=1}^s$ with $\gamma_i = \pm 1$ $(1 \le i \le s)$ let $g_{\bar{\gamma}} \in U_n \backslash \{0\}$ be such that (2.3), (2.4) and (2.5$'$) hold for it. Set

$$A = \{((Lg_{\bar{\gamma}})(y_i))_{i=1}^s : \bar{\gamma} = (\gamma_i)_{i=1}^s, \ \gamma_i = \pm 1 \ (1 \le i \le s)\} \subseteq \mathbf{R}^s.$$

If $\bar{0} \notin \operatorname{co} A$, then there exists $\bar{\eta} = (\eta_i)_{i=1}^s \in \mathbf{R}^s$ such that $(\bar{a}, \bar{\eta}) > 0$ for all $\bar{a} \in A$ which (in view of (2.5$'$)) contradicts the definition of A. Thus $\bar{0} \in \operatorname{co} A$, i.e., there exist $\bar{\gamma}^1, \dots, \bar{\gamma}^l$ and positive numbers $\alpha_1, \dots, \alpha_l$ so that for $\tilde{g}_1 = \sum_{i=1}^l \alpha_i g_{\bar{\gamma}^i} \in U_n \backslash \{0\}$, (2.3)–(2.5) hold.

Now it suffices to establish the existence of $g_1 \in U_n \backslash \{0\}$ satisfying (2.3), (2.4) and (2.5$'$). Let $g \in U_n \backslash \{0\}$, σ and $\{y_i\}_{i=1}^s$ be as required in (ii) and assume without loss of generality that γ_i in (2.5′) are chosen so that $\gamma_i = 1$ $(1 \le i \le m)$ and $\gamma_i = -1$ $(m+1 \le i \le s)$ for some $0 \le m \le s$. Assume that no $g_1 \in U_n \backslash \{0\}$ satisfies (2.3), (2.4) and (2.5′).

Set $\widetilde{U} = \{p \in U_n : p = 0 \text{ a.e. on } Z_0(g)\}$ and let Q be a complementary subspace of $\widetilde{U}$ in U_n. Thus U_n possesses a basis $\{p_i\}_{i=1}^n$ such that $\{p_i\}_{i=1}^l$ and $\{p_i\}_{i=l+1}^n$ are bases for $\widetilde{U}$ and Q, respectively. By the Liapunoff theorem [10] we can choose $\tilde{h} \in L^\infty(Z_0(g))$ such that $|\tilde{h}| \equiv 1$ on $Z_0(g)$ and $\int_{Z_0(g)} \tilde{h} q = 0$ for every $q \in Q$. Let $h = \sigma$ on $\operatorname{supp} g$ and $h = \tilde{h}$ on $Z_0(g)$. Then $|h| \equiv 1$. Set

$$\mathscr{P} = \left\{ \left(\int_a^b h p_k w\, dx + \sum_{i=1}^m \alpha_i (Lp_k)(y_i) - \sum_{i=m+1}^s \alpha_i (Lp_k)(y_i) \right)_{k=1}^n : \right.$$

$$\left. w \in W,\ \alpha_i > 0\ (1 \le i \le s) \right\} \subseteq \mathbf{R}^n .$$

$\mathscr{P}$ is a convex cone in $\mathbf{R}^n$ with $\bar{0} \in \overline{\mathscr{P}}$.

Assume that $\bar{0} \notin \mathscr{P}$, i.e., $\bar{0} \in \operatorname{Bd}\mathscr{P}$. Let $\bar{a} = (a_k)_{k=1}^n \in \mathbf{R}^n \backslash \{0\}$ define a hyperplane supporting $\mathscr{P}$ at $\bar{0}$. Then setting $g^* = \sum_{k=1}^n a_k p_k \in U_n \backslash \{0\}$ we have

$$\int_a^b h g^* w\, dx + \sum_{i=1}^m \alpha_i (Lg^*)(y_i) - \sum_{i=m+1}^s \alpha_i (Lg^*)(y_i) \ge 0 \tag{2.9}$$

for every $w \in W$ and $\alpha_i > 0$, $1 \le i \le s$. Letting $g^* = p^* + q$ where $p^* \in \widetilde{U}$ and $q \in Q$, (2.9) yields that $hg^* \ge 0$ a.e. on $[a, b]$; hence $hq \ge 0$ a.e. on $Z_0(g)$. By definition of h, this is possible only if $q = 0$ a.e. on $Z_0(g)$, i.e. $q \equiv 0$ (by definition of $\widetilde{U}$). Thus $g^* = p^* \in \widetilde{U}$ and $\sigma g^* \ge 0$ on $\operatorname{supp} g$. In addition, (2.9) also implies that $(Lg^*)(y_i) \ge 0$ if $1 \le i \le m$ and $(Lg^*)(y_i) \le 0$ if $m+1 \le i \le s$. Thus g^* satisfies (2.3), (2.4) and (2.5′), contradicting our assumption. Thus $\bar{0} \in \mathscr{P}$, i.e., for some $w^* \in W$ and $\alpha_i^* > 0$ $(1 \le i \le s)$ we have

$$\int_a^b h p w^* dx + \sum_{i=1}^m \alpha_i^* (Lp)(y_i) - \sum_{i=m+1}^s \alpha_i^* (Lp)(y_i) = 0 \tag{2.10}$$

for every $p \in U_n$.

Set $f^* = |g|h$. Since $h = \sigma$ on $\operatorname{supp} g$, $f^* \in C[a, b]$. Choose disjoint closed intervals $[\alpha_i, \beta_i] \subseteq K$ $(1 \le i \le s)$ so that $y_i \in [\alpha_i, \beta_i]$ $(1 \le i \le s)$ where $\alpha_i < \beta_i$ if $y_i \in \operatorname{Bd} K$ and $\alpha_i < y_i < \beta_i$ if $y_i \in \operatorname{Int} K$. Define for $r = 0$ or 1 and $1 \le i \le m$ a function v_r on $[\alpha_i, \beta_i]$ by

$$v_r(y) = \begin{cases} -|(Lg)(y)| - (y - y_i)^2, & \text{if } r = 0, \\ -\operatorname{sgn}(y - y_i) \int_{y_i}^y [|(Lg)'(t)| + (t - y_i)^2]\, dt, & \text{if } r = 1. \end{cases}$$

Since $y_i \in Z_r(Lg)$ it follows that $v_r \in C^r([\alpha_i, \beta_i])$ $(1 \le i \le m)$. In addition, $-|Lg| > v_r$ on $[\alpha_i, \beta_i] \backslash \{y_i\}$ and $v_r < 0$ on $[\alpha_i, \beta_i]$ unless $y = y_i$ $(1 \le i \le m)$. Now we can extend v_r to K so that $v_r \in C^r(K)$ and $v_r < -|Lg|$ on K if $y \ne y_i$ $(1 \le i \le m)$. Similarly we can construct a function $u_r \in C^r(K)$

so that $u_r > |Lg|$ on $K\backslash\{y_i\}_{i=m+1}^s$ and $u_r(y_i) = 0$ $(m+1 \le i \le s)$. Since $\dim LU_n|_{\{y_i\}_{i=1}^s} = s$ we can choose $\tilde{g} \in U_n$ so that $(L\tilde{g})(y_i) > 0$ $(1 \le i \le m)$ and $(L\tilde{g})(y_i) < 0$ $(m+1 \le i \le s)$. A compactness argument yields that for $t > 0$ sufficiently small $v_r < L(t\tilde{g}) < u_r$ on K, i.e., $\operatorname{Int} \widetilde{U}_n(v_r, u_r) \ne \phi$.

Let us show that εg $(0 < \varepsilon < 1)$ is a best approximant to $f^* \in C_{w^*}[a, b]$ from $\widetilde{U}_n(v_r, u_r)$. For $0 < \varepsilon < 1$ we have

$$v_r \le -|Lg| \le L(\varepsilon g) \le |Lg| \le u_r,$$

yielding that $\varepsilon g \in \widetilde{U}_n(v_r, u_r)$. Furthermore, $Z_0(v_r - L(\varepsilon g)) = \{y_1, \dots, y_m\}$, $Z_0(u_r - L(\varepsilon g)) = \{y_{m+1}, \dots, y_s\}$ and since $h(f^* - \varepsilon g) = |g| - \varepsilon hg \ge 0$,

$$\int_a^b h(f^* - \varepsilon g) w^* \, dx = \int_a^b |f^* - \varepsilon g| w^* \, dx.$$

Thus by Theorem 2.1 and (2.10) εg is a best approximant to $f^* \in C_{w^*}[a, b]$ from $\widetilde{U}_n(v_r, u_r)$. Since $0 < \varepsilon < 1$ is arbitrary this contradicts (i).

It follows from Theorem 2.2 that uniqueness of best constrained L_1-approximation with C- or C^1-boundaries reduces to the study of L^0-A or L^1-A-spaces, respectively. L^r-A-spaces $(r = 0, 1)$ are A-spaces of special type which combine A-property with certain requirements with respect to operator L. A number of characterizations and descriptions of A-spaces are given in the literature. We shall cite now some of them which will be frequently used in this paper. The first structural characterization of A-spaces was given by Pinkus [9, 10]. The next result due to Wu Li [7] is a simplification of Pinkus' theorem. For $U_n \subset C[a, b]$ we set $Z(U_n) = \bigcap\{Z_0(p) \colon p \in U_n\}$.

Theorem 2.3. *Let U_n be an n-dimensional subspace of $C[a, b]$ such that $Z(U_n) \cap (a, b) = \phi$. Then U_n is an A-space if and only if U_n is a weak Chebyshev (WT-) space and satisfies the following "splitting" property: if $p \in U_n$ and $p \equiv 0$ on $[c, d]$ $(a < c < d < b)$, then $p\chi_{[a,d]}, p\chi_{[c,b]} \in U_n$.*

It follows easily from the above theorem that if U_n is an A-space with $Z(U_n) \cap (a, b) = \phi$, then there are finitely many endpoints $a = c_0 < c_1 < \cdots < c_m < c_{m+1} = b$ of zero intervals of functions in U_n and $U_n|_{(c_i, c_{i+1})}$ is a Haar space $(0 \le i \le m)$ (Pinkus [9, 10]). On the other hand Pinkus [9, 10] also showed that if U_n is an A-space and $z \in Z(U_n) \cap (a, b)$, then U_n is the direct sum of two A-spaces with supports contained in $[a, z)$ and $(z, b]$, respectively. This and Theorem 2.3 imply that if U_n is an A-space in $C[a, b]$ and no function in $U_n\backslash\{0\}$ vanishes on a nondegenerate subinterval of $[a, b]$, then U_n is a Haar space on (a, b) (Havinson [3]).

Results mentioned above completely describe the structure of A-spaces. An immediate consequence of Theorem 2.2 shows that the study of L^0-A-spaces can be reduced to the study of A-spaces, thus leading to a full structural description of L^0-A-spaces.

Corollary 2.4. *Let $L \colon U_n \to C^0(K)$. Then U_n is an L^0-A-space if and only if for every $\{y_i\}_{i=1}^s \subseteq K$ $(0 \le s \le n-1)$ such that $\dim LU_n|_{\{y_i\}_{i=1}^s} = s$ the set $G_{\{y_i\}_{i=1}^s} = \{p \in U_n \colon (Lp)(y_i) = 0 \ (1 \le i \le s)\}$ is an A-space.*

The above corollary leads to a simple and elegant characterization of uniqueness sets for constraints imposed by linear functionals.

Let $U_n \subset C[a,b]$ be an n-dimensional space and consider linearly independent functionals $\varphi_1, \dots, \varphi_s \in U_n^*$. Set $\widetilde{U}_n(\bar{a}, \bar{b}) = \{p \in U_n : a_i \le \varphi_i(p) \le b_i \ (1 \le i \le s)\}$, where $\bar{a} = (a_i)_{i=1}^s$, $\bar{b} = (b_i)_{i=1}^s \in \mathbf{R}^s$ and $\bar{a} < \bar{b}$, i.e., $a_i < b_i \ (1 \le i \le s)$. Obviously, $\widetilde{U}_n(\bar{a}, \bar{b}) = \widetilde{U}_n(v, u)$, where $\widetilde{U}_n(v, u)$ is defined as above by $L: U_n \to C^0(K)$ with $K = \{1, 2, \dots, s\}$, $v(i) = a_i$, $u(i) = b_i$, $(Lp)(i) = \varphi_i(p) \ (1 \le i \le s)$. Furthermore, for arbitrary $\{s_i\}_{i=1}^m \subseteq \{1, 2, \dots, s\} = K$, we have $\dim LU_n|_{\{s_i\}_{i=1}^m} = m$ and $G_{\{s_i\}_{i=1}^m} = \{p \in U_n : (Lp)(s_i) = 0 \ (1 \le i \le m)\} = \bigcap_{i=1}^m \ker \varphi_{s_i} \ (G_\phi = U_n)$.
Thus we obtain the next

Corollary 2.5. *Let $U_n \subset C[a,b]$ and $\varphi_i \in U_n^* \ (1 \le i \le s)$ be linearly independent. Then in order that for every $w \in W$ and $\bar{a}$, $\bar{b} \in \mathbf{R}^s$ such that $\operatorname{Int} \widetilde{U}_n(\bar{a}, \bar{b}) \ne \phi$, $\widetilde{U}_n(\bar{a}, \bar{b})$ be a uniqueness set in $C_w[a,b]$ it is necessary and sufficient that U_n and $\bigcap_{i=1}^m \ker \varphi_{s_i}$ be A-spaces for every $\{s_1, \dots, s_m\} \subseteq \{1, 2, \dots, s\}$.*

For the special case of φ_i's being the coefficient functionals the above corollary is essentially proved in [11].

Although Corollary 2.4 shows that there is an intimate relationship between the L^0-A-property and the A-property, it appears that there is no similar clear connection between the L^1-A-spaces and A-property. However, we can derive from Theorem 2.2 some useful results which provide a partial characterization of L^1-A-spaces via A-spaces.

For $L: U_n \to C^1(K)$ and $\{y_i\}_{i=1}^s \subseteq K$, set $G'_{\{y_i\}_{i=1}^s} = \{p \in U_n : (Lp)(y_i) = 0 \ (1 \le i \le s)$ and $(Lp)'(y_i) = 0$ if $y_i \in \operatorname{Int} K \ (1 \le i \le s)\}$. Evidently, $G'_{\{y_i\}_{i=1}^s} \subseteq G_{\{y_i\}_{i=1}^s}$. Now from Theorem 2.2 we easily derive the following

Corollary 2.6. *Let $L: U_n \to C^1(K)$ and assume that U_n is an L^1-A-space. If for given $\{y_i\}_{i=1}^s \subseteq K \ (0 \le s \le n-1)$ such that $\dim LU_n|_{\{y_i\}_{i=1}^s} = s$ we have $G_{\{y_i\}_{i=1}^s} = G'_{\{y_i\}_{i=1}^s}$, then $G_{\{y_i\}_{i=1}^s}$ is an A-space.*

Let us note that $G_{\{y_i\}_{i=1}^s} = G'_{\{y_i\}_{i=1}^s}$ for every $\{y_i\}_{i=1}^s \subseteq \operatorname{Bd} K$ yielding that $G_{\{y_i\}_{i=1}^s}$ is an A-space if U_n satisfies the L^1-A-property. In what follows we shall frequently apply the above necessary condition for L^1-A-spaces. Let us also mention a useful sufficient condition for the L^1-A-property related to A-spaces.

Corollary 2.7. *Let $L: U_n \to C^1(K)$. If for every $\{y_i\}_{i=1}^s \subseteq K$ such that*

$$\dim LU_n|_{\{y_i\}_{i=1}^s} = s,$$

$G'_{\{y_i\}_{i=1}^s}$ is an A-space, then U_n is an L^1-A-space.

The converse of Corollary 2.7 does not hold in general. From Theorem 4.1, it will follow that the space Π_2 of polynomials of degree 2 or less is an L^1-A-space on $[-1, 1]$ with L the identity operator, however, $G'_{\{0\}} = \operatorname{span}\{x^2\}$ is not an A-space on $[-1, 1]$.

We conclude this section by citing the analogies between the Haar-type theories for constrained L_1- and L_∞-approximation. The L-Haar and L'-Haar properties were found in [6] to completely characterize those subspaces U_n of

$C[a, b]$ for which $\widetilde{U}_n(v, u)$ admits unique best uniform approximations to all $f \in C[a, b]$ for all continuous or smooth, respectively, boundary functions v and u with $\operatorname{Int} \widetilde{U}_n(v, u) \neq \phi$. Among other conditions, the L-Haar property was found to be equivalent to $G_{\{y_i\}_{i=1}^s}$ being a Haar space for all $\{y_i\}_{i=1}^s \subseteq K$ with $\dim LU_n|_{\{y_i\}_{i=1}^s} = s$. Hence, every L-Haar space is an L^0-A-space. Correspondingly, an analog of Havinson's result holds. That is, if U_n is an L^0-A-space on $[a, b]$ and no nontrivial element of U_n vanishes on a subinterval of $[a, b]$, then U_n is an L-Haar space on every $[\alpha, \beta] \subseteq (a, b)$. As in the context of this paper, the L'-Haar property was not found to have a simple equivalent as that above for the L-Haar property; however, useful criteria similar to Corollaries 2.6 and 2.7 were obtained.

3. Applications for C-boundaries (L^0-A-spaces)

As we have seen in the previous section the C-boundary independent uniqueness of constrained L_1-approximation is completely characterized by the L^0-A-property. In this section we shall consider some examples of L^0-A-spaces. Throughout this section we set $K = [a, b]$. It turns out that L^0-A-property is quite restrictive and we shall illustrate this in §3.1 by providing a "piecewise" bound on the dimension of L^0-A-spaces. Never-the-less, in L_1-approximation the C-boundary case is not as restrictive as in uniform approximation (see [6]). We shall present in §3.2 a certain family of spline functions providing a useful example of L^0-A-spaces.

3.1. A negative result on L^0-A-spaces. An operator $L: U_n \to C[a, b]$ is called a k-Rolle operator ($k \geq 0$) if whenever $p \in U_n$ and $p(x_i) = 0$ $(1 \leq i \leq k+1)$ for some $a \leq x_1 < \cdots < x_{k+1} \leq b$, then $(Lp)(y) = 0$ for some $y \in [x_1, x_{k+1}]$. Obviously, the identity operator (I) is a 0-Rolle operator. It is known [6] that if $U_n \subset C^k[a, b]$ and $\alpha_i \in C^{i-1}[a, b]$ $(1 \leq i \leq k)$, then $L = (D + \alpha_1(x)I) \cdots (D + \alpha_k(x)I)$ is k-Rolle. (Here and in what follows D denotes the differentiation operator.) It is easy to see that any k-Rolle operator L satisfies the following property: if $p, q \in U_n$ and $p \equiv q$ on $[c, d]$ $(a \leq c < d \leq b)$, then $Lp \equiv Lq$ on $[c, d]$. Therefore if $L: U_n \to C[a, b]$ is k-Rolle and $a \leq c < d \leq b$, then the operator $\widetilde{L}: U_n|_{[c,d]} \to C[c, d]$ given by $\widetilde{L}p|_{[c,d]} = (Lp)|_{[c,d]}$ is well-defined. In particular Corollary 2.4 immediately implies that if $L: U_n \to C[a, b]$ is k-Rolle and U_n is an L^0-A-space, then $U_n|_{[c,d]}$ is an $\widetilde{L}^0$-A-space for any $a \leq c < d \leq b$.

Our next result restricts the dimension of L^0-A-spaces related to k-Rolle operators.

Theorem 3.1. *Let $L: U_n \to C[a, b]$ be a k-Rolle operator and assume that U_n satisfies the L^0-A-property. Consider an arbitrary interval $[c, d] \subseteq [a, b]$ $(c < d)$ such that whenever $p \in U_n$ vanishes on a nondegenerate subinterval of $[c, d]$ we have $p \equiv 0$ on $[c, d]$. Then either $LU_n|_{[c,d]} = 0$ or $U_n|_{[c,d]}$ is a Haar space on (c, d) of dimension at most $k+1$.*

Proof. Since U_n is an A-space, according to the description of A-spaces given at the end of §2, $U_n|_{[c,d]}$ is a Haar space on (c, d). Suppose that $LU_n|_{[c,d]} \neq 0$ and $\dim U_n|_{[c,d]} = m \geq k+2$. Choose $(\alpha, \beta) \subset (c, d)$ such that for some $\tilde{g} \in U_n$ we have $L\tilde{g} \neq 0$ on (α, β). Let $x_1, \ldots, x_{m-1} \in (\alpha, \beta)$ be arbitrary.

Then there exists $p \in U_n$ not identically zero on $[c, d]$ such that $p(x_i) = 0$, $1 \le i \le m-1$. Since $m-1 \ge k+1$ it follows from the k-Rolle property, that $(Lp)(\eta) = 0$ for some $\eta \in (\alpha, \beta)$. Furthermore, in view of our choice of interval (α, β), $\dim LU_n|_{\{\eta\}} = 1$. Hence by Corollary 2.4, $G = G_{\{\eta\}} = \{g \in U_n\colon (Lg)(\eta) = 0\}$ is an A-space. Moreover, if $g \in G$ vanishes on a nondegenerate subinterval of $[c, d]$, then $g \equiv 0$ on $[c, d]$, yielding that G is a Haar space on (c, d). On the other hand $\dim G|_{[c,d]} = \dim U_n|_{[c,d]} - 1 = m-1$ and $p \in G|_{[c,d]} \backslash \{0\}$ has $m-1$ distinct zeros in (c, d), a contradiction.

When $k = 0$ (i.e. $L = I$) the above theorem reduces to a known result by Pinkus and Strauss [11], concerning restricted range approximation.

Corollary 3.2. *If U_n is an L^0-A-space with L the identity operator, then U_n decomposes into direct sum of one-dimensional A-spaces having disjoint supports.*

Proof. Since U_n is, in particular, an A-space, we have $U_n = U_n^1 \oplus \cdots \oplus U_n^l$, where the U_n^j are A-spaces having disjoint interval supports $J_j = [a_j, b_j]$ $(1 \le j \le l,\ Z(U_n^j) \cap (a_j, b_j) = \phi)$. Now let $a_j = c_0^j < c_1^j < \cdots < c_{m_j}^j < c_{m_j+1}^j = b_j$ be the endpoints of zero intervals of functions in U_n^j. Then $U_n|_{(c_i^j, c_{i+1}^j)} = U_n^j|_{(c_i^j, c_{i+1}^j)}$ is a Haar space of dimension at most 1. (We apply here Theorem 3.1 with $k = 0$.) This in turn implies that $m_j = 0$, since otherwise $c_1^j \in Z(U_n)$ $(1 \le j \le l)$. Hence for every $1 \le j \le l$, U_n^j is a Haar space on (a_j, b_j) of dimension 1.

Example 1. Let $L = D^k$ and $U_{n+1} = \Pi_n$, the set of algebraic polynomials of degree at most n. If $k = n$, then $D^n\Pi_n$ consists of constant functions. Thus applying Corollary 2.4 for $s = 0$ and 1 and noting that Π_n and Π_{n-1} are A-spaces, we obtain that Π_n is an L^0-A-space. On the other hand if $0 \le k \le n-2$, then $\dim \Pi_n = n+1 \ge k+2$ and it follows from Theorem 3.1 that Π_n is not an L^0-A-space.

Example 2. For given $a = c_0 < c_1 < \cdots < c_l < c_{l+1} = b$ denote by $\mathscr{S}_{m,l}$ the set of spline functions of order m with l simple fixed knots $c_1, \ldots, c_l$. Then $\mathscr{S}_{m,l} \subset C^{m-2}[a, b]$, $\mathscr{S}_{m,l} \not\subset C^{m-1}[a, b]$, and $\mathscr{S}_{m,l}|_{[c_i, c_{i+1}]} = \Pi_{m-1}$ $(0 \le i \le l)$. Let $L = D^k$, where we need to assume that $0 \le k \le m-2$ in order that $D^k\colon \mathscr{S}_{m,l} \to C[a, b]$. Then $\dim \mathscr{S}_{m,l}|_{[c_i, c_{i+1}]} = m \ge k+2$; hence, by Theorem 3.1, $\mathscr{S}_{m,l}$ is not an L^0-A-space.

3.2. A nontrivial example of an L^0-A-space. Simple examples of L^0-A-spaces can be provided by applying Corollary 2.5, i.e., letting L be a finite collection of linear functionals, for instance, coefficient evaluation functionals. In this section we shall present a spline space satisfying the L^0-A-property for $L = D^k$. First we give a general approach to constructing L^0-A-spaces for $L = D^k$ and then apply it in order to obtain spline spaces with the required properties.

Let $d_0 = a \le c_1 < d_1 \le c_2 < d_2 \le \cdots \le c_l < d_l \le b = c_{l+1}$. For $1 \le i \le l$ let $u_i \in C^k[a, b]$ satisfy the following properties:

$$u_i = 0 \quad \text{on } [a, c_i], \tag{3.1}$$

$$D^k u_i = 0 \quad \text{on } [a, c_i] \cup [d_i, b], \tag{3.2}$$

$$D^k u_i > 0 \quad \text{on } (c_i, d_i). \tag{3.3}$$

Let $U^*_{k+l} = \Pi_{k-1} \oplus \operatorname{span}\{u_1, \dots, u_l\}$.

Theorem 3.3. *For $L = D^k$ the space U^*_{k+l} satisfies the L^0-A-property ($k \geq 1$, $l \geq 1$).*

Proof. Let us verify at first that U^*_{k+l} is an A-space. Note that $\dim U^*_{k+l} = k+l$ and since $1 \in U^*_{k+l}$, $Z(U^*_{k+l}) \cap [a, b] = \phi$.

We need to show that U^*_{k+l} is a WT-space. Assume, to the contrary, that $u \in U^*_{k+l}$ has a strong sign alternation of length $k+l+1$. Then $D^k u$ has a strong sign alternation of length $l+1$, where $D^k u \in D^k U^*_{k+l} = \operatorname{span}\{D^k u_1, \dots, D^k u_l\}$. But by (3.2) and (3.3) no function in $D^k U^*_{k+l}$ can have a strong sign alternation of length greater than l. Thus U^*_{k+l} is a WT-space.

To demonstrate the splitting property for U^*_{k+l} (see Theorem 2.3), note that the restriction of U^*_{k+l} to (c_i, d_i) is the Haar space $\Pi_{k-1} \bigoplus \operatorname{span}\{u_i\}$, while its restriction to a nondegenerate interval of the form (d_i, c_{i+1}) $(0 \leq i \leq l)$ is the Haar space Π_{k-1} (see (3.2)). Thus it suffices to prove the splitting property for $u = p + \sum_{i=1}^{l} \alpha_i u_i (p \in \Pi_{k-1})$ vanishing on (c_j, d_j) $(1 \leq j \leq l)$ or on (d_j, c_{j+1}) $(0 \leq j \leq l, d_j < c_{j+1})$. If $u \equiv 0$ on (c_j, d_j), then by (3.1) $(p + \sum_{i=1}^{j-1} \alpha_i u_i) + \alpha_j u_j \equiv 0$ on (c_j, d_j). Thus $p + \sum_{i=1}^{j-1} \alpha_i u_i \equiv 0$ on $[c_j, b]$ and $\alpha_j = 0$, because $p + \sum_{i=1}^{j-1} \alpha_i u_i$ is a polynomial of degree $\leq k-1$ on $(c_j, b]$ and $u_j|_{(c_j, d_j)} \notin \Pi_{k-1}$. Thus $u\chi_{[a, d_j]} = p + \sum_{i=1}^{j-1} \alpha_i u_i \in U^*_{k+l}$ and $u\chi_{[c_j, b]} = \sum_{i=j+1}^{l} \alpha_i u_i \in U^*_{k+l}$. If $u \equiv 0$ on (d_j, c_{j+1}) where $d_j < c_{j+1}$, then $p + \sum_{i=1}^{j} \alpha_i u_i \equiv 0$ on $[d_j, c_{j+1}]$ and thus on $[d_j, b]$, too. So $u\chi_{[a, c_{j+1}]} = p + \sum_{i=1}^{j} \alpha_i u_i \in U^*_{k+l}$ and $u\chi_{[d_j, b]} = \sum_{i=j+1}^{l} \alpha_i u_i \in U^*_{k+l}$. Hence by Theorem 2.3 U^*_{k+l} is an A-space.

Now we can prove that U^*_{k+l} satisfies the L^0-A-property with $L = D^k$.

Obviously, $D^k U^*_{k+l} = \operatorname{span}\{D^k u_1, \dots, D^k u_l\}$ and any subset $\{y_1, \dots, y_s\} \subset [a, b]$ for which $\dim D^k U^*_{k+l}|_{\{y_i\}_{i=1}^s} = s$ $(0 \leq s \leq l)$ should satisfy the property that all y_j's belong to distinct intervals (c_i, d_i) $(1 \leq j \leq s, q \leq i \leq l)$. Let $1 \leq i_1 < \cdots < i_s \leq l$ be such that $y_j \in (c_{i_j}, d_{i_j})$, $1 \leq j \leq s$. Then, evidently,

$$\begin{aligned} G_{\{y_i\}_{i=1}^s} &= \{p \in U^*_{k+l} : (D^k p)(y_i) = 0, \ 1 \leq i \leq s\} \\ &= \Pi_{k-1} \oplus \operatorname{span}\{u_k : 1 \leq k \leq l, \ k \neq i_j, \ 1 \leq j \leq s\}, \end{aligned}$$

and repeating the above argument for $G_{\{y_i\}_{i=1}^s}$, we obtain that $G_{\{y_i\}_{i=1}^s}$ is an A-space. Thus it follows from Corollary 2.4 that U^*_{k+l} is an L^0-A-space for $L = D^k$.

The above theorem provides a simple method for constructing L^0-A-spaces if $L = D^k$. It also shows that an L^0-A-space for a k-Rolle operator L with $k \geq 1$ can have arbitrary high dimension and yet not decompose to direct sum of spaces with disjoint support, as occurs for $k = 0$.

Let us now apply Theorem 3.3 for a certain space of splines with fixed knots.

Consider the space $\mathscr{S}_{k+2, 2m-1}$ of splines of order $k+2$ with $2m-1$ simple knots $c_0 = a < c_1 < \cdots < c_{2m-1} < b = c_{2m}$ $(k, m \geq 1)$. Set $\widetilde{\mathscr{S}}_{k+2, 2m-1} =$

$\{s \in \mathcal{S}_{k+2,2m-1} : D^k s(c_{2i}) = 0,\ 0 \le i \le m\}$, $\dim \widetilde{\mathcal{S}}_{k+2,2m-1} = k+m$. Then $D^k\widetilde{\mathcal{S}}_{k+2,2m-1} = \operatorname{span}\{v_1, \dots, v_m\}$, where $v_i \equiv 0$ on $[a, b]\backslash(c_{2i-2}, c_{2i})$, $v_i(c_{2i-1}) = 1$ and v_i is linear on (c_{2i-2}, c_{2i-1}) and on (c_{2i-1}, c_{2i}) $(1 \le i \le m)$. Letting $u_i(x) = \int_a^x \int_a^{t_1} \cdots \int_a^{t_{k-1}} v_i(t_k)\,dt_k \cdots dt_1$, $1 \le i \le m$, we have $\widetilde{\mathcal{S}}_{k+2,2m-1} = \Pi_{k-1} \bigoplus \operatorname{span}\{u_1, \dots, u_m\}$, where u_i $(1 \le i \le m)$ satisfy (3.1)–(3.3). Hence, by Theorem 3.3, $\widetilde{\mathcal{S}}_{k+2,2m-1}$ is an L^0-A-space for $L = D^k$.

As was previously mentioned, the L^0-A-property is restrictive but not as restrictive as the L-Haar property for uniform approximation. In [6], it was found that if L is a nontrivial k-Rolle operator on U_n and U_n is L-Haar, then $\dim U_n = n \le k+1$. Although the "local dimensions" of L^0-A-spaces are bounded by $k+1$ when L is k-Rolle, the example of this section demonstrates that for $L = D^k$ $(k \ge 1)$ there are L^0-A-spaces of arbitrarily large dimension. Moreover, these spaces can be chosen to be uniformly dense in $C[a, b]$.

4. Applications for C^1-boundaries (L^1-A-spaces)

For the operator $L = D^k$ $(k \ge 1)$, §3.2 provides spaces of spline functions of arbitrarily large dimension that are L^0-A-spaces. Theorem 3.1 and the remarks following Theorem 2.3 suggest that all high dimensional L^0-A-spaces are necessarily "spline-like." That is, if U_n is an L^0-A-space on $[a, b]$, then there exist points $a = c_0 < c_1 < \cdots < c_{l+1} = b$ where $U_n|_{(c_{i-1}, c_i)}$ is a Haar space of dimension $k+1$ or less $(1 \le i \le l)$. We shall see in this section that the development of L^1-A-spaces takes a different direction as we obtain L^1-A-spaces of arbitrarily large "local dimension."

In §4.1, we consider approximation by polynomials with restrictions on the range and several derivatives, and in §4.2, we study lacunary polynomial spaces with constraints on the kth derivative. In both of these applications, Birkhoff interpolation plays a significant role. We refer the reader to chapter 1 of the text [8] for the definitions of an interpolation matrix and regularity and for the Atkinson-Sharma regularity theorem. In §4.3, we consider approximation by polynomials with constraints involving the linear operator $D - \alpha I$. In the previous three cases, the nature of uniqueness for uniform approximation is completely understood, and we make appropriate comparisons. Finally, in §4.4, we demonstrate that the space of spline functions of order m $(m \ge 4)$ with simple knots is an L^1-A-space although it is not an L^0-A-space $(L = D^{m-3})$.

4.1. Polynomials with several derivatives. Given integers $0 \le k_1 < k_2 < \cdots < k_l$, we consider the problem of approximating continuous functions by polynomials $p \in \Pi_m$ satisfying constraints of the form $v_i \le D^{k_i} p \le u_i$ $(1 \le i \le l)$ where each v_i, $u_i \in C^1[a, b]$. To put this problem into the context of this paper, let $K = \bigcup_{i=1}^{l} [a_i, b_i]$ where the intervals $[a_i, b_i]$ $(i = 1, \dots, l)$ are pairwise disjoint and are copies of $[a, b]$ (that is, $b_i - a_i = b - a$). Define $L = D^{k_1} \cdots D^{k_l} \colon \Pi_m \to C^1(K)$ by $Lp = D^{k_i} p_i$ on $[a_i, b_i]$ where $p_i(x) = p(x - a_i + a)$ is the translate of $p|_{[a,b]}$ to $[a_i, b_i]$. We have

Theorem 4.1. *For any $0 \le k_1 < k_2 < \cdots < k_l$ the space Π_m of polynomials of degree m or less is an L^1-A-space for $L = D^{k_1} \cdots D^{k_l}$ $(m \ge 0)$.*

In Example 1, we saw that Π_m $(m \ge k+1)$ is not an L^0-A-space with $L = D^k$, however, Theorem 4.1 implies that Π_m is an L^1-A-space.

We note that in the uniform norm setting analogous results hold when $k_1 > 0$ [6]. When $k_1 = 0$, uniqueness of a best constrained uniform approximation requires that the function f being approximated satisfy the constraint $v_1 \leq f \leq u_1$ (see [2, 12]). This is not the case for L^1-approximation.

Proof of Theorem 4.1. We employ Theorem 2.2. Let $g \in \Pi_m \backslash \{0\}$. By discarding some k_i's, if necessary, we may assume that $k_l < \deg g \leq m$. Let $\{y_1, \ldots, y_s\} \subseteq Z_1(Lg)$ and σ be a continuous sign function defined on $\operatorname{supp}(g)$. Let $x_1, \ldots, x_\mu$ denote the points of sign change of σ. Evidently, $\{x_1, \ldots, x_\mu\} \subseteq Z(g) \cap (a, b)$. For $y \in [a_i, b_i]$ let $\tilde{y} = y - a_i + a$ be the corresponding point in $[a, b]$. For $p \in \Pi_m$, we consider the interpolation conditions

$$p(x_i) = 0 \qquad (1 \leq i \leq \mu), \tag{4.1a}$$

$$D^{k_j} p(\tilde{y}_i) = D^{k_j+1} p(\tilde{y}_i) = 0 \qquad (1 \leq i \leq s,\ y_i \in (a_j, b_j),\ 1 \leq j \leq l), \tag{4.1b}$$

$$D^{k_j} p(\tilde{y}_i) = 0 \qquad (1 \leq i \leq s,\ y_i \in \{a_j, b_j\}\ 1 \leq j \leq l). \tag{4.1c}$$

Overlaps in (4.1a) and (4.1b) can occur. We construct an "alternate interpolation problem" by possibly removing some of the conditions in (4.1b). If (4.1a) and (4.1b), impose conditions of the form

$$p(x_i) = Dp(x_i) = \cdots = D^{k_j} p(x_i) = D^{k_j+1} p(x_i) = 0$$

and do not impose a condition on $D^{k_j+2} p(x_i)$, we discard the condition $D^{k_j+1} p(x_i) = 0$ if k_j is even and retain it if k_j is odd. The resulting sequence of conditions has odd length. Assume that (4.1b) imposes conditions of the form

$$D^{k_j} p(y) = D^{k_j+1} p(y) = \cdots = D^{k_\nu} p(y) = D^{k_\nu+1} p(y) = 0$$

where no condition is imposed on $D^{k_\nu+2} p(y)$ and either $k_1 = 0$ and $y \notin \{x_1, \ldots, x_\mu\}$ or $k_j \geq 1$ with no condition being placed on $D^{k_j-1} p(y)$. Then we discard the condition $D^{k_\nu+1} p(y) = 0$ if $k_\nu - k_j$ is odd and retain it if $k_\nu - k_j$ is even. The resulting sequence of conditions has even length. Further by the choice of the discarded conditions, the alternate interpolation problem includes the condition $p(x_i) = 0$ $(1 \leq i \leq \mu)$ and $(Lp)(y_i) = 0$ $(1 \leq i \leq s)$.

Let E be the interpolation matrix for the alternate interpolation problem with infinitely many augmented zero columns and where the column index starts with 0. By the discarding process, E has no odd supported sequences of ones. Let j be the smallest index where the number of ones in columns $0-j$ of E is less than $j+1$ (j could be 0). Then column j of E is a zero column. Let E' be the matrix consisting of columns $0-(j-1)$ of E. Then E' contain j ones, has no odd support sequences, and satisfies the Polya condition, and thus E' is order regular (see [8, p. 10]). If $j - 1 \geq m$, then $g \in \Pi_{j-1}$ would satisfy the homogeneous conditions corresponding to E' and thus $g = 0$, a contradiction. Thus $j \leq m$. Let E'' be the matrix consisting of columns $0 - j$ of E. Since E'' corresponds to j conditions and $\dim \Pi_j = j + 1$, there exists $p \in \Pi_j \backslash \{0\}$ satisfying the homogeneous conditions of the alternate interpolation problem corresponding to E''. Further since $D^\nu p = 0$ for $\nu > j$, p satisfies all of the conditions of the alternate interpolation problem. In particular, $(Lp)(y_j) = 0$

$(1 \le j \le s)$. Finally, to verify that $\sigma p \ge 0$ or $\sigma(-p) \ge 0$ on $\operatorname{supp} g$, we show that p changes sign precisely at the points x_i $(1 \le i \le \mu)$. If p fails to change sign at some x_i, then p would have a zero of even multiplicity at x_i. Since the sequence of ones in E'' starting with column 0 in the row corresponding to x_i is odd, then p would have an extra zero at x_i that is not specified by E''. So p would satisfy a homogeneous Birkhoff interpolation problem in Π_j with an order regular matrix, and hence $p = 0$, a contradiction. If p changes sign at some $y \in (a, b)\backslash\{x_1, \dots, x_\mu\}$, then p would have a zero of odd multiplicity at y. By the construction of the alternate interpolation problem, p would satisfy a homogeneous Birkhoff interpolation problem in Π_j with an order regular matrix so that $p = 0$, a contradiction. Thus by Theorem 2.2, Π_m is an L^1-A-space.

4.2. Lacunary polynomials. We consider the lacunary polynomial space $P_n = \operatorname{span}\{x^{k_1} = 1, x^{k_2}, \dots, x^{k_n} = x^N\}$ where $0 = k_1 < k_2 < \cdots < k_n = N$ are integers, $[a, b] = K = [-1, 1]$, and $L = D^k$. Throughout this section, we assume that $N \ge k + 1$ so that L is not a linear functional on P_n.

Theorem 4.2. *Let* $[a, b] = K = [-1, 1]$. *Then* P_n *is an* L^1-A-*space with* $L = D^k$ $(1 \le k \le N - 1)$ *if and only if*

(a) $k_{i+1} - k_i$ *is odd* $(1 \le i \le n - 1)$, *and*
(b) $x^k \notin P_n$ *or* x^k, $x^{k+1} \in P_n$.

We note that for uniform approximation (a) and (b) are the necessary and sufficient conditions for the lacunary polynomial space P_n to be L'-Haar. Thus in this case P_n is L^1-A if and only if it is L'-Haar.

Before proving Theorem 4.2, we state a lemma on lacunary polynomial spaces. We say that a subspace V_l (of dimension l) of polynomials is a *Haar space of order* r on a set $S \subseteq \mathbf{R}$ if the only polynomial $p \in V_l$ that has l zeros in S counting multiplicities up to order r is $p = 0$.

Lemma 4.3. *Let* $V_l = \operatorname{span}\{x^{m_1}, x^{m_2}, \dots, x^{m_l}\}$ *where* $0 \le m_1 < m_2 < \cdots < m_l$ *are integers.*

(i) V_l *is a Haar space on* $(-1, 1)$ *if and only if* $m_1 = 0$ *and* $m_{i+1} - m_i$ *is odd* $(1 \le i \le l - 1)$.

(ii) V_l *is a Haar space on* $(-1, 0) \cup (0, 1)$ *if and only if* $m_{i+1} - m_i$ *is odd* $(1 \le i \le l - 1)$.

(iii) *If* $m_1 = 0$, $m_2 = 1$, *and* $m_{i+1} - m_i$ *is odd* $(1 \le i \le l - 1)$, *then* V_l *is a Haar space of order* 2 *on* $[-1, 1]$.

(iv) *If* $m_1 > 0$ *and* $m_{i+1} - m_i$ *is odd* $(1 \le i \le l - 1)$, *then* V_l *is a Haar space of order* 2 *on* $[-1, 0) \cup (0, 1]$.

Statements (i), (ii), and (iii) are essentially given in [8, pp. 131–132], and (iv) readily follows from (ii).

Proof of Theorem 4.2. Let $0 = k_1 < \cdots < k_m < k \le k_{m+1} < \cdots < k_n = N$. We first prove necessity. Assume that P_n is an L^1-A-space. By Corollary 2.6 with $s = 0$, $G_\phi = P_n$ is an A-space. Since no function in $P_n\backslash\{0\}$ vanishes identically on a subinterval of $[-1, 1]$, Havinson's theorem implies that P_n is a Haar space on $(-1, 1)$. By Lemma 4.3(i), (a) holds.

Suppose that (b) fails. Then $x^k \in P_n$ and $x^{k+1} \notin P_n$. Then $\dim D^k P_n|_{\{0\}} = 1$ and $G_{\{0\}} = G'_{\{0\}} = \operatorname{span}\{x^{k_1}, \dots, x^{k_m}, x^{k_{m+2}}, \dots, x^{k_n}\}$. Since $k_{m+2} - k_m$ is even, $G_{\{0\}}$ is not a Haar space on $(-1, 1)$ and by Havinson's theorem is not an A-space. This contradicts Corollary 2.6, and thus (b) holds.

We employ Corollary 2.7 to prove sufficiency. Suppose that (a) and (b) hold, let $\{y_1, \dots, y_s\} \subseteq [-1, 1]$ where $\dim D^k P_n|_{\{y_1,\dots,y_s\}} = s$, and let $G' = G'_{\{y_1,\dots,y_s\}}$. We show that G' is a Haar space on $[-1, 1]$ and hence is an A-space. If $s = 0$, then $G' = P_n$ is a Haar space by Lemma 4.3(i). Let $r' = \#\{i\colon 1 \le i \le s,\ y_i \in (-1, 1)\}$ and $r'' = s - r'$. Note also that if $x^k \notin P_n$, then $\{y_1, \dots, y_s\} \subseteq [-1, 0) \cup (0, 1]$. Now if $2r' + r'' \ge n - m$, then by Lemma 4.3(iii) or (iv), $G' = \ker D^k|_{P_n} = \operatorname{span}\{x^{k_1}, \dots, x^{k_m}\}$ is a Haar space on $[-1, 1]$. We assume that $s \ge 1$ and $2r' + r'' < n - m$. Again by Lemma 4.3(iii) or (iv), $\dim G' = n - 2r' - r''$. Suppose $g \in G'\backslash\{0\}$ and g has $\mu = n - 2r' - r''$ zeros x_i $(1 \le i \le \mu)$. Now g satisfies the following interpolation conditions

(4.2a) $$g(x_i) = 0 \qquad (1 \le i \le \mu),$$

(4.2b) $$D^j g(0) = 0 \qquad (1 \le j \le N-1,\ j \ne k_i\ (2 \le i \le n-1)),$$

(4.2c) $$D^k g(y_i) = D^{k+1} g(y_i) = 0 \qquad (1 \le i \le r'),$$

(4.2d) $$D^k g(y_i) = 0 \qquad (r'+1 \le i \le s)$$

where we assume that $y_i \in (-1, 1)$ $(1 \le i \le r')$ and $y_i \in \{-1, 1\}$ $(r'+1 \le i \le s)$. If $x^k \notin P_n$, then $0 \notin \{y_1, \dots, y_s\}$ and if x^k, $x^{k+1} \in P_n$, then $k_{m+1} = k$ and $k_{m+2} = k+1$. As a result, conditions (4.2b) and (4.2c) do not overlap, and thus (4.2) constitutes $\mu + N + 1 - n + 2r' + r'' = N + 1$ distinct conditions. Let E be the interpolation matrix for (4.2) with columns indexed $0 - N$. By (a) and the nonoverlapping of (4.2b) and (4.2c), E has no odd supported sequences. Now E fails to satisfy the Polyá condition; otherwise, E would be order regular [8, p. 10] and we would then have $g \equiv 0$. For $0 \le j \le N$, let w_j be the number of "ones" in columns $0 - j$ of E. For $0 \le j \le k-1$, $w_j \ge \mu + j + 1 - m = n - 2r' - r'' + j + 1 - m \ge j + 1$. Also, $w_k \ge w_{k-1} + s \ge k + 1$. So for some $k+1 \le j \le N$, $w_j \le j$. But for $k+1 \le j \le N$, w_j increases by at most one from column to column. Thus $w_N \le N$, a contradiction. Thus in this last case, G' is a Haar space on $[-1, 1]$. By Corollary 2.7, P_n is an L^1-A-space.

4.3. The operator $D - \alpha I$. In this section, the approximating space is Π_m and the constraints are defined by the operator $L = D - \alpha I$ $(\alpha \ne 0)$. Throughout $[a, b] = K = [-1, 1]$. By a simple translation, all results can be obtained for arbitrary $[a, b]$. It is known that L is a 1-Rolle operator [6] so that if $m \ge 2$, then Π_m is not an L^0-A-space, (see Theorem 3.1). We determine the parameters α for which Π_m is an L^1-A-space.

Theorem 4.4. *Let* $[a, b] = K = [-1, 1]$ *and* $m \ge 1$. *Then* Π_m *is an* L^1-A-*space with* $L = D - \alpha I$ *if and only if* $|\alpha| \le m/2$.

In [6], the question of when Π_m is an L'-Haar space with $L = D - \alpha I$ was studied. The same result was obtained except when $m = 1, 2$. When $m = 1, 2$, Π_m is not L'-Haar for $\alpha = m/2$.

To prove Theorem 4.4, we shall use two lemmas on interpolation involving $L = D - \alpha I$.

Lemma 4.5. *Let* $\{y_1, \dots, y_s\} \subseteq [-1, 1]$, $v_1, \dots, v_s$ *be positive integers where* v_i *is even if* $y_i \in (-1, 1)$, *and let* $l = v_1 + \cdots + v_s$. *If* $g \in \Pi_m \backslash \{0\}$ *and*

$$(4.3) \qquad (Lg)(y_i) = (Lg)'(y_i) = \cdots = (Lg)^{(v_i - 1)}(y_i) = 0 \qquad (1 \le i \le s)$$

then g *has at most* $m + 1 - l$ *zeros in* $[-1, 1]$. *Moreover, if* g *has* $m + 1 - l$ *zeros in* $(-1, 1)$, *then these zeros are all sign changes of* g.

Proof. Suppose that g has $m+2-l$ zeros in $[-1, 1]$. In [6, Lemma 3.2], it was shown that strictly between two zeros of g, Lg changes sign or is identically zero. Since $g \ne 0$, $Lg \ne 0$ and so Lg has a sign change between successive zeros of g. If such a sign change coincides with some $y_i \in (-1, 1)$, then since v_i is even we have that $(Lg)(y_i) = \cdots = (Lg)^{(v_i)}(y_i) = 0$. So Lg has $m+1-l$ zeros in addition to those specified in (4.3). Thus Lg has $m + 1$ zeros, and since $Lg \in \Pi_m$, $Lg = 0$. This is a contradiction.

Suppose g has $m+1-l$ zeros in $(-1, 1)$. As above, Lg has m zeros. We argue that if a zero of g is not a sign change of g, then Lg picks up an extra zero. Suppose that $g(x) = 0$, $x \in (-1, 1)$, and g does not change sign at x. If $x \ne y_i$ $(1 \le i \le s)$, then $g'(x) = 0$ and $(Lg)(x) = 0$. This zero of Lg was not counted in the argument above. If $x = y_i \in (-1, 1)$, then $g(y_i) = 0$ and (4.3) imply that $g(y_i) = g'(y_i) = \cdots = g^{(v_i)}(y_i) = 0$. Since v_i is even and g does not change sign at y_i, $g^{(v_i+1)}(y_i) = 0$ and $(Lg)^{(v_i)}(y_i) = 0$. Again this extra zero of Lg was not counted in the argument above. Either way, Lg has $m + 1$ zeros, a contradiction.

As mentioned before, uniform approximation with constraints defined by $L = D - \alpha I$ was studied by the authors. The following lemma is given in [6, Lemma 3.7].

Lemma 4.6. (i) *If* $\alpha \ge -m/2$ *and* $p \in \Pi_m \backslash \{0\}$ *has* m *zeros in* $(-1, 1]$ *with at least one of them in* $(-1, 1)$, *then* $(Lp)(-1) \ne 0$.

(ii) *If* $\alpha \le m/2$ *and* $p \in \Pi_m \backslash \{0\}$ *has* m *zeros in* $[-1, 1)$ *with at least one of them in* $(-1, 1)$, *then* $(Lp)(1) \ne 0$.

(iii) *If* $|\alpha| \le m/2$ *and* $p \in \Pi_m \backslash \{0\}$ *has* $m - 1$ *zeros in* $(-1, 1)$, *then* $(Lp)(1) \ne 0$ *or* $(Lp)(-1) \ne 0$.

Proof of Theorem 4.4. For necessity, assume $|\alpha| > m/2$. Without loss of generality, $\alpha > m/2$. We can choose $-1 < x_1 < \cdots < x_m < 1$ so that for $p(x) = \prod_{i=1}^m (x - x_i)$ we have

$$\frac{p'(1)}{p(1)} = \sum_{i=1}^m \frac{1}{1 - x_i} = \alpha.$$

Then $G_{\{1\}} = G'_{\{1\}}$ has dimension m and contains a nonzero function p having m zeros in $(-1, 1)$. So $G_{\{1\}}$ is not a Haar space on $(-1, 1)$. Since nontrivial functions in $G_{\{1\}}$ have no zero intervals, Havinson's theorem implies that $G_{\{1\}}$ is not an A-space. By Corollary 2.6, Π_m is not an L^1-A-space.

For sufficiency, suppose that $|\alpha| \le m/2$. Let $g \in \Pi_m \backslash \{0\}$, $\{y_1, \dots, y_s\} \subseteq Z_1(Lg)$, and $\sigma\colon \operatorname{supp} g \to \{-1, 1\}$ be continuous. Let $r' = \#\{i : 1 \le i \le s, y_i \in (-1, 1)\}$ and $r'' = s - r'$. Letting $v_i = 2$ if $y_i \in (-1, 1)$ and $v_i = 1$ if $y_i \in \{-1, 1\}$, Lemma 4.5 implies that g has at most $m + 1 - (2r' + r'')$ zeros in $(-1, 1)$. Let σ have μ points of sign change $x_1, \dots, x_\mu$ in $(-1, 1)$. Since $\{x_1, \dots, x_\mu\} \subseteq Z(g)$, $0 \le \mu \le m + 1 - (2r' + r'')$.

If $\mu = m + 1 - (2r' + r'')$, then by Lemma 4.5, g changes sign precisely at the points x_i $(1 \le i \le \mu)$ so that $\sigma(\pm g) \ge 0$ on $\operatorname{supp} g$.

Suppose now that $\mu \le m-(2r'+r'')$. Assume that $y_i \in (-1, 1)$ for $1 \le i \le r'$ and $y_i = x_i$ for $1 \le i \le \mu_0$ $(0 \le \mu_0 \le \min(\mu, r'))$. We choose $p \in \Pi_m \backslash \{0\}$ to satisfy m linear conditions below. The first $\mu + 2r' + r''$ conditions are

(4.4a) $$p(x_i) = 0 \qquad (\mu_0 + 1 \le i \le \mu),$$

(4.4b) $$p(y_i) = p'(y_i) = 0 \qquad (\mu_0 + 1 \le i \le r'),$$

(4.4c) $$p(y_i) = p'(y_i) = p''(y_i) = 0 \qquad (1 \le i \le \mu_0),$$

(4.4d) $$(Lp)(y_i) = 0 \qquad (r' + 1 \le i \le s).$$

The remaining $\nu = m - \mu - (2r' + r'')$ conditions are as follows. If $r'' = 0$ or 1, choose $z \in \{-1, 1\} \backslash \{y_s\}$ and impose the conditions

(4.4e) $$p(z) = p'(z) = \cdots = p^{(\nu-1)}(z) = 0.$$

If $r'' = 2$, we impose

(4.4f) $$p(1) = p'(1) = \cdots = p^{(\nu)}(1) = 0.$$

In case $r'' = 2$, we drop condition (4.4d) with $y_i = 1$ as it is redundant on (4.4f).

Finally, we observe that p changes sign precisely at the point $x_1, \dots, x_\mu$. If p has a sign change in $(-1, 1) \backslash \{x_1, \dots, x_\mu\}$ or fails to change sign at some x_i, then p would have a zero in $(-1, 1)$ in addition to those specified by (4.4a, b, c). If $r'' = 0$, then p would satisfy a Hermite problem with $m + 1$ homogeneous condition so that $p = 0$. If $r'' = 1$, then p would have m zeros in $(-1, 1]$ if $y_s = -1$ or in $[-1, 1)$ if $y_s = 1$ with the additional zero in $(-1, 1)$. Lemma 4.6 and (4.4d) would yield a contradiction. Finally, if $r'' = 2$, Lemma 4.6 and (4.4d) would also lead to a contradiction. Thus p changes sign precisely at $x_1, \dots, x_\mu$. Thus $\sigma(\pm p) \ge 0$ on $\operatorname{supp} g$.

In all cases above, $(Lp)(y_i) = 0$ $(1 \le i \le s)$. Hence, by Theorem 2.2, Π_m is an L^1-A-space.

4.4. Splines and smooth boundaries. Let $a = c_0 < c_1 < \cdots < c_l < c_{l+1} = b$ and $\mathscr{S}_{m,l}$ denote the space of mth order spline functions with simple knots $\{c_i\}_{i=1}^l$. We have that $\mathscr{S}_{m,l} \subseteq C^{m-2}[a, b]$ but $\mathscr{S}_{m,l} \not\subseteq C^{m-1}[a, b]$. In §3, we saw that $\mathscr{S}_{m,l}$ is not a D^k-A-space where $k \le m - 2$. In order to consider smooth constraint boundaries we require that $D^k : \mathscr{S}_{m,l} \to C^1[a, b]$ so that $k \le m - 3$. In this section, we show that if $m \ge 4$, then $\mathscr{S}_{m,l}$ is an L^1-A-space with $L = D^{m-3}$.

Theorem 4.7. *Let $K = [a, b]$ and $m \ge 4$. Then $\mathscr{S}_{m,l}$ is an L^1-A-space with $L = D^{m-3}$.*

The main step in our proof of Theorem 4.7 involves proving that a certain subspace of the space $\mathscr{S}_{3,l}$ of quadratic splines is a WT-space. For $\{y_i\}_{i=1}^s \subseteq [a, b]$, let $\mathscr{S}_{\{y_i\}_{i=1}^s} = \{p \in \mathscr{S}_{3,l} : p(y_i) = 0$ $(1 \le i \le s,\ y_i \in \{a, b\})$ and $p(y_i) = p'(y_i) = 0$ $(1 \le i \le s,\ y_i \in (a, b))\}$.

Lemma 4.8. *For any* $\{y_i\}_{i=1}^s \subseteq [a, b]$, $\mathscr{S}_{\{y_i\}_{i=1}^s}$ *is a* WT*-space.*

The proof of Lemma 4.8 is somewhat technical, and we first give the proof of Theorem 4.7 and then prove Lemma 4.8.

Proof of Theorem 4.7. Let $\{y_1, \dots, y_s\} \subseteq [a, b]$ and $G' = G'_{\{y_i\}_{i=1}^s} = \{p \in S_{m,l} : (D^{m-3}p)(y_i) = 0 \ (1 \leq i \leq s, \ y_i \in \{a, b\})$ and $(D^{m-3}p)(y_i) = (D^{m-2}p)(y_i) = 0 \ (1 \leq i \leq s, \ y_i \in (a, b))\}$. By Corollary 2.7, it suffices to prove that G' is an A-space. Since $m \geq 4$, $1 \in G'$ so that $Z(G') \cap (a, b) = \phi$. Further, G' clearly satisfies the splitting property. Finally, we show that G' is a WT-space. Set $n = \dim G'$ and suppose that $p \in G'$ has a strong alternation of length $n+1$. Since $1, x, \dots, x^{m-4} \in G'$, $D^{m-3}G' = \mathscr{S}_{\{y_i\}_{i=1}^s}$ has dimension $n - m + 3$. But $D^{m-3}p$ has a strong alternation of length $n + 1 - (m-3) = n - m + 4$ which contradicts Lemma 4.8. Thus G' is a WT-space, and by Theorem 2.3, G' is an A-space. Corollary 2.7 now implies that $\mathscr{S}_{m,l}$ is an L^1-A-space.

To prove Lemma 4.8, we require an additional lemma.

Lemma 4.9. *Let* U *be an n-dimensional* WT*-space in* $C^1[a, b]$. *Then*

(i) $U_a^0 := \{p \in U : p(a) = 0\}$ *and* $U_a^{00} = \{p \in U : p(a) = p'(a) = 0\}$ *are* WT*-spaces, and*

(ii) *if* $y \in (a, b)$ *and for some* $q \in U$, $q(y) = 0$ *and* $q'(y) \neq 0$, *then* $U_y^{00} := \{p \in U : p(y) = p'(y) = 0\}$ *is a* WT*-space.*

Proof. If $U_a^0 = U$, then U_a^0 is a WT-space. Suppose $U_a^0 \neq U$ so that $\dim U_a^0 = n - 1$. Suppose $p \in U_a^0$ has a strong alternation $x_1 < \cdots < x_n$ of length n. Since $p \in U_a^0$, $x_1 > a$ and we may assume that $p(x_1) > 0$. Now choose $\tilde{p} \in U$ so that $\tilde{p}(a) = 1$. Then for $\varepsilon > 0$ sufficiently small, $p - \varepsilon\tilde{p}$ has a strong alternation $a = x_0 < x_1 < \cdots < x_n$ of length $n+1$ contradicting the fact that U is a WT-space. Thus U_a^0 is a WT-space.

If $U_a^{00} = U_a^0$, then by the previous case, U_a^{00} is a WT-space. Suppose that $U_a^{00} \neq U_a^0$. Let $m = \dim U_a^0$ so that $\dim U_a^{00} = m - 1$, and suppose that $p \in U_a^{00}$ has a strong alternation $x_1 < \cdots < x_m$ of length m. Then $x_1 > a$ and we may assume that $p(x_1) > 0$. Now choose $\tilde{p} \in U_a^0$ so that $\tilde{p}'(a) = 1$. Then it is easy to see that for $\varepsilon > 0$ sufficiently small, $p - \varepsilon\tilde{p}$ has a strong alternation $a < x_0 < x_1 < \cdots < x_m$ of length $m+1$ which contradicts the fact that U_a^0 is a WT-space. Thus U_a^{00} is a WT-space.

For (ii), $\dim U_y^{00} = m \geq n - 2$. Given $p \in U_y^{00}$ having strong alternation of length $m+1$, the approach in the previous case yields a function in U having strong alternation of length $m + 3 \geq n + 1$, a contradiction.

Proof of Lemma 4.8. In view of Lemma 4.9(i), it suffices to only consider the case where $\{y_1, \dots, y_s\} \subseteq (a, b)$. The proof is by induction on s. When $s = 0$, $\mathscr{S}_{\{y_1, \dots, y_s\}} = \mathscr{S}_{3,l}$ is well known to be a WT-space.

Suppose that $\mathscr{S}_{\{y_i\}_{i=1}^s}$ is a WT-space whenever $\{y_i\}_{i=1}^s \subseteq (a, b)$ and $s \leq k-1$. Let $a < y_1 < \cdots < y_k < b$. Choose the index ν so that $c_\nu < y_k \leq c_{\nu+1}$. We consider two cases.

Case 1. Suppose $p \equiv 0$ on $[c_\nu, c_{\nu+1}]$ for all $p \in \mathscr{S}_{\{y_i\}_{i=1}^s}$. Then $\mathscr{S}_{\{y_i\}_{i=1}^s} = \mathscr{S}^1 \oplus \mathscr{S}^2$ where $\mathscr{S}^1 = \{p \in \mathscr{S}_{3,l} : p(y_i) = p'(y_i) = 0 \ (1 \leq i \leq k-1)$ and

$p \equiv 0$ on $[c_\nu, b]\}$ and $\mathscr{S}^2 = \{p \in \mathscr{S}_{3,l} : p \equiv 0$ on $[a, c_{\nu+1}]\}$. By the induction hypothesis and Lemma 4.9(i), $\mathscr{S}^1|_{[a,c_\nu]}$ is a WT-space and thus $\mathscr{S}^1$ is a WT-space. Similarly, $\mathscr{S}^2$ is a WT-space and the direct sum $\mathscr{S}_{\{y_i\}_{i=1}^s} = \mathscr{S}^1 \oplus \mathscr{S}^2$ is a WT-space.

Case 2. Suppose that for some $p \in \mathscr{S}_{\{y_i\}_{i=1}^k}$, $p \not\equiv 0$ on $[c_\nu, c_{\nu+1}]$. Let us first suppose that $\nu \geq 1$. Since $y_k \in [c_\nu, c_{\nu+1}]$, it follows that $p \neq 0$ on $[c_\nu, c_{\nu+1}]\backslash\{y_k\}$ and $\operatorname{sgn} p(c_\nu) = -\operatorname{sgn} p'(c_\nu)$. Since $p \neq 0$ on $[c_\nu, y_k)$, $y_{k-1} \notin [c_\nu, y_k)$. Also, if $y_{k-1} \in [c_{\nu-1}, c_\nu)$, it would follow that $\operatorname{sgn} p(c_\nu) = \operatorname{sgn} p'(c_\nu)$ which is false. Thus $\{y_i\}_{i=1}^{k-1} \cap [c_{\nu-1}, y_k) = \phi$. Now consider the function

$$q(x) = (x - c_\nu)_+^2 - \frac{(y_k - c_\nu)^2}{(y_k - c_{\nu-1})^2}(x - c_{\nu-1})_+^2 .$$

We have that $q \in \mathscr{S}_{\{y_i\}_{i=1}^{k-1}}$, $q(y_k) = 0$, and $q'(y_k) \neq 0$. By the induction hypothesis $\mathscr{S}_{\{y_i\}_{i=1}^{k-1}}$ is a WT-space, and by Lemma 4.9(ii), $\mathscr{S}_{\{y_i\}_{i=1}^k}$ is a WT-space. Finally, if $\nu = 0$, then as above $k = 1$ and $q(x) = x - y_k$ suffices in the use of Lemma 4.9(ii).

Finally, the authors conjecture that Theorem 4.7 holds for $L = D^k$ with any $1 \leq k \leq m - 3$ as well, i.e., $\mathscr{S}_{m,l}$ is an L^1-A-space in this case.

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