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ON THE p-ADIC COMPLETIONS OF NONNILPOTENT SPACES

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ABSTRACT. This paper deals with the p-adic completion $F_{p\infty}X$ developed by Bousfield-Kan for a space X and prime p. A space X is called F_p-good when the map $X \to F_{p\infty}X$ is a mod-p homology equivalence, and called F_p-bad otherwise. General examples of F_p-good spaces are established beyond the usual nilpotent or virtually nilpotent ones. These include the polycyclic-by-finite spaces. However, the wedge of a circle with a sphere of positive dimension is shown to be F_p-bad. This provides the first example of an F_p-bad space of finite type and implies that the p-profinite completion of a free group on two generators must have nontrivial higher mod-p homology as a discrete group. A major part of the paper is devoted to showing that the desirable properties of nilpotent spaces under the p-adic completion can be extended to the wider class of p-seminilpotent spaces.

1. INTRODUCTION

In this paper we shall study the *F_p-completion* (or *p-adic completion*) $F_{p\infty}X$ developed by Bousfield-Kan [6] for a space X and prime p. This completion is well understood when X is nilpotent, and we shall consider other more general spaces. Bousfield-Kan were originally interested in $F_{p\infty}X$ because $\pi_*F_{p\infty}X$ served as the natural target of the unstable Adams spectral sequence for X. At about the same time, in work on the Adams conjecture, Sullivan [26] developed his p-profinite completion of X, which agrees up to homotopy with $F_{p\infty}X$ when $H_*(X; F_p)$ is of finite type. Friedlander and others subsequently used $F_{p\infty}X$ in algebraic K-theory and étale homotopy theory [13]. More recently, Miller revived interest in $F_{p\infty}X$ by using it in his proof of the Sullivan conjecture [18]. Now, with the confirmation of the generalized Sullivan conjecture, $F_{p\infty}X$ has acquired new significance (see [12, 17]).

A space X is called *F_p-good* when the map $X \to F_{p\infty}X$ is a mod-p homology equivalence, and called *F_p-bad* otherwise. Also, X is called *F_p-complete* when $X \to F_{p\infty}X$ is a weak equivalence. By [6, p. 24], a space X is F_p-good if and only if $F_{p\infty}X$ is F_p-complete, and consequently the functor $F_{p\infty}$ acts idempotently on the homotopy category of F_p-good spaces. Known examples of F_p-good spaces include:

(i) the nilpotent spaces [6, p. 184];

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(ii) the spaces with finite fundamental groups [6, p. 215] and other virtually nilpotent spaces [11]; and
(iii) the spaces with p-perfect fundamental groups [6, p. 206]. (A group G is *p-perfect* when $H_1(G; F_p) = 0$.)

Bousfield-Kan observed that an infinite wedge of circles was F_p-bad [6, p. 114], but continued to hope that all spaces of finite type might be F_p-good. In this paper, we deflate that hope by showing that $S^n \vee S^1$ is F_p-bad for $n \geq 1$. However, we also establish the F_p-goodness of a wide class of finite type complexes, including the *polycyclic-by-finite* spaces. These are the connected spaces whose fundamental groups have polycyclic normal subgroups of finite index and whose higher homotopy groups are finitely generated. To deal with such spaces and others, we devote the first part of the paper to showing that the desirable properties of nilpotent spaces under the p-adic completion actually hold for more general spaces which we call *p-seminilpotent.*

This paper is organized as follows. In §2 we introduce the p-seminilpotent group actions. In §3 we develop the consequent notions of p-seminilpotent groups, spaces, and fibrations. In §4 we discuss the functor $F_{p\infty}$, and show that it acts idempotently on the homotopy category of p-seminilpotent spaces. In §5 we discuss the group theoretic p-adic completion functor $(\ )^\wedge_p$ and introduce its derived functors. In §6 we show that the groups $\pi_i F_{p\infty} X$ for p-seminilpotent spaces are usually given by $(\pi_i X)^\wedge_p$, and can always be expressed using derived p-adic completions. In §7 we prove an F_p-goodness theorem which applies to the polycyclic-by-finite spaces and other "virtually p-seminilpotent" spaces. Here, we use techniques developed by Dror-Dwyer-Kan [11] in their work on virtually nilpotent spaces. In §8 we prepare for the next section by introducing the *p-adic G-completion* for modules over a group G, and showing that it is particularly well behaved when G is p-seminilpotent polycyclic. This follows from Roseblade's "Artin-Rees lemma" [24]. In §9 we establish a partial F_p-goodness theorem which applies to many spaces of finite type, like $S^n \vee S^1$, with nonfinitely generated homotopy groups. When such a space X is p-seminilpotent polycyclic below some dimension $n \geq 2$, we show that $H_i(F_{p\infty}X; F_p) \cong H_i(X; F_p)$ for $i \leq 2n-1$ and that $\pi_i F_{p\infty} X$ is the p-adic $\pi_1 X$-completion of $\pi_i X$ for $i \leq 2n-2$. This is closely related to a result of Dror-Dwyer [10] giving a stable range for integral homology localizations, although we have had to use different techniques to reach our top dimension and cope with mod-p problems. In §10 we show that $S^n \vee S^1$ is F_p-bad with $H_{2n}(F_{p\infty}(S^1 \vee S^1); F_p)$ uncountable for $n \geq 2$. Finally, in §11, we deduce that $S^1 \vee S^1$ is F_p-bad with $H_m(F_{p\infty}(S^1 \vee S^1); F_p)$ uncountable for $m = 2$ or $m = 3$ or possibly both. This implies that the p-profinite completion of a free group on two generators must have uncountable higher mod-p homology as a discrete group.

While the known examples of F_p-good spaces X still seem very diverse, we remark that their completions $F_{p\infty}X$ are all p-seminilpotent. Thus, all known examples of F_p-complete spaces are p-seminilpotent.

In [3] the author circumvented the problem of bad spaces by constructing localizations of arbitrary spaces with respect to homology theories. For a space X this gives an $H_*(\ ; F_p)$-localization map $X \to X_{F_p}$ which is the homotopically terminal example of an $H_*(\ ; F_p)$-equivalence out of X. From the standpoint of [3], $F_{p\infty}X$ is always $H_*(\ ; F_p)$-local, and X is F_p-good if and only if the canonical map $X_{F_p} \longrightarrow F_{p\infty}X$ is an equivalence. In general, although $F_{p\infty}X$ is of independent interest, it may be viewed as an initial stage of a transfinite towerwise construction of X_{F_p} (see [9]).

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We work simplicially and generally follow the terminology of [6], so that "space" will mean "simplicial set." Throughout this paper, p will denote a fixed prime.

2. BASIC PROPERTIES OF p-SEMINILPOTENT GROUP ACTIONS

Before introducing p-seminilpotent groups, spaces, and fibrations in §3, we must deal with group actions.

2.1. **The p-seminilpotent ZG-modules.** For a group G, recall that a ZG-module M is *nilpotent* when $(IG)^n M = 0$ for some $n \geq 0$ where $IG \subset ZG$ is the augmentation ideal. This is equivalent to saying that M has a finite filtration by ZG-submodules with trivial G-action on the associated quotients. A ZG-module M will be called *p-seminilpotent* when the ZG-modules $M \otimes Z/p$ and $\operatorname{Tor}(M, Z/p)$ are both nilpotent. A nilpotent ZG-module is clearly p-seminilpotent for all p. Moreover, if G is a finite p-group, then each ZG-module is p-seminilpotent.

2.2. **The p-seminilpotent group actions.** Now let A be a *G-group*, i.e. a group with a homomorphism $G \longrightarrow \operatorname{Aut} A$. A ZG-series for A is a finite filtration

$$A = A_1 \supset A_2 \supset \cdots \supset A_n = \{1\}$$

of A by G-invariant subgroups such that each A_{i+1} is normal in A_i with A_i/A_{i+1} abelian, and thus with A_i/A_{i+1} a ZG-module. The action of G on A will be called *p-seminilpotent* when there exists a ZG-series $\{A_i\}$ for A such that the ZG-modules A_i/A_{i+1} are all p-seminilpotent. Such a filtration $\{A_i\}$ will be called a *p-seminilpotent* ZG-series for A. Note that the action of a group G on an abelian group A is p-seminilpotent if and only if A is p-seminilpotent as a ZG-module.

Proposition 2.3. *Let $f\colon A \to B$ be a homomorphism of G-groups. If the G-action on A and B is p-seminilpotent, then so is the G-action on* $\operatorname{im} f$ *and* $\ker f$, *and also on* $\operatorname{coker} f$ *when* $\operatorname{im} f$ *is normal in B.*

Proof. This follows easily when A and B are abelian. In general, let $\{A_i\}$ and $\{B_j\}$ be p-seminilpotent ZG-series for A and B. Then an induction, using Lemma 2.4 below, shows that the ZG-modules $(A_i \cap \ker f)A_{i+1}/A_{i+1}$ and $B_j/(\operatorname{im} f \cap B_j)B_{j+1}$ are p-seminilpotent. Thus $\{A_i \cap \ker f\}$ is a p-seminilpotent ZG-series for $\ker f$, and so is $\{(\operatorname{im} f)B_j\}$ for $\operatorname{coker} f$ when $\operatorname{im} f$ is normal in B. Likewise, $\{f A_i\}$ is a p-seminilpotent ZG-series for $\operatorname{im} f$.

Lemma 2.4. *If A and B are G-groups with ZG-series $\{A_i\}$ and $\{B_j\}$, then a homomorphism $f\colon A \to B$ induces an exact sequence*

$$\begin{aligned} 0 \to \frac{(A_i \cap f^{-1}B_{j+1})A_{i+1}}{A_{i+1}} &\xrightarrow{u} \frac{(A_i \cap f^{-1}B_j)A_{i+1}}{A_{i+1}} \\ \xrightarrow{f} \frac{B_j}{(fA_{i+1} \cap B_j)B_{j+1}} &\xrightarrow{v} \frac{B_j}{(fA_i \cap B_j)B_{j+1}} \to 0 \end{aligned}$$

of ZG-modules for each i and j.

Proof. This follows since f induces an isomorphism

$$\frac{A_i \cap f^{-1}B_j}{(A_i \cap f^{-1}B_{j+1})(A_{i+1} \cap f^{-1}B_j)} \cong \frac{fA_i \cap B_j}{(fA_i \cap B_{j+1})(fA_{i+1} \cap B_j)}$$

from $\operatorname{coker} u$ to $\ker v$.

Proposition 2.5. *Let $A \rightarrowtail B \twoheadrightarrow C$ be a short exact sequence of G-groups. If the G-action on A and C is p-seminilpotent, then so is the G-action on B.*

This is immediate, but its converse is false. For example, $G = Z/2$ acts p-seminilpotently on Q by negation, but not on Z or Q/Z when p is odd. This difficulty is often avoided by

Lemma 2.6. *Let $A \subset B$ be ZG-modules such that the increasing sequence $\{x \in B | p^i x \in A\}$ for $i \geq 0$ attains a maximum $\overline{A}$. If B is p-seminilpotent, then so are A and B/A.*

Proof. Since the exactness of $\overline{A} \rightarrowtail B \twoheadrightarrow B/\overline{A}$ is preserved by $Z/p \otimes -$ and $\operatorname{Tor}(Z/p, -)$, $\overline{A}$ is p-seminilpotent. A downward induction now shows that each $\{x \in B | p^i x \in A\}$ is p-seminilpotent.

Recall that a group B is called *polycyclic* when there exists a finite filtration

$$B = B_1 \supset B_2 \supset \cdots \supset B_n = \{1\}$$

of B by subgroups such that each B_{i+1} is normal in B_i with B_i/B_{i+1} cyclic. The polycyclic groups are closed under the formation of subgroups, quotient groups, and extension groups.

Proposition 2.7. *Let $A \subset B$ be G-groups with B polycyclic (as a group). If the G-action on B is p-seminilpotent, then so is the G-action on A, and also on B/A when A is normal in B.*

Proof. Using a p-seminilpotent ZG-series $\{B_i\}$ for B, we obtain a p-seminilpotent G-series $\{A \cap B_i\}$ for A by 2.6.

In our applications, the action of a group G on a group A will usually *include the inner automorphisms of* A, i.e. for each $a \in A$ there will exist $g \in G$ such that $axa^{-1} = gx$ for all $x \in A$. This ensures that the G-invariant subgroups of A are normal.

Proposition 2.8. *Suppose that the action of a group G on a group A is p-seminilpotent and includes the inner automorphisms of A. Then*

(i) *G acts nilpotently on each $H_n(A; F_p)$;*
(ii) *the derived series $\{D^j A\}$ of A is a p-seminilpotent ZG-series for A.*

Proof. We choose a p-seminilpotent ZG-series $\{A_i\}$ for A and note that G acts nilpotently on each $H_n(A_i/A_{i+1}; F_p)$. Using the Serre spectral sequence, we inductively deduce that G acts nilpotently on each $H_n(A/A_i, F_p)$ and thus on each $H_n(A; F_p)$. Hence G acts p-seminilpotently on $H_1(A; Z) \cong A/D^1A$ and also on D^1A by 2.3. This argument may be repeated inductively to show that G acts p-seminilpotently on each $D^jA/D^{j+1}A$ as required.

3. THE p-SEMINILPOTENT GROUPS, SPACES AND FIBRATIONS

The familiar concept of nilpotency for groups, spaces, and fibrations (see [6 or 15]) will now be generalized to p-seminilpotency.

3.1. **The p-seminilpotent groups.** A group G is called *p-seminilpotent* when the inner automorphism action of G on itself is p-seminilpotent. This is equivalent to saying that G has a finite filtration

$$G = G_1 \supset G_2 \supset \cdots \supset G_n = \{1\}$$

by normal subgroups with abelian quotients G_i/G_{i+1} which are p-seminilpotent as ZG-modules. Such a filtration is called a *p-seminilpotent series* for G. Clearly each nilpotent group is p-seminilpotent, and each p-seminilpotent group is solvable. Moreover, by 2.8, a solvable group G is p-seminilpotent if and only if G acts p-seminilpotently on each derived series quotient $D^jG/D^{j+1}G$.

The following four propositions may be deduced from 2.3, 2.5, and 2.7.

Proposition 3.2. *If $f: G \longrightarrow H$ is a homomorphism of p-seminilpotent groups, then* $\operatorname{im} f$ *and* $\ker f$ *are also p-seminilpotent.*

Proposition 3.3. *For a short exact sequence $G' \rightarrowtail G \twoheadrightarrow G''$ of groups, any two of the following conditions imply the third:*

(i) *G is p-seminilpotent;*
(ii) *G'' is p-seminilpotent;*
(iii) *the action of G on G' is p-seminilpotent.*

Proposition 3.4. *If $\{G_i\}$ is a p-seminilpotent series for a group G, then the quotient groups G_i/G_{i+j} are all p-seminilpotent.*

In general, a subgroup or quotient group of a p-seminilpotent group need not be p-seminilpotent. For example, the semidirect product $Q(Z) \rtimes Z$ is p-seminilpotent while $Z(Z) \rtimes Z$ and $(Q(Z) \rtimes Z)/Z(Z)$ are not. However

Proposition 3.5. *If G is a p-seminilpotent polycyclic group, then so is each subgroup and each quotient group of G.*

The p-seminilpotent (or "p-nilpotent") polycyclic groups have previously arisen in work of Roseblade [24] on the Artin-Rees property for group rings (see 8.4). In his exposition, Passman [19, p. 497] defines them by the conditions of

Proposition 3.6. *A polycyclic group is p-seminilpotent if and only if each of its finite quotient groups is p-seminilpotent. A polycyclic finite group is p-seminilpotent if and only if it has a normal p-complement (i.e. its elements of order prime to p form a subgroup).*

Proof. Let G be a polycyclic group which is not p-seminilpotent, and choose a finite filtration $\{G_i\}$ of normal subgroups with abelian quotients G_i/G_{i+1}.

Then G acts nonnilpotently on $G_i/G_{i+1}\otimes Z/p$ or $\mathrm{Tor}(G_i/G_{i+1}, Z/p)$ for some i. Since each polycyclic group is residually finite, there is a finite quotient group $\overline{G}$ of G with a corresponding nonilpotent action, and $\overline{G}$ is not p-seminilpotent. Next let B be a finite p-seminilpotent group. The intersection of the mod-p derived series for B gives a p-perfect normal subgroup $N \subset B$ with index a power of p. This N is p-seminilpotent since B is, and $|N|$ is prime to p by

Lemma 3.7. *If N is a p-perfect p-seminilpotent finite group, then $|N|$ is prime to p.*

Proof. Let $\{N_i\}$ be a p-seminilpotent series for N, and assume inductively that $|N/N_i|$ is prime to p. Using the $H_*(\ ; F_p)$-spectral sequence for

$$N_i/N_{i+1} \rightarrowtail N/N_{i+1} \twoheadrightarrow N/N_i$$

and the nilpotent action of N/N_i on $Z/p \otimes N_i/N_{i+1}$, one finds that $Z/p \otimes N_i/N_{i+1} = 0$. Thus $|N/N_{i+1}|$ is prime to p.

We shall need the following technical result later.

Proposition 3.8. *Let $f\colon G' \rightarrowtail G$ be a monomorphism and $g\colon G \twoheadrightarrow G''$ be an epimorphism of p-seminilpotent groups. If $\{G_i\}$ is a p-seminilpotent series for G, then $\{f^{-1}(G_i)\}$ and $\{g(G_i)\}$ are p-seminilpotent series for G' and G''.*

Proof. This follows using 3.3 since the image of $f\colon G' \to G/G_i$ is p-seminilpotent by 3.2, and since G acts p-seminilpotently on the image of $g\colon G_i \to G''$ by 2.3.

3.9. **The p-seminilpotent spaces and fibrations.** A space X is called *p-seminilpotent* when it is connected, $\pi_1 X$ is a p-seminilpotent group, and the action of $\pi_1 X$ on $\pi_i X$ is p-seminilpotent for each $i \geq 2$ (after a basepoint is chosen for X). For instance, a Klein bottle or an even dimensional real projective space is p-seminilpotent for $p = 2$ but not for p odd. More generally, a fibration $f\colon X \to Y$ of connected spaces is called *p-seminilpotent* when its fiber E is connected and the action of $\pi_1 X$ on $\pi_i E$ is p-seminilpotent for each $i \geq 1$ (after a basepoint is chosen for X). Thus a space X is p-seminilpotent if and only if the fibration $X \longrightarrow *$ is p-seminilpotent. Each nilpotent space or fibration is automatically p-seminilpotent for all p. Clearly

Proposition 3.10. *For a p-seminilpotent fibration $X \to Y$ and map $A \to Y$ of connected spaces, the induced fibration over A is p-seminilpotent.*

From 2.3 and 2.5, we deduce

Proposition 3.11. *Let $f\colon X_2 \to X_1$ and $g\colon X_1 \to X_0$ be fibrations of connected spaces with connected fibers. If any two of f, g and gf are p-seminilpotent, then so is the third.*

Thus, for a fibration $f\colon X \to Y$ of connected spaces with connected fiber, if any two of X, Y, and f are p-seminilpotent, then so is the third.

Finally, we establish a crucial homological interpretation of p-seminilpotency.

Theorem 3.12. *Let $f\colon X \to Y$ be a fibration of pointed connected spaces with fiber E. Then f is p-seminilpotent if and only if E is p-seminilpotent and $\pi_1 Y$ acts nilpotently on each $H_n(E; F_p)$.*

Proof. The "only if" part follows as in [6, p. 63], and we assume the "if" hypotheses. Then $\pi_1 X$ acts p-seminilpotently on $H_1(E; Z) \cong \pi_1 E/D^1\pi_1 E$.

To show that it does so on $D^1\pi_1E/D^2\pi_1E$, we use the generalized Moore-Postnikov construction [5, 5.1] to factor f as a composition $f = hg$ of a fibration $g\colon X \to \overline{X}$ with connected fiber $\widetilde{E}$ and a fibration $h\colon \overline{X} \to Y$ with connected fiber $\overline{E}$ such that $\pi_i\overline{E} = 0$ for $i \geq 2$ and $\pi_1E \to \pi_1\overline{E}$ is onto with kernel $D^1\pi_1E$. The fibrations $\overline{X} \to Y$ and $E \to \overline{E}$ are p-seminilpotent. Thus π_1Y and $\pi_1\overline{E}$ respectively act nilpotently on each $H_n(\overline{E}; F_p)$ and $H_n(\widetilde{E}; F_p)$ by the "only if" part. Hence, $\pi_1\overline{X}$ acts nilpotently on each $H_n(\widetilde{E}; F_p)$ by 3.13 below, and π_1X acts p-seminilpotently on $H_1(\widetilde{E}; Z) \cong D^1\pi_1E/D^2\pi_1E$. This argument may be repeated inductively to prove the theorem.

Lemma 3.13. *Let $g\colon X_2 \to X_1$ and $h\colon X_1 \to X_0$ be fibrations of pointed connected spaces with connected fibers E_2 and E_1 respectively, and let E be the fiber of hg. If π_1E_1 acts nilpotently on each $H_n(E_2; F_p)$, and if π_1X_0 acts nilpotently on each $H_n(E_1; F_p)$ and $H_n(E; F_p)$, then π_1X_1 acts nilpotently on each $H_n(E_2; F_p)$.*

Proof. Suppose inductively that π_1X_1 acts nilpotently on $H_t(E_2, F_p)$ for each $t \leq n-1$. Then π_1X_2 acts nilpotently on $H_s(E_1; H_t(E_2; F_p))$ for $s \geq 0$ and $t \leq n-1$, and on $H_n(E; F_p)$. Thus by Serre spectral sequence argument, π_1X_2 acts nilpotently on $H_0(E_1; H_n(E_2; F_p))$, which equals $H_n(E_2; F_p)/IH_n(E_2; F_p)$ where $I \subset F_p\pi_1E_1$ is the augmentation ideal. Also, π_1X_2 acts nilpotently through π_1X_0 on $I/I^2 \cong H_1(E_1; F_p)$. Thus by the epimorphism

$$\frac{I}{I^2} \otimes \frac{H_n(E_2, F_p)}{IH_n(E_2; F_p)} \to \frac{IH_n(E_2; F_p)}{I^2H_n(E_2; F_p)},$$

π_1X_2 acts nilpotently on $IH_n(E_2; F_p)/I^2H_n(E_2; F_p)$. This argument may be repeated inductively to show that π_1X_2 acts nilpotently on the successive quotients of $\{I^jH_n(E_2; F_p)\}$, and thus on $H_n(E_2; F_p)$.

4. ON THE F_p-COMPLETION OF A SPACE

We now return to the Bousfield-Kan F_p-completion (or p-adic completion) $F_{p\infty}X$ of a space X, and show that it is particularly well behaved when X is p-seminilpotent. We begin with a "Whitehead theorem" which unifies results of [6, pp. 30 and 113].

Proposition 4.1. *For $k \geq 0$, let $f\colon X \to Y$ be a map of pointed connected spaces such that $f_*\colon H_i(X; F_p) \to H_i(Y; F_p)$ is an isomorphism for $i \leq k$ and onto for $i = k+1$. Then $f_*\colon \pi_iF_{p\infty}X \to \pi_iF_{p\infty}Y$ is also an isomorphism for $i \leq k$ and onto for $i = k+1$.*

Proof. Recall that $F_{p\infty}X$ is the inverse limit of a canonical tower $\{(F_p)_sX\}$ of fibrations under X, and let J_sX be the fiber of $(F_p)_sX \to (F_p)_{s-1}X$. For each s, $f_*\colon \pi_iJ_sX \to \pi_iJ_sY$ is an isomorphism for $i \leq k$ and onto for $i = k+1$ by [6, p. 32]. Letting K_s denote the homotopy fiber of $(F_p)_sX \to (F_p)_sY$, we note that the maps $K_s \to K_{s-1}$ and $J_sX \to J_sY$ have equivalent k-connected homotopy fibers. Hence the homotopy limit K_∞ of the tower $\{K_s\}$ is also k-connected. Since K_∞ is the homotopy fiber of $F_{p\infty}X \to F_{p\infty}Y$, the result follows.

Although $F_{p\infty}$ carries each "mod-p homology type" to a single homotopy type, the resulting homotopy type may sometimes have extraneous mod-p homology. Recall that a space X is called *F_p-good* when the map $X \to F_{p\infty}X$ induces an isomorphism $H_*(X; F_p) \cong H_*(F_{p\infty}X; F_p)$ and is called *F_p-bad* otherwise. By [6, p. 26], the functor $F_{p\infty}$ has a triple (or monad) structure, $\iota\colon \mathrm{Id} \to F_{p\infty}$ and $\mu\colon F_{p\infty}F_{p\infty} \to F_{p\infty}$, on the category of spaces and hence on the pointed homotopy category of spaces. Moreover, the functor $F_{p\infty}$ acts as an idempotent functor on the pointed homotopy category of F_p-good spaces. The following result may be used to build examples of F_p-good spaces.

Proposition 4.2. *Let $E \to X \to B$ be a homotopy fiber sequence of pointed connected spaces such that the action of $\pi_1 B$ on each $H_i(E; F_p)$ is nilpotent. Then*

(i) *$F_{p\infty}E \to F_{p\infty}X \to F_{p\infty}B$ is a homotopy fiber sequence;*
(ii) *if E is F_p-good, then the action of $\pi_1 F_{p\infty}B$ on $H_i(F_{p\infty}E; F_p)$ is nilpotent;*
(iii) *if E and B are F_p-good, then so is X.*

Proof. Part (i) follows from [6, p. 62], part (ii) from [6, p. 91], and part (iii) from the preceding parts using the Serre spectral sequence.

Theorem 4.3. *If X is a p-seminilpotent space, then X is F_p-good and $F_{p\infty}X$ is p-seminilpotent.*

Proof. The Postnikov tower of X can be refined to a tower of p-seminilpotent fibrations with abelian Eilenberg-Mac Lane spaces as fibers. By [6, pp. 183–184], these fibers are F_p-good and their p-adic completions are nilpotent. The result now follows by 3.12, 4.1, and 4.2.

Theorem 4.4. *If $f\colon X \to Y$ is a p-seminilpotent fibration of pointed connected spaces with homotopy fiber E, then $F_{p\infty}f\colon F_{p\infty}X \to F_{p\infty}Y$ is a p-seminilpotent fibration with homotopy fiber $F_{p\infty}E$.*

Proof. This follows by 3.12, 4.2, and 4.3.

Note that f is automatically p-seminilpotent by 3.11 when X and Y are. We conclude that $F_{p\infty}$ acts as an idempotent functor on the pointed homotopy category of p-seminilpotent spaces, and $F_{p\infty}$ preserves homotopy fiber sequences of such spaces. Finally, we must sometimes view $F_{p\infty}X$ from the "homological localization" standpoint [3].

4.5. **On $F_{p\infty}X$ as an $H_*(\ ; F_p)$-local space.** The results of [6, p. 205] show that $F_{p\infty}X$ is always $H_*(\ ; F_p)$-local in the sense of [3]. Moreover, for a pointed space X with $H_*(\ ; F_p)$-localization X_{F_p}, there is a canonical map $X_{F_p} \to F_{p\infty}X$ in the pointed homotopy category. This gives an equivalence $X_{F_p} \simeq F_{p\infty}X$ if and only if X is F_p-good, since each $H_*(\ ; F_p)$-equivalence of $H_*(\ ; F_p)$-local spaces is a weak equivalence.

4.6. **On $\pi_i F_{p\infty}X$ as an HF_p-local group.** Since $F_{p\infty}X$ is $H_*(\ ; F_p)$-local for a pointed space X, $\pi_i F_{p\infty}X$ is an *HF_p-local group* for $i \geq 1$ by [3, p. 138]. We refer the reader to [3 or 4] for an account of HF_p-local groups. By [4, p. 13] they form the smallest class of groups which:

(i) contain the trivial group,
(ii) are closed under central F_p-module extensions, and
(iii) are closed under arbitrary inverse limits.

Moreover, an abelian group G is HF_p-local if and only if G is p-cotorsion (or Ext-p-complete in the sense of [6]), i.e.,

$$\operatorname{Hom}(Z[1/p], G) \cong 0 \cong \operatorname{Ext}(Z[1/p], G).$$

5. THE p-ADIC GROUP COMPLETION AND ITS DERIVED FUNCTORS

In preparation for further work on the homotopy groups $\pi_i F_{p\infty} X$, we now study the p-adic completion functor and its derived functors on groups. The examples in 5.7 show the interesting diversity of these derived functors.

5.1. **The p-adic completion of a group.** For a group A, let $\{\Gamma_n^p A\}$ be the lower p-central series with $\Gamma_1^p A = A$ and with $\Gamma_{n+1}^p A$ generated by elements of the form $xyx^{-1}y^{-1}z^p$ for $x \in A$ and $y, z \in \Gamma_n^p A$. This is the fastest descending central series in A with F_p-module factors. The *p-adic completion* of A is defined as in [6, p. 103] to be $A_p^\wedge = \varprojlim A/\Gamma_n^p A$ viewed as a discrete group. Note that the tower $\{A/\Gamma_n^p A\}$ is characterized up to pro-isomorphism by its property of cofinality in the system of all p-torsion nilpotent groups of finite exponent under A. Thus, in the construction of $A_p^\wedge$, we can use alternative versions of the lower p-central series as in [23]. Note also that if $H_1(A; F_p)$ is finite, then the p-adic completion of A agrees with the p-profinite completion. The close relationship between p-adic completions of groups and of spaces is indicated by the equivalence $\overline{W}(GX)_p^\wedge \simeq F_{p\infty}X$ shown in [6, p. 109] for a pointed connected space X, where $\overline{W}$ is the simplicial classifying space functor and GX is Kan's free simplicial loop group of X. We now give some basic properties of $(\ )_p^\wedge$.

Lemma 5.2. *For a homomorphism $f: A \to B$ of groups, $f_p^\wedge: A_p^\wedge \to B_p^\wedge$ is onto if and only if $f_*: H_1(A; F_p) \to H_1(B; F_p)$ is onto. If $f_*: H_i(A; F_p) \to H_i(B; F_p)$ is an isomorphism for $i = 1$ and onto for $i = 2$, then $f_p^\wedge: A_p^\wedge \cong B_p^\wedge$ and $f: A/\Gamma_n^p A \cong B/\Gamma_n^p B$ for all $n \geq 1$.*

Proof. First, let $f_*: H_1(A; F_p) \to H_1(B; F_p)$ be onto. Then

$$f_*: \Gamma_n^p A/\Gamma_{n+1}^p A \to \Gamma_n^p B/\Gamma_{n+1}^p B$$

is onto for all $n \geq 1$ by [4, p. 51], and we obtain a short exact sequence of group towers

$$\{K_n\} \rightarrowtail \{A/\Gamma_n^p A\} \twoheadrightarrow \{B/\Gamma_n^p B\}$$

with each $K_{n+1} \to K_n$ onto. Hence $f_p^\wedge: A_p^\wedge \to B_p^\wedge$ is onto. The converse is clear since $H_1(B; F_p)$ is a quotient of $B_p^\wedge$, and the final statement follows by Stalling's argument [25].

5.3. **Idempotency properties.** The p-adic completion of groups has a triple (or monad) structure given by the obvious homomorphisms $\imath: A \to A_p^\wedge$ and $\mu: (A_p^\wedge)_p^\wedge \to A_p^\wedge$. By 5.2, $\imath_p^\wedge: A_p^\wedge \to (A_p^\wedge)_p^\wedge$ is onto if and only if $\imath_*: H_1(A; F_p) \to H_1(A_p^\wedge; F_p)$ is onto. Thus, the p-adic completion acts idempotently on A if and only if $\imath_*: H_1(A; F_p) \to H_1(A_p^\wedge; F_p)$ is onto. (Note that $\imath_*$ is always monic.) By [4, p. 57], this idempotency holds whenever $H_1(A; F_p)$ is finite, and thus whenever A is finitely generated. By 5.5 below, it also holds whenever $K(A, 1)$ is F_p-good, e.g., when A is p-seminilpotent. However, by [6, p. 114], it fails when A is a free group on an infinite set of generators.

5.4. **Derived functors of the p-adic completion.** For a group A and $n \geq 0$, we let $c^p_n(A)$ denote the group $\pi_{n-1}F_{p\infty}K(A, 1)$, which is always HF_p-local and is p-cotorsion abelian for $n \geq 1$ by 4.6. There is a natural isomorphism

$$c^p_n(A) \cong \pi_n(GK(A, 1))^\wedge_p$$

by 5.1, and we may view c^p_n as the nth left derived functor of the p-adic completion functor, since $GK(A, 1)$ is a free simplicial group with

$$\pi_i GK(A, 1) \cong \begin{cases} A & \text{for } i = 0, \\ 0 & \text{for } i \geq 1. \end{cases}$$

By 4.2, if $A \rightarrowtail B \twoheadrightarrow C$ is a short exact sequence of groups such that the action by C on $H_i(A; F_p)$ is nilpotent for $i \geq 0$, then there is an induced homotopy fiber sequence

$$F_{p\infty}K(A, 1) \to F_{p\infty}K(B, 1) \to F_{p\infty}K(C, 1)$$

which determines a long exact sequence

$$\begin{aligned} \cdots \to c^p_{n+1}(C) \to c^p_n(A) \to c^p_n(B) \to c^p_n(C) \\ \to \cdots \to c^p_0(B) \to c^p_0(C) \to \{1\} \end{aligned}$$

of groups. By 2.8 this applies automatically when the action of B on A is p-seminilpotent. By 4.1 we have

Proposition 5.5. *If X is a pointed connected space, then there is a natural isomorphism $\pi_1 F_{p\infty}X \cong c^p_0(\pi_1 X)$.*

5.6. **Properties of c^p_0.** The functor c^p_0 has a triple (or monad) structure, ι: Id $\to c^p_0$ and $\mu: c^p_0 c^p_0 \to c^p_0$, induced by that of $F_{p\infty}$, or equivalently by that of $\overline{W}(G(\ ))^\wedge_p$. By the argument of 5.3, c^p_0 acts idempotently on a group A if and only if $\iota: H_1(A; F_p) \to H_1(c^p_0 A; F_p)$ is onto. Thus c^p_0 acts idempotently on A when $K(A, 1)$ is F_p-good, e.g. when A is p-seminilpotent. Whenever c^p_0 acts idempotently on A, then so does $(\ )^\wedge_p$ by our homological criteria, because there is a natural epimorphism $\gamma: c^p_0 A \twoheadrightarrow A^\wedge_p$ induced by the isomorphisms

$$\pi_0(G/\Gamma^p_n G)K(A, 1) \cong A/\Gamma^p_n A.$$

By [6, p. 254], γ has kernel

$$\lim{}^1 \pi_1(G/\Gamma^p_n G)K(A, 1) \cong \lim{}^1 \pi_2(F_{pn}K(A, 1))$$

which is p-cotorsion abelian (see 4.6). If A is a group with $H_1(A; F_p)$ and $H_2(A; F_p)$ both finite, then this $\lim^1$ term vanishes, and thus $\gamma: c^p_0 A \cong A^\wedge_p$. In particular, if A is finitely presented, then $c^p_0 A \cong A^\wedge_p$.

5.7. **Examples.** To illustrate the interesting diversity of the groups $c^p_n A$, we observe:

(i) if A is free, then $c^p_0 A \cong A^\wedge_p$ and $c^p_n A = 0$ for $n \geq 1$ by [6, p. 114];

(ii) if A is nilpotent, then $c^p_0 A \cong \mathrm{Ext}(Z_{p^\infty}, A)$, $c^p_1 A \cong \mathrm{Hom}(Z_{p^\infty}, A)$, and $c^p_n A = 0$ for $n \geq 2$ by [6, p. 167]; if A is p-seminilpotent, there is a similar result by 6.1 below;

(iii) for the infinite general linear group $GL(\Lambda)$ over a ring Λ with identity, $c^p_n GL(\Lambda)$ is the p-adic algebraic K-group $\pi_{n+1}F_{p\infty}(BGL(\Lambda)^+)$ of Λ for $n \geq 0$ by [22];

(iv) for the infinite symmetric group Σ_∞, $c_n^p\Sigma_\infty$ is the p-torsion subgroup of the stable homotopy group of spheres $\pi_{n+1}S$ for $n \geq 0$ by a theorem of Priddy as noted in [6, p. 207];

(v) for the symmetric group Σ_3, there is a short exact sequence

$$0 \to \pi_{n+1}S^3 \otimes Z/3 \to c_n^3\Sigma_3 \to \operatorname{Tor}(\pi_n S^3, Z/3) \to 0$$

for $n \geq 0$ by [6, p. 213];

(vi) for a finite group A, $c_n^p A$ is finite p-group for $n \geq 0$ by [6, p. 212], and is trivial for p prime to $|A|$.

6. THE GROUPS $\pi_* F_{p\infty} X$ FOR p-SEMINILPOTENT SPACES

As in the nilpotent case [6, p. 183], we shall express $\pi_* F_{p\infty} X$ in terms of $\pi_* X$ using:

6.1. **The functors** $\operatorname{Ext}(Z_{p^\infty}, G)$ **and** $\operatorname{Hom}(Z_{p^\infty}, G)$. For an abelian group A, there are natural isomorphisms $c_0^p A \cong \operatorname{Ext}(Z_{p^\infty}, A)$, $c_1^p A \cong \operatorname{Hom}(Z_{p^\infty}, A)$, and $c_n^p A \cong 0$ for $n \geq 2$ by [6, p. 167]. For a p-seminilpotent group G, we deduce that $c_n^p G = 0$ for $n \geq 2$ by 5.4, and we let $\operatorname{Ext}(Z_{p^\infty}, G)$ and $\operatorname{Hom}(Z_{p^\infty}, G)$ respectively denote $c_0^p G$ and $c_1^p G$. When G is nilpotent, these groups have been extensively studied in [6 and 16]. Our notation follows [6], and the reader is warned that $\operatorname{Ext}(Z_{p^\infty}, G)$ and $\operatorname{Hom}(Z_{p^\infty}, G)$ do not in general indicate sets of group extensions or homomorphisms. By 4.3 and 4.6, $\operatorname{Ext}(Z_{p^\infty}, G)$ is an HF_p-local p-seminilpotent group, while $\operatorname{Hom}(Z_{p^\infty}, G)$ is a p-cotorsion abelian group. By 5.4, a short exact sequence $G' \rightarrowtail G \twoheadrightarrow G''$ of p-seminilpotent groups induces an exact sequence of groups

$$\begin{aligned} 0 &\to \operatorname{Hom}(Z_{p^\infty}, G') \to \operatorname{Hom}(Z_{p^\infty}, G) \to \operatorname{Hom}(Z_{p^\infty}, G'') \\ &\to \operatorname{Ext}(Z_{p^\infty}, G') \to \operatorname{Ext}(Z_{p^\infty}, G) \to \operatorname{Ext}(Z_{p^\infty}, G'') \to 0. \end{aligned}$$

Using 3.8, this implies that each monomorphism $H \rightarrowtail G$ of p-seminilpotent groups induces a monomorphism $\operatorname{Hom}(Z_{p^\infty}, H) \rightarrowtail \operatorname{Hom}(Z_{p^\infty}, G)$. The p-cotorsion abelian group $\operatorname{Hom}(Z_{p^\infty}, G)$ is torsion free by induction from the case of G abelian. For a p-seminilpotent group G, the natural map $\iota: G \to \operatorname{Ext}(Z_{p^\infty}, G)$ will be called the *p-cotorsion completion* of G. It is part of an idempotent triple on the category of p-seminilpotent groups by 5.6. We say that a p-seminilpotent group G has *serially bounded p-torsion* when G has a p-seminilpotent series $\{G_i\}$ such that the p-torsion subgroup of each G_i/G_{i+1} is of finite exponent. For instance, a p-seminilpotent polycyclic group must have serially bounded p-torsion, and a nilpotent group G has serially bounded p-torsion if and only if its p-torsion subgroup is of finite exponent.

Proposition 6.2. *If G is a p-seminilpotent group with serially bounded p-torsion, then* γ: $\operatorname{Ext}(Z_{p^\infty}, G) \cong G_p^\wedge$ *and* $\operatorname{Hom}(Z_{p^\infty}, G) \cong 0$.

Proof. By 5.6 it suffices to show that the tower $\{\pi_2 F_{pn} K(G, 1)\}$ is pro-trivial and $\operatorname{Hom}(Z_{p^\infty}, G) = 0$. This holds when G is abelian by [6, p. 170] and holds in general by an induction using [6, p. 91].

Theorem 6.3. *For a p-seminilpotent space X, there is a splittable natural short exact sequence*

$$0 \to \operatorname{Ext}(Z_{p^\infty}, \pi_n X) \to \pi_n F_{p\infty} X \to \operatorname{Hom}(Z_{p^\infty}, \pi_{n-1} X) \to 0.$$

Proof. This follows by applying 4.4 to the Postnikov tower of X. The splittability is trivial for $n = 1$ and holds as in [6, p. 183] for $n \geq 2$ since $\operatorname{Ext}(Z_{p^\infty}, \pi_n X)$ is p-cotorsion and $\operatorname{Hom}(Z_{p^\infty}, \pi_{n-1} X)$ is torsion free.

Corollary 6.4. *For a p-seminilpotent space X whose homotopy groups have serially bounded p-torsion, there is a natural isomorphism $\pi_n F_{p\infty} X \cong (\pi_n X)^\wedge_p$ for $n \geq 1$.*

6.5. **The p-cotorsion groups.** Generalizing the abelian terminology, we call a p-seminilpotent group G *p-cotorsion* when $\iota \colon G \cong \operatorname{Ext}(Z_{p^\infty}, G)$ and $\operatorname{Hom}(Z_{p^\infty}, G) = 0$. Applying 6.3 to the F_p-complete space $X = F_{p\infty} X(G, 1)$, we find that the groups $\operatorname{Ext}(Z_{p^\infty}, G)$ and $\operatorname{Hom}(Z_{p^\infty}, G)$ are always p-cotorsion. Thus, in the above definition, the condition "$\operatorname{Hom}(Z_{p^\infty}, G) = 0$" is superfluous. For a nilpotent group, our p-cotorsion conditions agree with those of [16 and 6], where the term "Ext-p-complete" is used. For a homomorphism $f \colon G \to H$ of p-cotorsion p-seminilpotent groups, $\ker f$ and $\operatorname{im} f$ are also p-cotorsion p-seminilpotent by 3.2 and [4, 1.5 and 2.10]. In a short exact sequence $G' \rightarrowtail G \twoheadrightarrow G''$ of p-seminilpotent groups, if any two groups are p-cotorsion, then so is the third by a 5-lemma argument.

Proposition 6.6. *A p-seminilpotent space X is F_p-complete if and only if each $\pi_n X$ is p-cotorsion.*

This follows by 6.3. The p-seminilpotent F_p-complete spaces are the only known examples of F_p-complete spaces. We now turn to the $H_*(\ ; F_p)$-acyclic spaces and first show:

Proposition 6.7. *For a group G, the following conditions are equivalent*:

(i) *G is p-seminilpotent and $H_i(G; F_p) = 0$ for all $i \geq 1$;*

(ii) *G is p-seminilpotent and $H_i(G; F_p) = 0$ for $i = 1, 2$;*

(iii) *G is p-seminilpotent,* $\operatorname{Ext}(Z_{p^\infty}, G) = 0$, *and* $\operatorname{Hom}(Z_{p^\infty}, G) = 0$;

(iv) *G is solvable and its derived series quotients $D^n G / D^{n+1} G$ are uniquely p-divisible for each n;*

(v) *G has a finite decreasing filtration by normal subgroups $\{G_n\}$ such that G_n / G_{n+1} is uniquely p-divisible abelian for each n.*

Proof. Clearly (i) implies (ii); (ii) implies (iii) by 4.1; and (iii) implies $F_{p\infty} K(G, 1) \simeq *$, which implies (i). Thus (iii) implies that $H_1(G; Z)$ is uniquely p-divisible, which implies (iv) by induction; (iv) clearly implies (v); and (v) implies (iii) by induction.

A p-seminilpotent group G will be called *p-trivial* when it satisfies the above conditions. When G is nilpotent, this is equivalent to asserting the existence of unique pth roots in G. However, in general, this does not imply either the existence or uniqueness of pth roots (see [2, pp. 247–248]). By 6.3

Proposition 6.8. *A p-seminilpotent space X is $H_*(\ ; F_p)$-acyclic if and only if each $\pi_n X$ is p-trivial.*

7. THE F_p-GOODNESS OF POLYCYCLIC-BY-FINITE SPACES

Using methods of Dror-Dwyer-Kan [11], we shall prove the F_p-goodness of polycyclic-by-finite spaces and other "virtually p-seminilpotent" spaces. Recall that a group G is *polycyclic-by-finite* when it contains a polycyclic normal subgroup of finite index. This is equivalent to saying that G is *poly-(cyclic or finite)*, i.e., G has a finite filtration $G = G_1 \supset G_2 \supset \cdots \supset G_m = \{1\}$ such that G_{i+1} is a normal subgroup of G_i with G_i/G_{i+1} cyclic or finite for each i. A space X is called *polycyclic-by-finite* when it is connected with $\pi_1 X$ polycyclic-by-finite and $\pi_i X$ finitely generated for each $i \geq 2$. An action by a group G on a group M is called *virtually p-seminilpotent* when G has a normal subgroup of finite index which acts p-seminilpotently on M. Our main F_p-goodness theorem is:

Theorem 7.1. *If X is a pointed connected space with $\pi_1 X$ polycyclic-by-finite and with $\pi_1 X$ acting virtually p-seminilpotently on $\pi_n X$ for $n \geq 2$, then X is F_p-good and $F_{p\infty} X$ is p-seminilpotent.*

This will be proved in 7.12. It applies to any polycyclic-by-finite space X since $\pi_1 X$ automatically acts virtually p-seminilpotently on $\pi_n X$ when $\pi_n X \otimes Z/p$ and $\operatorname{Tor}(\pi_n X, Z/p)$ are both finite. We shall deal explicitly with this case in 7.2.

A group A is called *p-adically polycyclic* when A has a finite filtration $A = A_1 \supset A_2 \supset \cdots \supset A_m = \{1\}$ such that each A_{i+1} is a normal subgroup of A_i with A_i/A_{i+1} p-adically cyclic, i.e., isomorphic to $Z_p^\wedge$ or to Z/p^j for some $j \geq 0$. When A is abelian, this simply means that A is isomorphic to a finitely generated $Z_p^\wedge$-module. A space Y is called *p-adically polycyclic* when Y is connected and $\pi_n Y$ is p-adically polycyclic for $n \geq 1$. In 7.13, we shall prove:

Theorem 7.2. *If X is a pointed polycyclic-by-finite space, then X is F_p-good and $F_{p\infty} X$ is p-adically polycyclic with $\pi_1 F_{p\infty} X \cong c_0^p(\pi_1 X) \cong (\pi_1 X)_p^\wedge$.*

As illustrated by 5.7(v), a higher group $\pi_i F_{p\infty} X$ may differ profoundly from $(\pi_i X)_p^\wedge$. For later use, we note

Theorem 7.3. *If X is a polycyclic-by-finite space, then X is of finite type, i.e., weakly equivalent to a complex with finitely many cells in each dimension.*

This follows from [27, p. 61] using the result [14]:

Theorem 7.4. *If G is a polycyclic-by-finite group and R is a commutative Noetherian ring, then G is finitely presented and the group ring RG is (left) Noetherian.*

In order to prove 7.1 and 7.2, we require a series of lemmas.

Lemma 7.5. *For a group A, the following conditions are equivalent:*

(i) *A is p-adically polycyclic;*
(ii) *A is a p-cotorsion p-seminilpotent group with $H_j(A; F_p)$ finite for each j;*
(iii) *A is a p-cotorsion p-seminilpotent group with $H_1(A; F_p)$ finite;*
(iv) *A has a p-seminilpotent series $\{A_k\}$ with finitely generated $Z_p^\wedge$-module quotients A_k/A_{k+1}.*

Proof. To show (i) $\Rightarrow$ (ii), we let $\{A_k\}$ be a finite filtration of A with p-adically cyclic quotients A_k/A_{k+1}, and we assume inductively that A_{k+1} is p-seminilpotent. The group A_k/A_{k+1} must act nilpotently on each $H_j(A_{k+1}; F_p)$ since $H_j(A_{k+1}; F_p)$ is a finite F_p-module, and since each finite quotient of A_k/A_{k+1} is a p-group. Thus A_k is p-seminilpotent by 3.3 and 3.12, and (ii) follows. To show (iii) $\Rightarrow$ (iv), we choose a p-seminilpotent series $\{A_k\}$ for

A with p-cotorsion quotients A_k/A_{k+1}, for instance, by letting A_{k+1} be the kernel of $A \to \mathrm{Ext}(Z_{p^\infty}, A/D^kA)$. We assume inductively that A/A_k has finite F_p-homology groups and that A_{k-1}/A_k is a finitely generated $Z_p^\wedge$-module. Since A/A_k acts nilpotently on $H_1(A_k/A_{k+1}; F_p)$, and since $H_1(A/A_{k+1}; F_p)$ is finite, we conclude that $H_1(A_k/A_{k+1}; F_p)$ is finite. Thus, since A_k/A_{k+1} is p-cotorsion abelian, it must be a finitely generated $Z_p^\wedge$-module, and (iv) follows. The implications (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i) are obvious.

Lemma 7.6. *A pointed connected space Y is p-adically polycyclic if and only if Y is an F_p-complete p-seminilpotent space with each $H_j(Y; F_p)$ finite.*

This follows from 7.5 using

Lemma 7.7. *A p-adically polycyclic group G must act p-seminilpotently on a ZG-module M when $M \otimes Z/p$ and* $\mathrm{Tor}(M, Z/p)$ *are finite.*

Proof. Since G is HF_p-local by 4.6 and 7.5, it has unique qth roots for each prime $q \neq p$. Thus each finite quotient of G is a p-group, and G must act nilpotently on $M \otimes Z/p$ and $\mathrm{Tor}(M, Z/p)$.

In view of 3.12, the "nilpotent action lemma" of Dror-Dwyer-Kan [11, 5.1] becomes

Lemma 7.8. *For a pointed connected space X, let $X \to K(\varphi, 1)$ and $X \to K(\psi, 1)$ be fibrations with fibers L and M respectively, such that the induced map $\pi_1 X \to \varphi \times \psi$ is onto. If the group ψ and space L are both p-seminilpotent, then ψ acts nilpotently on each $H_n(M; F_p)$.*

The "pre-nilpotency lemma" of Dror-Dwyer-Kan [11, 52] becomes

Lemma 7.9. *For a pointed connected space Y and p-perfect group φ, if $Y \to K(\varphi, 1)$ is a fibration whose fiber N is p-seminilpotent with $H_1(N; F_p)$ finite, then Y is F_p-good and $F_{p\infty}Y$ is p-seminilpotent.*

Proof. First apply the fiberwise F_p-completion [6, p. 40] to give a fibration $Y' \to K(\varphi, 1)$ with fiber $F_{p\infty}N$. Then $H_*(Y; F_p) \cong H_*(Y'; F_p)$ and $(\pi_1 Y)_{F_p} \cong (\pi_1 Y')_{F_p}$, where $(\ )_{F_p}$ denotes the HF_p-localization of [3 or 4]. Thus there is a natural map $Y' \to K((\pi_1 Y)_{F_p}, 1)$ such that $H_i(Y'; F_p) \to H_i((\pi_1 Y)_{F_p}; F_p)$ is iso for $i = 1$ and onto for $i = 2$. Since $H_1(\varphi; F_p) = 0$, the map

$$H_1(\pi_1 F_{p\infty}N; F_p) \to H_1((\pi_1 Y)_{F_p}; F_p)$$

is onto, and thus $\pi_1 F_{p\infty}N \to (\pi_1 Y)_{F_p}$ is onto by [4, 2.13]. Since $\pi_1 F_{p\infty}N$ is p-adically polycyclic by 7.5, so is $(\pi_1 Y)_{F_p}$ by 7.10 below. Thus 7.8 shows that $(\pi_1 Y)_{F_p}$ acts nilpotently on each $H_n(M; F_p)$ where M is the homotopy fiber of $Y' \to K((\pi_1 Y)_{F_p}, 1)$. Since $H_1(M; F_p) = 0$, 4.2 shows that Y', and hence Y, is F_p-good. Moreover, the space $F_{p\infty}Y' \simeq F_{p\infty}Y$ is p-seminilpotent by 3.12.

We have used

Lemma 7.10. *If $f\colon A \twoheadrightarrow B$ is a homomorphism from a p-adically polycyclic group A onto an HF_p-local group B, then B is p-adically polycyclic.*

Proof. Let $\{A_k\}$ be a finite filtration of A by subgroups such that each A_{k+1} is normal in A_k with A_k/A_{k+1} p-adically cyclic. Then each $B_k = f(A_k)$ is HF_p-local by [4, 2.12]. Thus the maximal p-perfect subgroup of B_k/B_{k+1} is trivial, and B_k/B_{k+1} maps onto its HF_p-localization by [4, 2.11 and 2.12]. Hence, B_k/B_{k+1} is a p-cotorsion quotient of A_k/A_{k+1}, and must therefore be p-adically cyclic.

Lemma 7.11. *For a polycyclic-by-finite group G, there exists a p-seminilpotent polycyclic normal subgroup $N \subset G$ of finite index.*

Proof. Let $M \subset G$ be a polycyclic normal subgroup of finite index. Since $D^rM/D^{r+1}M$ is finitely generated for $r \geq 1$, there is a normal subgroup $K \subset G$ of finite index which acts p-seminilpotently on each $D^rM/D^{r+1}M$. Since K acts p-seminilpotently on M, it does so on $M \cap K$ by 2.7. Thus we may let $N = M \cap K$.

7.12. *Proof of* 7.1. By 4.1 we may assume that X is a Postnikov space with only finitely many nontrivial homotopy groups. Then by 7.11 there is a finite group θ and a map $X \to K(\theta, 1)$ with p-seminilpotent homotopy fiber N. Let $\varphi \subset \theta$ be the maximal p-perfect subgroup, and note that θ/φ is a finite p-group. Applying 7.9, 4.2, and 3.12 to the associated homotopy fiber sequences

$$N \to X' \to K(\varphi, 1), \quad X' \to X \to K(\theta/\varphi, 1)$$

we deduce that X' and X are F_p-good, while $F_{p\infty}X'$ and $F_{p\infty}X$ are p-seminilpotent.

7.13. *Proof of* 7.2. By 7.1, X is F_p-good and $F_{p\infty}X$ is p-seminilpotent. Thus by 7.6, $F_{p\infty}X$ is p-adically polycyclic. The isomorphisms follow by 5.5 and 5.6.

8. THE p-ADIC G-COMPLETION

To prepare for the statement and proof of our next F_p-goodness theorem (9.1), we now introduce

8.1. **The p-adic G-completion.** For a group ring ZG, let

$$I_p = \ker(ZG \to Z/p)$$

denote the p-adic augmentation ideal. The *p-adic G-completion* of a ZG-module M is given by the ZG-module

$$M_G^{\wedge p} = \varprojlim{}_k M/(I_p)^k M.$$

We may regard $M_G^{\wedge p}$ as a $(ZG)_G^{\wedge p}$-module with the obvious multiplication. The towerwise *p-adic G-completion* of a ZG-module M is given by the tower $\{M/(I_p)^k M\}_{k\geq 0}$ viewed as a pro-ZG-module (see [1]). This tower is determined up to pro-isomorphism by its property of cofinality in the system of all nilpotent ZG-modules under M which are annihilated by powers of p. Note that when G is trivial, the above completions reduce to the *p-adic completion* $M_p^\wedge = \varprojlim{}_k M/p^k M$ and to the *towerwise p-adic completion* $\{M/p^k M\}_{k\geq 0}$. In general:

Proposition 8.2. *If the action of G on M is p-seminilpotent, then the natural homomorphism $M_p^\wedge \to M_G^{\wedge p}$ is an isomorphism and $\{M/p^k M\}_{k\geq 0} \to \{M/(I_p)^k M\}_{k\geq 0}$ is a pro-isomorphism.*

Proof. If the action of G on M is p-seminilpotent, then each M/p^kM is a nilpotent ZG-module and $\{M/p^kM\}_{k\geq 0}$ shares the cofinality property of $\{M/(I_p)^kM\}_{k\geq 0}$.

When N is an F_pG-module, we may use the augmentation ideal $I = \ker(F_pG \to F_p)$ to construct the p-adic G-completions $N^{\wedge p}_G = \varprojlim_k N/I^kN$ and $\{N/(I_p)^kN\}_{k\geq 0} = \{N/I^kN\}_{k\geq 0}$. The following result will allow us to use other towers in place of $\{N/I^kN\}_{k\geq 0}$.

Proposition 8.3. *If $\{N_k\}_{k\geq 0}$ is a tower of nilpotent F_pG-modules under an F_pG-module N such that $H_i(G; N) \to \{H_i(G; N_k)\}_{k\geq 0}$ is a pro-isomorphism for $i = 0$ and pro-epimorphism for $i = 1$, then $\{N_k\}_{k\geq 0}$ is naturally pro-isomorphic to $\{N/I^kN\}_{k\geq 0}$.*

Proof. It suffices to show that $N/I^mN \to \{N_k/I^mN_k\}_{k\geq 0}$ is a pro-isomorphism for each $m \geq 1$, and this follows by induction using $H_0(G; N) \cong N/IN$ and the exact sequence

$$H_1(G; N) \to H_1(G; N/I^mN) \to I^mN/I^{m+1}N \to 0.$$

As in [8], for our applications of the p-adic G-completion, we need an "Artin-Rees theorem." First, recall from 7.4 that the group ring RG is (left) Noetherian for a commutative Noetherian ring R and polycyclic group G. A two sided ideal $J \subset RG$ is said to have the *weak Artin-Rees property* when for each finitely generated RG-module M and submodule $M' \subset M$, there exists $r \geq 0$ (depending on M and M') with $J^rM \cap M' \subset JM'$. This inductively implies that the neighborhood systems $\{J^kM'\}_{k\geq 0}$ and $\{M' \cap J^kM\}_{k\geq 0}$ determine the same topology on M', or equivalently, that the J-adic topology on M' is the restriction of the J-adic topology on M. As explained in [7, p. 89] and [19], the following theorem (among other more general results) is due to Roseblade [24].

Theorem 8.4. *For p prime, let R be a commutative Noetherian ring such that R/pR is a field. If G is a p-seminilpotent polycyclic group, then the ideal $I_p = \ker(RG \to R/pR)$ has the weak Artin-Rees property.*

8.5. **Exactness of the p-adic G-completion.** Let G be a p-seminilpotent polycyclic group. By 8.4, on the category of finitely generated ZG-modules, the towerwise p-adic G-completion $\{M/(I_p)^kM\}_{k\geq 0}$ is pro-exact, and the p-adic G-completion $M^{\wedge p}_G$ is exact. Consequently, for a finitely generated ZG-module M or finitely generated F_pG-module N, the natural homomorphisms

$$(ZG)^{\wedge p}_G \otimes_{ZG} M \to M^{\wedge p}_G, \qquad (F_pG)^{\wedge p}_G \otimes_{F_pG} N \to N^{\wedge p}_G,$$

are isomorphisms. Thus $(ZG)^{\wedge p}_G$ and $(F_pG)^{\wedge p}_G$ are flat as right modules (and similarly as left modules) over ZG and F_pG respectively.

9. A PARTIAL F_p-GOODNESS THEOREM

Our earlier F_p-goodness theorem (7.2) applies to many spaces of finite type, but not to those like $S^n \vee S^1$ for $n \geq 2$ with nonfinitely generated homotopy groups. In fact, $S^n \vee S^1$ is F_p-bad by 10.1 below. Here, we establish a partial F_p-goodness theorem (9.1) showing that when a space X of finite type is p-seminilpotent polycyclic below some dimension $n \geq 2$, then X is F_p-good below dimension $2n$. This is closely related to a result of Dror-Dwyer [10] giving a stable range for integral homology localizations.

A space Y is called *p-seminilpotent polycyclic* when it is p-seminilpotent and the groups $\pi_i Y$ are polycyclic for $i \geq 1$. Thus a nilpotent space of finite type is always p-seminilpotent polycyclic.

Theorem 9.1. *If X is a pointed connected space of finite type whose $(n-1)$th Postnikov section $P^{n-1}X$ is p-seminilpotent polycyclic for some $n \geq 2$, then the p-adic completion $X \to F_{p\infty}X$ induces isomorphisms*

$$\begin{aligned} H_i(X; F_p) &\cong H_i(F_{p\infty}X, F_p) && \text{for } i \leq 2n-1, \\ (\pi_i X)_p^{\wedge} &\cong \pi_i F_{p\infty}X && \text{for } 1 \leq i \leq n-1, \\ (\pi_i X)_{\pi_1 X}^{\wedge p} &\cong \pi_i F_{p\infty}X && \text{for } 2 \leq i \leq 2n-2. \end{aligned}$$

Note that 8.2 already guarantees that $(\pi_i X)_p^{\wedge} \cong (\pi_i X)_{\pi_1 X}^{\wedge p}$ for $2 \leq i \leq n-1$. We devote the rest of this section to proving 9.1.

Lemma 9.2. *If Y is a p-seminilpotent polycyclic space and N is a finitely generated $F_p\pi_1 Y$-module, then for each m,*

(i) *$H_m(Y; N)$ is finite;*

(ii) $H_m(Y; N) \cong H_m(Y; N_{\pi_1 Y}^{\wedge p}) \cong \varprojlim_k H_m(Y; N/I^k N)$;

(iii) *the map $H_m(Y; N) \to \{H_m(Y; N/I^k N)\}_{k \geq 0}$ is a pro-isomorphism.*

Proof. By 7.4 and 8.5, it suffices to assume that $N = F_p\pi_1 Y$. Now (i) follows since $H_*(Y; N) \cong H_*(\widetilde{Y}; F_p)$ where $\widetilde{Y}$ is the universal covering of Y, and the second isomorphism of (ii) follows since Y is of finite type by 7.3. Since N and $N_{\pi_1 Y}^{\wedge p}$ are flat left $F_p\pi_1 Y$-modules by 8.5, it suffices for the first isomorphism of (ii) to show

$$H_*(\widetilde{Y}; F_p) \otimes_{F_p\pi_1 Y} N \cong H_*(\widetilde{Y}; F_p) \otimes_{F_p\pi_1 Y} N_{\pi_1 Y}^{\wedge p}.$$

Since Y is p-seminilpotent, $\pi_1 Y$ acts nilpotently on each $H_i(\widetilde{Y}; F_p)$ by 3.12, and it suffices to show

$$F_p \otimes_{F_p\pi_1 Y} N \cong F_p \otimes_{F_p\pi_1 Y} N_{\pi_1 Y}^{\wedge p}.$$

This follows since

$$F_p \otimes_{F_p\pi_1 Y} N_{\pi_1 Y}^{\wedge p} \cong \varprojlim_k F_p \otimes_{F_p\pi_1 Y} N/I^k N \cong N/IN.$$

Finally, part (iii) follows easily from (i) and (ii).

For X as in Theorem 9.1, let

$$X' \to X \to P^{n-1}X$$

be the Postnikov fiber sequence.

Lemma 9.3. *For a commutative Noetherian ring R, each $H_i(X'; R)$ is finitely generated as an $R\pi_1 X$-module. If $i \leq 2n-2$ then $\pi_i X'$ is finitely generated as a $Z\pi_1 X$-module.*

Proof. Since X is of finite type, its universal covering space $\widetilde{X}$ has an R-chain complex of finite type over $R\pi_1 X$, and each $H_i(\widetilde{X}; R)$ is a finitely generated $R\pi_1 X$-module. Moreover, each $H_j(P^{n-1}\widetilde{X}; R)$ is a finitely generated R-module. The homology result now follows by induction on i, using the Serre spectral sequence of $X' \to \widetilde{X} \to P^{n-1}\widetilde{X}$. The stable homotopy result follows easily from the Z-homology result.

Using the functors $(F_p)_k$ of [6, p. 21] for $k \geq 0$, let

$$X'_k \to (F_p)_k X \to (F_p)_k P^{n-1} X$$

be the fiber sequence induced by $X \to P^{n-1}X$.

Lemma 9.4. *For each* i

$$H_i(X'; F_p) \to \{H_i(X'_k; F_p)\}_{k\geq 0}$$

is pro-isomorphic to the towerwise p*-adic* $\pi_1 X$*-completion of* $H_i(X'; F_p)$.

Proof. Suppose inductively that this holds for each $i \leq m$. Then for each $i \leq m$ and j,

$$H_j(P^{n-1}X; H_i(X'; F_p)) \to \{H_j(P^{n-1}X; H_i(X'_k; F_p))\}_k$$

is a pro-isomorphism by 9.2 and 9.3. Also for each i and j,

$$\{H_j(P^{n-1}X; H_i(X'_k; F_p))\}_k \to \{H_j(F_{pk}P^{n-1}X; H_i(X'_k; F_p))\}_k$$

is a pro-isomorphism since $\pi_1 F_{pk} P^{n-1} X$ acts nilpotently on $H_i(X'_k; F_p)$. Thus

$$H_j(P^{n-1}X; H_{m+1}(X'; F_p)) \to \{H_j(F_{pk}P^{n-1}X; H_{m+1}(X'_k; F_p))\}_k$$

is a pro-isomorphism for $j = 0$ and pro-epimorphism for $j = 1$ by the spectral sequence comparison lemma of [6, p. 92]. Hence $H_{m+1}(X'; F_p) \to \{H_{m+1}(X'_k; F_p)\}_k$ is pro-isomorphic to the towerwise p-adic $\pi_1 X$-completion by 8.3.

Lemma 9.5. *For each* $i \geq 1$,

$$H_i(X'; Z) \to \{H_i(X'_k; Z)\}_{k\geq 0}$$

is pro-isomorphic to the towerwise p*-adic* $\pi_1 X$*-completion of* $H_i(X'; Z)$.

Proof. An induction starting with 9.4 and using Bockstein exact sequences shows that

$$H_i(X'; Z/p^m) \to \{H_i(X'_k; Z/p^m)\}_k$$

is pro-isomorphic to the towerwise p-adic completion of $H_i(X'; Z/p^m)$ for each $i \geq 1$ and $m \geq 1$. Since $H_i(X'; Z)$ is a finitely generated $Z\pi_1 X$-module, its p-torsion is of finite exponent by an ascending chain argument, and the map

$$\{H_i(X'; Z) \otimes Z/p^m\}_m \to \{H_i(X'; Z/p^m)\}_m$$

is a pro-isomorphism. Since $H_i(X'_k; Z)$ is a finite p-torsion group, the map

$$H_i(X'_k; Z) \to \{H_i(X'_k; Z/p^m)\}_m$$

is a pro-isomorphism, and the lemma follows.

Lemma 9.6. *For each* $i \leq 2n-2$, $\pi_i X' \to \{\pi_i X'_k\}_{k\geq 0}$ *is pro-isomorphic to the towerwise* p*-adic* $\pi_1 X$*-completion of* $\pi_i X'$.

Proof. Since the spaces X' and X'_k are $(n-1)$-connected by [6, p. 30], this stable result follows easily from 9.5.

9.7. **Determination of** $\pi_i F_{p\infty} X$. The results on $\pi_i F_{p\infty} X$ in Theorem 9.1 follow by using 6.4 and 9.6 to show that the fiber sequence

$$X'_\infty \to F_{p\infty} X \to F_{p\infty} P^{n-1} X$$

has $\pi_i X'_\infty = 0$ for $i \leq n$, $\pi_i X'_\infty \cong (\pi_i X)^{\wedge p}_{\pi_i X}$ for $n \leq i \leq 2n-2$, $\pi_i F_{p\infty} P^{n-1} X \cong (\pi_i X)^\wedge_p$ for $i \leq n-1$, and $\pi_i F_{p\infty} P^{n-1} X = 0$ for $i \geq n$.

The results on $H_i(F_{p\infty} X; F_p)$ in 9.1 will be proved using another series of lemmas.

Lemma 9.8. *For $n \geq 2$, let Y_∞ be the homotopy inverse limit of a tower $\{Y_k\}_{k \geq 0}$ of $(n-1)$-connected pointed spaces with finite homotopy groups. Then $H_i(Y_\infty; F_p) \to \varprojlim_k H_i(Y_k; F_p)$ is an isomorphism for $i \leq 2n-2$ and onto for $i = 2n-1$.*

Proof. This follows using the natural fiber sequences

$$Y'_k \to Y_k \to K(\pi_n Y_k, n)$$

since

$$H_i(K(\pi_n Y_\infty, n); F_p) \cong \varprojlim_k H_i(K(\pi_n Y_k, n); F_p)$$

for $i \leq 2n-1$, and since we may inductively assume that $H_i(Y'_\infty; F_p) \to \varprojlim_k H_i(Y'_k; F_p)$ is isomorphic for $i \leq 2n-2$ and onto for $i = 2n-1$.

Lemma 9.9. *For a group G with finite $H_1(G; F_p)$, the towers $\{F_p(G/\Gamma^p_k G)\}_{k \geq 0}$ and $\{F_p G / I^k F_p G\}_{k \geq 0}$ are pro-isomorphic under $F_p G$.*

This follows by [21] and implies that a nilpotent action by G on an $F_p G$-module must factor through some lower p-central series quotient of G. Thus the actions in the following lemma are well defined.

Lemma 9.10. *For a p-seminilpotent polycyclic space Y and tower $\{N_n\}_{n \geq 0}$ of finite nilpotent $F_p \pi_1 Y$-modules with inverse limit N_∞, the maps*

$$\begin{tikzcd}
H_*(Y; N_\infty) \arrow[r] \arrow[d] & H_*(F_{p\infty} Y; N_\infty) \arrow[d] \\
\varprojlim_n H_*(Y; N_n) \arrow[r] & \varprojlim_n n H_*(F_{p\infty} Y; N_n)
\end{tikzcd}$$

are all isomorphisms.

Proof. The left isomorphism follows since Y is of finite type by 7.3 and each N_n is finite. The bottom isomorphism follows since Y is F_p-good by 4.3 and each N_n is nilpotent. For the top isomorphism, first assume $Y = K(Z, 1)$. Since the action of $Z^\wedge_p / Z$ on each $H_*(Z; N_n) \cong H_*(Z^\wedge_p; N_n)$ is trivial, the inverse limit action of $Z^\wedge_p / Z$ on $H_*(Z; N_\infty)$ is also trivial. Hence $H_*(Z; N_\infty) \cong H_*(Z^\wedge_p; N_\infty)$ by the Serre spectral sequence. Also,

$$H_*(Y; N_\infty) \cong H_*(F_{p\infty} Y; N_\infty)$$

for $Y = K(Z/p, 1)$ or for $Y = K(Z/q, 1)$ at a prime $q \neq p$, since a nilpotent action by Z/q on an F_p-module must be trivial. We can now assume that Y lies in a fiber sequence $Y' \to Y \to Y''$ of p-seminilpotent polycyclic spaces such that the lemma holds for Y' and Y''. Then

$$H_*(Y''; H_*(Y'; N_\infty)) \cong H_*(F_{p\infty} Y''; H_*(F_{p\infty} Y'; N_\infty)),$$

and we deduce $H_*(Y; N_\infty) \cong H_*(F_{p\infty} Y; N_\infty)$ by the Serre spectral sequence.

Now 9.2 and 9.10 imply

Lemma 9.11. *If Y is a p-seminilpotent polycyclic space and N is a finitely generated $F_p\pi_1 Y$-module, then*

$$H_*(Y; N) \cong H_*(F_{p\infty}Y; N^{\wedge p}_{\pi_1 Y}).$$

Lemma 9.12. *For a p-seminilpotent polycyclic group G and finitely generated ZG-module M with $n \geq 2$, the map*

$$H_i(K(M^{\wedge p}_G, n); F_p) \longrightarrow \varprojlim{}_k H_i(K(M/I^k_p M, n); F_p)$$

is an isomorphism for $i \leq 2n-1$, and is carried by $H_0(G; -)$ to an epimorphism for $i = 2n$.

Proof. This is elementary for $i \leq 2n-1$ and follows for $i = 2n$ by showing that $H_0(G; -)$ carries

$$M^{\wedge p}_G \otimes M^{\wedge p}_G \otimes F_p \longrightarrow \varprojlim{}_k (M/I^k_p M \otimes M/I^k_p M \otimes F_p)$$

to an epimorphism. After reduction to the case $M = ZG$, this follows since $H_0(G; -)$ carries

$$\varprojlim{}_k F_p(G/\Gamma^p_k G) \otimes \varprojlim{}_k F_p(G/\Gamma^p_k G) \longrightarrow \varprojlim{}_k (F_p(G/\Gamma^k_p G) \otimes F_p(G/\Gamma^p_k G))$$

to an epimorphism.

9.13. *Proof of Theorem* 9.1. It remains to prove $H_i(X; F_p) \cong H_i(F_{p\infty}X; F_p)$ for $i \leq 2n-1$. By 9.4 and 9.8

$$H_i(X'_\infty; F_p) \cong H_i(X'; F_p)^{\wedge p}_{\pi_1 X}$$

for $i \leq 2n-2$. Thus by 9.11,

$$H_*(P^{n-1}X; H_i(X'; F_p)) \cong H_*(F_{p\infty}P^{n-1}X; H_i(X'_\infty; F_p))$$

for $i \leq 2n-2$, and it suffices to show that the natural map α in

$$\begin{tikzcd}
H_0(P^{n-1}X; H_{2n-1}(X'; F_p)) \arrow[r, "\alpha"] \arrow[dr, "\gamma"] & H_0(F_{p\infty}P^{n-1}X; H_{2n-1}(X'_\infty; F_p)) \arrow[d, "\beta"] \\
 & H_0(F_{p\infty}P^{n-1}X; H_{2n-1}(X'; F_p)^{\wedge p}_{\pi_1 X})
\end{tikzcd}$$

is isomorphic. Since γ is isomorphic by 9.11, it suffices to show that β is isomorphic. By 9.6 and 9.12, the map

$$H_j(K(\pi_n X'_\infty, n); F_p) \longrightarrow \varprojlim{}_k H_j(K(\pi_n X'_k, n); F_p)$$

is isomorphic for $j \leq 2n-1$ and is carried by $H_0(F_{p\infty}P^{n-1}X; -)$ to an epimorphism for $j = 2n$. In the tower of fiber sequences

$$X''_k \longrightarrow X'_k \longrightarrow K(\pi_n X'_k, n)$$

the map $H_j(X''_\infty, F_p) \longrightarrow \varprojlim{}_k H_j(X''_k; F_p)$ is also isomorphic for $j \leq 2n$ by 9.8 and thus

$$H_{2n-1}(X'_\infty; F_p) \longrightarrow \varprojlim{}_k H_{2n-1}(X'_k; F_p) \cong H_{2n-1}(X'; F_p)^{\wedge p}_{\pi_1 X}$$

is carried by $H_0(F_{p\infty}P^{n-1}X; -)$ to an isomorphism as required.

10. ON THE F_p-BADNESS OF $S^n \vee S^1$

For $n \geq 2$, the p-adic completion $S^n \vee S^1 \to F_{p\infty}(S^n \vee S^1)$ induces an $H_i(\ ; F_p)$-isomorphism for $i \leq 2n-1$ by Theorem 9.1. This result is best possible by

Theorem 10.1. *For* $n \geq 2$, *the group* $H_{2n}(F_{p\infty}(S^n \vee S^1); F_p)$ *is uncountable.*

This provides the first example of a finite complex which is F_p-bad, and we devote the rest of this section to the proof.

By the F_p-nilpotent tower lemma of [6, p. 88] for $n \geq 2$, $F_{p\infty}(S^n \vee S^1)$ is equivalent to the homotopy inverse limit of the tower

$$\{M(Z/p^k, n) \vee K(Z/p^k, 1)\}_{k \geq 1}$$

under $S^n \vee S^1$. Since the homotopy fiber of the pinching (or Postnikov) map

$$M(Z/p^k, n) \vee K(Z/p^k, 1) \longrightarrow K(Z/p^k, 1)$$

is equivalent to the Moore space $M(Z/p^k(Z/p^k), n)$, the homotopy inverse limit construction gives

Lemma 10.2. *For* $n \geq 2$, *the fiber of the Postnikov map* $F_{p\infty}(S^n \vee S^1) \to F_{p\infty}S^1$ *is equivalent to the homotopy inverse limit of* $\{M(Z/p^k(Z/p^k), n)\}_{k \geq 1}$.

In general, let M_∞ be the homotopy inverse limit of a tower $\{M(G_k, n)\}_{k \geq 1}$ of Moore spaces with $n \geq 2$ and G_k finite abelian. By 9.8, the natural homomorphism

$$H_i(M_\infty; F_p) \longrightarrow \varprojlim{}_k H_i(M(G_k, n); F_p)$$

is an isomorphism for $i \leq 2n-2$ and onto for $i = 2n-1$.

Lemma 10.3. *The kernel of the above homomorphism for* $i = 2n-1$ *is isomorphic to the cokernel of*

$$H_{2n}(K(G_\infty, n); F_p) \longrightarrow \varprojlim{}_k H_{2n}(K(G_k, n); F_p)$$

where $G_\infty = \varprojlim{}_k G_k$.

Proof. We form the homotopy fiber sequences

$$\widetilde{M}(G_k, n) \longrightarrow M(G_k, n) \longrightarrow K(G_k, n)$$

with inverse limit

$$\widetilde{M}_\infty \longrightarrow M_\infty \longrightarrow K(G_\infty, n),$$

and we consider the exact sequence

$$\begin{aligned} H_{2n}K(G_\infty, n); F_p) &\longrightarrow H_{2n-1}(\widetilde{M}_\infty; F_p) \longrightarrow H_{2n-1}(M_\infty; F_p) \\ &\longrightarrow H_{2n-1}(K(G_\infty, n); F_p) \longrightarrow H_{2n-2}(\widetilde{M}_\infty; F_p). \end{aligned}$$

Using 9.8 we find that the last map is an isomorphism for $n \geq 3$ and corresponds to $\varprojlim{}_k \operatorname{Tor}(G_k, F_p) \longrightarrow 0$ for $n = 2$. Thus, by the above exact sequence, the desired kernel is isomorphic to the cokernel of

$$\partial\colon H_{2n}(K(G_\infty, n); F_p) \longrightarrow H_{2n-1}(\widetilde{M}_\infty; F_p).$$

The lemma now follows from the isomorphisms

$$H_{2n-1}(\widetilde{M}_\infty; F_p) \cong \varprojlim_k H_{2n-1}(\widetilde{M}(G_k, n); F_p),$$
$$\partial\colon H_{2n}(K(G_k, n); F_p) \cong H_{2n-1}(\widetilde{M}(G_k, n); F_p).$$

For an abelian group A and integer n, let $D_n(A) = (A \otimes A)/R$ where R is the subgroup generated by all $x \otimes y - (-1)^n y \otimes x$ for all $x, y \in A$.

Lemma 10.4. *The cokernel in* 10.3 *is isomorphic to the cokernel of*

$$D_n(G_\infty \otimes F_p) \longrightarrow \varprojlim{}_k\, D_n(G_k \otimes F_p).$$

Proof. For each k, there is a natural exact sequence

$$0 \to G_k \otimes F_p \otimes F_2 \longrightarrow D_n(G_k \otimes F_p) \longrightarrow H_{2n}(K(G_k, n); F_p)$$
$$\to H_{2n+1}(K(G_k, n+1); F_p) \longrightarrow 0$$

obtained using the Pontrjagin product and suspension. This sequence for $k = \infty$ maps to the inverse limit of these sequences for $i \le k < \infty$, which is also exact. Since $G_\infty \otimes F_p \otimes F_2$ and $H_{2n+1}(K(G_\infty, n+1); F_p)$ map by isomorphisms, the lemma follows.

10.5. **Reduction of Theorem 10.1 to a lemma.** By 10.2 for $n \ge 2$, there is a homotopy fiber sequence

$$M_\infty \longrightarrow F_{p\infty}(S^n \vee S^1) \longrightarrow F_{p\infty}S^1$$

where M_∞ is the homotopy inverse limit of $\{M(Z/p^k(Z/p^k), n)\}_{k\ge 1}$. By 9.8 the only nontrivial groups $H_i(M_\infty; F_p)$ for $i \le 2n-2$ are $H_0(M_\infty; F_p) \cong F_p$ and $H_n(M_\infty; F_p) \cong \varprojlim{}_k\, F_p(Z/p^k)$. Moreover, by 9.9 and 9.11, the only nontrivial groups

$$H_m(F_{p\infty}S^1; H_i(M_\infty; F_p))$$

for $i \le 2n-2$ are copies of F_p when (m, i) is $(0,0)$, $(1,0)$, or $(0,n)$. Thus for Theorem 10.1, it suffices to show that $H_1(F_{p\infty}S^1; H_{2n-1}(M_\infty; F_p))$ is uncountable. Using 10.3 and 10.4, we obtain a natural short exact sequence

$$0 \longrightarrow D_n(\varprojlim{}_k\, F_p(Z/p^k)) \longrightarrow \varprojlim{}_k\, D_n(F_p(Z/p^k)) \longrightarrow H_{2n-1}(M_\infty; F_p) \longrightarrow 0,$$

and it now suffices to show:

Lemma 10.6. *The natural homomorphism*

$$H_0(Z_p^\wedge; D_n(\varprojlim F_p(Z/p^k))) \longrightarrow H_0(Z_p^\wedge; \varprojlim D_n(F_p(Z/p^k)))$$

has uncountable kernel.

Proof. It suffices to show that

$$(\otimes^2 \varprojlim F_p(Z/p^k))_{Z/2\times Z_p^\wedge} \longrightarrow (\varprojlim \otimes^2 F_p(Z/p^k))_{Z/2\times Z_p^\wedge}$$

has uncountable kernel, where $Z/2$ acts by $x \otimes y \mapsto (-1)^n y \otimes x$ and where $Z_p^\wedge$ acts diagonally. This is equivalent to showing that

$$\mu_{Z/2}\colon (\varprojlim F_p(Z/p^k) \otimes_{F_p Z_p^\wedge} \varprojlim F_p(Z/p^k))_{Z/2} \longrightarrow (\varprojlim F_p(Z/p^k))_{Z/2}$$

has uncountable kernel, where μ is the multiplication map for the algebra $\varprojlim F_p(Z/p^k)$ and where $Z/2$ now acts by $x \otimes y \mapsto (-1)^n ay \otimes ax$ on the domain and by $x \mapsto (-1)^n ax$ on the target of μ, using the antipodal automorphism

$$a\colon \varprojlim F_p(Z/p^k) \longrightarrow \varprojlim F_p(Z/p^k).$$

Since the target $\varprojlim F_p(Z/p^k)$ of μ has

$$H_1(Z/2;\ \varprojlim F_p(Z/p^k)) \cong \varprojlim H_1(Z/2; F_p(Z/p^k)) \cong \begin{cases} Z/2 & \text{for } p = 2, \\ 0 & \text{for } p \text{ odd}, \end{cases}$$

it now suffices to show that $(\ker\mu)_{Z/2}$ is uncountable. Clearly μ is a $Z/2$-equivariant $F_pZ_p^\wedge$-module map, where the action of the generator $t \in Z/2$ commutes with the action of each $r \in F_pZ_p^\wedge$ via $tr = (ar)t$. Note that $F_pZ_p^\wedge \subset \varprojlim F_p(Z/p^k) \cong F_p[\![x]\!]$, and let K denote the field of fractions of the integral domain $F_pZ_p^\wedge$. By 10.7 below, there exists an element $\sigma \in \varprojlim F_p(Z/p^k)$ with $\sigma \notin K$. Hence the element $\xi = \sigma \otimes 1 - 1 \otimes \sigma$ in $\ker\mu$ is nonzero in $K \otimes_{F_pZ_p^\wedge} \ker\mu$ and $F_pZ_p^\wedge \cong F_pZ_p^\wedge\xi \subset \ker\mu$. Thus $\ker\mu$ is uncountable, and this implies that $(\ker\mu)_{Z/2}$ is uncountable when $p = 2$. For p odd, the elements $\xi^+ = \frac{1}{2}(\xi + t\xi)$ and $\xi^- = \frac{1}{2}(\xi - t\xi)$ are in $\ker\mu$, and at least one of them is nonzero in $K \otimes_{F_pZ_p^\wedge} \ker\mu$ because $\xi = \xi^+ + \xi^-$. Either

$$F_pZ_p^\wedge \cong F_pZ_p^\wedge\xi^+ \subset \ker\mu \quad \text{or} \quad F_pZ_p^\wedge \cong F_pZ_p^\wedge\xi^- \subset \ker\mu,$$

and thus at least one of the groups $(F_pZ_p^\wedge\xi^+)_{Z/2}$ and $(F_pZ_p^\wedge\xi^-)_{Z/2}$ is uncountable. Since $(\ )_{Z/2}$ is exact on $F_p(Z/2)$-modules for p odd, $(\ker\mu)_{Z/2}$ is also uncountable.

We have used

Lemma 10.7. *There exists an element $\sigma \in \varprojlim F_p(Z/p^k)$ with $\sigma \notin K$, where K is the field of fractions of $F_pZ_p^\wedge$.*

Proof. For each nonzero element $c \in \varprojlim F_p(Z/p^k)$, let $v(c)$ be the nonnegative integer with $c \in I^{v(c)}$ and $c \notin I^{v(c)+1}$ where $I \subset \varprojlim F_p(Z/p^k)$ is the argumentation ideal. For nonzero elements, $c, d \in \varprojlim F_p(Z/p^k)$, note that $c/d \in \varprojlim F_p(Z/p^k)$ if and only if $v(c) \geq v(d)$. Let

$$D = (r, s, m, n, a_1, \dots, a_r, b_1, \dots, b_s)$$

be a list of integers $r, s \geq 1$ and $m \geq n \geq 0$ together with elements $a_1, \dots, a_r \in F_p$ and $b_1, \dots, b_s \in F_p$. Let $S_D \subset \varprojlim F_p(Z/p^k)$ be the subset given by all

$$\frac{a_1x_1 + \cdots + a_rx_r}{b_1y_1 + \cdots + b_sy_s}$$

with $x_1, \dots, x_r \in Z_p^\wedge$ and $y_1, \dots, y_s \in Z_p^\wedge$ where

$$v(a_1x_1 + \cdots + a_rx_r) = m \quad \text{and} \quad v(b_1y_1 + \cdots + b_sy_s) = n.$$

Now S_D is closed in $\varprojlim F_p(Z/p^k)$ with respect to the profinite topology. The annihilating ideal of the image element of $b_1y_1 + \cdots + b_sy_s$ in $F_p(Z/p^k)$ has

p^n elements, and thus the image of S_D in $F_p(Z/p^k)$ contains at most $p^{k(r+s)+n}$ elements. Since $F_p(Z/p^k)$ contains p^{p^n} elements and

$$\lim_{k\to\infty}(p^{k(r+s)+n}/p^{p^k}) = 0,$$

S_D has empty interior in $\varprojlim F_p(Z/p^k)$. Thus the countable union $\bigcup_D S_D$ has empty interior by the Baire theorem. Since this union gives the nonzero elements of $K \cap \varprojlim F_p(Z/p^k)$, there exists $\sigma \in \varprojlim F_p(Z/p^k)$ with $\sigma \notin K$.

11. On the F_p-badness of $S^1 \vee S^1$

For a free group A on $n < \infty$ generators, the space

$$K(A, 1) \simeq S^1 \vee \cdots \vee S^1$$

has F_p-completion

$$F_{p\infty}K(A, 1) \simeq K(A_p^\wedge, 1)$$

by [6, p. 114], where the p-adic completion $A_p^\wedge$ equals the p-profinite completion of A with the topology forgotten as in 5.1. We showed in [4, p. 57] that $H_1(A_p^\wedge; F_p) \cong (F_p)^n$. With similar methods we can show $H_1(A_p^\wedge; Z) \cong (Z_p^\wedge)^n$, and it is conceivable that $H_2(A_p^\wedge; F_p)$, like $H_2(A; F_p)$, is always trivial. However, using 10.1, we shall deduce

Theorem 11.1. *For a free group A on at least two generators, the group*

$$H_m(F_{p\infty}K(A, 1); F_p) \cong H_m(A_p^\wedge; F_p)$$

is uncountable for $m = 2$ or $m = 3$ or both. In particular, the space $S^1 \vee S^1$ is F_p-bad.

Proof. Let J denote the free simplicial group $J = G(S^2 \vee S^1)$ and recall from 5.1 that $\overline{W}J_p^\wedge \simeq F_{p\infty}(S^2 \vee S^1)$. Thus by [6, p. 108], there is a natural first quadrant spectral sequence $\{E_{i,j}^r\}$ converging to $H_{i+j}(F_{p\infty}(S^2 \vee S^1); F_p)$ with

$$E_{i,j}^1 = H_j((J_i)_p^\wedge; F_p), \quad d_r\colon E_{i,j}^r \longrightarrow D_{i-r,j+r-1}^r.$$

Since $J_0 \cong Z$, $H_0((J_i)_p^\wedge; F_p) \approx F_p$, and $H_1((J_i)_p^\wedge; F_p) \cong abJ_i \otimes F_p$, we find that $E_{0,j}^2 = 0$ for $j \geq 2$, $E_{i,0}^2 = 0$ for $i \geq 1$, and $E_{i,1}^2 = 0$ for $i \geq 2$. Thus, since $H_4(F_{p\infty}(S^2 \vee S^1); F_p)$ is uncountable by 10.1, $H_2((J_2)_p^\wedge; F_p)$ or $H_3((J_1)_p^\wedge; F_p)$ must be uncountable, where J_i is a free group on $i+1$ generators. The theorem now follows from

Lemma 11.2. *If $H_2((J_2)_p^\wedge; F_p)$ is uncountable, then so is $H_2((J_1)_p^\wedge; F_p)$.*

Proof. Choose a normal subgroup $N \subset J_1$, such that J_1/N is a finite p-group and N is free on ≥ 3 generators. Then the sequence

$$N_p^\wedge \rightarrowtail (J_1)_p^\wedge \longrightarrow J_1/N$$

is short exact by 4.2, and the lemma follows since $H_i(J_1/N; H_j(N_p^\wedge; F_p))$ is finite for $j < 2$ and uncountable for $(i, j) = (0, 2)$.

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