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MULTIPLIERS OF FAMILIES OF CAUCHY-STIELTJES TRANSFORMS

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This paper is dedicated to Glenn Schober

ABSTRACT. For $\alpha > 0$ let $\mathcal{F}_\alpha$ denote the class of functions defined for $|z| < 1$ by integrating $1/(1-xz)^\alpha$ against a complex measure on $|x| = 1$. A function g holomorphic in $|z| < 1$ is a multiplier of $\mathcal{F}_\alpha$ if $f \in \mathcal{F}_\alpha$ implies $gf \in \mathcal{F}_\alpha$. The class of all such multipliers is denoted by $\mathcal{M}_\alpha$. Various properties of $\mathcal{M}_\alpha$ are studied in this paper. For example, it is proven that $\alpha < \beta$ implies $\mathcal{M}_\alpha \subset \mathcal{M}_\beta$, and also that $\mathcal{M}_\alpha \subset H^\infty$. Examples are given of bounded functions which are not multipliers. A new proof is given of a theorem of Vinogradov which asserts that if f' is in the Hardy class H^1, then $f \in \mathcal{M}_1$. Also the theorem is improved to $f' \in H^1$ implies $f \in \mathcal{M}_\alpha$, for all $\alpha > 0$. Finally, let $\alpha > 0$ and let f be holomorphic in $|z| < 1$. It is known that f is bounded if and only if its Cesàro sums are uniformly bounded in $|z| \le 1$. This result is generalized using suitable polynomials defined for $\alpha > 0$.

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Let $\Delta = \{z : |z| < 1\}$ and $\Gamma = \{z : |z| = 1\}$, and let $\mathcal{M}$ denote the set of complex-valued Borel measures on Γ. For $\alpha > 0$, let $\mathcal{F}_\alpha$ denote the family of functions f for which there exists $\mu \in \mathcal{M}$ such that

$$f(z) = \int_\Gamma \frac{1}{(1-xz)^\alpha}\, d\mu(x), \qquad |z| < 1. \tag{1}$$

Here we choose the branch of $1/(1-z)^\alpha$ which equals 1 when $z = 0$.

This class of functions has been studied extensively in the case $\alpha = 1$ [1, 7, 8, 10, 15, 16]. More recently, the families $\mathcal{F}_\alpha$ $(\alpha \neq 1)$ were introduced in [13]. Closure properties of the families $\mathcal{F}_\alpha$ were studied by the present authors in [9].

The following two results were proven in [13], and will be useful here.

Theorem A. *For $\alpha > 0$, $f \in \mathcal{F}_\alpha$ if and only if $f' \in \mathcal{F}_{\alpha+1}$.*

Theorem B. *If $f \in \mathcal{F}_\alpha$ and $g \in \mathcal{F}_\beta$, then $fg \in \mathcal{F}_{\alpha+\beta}$.*

For $f \in \mathcal{F}_\alpha$, let

$$\|f\|_{\mathcal{F}_\alpha} = \inf\{\|\mu\| : \mu \in \mathcal{M} \text{ such that (1) holds}\}. \tag{2}$$

With this norm, $\mathcal{F}_\alpha$ is a Banach space. As an example, suppose that $f \in \mathcal{F}_\alpha$, μ is a positive measure, and (1) holds. Then $\|f\|_{\mathcal{F}_\alpha} = \|\mu\|$. In the case $\alpha = 1$,

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this was first observed by P. Bourdon and J. A. Cima, who showed in [1] that if $\nu \in \mathscr{M}$ is any other representing measure for f, then

$$\|\mu\| = \mu(\Gamma) = f(0) = \int_\Gamma 1\, d\nu(x) \le \|\nu\|.$$

We note that by an easy argument, the infimum in (2) is actually attained.

Let $\{f_n : n = 1, 2, \dots\}$ be a sequence of functions in $\mathscr{F}_\alpha$ and suppose that $f_n \to f$ in the norm (2). It is easy to show that this implies that $f_n \to f$ uniformly on compact sets. To see that the converse is false in the case $\alpha = 1$, let $f_n(z) = z^n$ for $|z| < 1$. Then f_n converges uniformly on compact sets to the function $f(z) = 0$. On the other hand, suppose that $\mu_n \in \mathscr{M}$ is any measure representing f_n. Then since

$$z^n = \int_\Gamma \frac{1}{1 - xz}\, d\mu_n(x),$$

it follows that

$$1 = \int_\Gamma x^n\, d\mu_n(x) \le \int_\Gamma 1\, d|\mu_n|(x) = \|\mu_n\|.$$

This shows that for each n, $\|f_n\|_{\mathscr{F}_1} \ge 1$, so that the sequence f_n does not converge to f in norm. In the case $\alpha \ne 1$, a similar example can be constructed.

Definition. Suppose that f is holomorphic in Δ. Then f is called a multiplier of $\mathscr{F}_\alpha$ if $g \in \mathscr{F}_\alpha \Rightarrow fg \in \mathscr{F}_\alpha$.

The family of all such multipliers is denoted by $\mathscr{M}_\alpha$.

Suppose that $f \in \mathscr{M}_\alpha$ for some $\alpha > 0$. An application of the Closed Graph Theorem shows that the map $\Lambda: \mathscr{F}_\alpha \to \mathscr{F}_\alpha$ defined by $\Lambda(g) = fg$ is continuous. Equivalently, Λ is a bounded operator on $\mathscr{F}_\alpha$, so that

$$\sup\{\|fg\|_{\mathscr{F}_\alpha} : g \in \mathscr{F}_\alpha,\ \|g\|_{\mathscr{F}_\alpha} \le 1\} < \infty.$$

This last quantity will be denoted by $\|f\|_{\mathscr{M}_\alpha}$, and with this norm $\mathscr{M}_\alpha$ is itself a Banach space.

This paper is concerned with the multiplier families $\mathscr{M}_\alpha$. The family $\mathscr{M}_1$ has been studied in [10], [15], and [16], and various properties of $\mathscr{M}_1$ which were developed there will be generalized to $\mathscr{M}_\alpha$ for $\alpha \ne 1$. For example, S. A. Vinogradov [16] has shown that if f' is in the Hardy space H^1, then $f \in \mathscr{M}_1$. We give a new proof of this result, and show that if $f' \in H^1$, then $f \in \mathscr{M}_\alpha$, for every $\alpha > 0$. Also we show that if $f \in \mathscr{M}_\alpha$, then f is bounded, and that f has a number of other properties. Examples are given of bounded functions which are not in any $\mathscr{M}_\alpha$ for $\alpha > 0$.

Finally, suppose that f is holomorphic in Δ, and let $f(z) = \sum_{n=0}^\infty a_n z^n$. Let

$$\sigma_n(z) = \sum_{j=0}^{n} \frac{n - j + 1}{n + 1} a_j z^j.$$

It is a classical result that f is bounded if and only if the Cesàro sums $\sigma_n(z)$ are uniformly bounded for $|z| \le 1$, and that in this case $\|\sigma_n\|_{H^\infty} \le \|f\|_{H^\infty}$,

$n = 0, 1, \ldots$. This result is generalized here where σ_n is replaced by suitable polynomials depending on $\alpha > 0$.

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In this section various properties of the families $\mathscr{M}_\alpha$ are studied. The following lemma will be useful.

Lemma 2.1. *Let f be holomorphic in Δ, and let $\alpha > 0$. Then $f \in \mathscr{M}_\alpha$ if and only if $f(z)/(1-xz)^\alpha \in \mathscr{F}_\alpha$ for every x with $|x| = 1$ and there exists a constant M such that $\|f(z)/(1-xz)^\alpha\|_{\mathscr{F}_\alpha} \leq M$ for $|x| = 1$.*

Proof. First suppose that $f \in \mathscr{M}_\alpha$. Then multiplication by f is a bounded operator on $\mathscr{F}_\alpha$, and there is a constant M such that

$$\|fg\|_{\mathscr{F}_\alpha} \leq M\|g\|_{\mathscr{F}_\alpha} \tag{3}$$

for all $g \in \mathscr{F}_\alpha$. In particular, (3) holds for all functions of the form $g(z) = 1/(1-xz)^\alpha$, where $|x| = 1$. Since $\|1/(1-xz)^\alpha\|_{\mathscr{F}_\alpha} = 1$, this implies that $\|f(z)/(1-xz)^\alpha\|_{\mathscr{F}_\alpha} \leq M$ for all $|x| = 1$.

For the converse, let $g \in \mathscr{F}_\alpha$. Then for some $\mu \in \mathscr{M}$,

$$g(z) = \int_\Gamma \frac{1}{(1-xz)^\alpha}\, d\mu(x).$$

To show that $fg \in \mathscr{F}_\alpha$, it is enough to consider the case in which μ is a probability measure. Then g is the limit in the topology of uniform convergence on compact subsets of Δ of functions of the form

$$h(z) = \sum_{k=1}^{n} \mu_k \frac{1}{(1-x_k z)^\alpha}$$

where $\mu_k \geq 0$, $\sum_{k=1}^n \mu_k = 1$, $|x_k| = 1$, and n is a natural number.

For such a function h,

$$f(z)h(z) = \sum_{k=1}^{n} \mu_k \frac{f(z)}{(1-x_k z)^\alpha}. \tag{4}$$

By the assumption, there is a measure $\nu_k \in \mathscr{M}$ with $\|\nu_k\| \leq M$ such that

$$\frac{f(z)}{(1-x_k z)^\alpha} = \int_\Gamma \frac{1}{(1-xz)^\alpha}\, d\nu_k(x).$$

Letting $\lambda = \sum_{k=1}^n \mu_k \nu_k$, (4) can be written as

$$f(z)h(z) = \int_\Gamma \frac{1}{(1-xz)^\alpha}\, d\lambda(x),$$

where $\lambda \in \mathscr{M}$ and $\|\lambda\| \leq \sum_{k=1}^n \mu_k \|\nu_k\| \leq M \sum_{k=1}^n \mu_k = M$.

Since $\{\lambda \in \mathscr{M} : \|\lambda\| \leq M\}$ is compact, an argument using subsequences now yields a measure $\sigma \in \mathscr{M}$ with $\|\sigma\| \leq M$ and $f(z)g(z) = \int_\Gamma 1/(1-xz)^\alpha\, d\sigma(x)$. Therefore $fg \in \mathscr{F}_\alpha$, and $f \in \mathscr{M}_\alpha$.

Theorem 2.2. *If* $0 < \alpha < \beta$, *then* $\mathscr{M}_\alpha \subset \mathscr{M}_\beta$.

Proof. Let $f \in \mathscr{M}_\alpha$. By 2.1, it is enough to show that $f(z)/(1-xz)^\beta \in \mathscr{F}_\beta$ for every x with $|x| = 1$, and to show that there is a constant N such that $\|f(z)/(1-xz)^\beta\|_{\mathscr{F}_\beta} \le N$, for $|x| = 1$.

Since $f \in \mathscr{M}_\alpha$, the lemma implies that there is a constant M with

$$\|f(z)/(1-xz)^\alpha\|_{\mathscr{F}_\alpha} \le M, \qquad \text{for } |x| = 1.$$

Equivalently, for any x with $|x| = 1$, there is a measure $\mu_x \in \mathscr{M}$ such that

$$\frac{f(z)}{(1-xz)^\alpha} = \int_\Gamma \frac{1}{(1-yz)^\alpha}\, d\mu_x(y) \tag{5}$$

and $\|\mu_x\| \le M$.

Since

$$\frac{f(z)}{(1-xz)^\beta} = \frac{f(z)}{(1-xz)^\alpha}\frac{1}{(1-xz)^{\beta-\alpha}},$$

(5) yields that

$$\frac{f(z)}{(1-xz)^\beta} = \left\{\int_\Gamma \frac{1}{(1-yz)^\alpha}\, d\mu_x(y)\right\}\frac{1}{(1-xz)^{\beta-\alpha}}$$
$$= \int_\Gamma \frac{1}{(1-yz)^\alpha}\frac{1}{(1-xz)^{\beta-\alpha}}\, du_x(y).$$

For every x and y with $|x| = |y| = 1$, there is a probability measure $\nu_{x,y}$ such that

$$\frac{1}{(1-yz)^\alpha}\frac{1}{(1-xz)^{\beta-\alpha}} = \int_\Gamma \frac{1}{(1-wz)^\beta}\, d\nu_{x,y}(w) \quad [2, \text{p. } 415].$$

Therefore,

$$\frac{f(z)}{(1-xz)^\beta} = \int_\Gamma\int_\Gamma \frac{1}{(1-wz)^\beta}\, d\nu_{x,y}(w)\, d\mu_x(y).$$

Because $\|\nu_{x,y}\| \le 1$ and $\|\mu_x\| \le M$, an argument as in the proof of Lemma 2.1 shows that there is a measure $\lambda \in \mathscr{M}$ with $\|\lambda\| \le M$ and such that

$$\frac{f(z)}{(1-xz)^\beta} = \int_\Gamma \frac{1}{(1-sz)^\beta}\, d\lambda(s).$$

This shows that $f(z)/(1-xz)^\beta \in \mathscr{F}_\beta$, and that $\|f(z)/(1-xz)^\beta\|_{\mathscr{F}_\beta} \le M$.

Next we obtain several properties of functions in $\mathscr{M}_\alpha$. First it is shown that such functions are bounded.

Theorem 2.3. *Let* $\alpha > 0$ *and let* $f \in \mathscr{M}_\alpha$. *Then* $f \in H^\infty$, *and* $\|f\|_{H^\infty} \le \|f\|_{\mathscr{M}_\alpha}$.

Proof. Let M be a constant with $\|f\|_{\mathscr{M}_\alpha} < M$. Let $z_0 = re^{i\theta}$ $(0 \le r < 1)$ and let $x = e^{-i\theta}$.

Since $f \in \mathscr{M}_\alpha$, there is a measure $\mu_x \in \mathscr{M}$ with $\|\mu_x\| < M$ and such that

$$\frac{f(z)}{(1-xz)^\alpha} = \int_\Gamma \frac{1}{(1-yz)^\alpha}\, d\mu_x(y).$$

It follows that

$$f(z) = \int_\Gamma \left(\frac{1-xz}{1-yz}\right)^\alpha d\mu_x(y). \tag{6}$$

Letting $z = z_0$ in (6) yields

$$(7) \qquad |f(re^{i\theta})| = \left| \int_\Gamma \left(\frac{1-r}{1-r\bar{x}y} \right)^\alpha d\mu_x(y) \right| \leq \int_\Gamma d|\mu_x|(y) < M .$$

Since (7) holds for all r and θ, it follows that $f \in H^\infty$ and $\|f\|_{H^\infty} < M$, for every M with $M > \|f\|_{\mathcal{M}_\alpha}$. Therefore, $\|f\|_{H^\infty} \leq \|f\|_{\mathcal{M}_\alpha}$.

Theorem 2.4. *Let $\alpha > 0$, and let $f \in \mathcal{M}_\alpha$. Then $f \in \mathcal{F}_\alpha$, and $\|f\|_{\mathcal{F}_\alpha} \leq \|f\|_{\mathcal{M}_\alpha}$.*

Proof. Let $I(z) = 1$ for $|z| < 1$. Since

$$I(z) = \int_\Gamma \frac{1}{(1-xz)^\alpha} \, dm(x),$$

where m denotes normalized Lebesgue measure, $I \in \mathcal{F}_\alpha$. Also, since m is a positive measure, the remark in §1 shows that

$$(8) \qquad \|I\|_{\mathcal{F}_\alpha} = \|m\| = 1 .$$

Since $f \in \mathcal{M}_\alpha$ and $I \in \mathcal{F}_\alpha$, it follows that $f = fI \in \mathcal{F}_\alpha$. Also, since

$$\|f\|_{\mathcal{F}_\alpha} = \|fI\|_{\mathcal{F}_\alpha} \leq \|f\|_{\mathcal{M}_\alpha} \|I\|_{\mathcal{F}_\alpha},$$

(8) implies that

$$(9) \qquad \|f\|_{\mathcal{F}_\alpha} \leq \|f\|_{\mathcal{M}_\alpha}. \quad \Box$$

We note that the inequality (9) is sharp, because $I \in \mathcal{M}_\alpha$ and $\|I\|_{\mathcal{F}_\alpha} = 1$.

As an application of Theorem 2.4, let

$$(10) \qquad \frac{1}{(1-z)^\alpha} = \sum_{n=0}^\infty A_n(\alpha) z^n \qquad (|z| < 1),$$

and suppose that $f \in \mathcal{M}_\alpha$ where $f(z) = \sum_{n=0}^\infty a_n z^n$ $(|z| < 1)$. The theorem asserts that for some $\mu \in \mathcal{M}$,

$$(11) \qquad f(z) = \int_\Gamma \frac{1}{(1-xz)^\alpha} \, d\mu(x).$$

Equations (10) and (11) imply that

$$a_n = A_n(\alpha) \int_\Gamma x^n \, d\mu(x).$$

Since $A_n(\alpha) = O(n^{\alpha-1})$, and since $|\int_\Gamma x^n \, d\mu(x)| \leq \|\mu\|$, this shows that the coefficients of f obey $|a_n| = O(n^{\alpha-1})$.

In the case $0 < \alpha < 1$, this coefficient estimate provides additional information on functions in $\mathcal{M}_\alpha$. Suppose that f is holomorphic in Δ, and that $f(z) = \sum_{n=0}^\infty a_n z^n$. In [16] it was shown that if $\sum_{n=0}^\infty |a_n| \log(n+2) < \infty$, then $f \in \mathcal{M}_1$. In particular, the function $f(z) = \sum_{n=0}^\infty (1/n^3) z^{2^n}$ is in $\mathcal{M}_1$, but for $m = 2^n$, $a_m \neq O(m^{\alpha-1})$, for each α $(0 < \alpha < 1)$. This shows that $f \notin \mathcal{M}_\alpha$ for $\alpha < 1$. The first author and E. A. Nordgren have shown that $\mathcal{M}_1 \neq \mathcal{M}_2$, and also that for $0 < \alpha < \beta < 1$, $\mathcal{M}_\alpha \neq \mathcal{M}_\beta$. It is an open question to determine if $\mathcal{M}_\alpha \neq \mathcal{M}_\beta$ for all $\alpha \neq \beta$.

It was shown in [9] that $\mathcal{F}_\alpha$ is closed under composition with disk automorphisms $z \to (z+\xi)/(1+\bar{\xi}z)$, where $|\xi| < 1$. This will be used in the proof of the next theorem, which asserts the same result for $\mathcal{M}_\alpha$.

Theorem 2.5. *Let* $\alpha > 0$. *If* $f \in \mathcal{M}_\alpha$, $|\xi| < 1$, *and* $g(z) = f((z+\xi)/(1+\overline{\xi}z))$, *then* $g \in \mathcal{M}_\alpha$.

Proof. Let $h \in \mathcal{F}_\alpha$, and let $k(z) = h((z-\xi)/(1-\overline{\xi}z))$. Since the map $w = (z-\xi)/(1-\overline{\xi}z)$ is an automorphism of Δ, the result in [9] quoted above shows that $k \in \mathcal{F}_\alpha$. Since $f \in \mathcal{M}_\alpha$, it follows that $m = fk \in \mathcal{F}_\alpha$. A second application of the result in [9] implies that $m((z+\xi)/(1+\overline{\xi}z)) \in \mathcal{F}_\alpha$. Since

$$m\left(\frac{z+\xi}{1+\overline{\xi}z}\right) = f\left(\frac{z+\xi}{1+\overline{\xi}z}\right) k\left(\frac{z+\xi}{1+\overline{\xi}z}\right) = g(z)h(z),$$

this shows that $g \in \mathcal{M}_\alpha$.

The following theorem generalizes a result in [16], which showed that if $f \in \mathcal{M}_1$, then f has finite radial variation in every direction.

Theorem 2.6. *For each* $\alpha > 0$ *there is a constant* A_α *such that if* $f \in \mathcal{M}_\alpha$, *then the radial variation of* f *in the direction* θ *obeys* $V(f, \theta) \le A_\alpha \|f\|_{\mathcal{M}_\alpha}$ *for all* θ.

Proof. Suppose that $f \in \mathcal{M}_\alpha$ for some $\alpha > 0$. If $|\xi| = 1$ then there is a measure μ_ξ such that

$$f(z)\frac{1}{(1-\xi z)^\alpha} = \int_\Gamma \frac{1}{(1-xz)^\alpha}\, d\mu_\xi(x). \tag{12}$$

Also, if $M = \|f\|_{\mathcal{M}_\alpha}$, and $\varepsilon > 0$, then $\|\mu_\xi\| \le M + \varepsilon$ for $|\xi| = 1$.

It follows from (12) that

$$f'(z) = \alpha \int_\Gamma \frac{(1-\xi z)^{\alpha-1}(x-\xi)}{(1-xz)^{\alpha+1}}\, d\mu_\xi(x),$$

and therefore

$$\int_0^1 |f'(r\overline{\xi})|\, dr \le \alpha \int_\Gamma \left[\int_0^1 \frac{(1-r)^{\alpha-1}|x-\xi|}{|1-rx\overline{\xi}|^{\alpha+1}}\, dr\right] d|\mu_\xi|(x). \tag{13}$$

Let I denote the inner integral on the right-hand side of (13). Because

$$\begin{aligned}|1-rx\overline{\xi}|^{\alpha+1} &= \{|1-rx\overline{\xi}|^2\}^{(\alpha+1)/2} = \{(1-r)^2 + r|1-x\overline{\xi}|^2\}^{(\alpha+1)/2} \\ &\ge \{(1-r)^2 + r^2|1-x\overline{\xi}|^2\}^{(\alpha+1)/2},\end{aligned}$$

it follows that

$$I \le \int_0^1 \frac{(1-r)^{\alpha-1} b}{\{(1-r)^2 + r^2 b^2\}^{(\alpha+1)/2}}\, dr \equiv J,$$

where $b = |1 - x\overline{\xi}|$. The change of variables $y = rb/(1-r)$ shows that $J = \int_0^\infty \frac{1}{(1+y^2)^{(\alpha+1)/2}}\, dy \equiv B_\alpha$. This integral converges since $\int_1^\infty 1/y^\beta\, dy$ converges for $\beta > 1$. Therefore (13) yields that

$$\int_0^1 |f'(r\overline{\xi})|\, dr \le \alpha \int_\Gamma B_\alpha\, d|\mu_\xi|(x) \le A_\alpha(M+\varepsilon),$$

where $A_\alpha = \alpha B_\alpha$. Let $\varepsilon \to 0$, the theorem is established. □

Let $f \in \mathcal{M}_\alpha$. As a consequence of Theorem 2.6, the radial limit $\lim_{r\to 1} f(re^{i\theta})$ exists for all θ. Also, note that the conclusion of the theorem implies that f is bounded.

As an application of Theorem 2.6, we next give a number of simple examples of bounded functions which are not in M_α for any $\alpha > 0$.

As a first example, let $f(z) = (1-z)^{-i}$, using the principal branch of the logarithm. Then f is holomorphic in Δ, and since $|f(z)| = e^{-\operatorname{Arg}(1-z)}$, it follows that $|f(z)| < e^{\pi/2}$ for $|z| < 1$. It is easy to verify that f maps the interval $[0, 1)$ onto the circle Γ covered infinitely often and hence the curve $w = f(r)$, $0 \le r < 1$, is not rectifiable. It follows by Theorem 2.6 that $f \notin \mathscr{M}_\alpha$ for any $\alpha > 0$.

In [9], it was shown that if f is holomorphic in $\overline{\Delta}$, then $f \in \mathscr{M}_\alpha$ for all $\alpha > 0$. In particular, this implies that a finite Blaschke product belongs to $\mathscr{M}_\alpha$ for $\alpha > 0$. Theorem 2.5 provides a second proof of this fact, as follows. Let $I(z) = z$ for $|z| < 1$. It is clear that $I \in \mathscr{M}_\alpha$ for $\alpha > 0$. If $|\xi| < 1$, then Theorem 2.5 implies that

$$I\left(\frac{z+\xi}{1+\overline{\xi}z}\right) = \frac{z+\xi}{1+\overline{\xi}z} \in \mathscr{M}_\alpha, \qquad \text{for } \alpha > 0.$$

Since the finite product of functions in $\mathscr{M}_\alpha$ is itself in $\mathscr{M}_\alpha$, this proves the assertion.

We next show that there are infinite Blaschke products which are not in $\mathscr{M}_\alpha$ for any $\alpha > 0$. Let $f(z) = \prod_{n=1}^{\infty}(a_n - z)/(1 - a_n z)$ where $a_n = 1 - 1/2^n$, $n = 1, 2, \ldots$. In [6] it was shown that there is a constant $A > 0$ such that if $\rho_n = \frac{1}{2}(a_n + a_{n+1})$ then $|f(\rho_n)| \ge A$ for $n = 1, 2, \ldots$. It follows that $\int_0^1 |f'(r)|\, dr = \infty$, so that by Theorem 2.6, $f \notin \mathscr{M}_\alpha$ for $\alpha > 0$.

We note that in [10], it was proved that an inner function belongs to $\mathscr{M}_1$ if and only if it is a Blaschke product with the sequence of zeros satisfying the Frostman condition.

The next example shows that a function holomorphic in Δ and continuous in $\overline{\Delta}$ need not be in $\mathscr{M}_\alpha$ for any $\alpha > 0$. In [17], L. Zalcman described a bounded region D such that ∂D is a Jordan curve, $z = 1 \in \partial D$, and $z = 1$ is not rectifiably accessible from the interior of D. Since ∂D is a Jordan curve, any conformal mapping of Δ onto D extends continuously to $\overline{\Delta}$. Let f be such a map with $f(1) = 1$. Then $f \notin \mathscr{M}_\alpha$, since the curve $w = f(r)$, $0 \le r \le 1$, is not rectifiable. The argument in [17] even shows that the power series for f is uniformly convergent on $\partial\Delta$. Hence even with this additional condition we can still have $f \notin \mathscr{M}_\alpha$ for all $\alpha > 0$.

The examples above give bounded functions for which the radial variation in one direction is infinite. A stronger result is presented in [14], where examples are given of infinite Blaschke products $B(z)$ for which the radial variation $V(B, \theta) = \infty$ for almost all θ. Also, [14] includes the construction of a function f holomorphic in Δ and continuous in $\overline{\Delta}$ for which $V(f, \theta) = \infty$ for almost all θ.

3

In this section a condition is shown to be sufficient for membership in $\mathscr{M}_\alpha$ for every $\alpha > 0$. Let H^1 denote the Hardy space of functions f that are holomorphic in Δ and such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta < \infty.$$

In [16, p. 20] it was proved by Vinogradov that if $f' \in H^1$ then $f \in \mathscr{M}_1$. This result is generalized to $f' \in H^1$ implies $f \in \mathscr{M}_\alpha$ for every $\alpha > 0$. This strengthens the result in [9] which asserts that if f is holomorphic in $\overline{\Delta}$ then $f \in \mathscr{M}_\alpha$ for every $\alpha > 0$.

We begin by giving a new proof of Vinogradov's theorem. It may have independent interest especially since it shows that this result is related to the class of functions of bounded mean oscillation [5, p. 222]. Let $\mathscr{B}$ denote the set of functions f holomorphic in Δ which can be expressed as $f = g + h$, where g and h are holomorphic in Δ, $\operatorname{Re} g$ is bounded in Δ, and $\operatorname{Im} h$ is bounded in Δ. If $f \in \mathscr{B}$ then $\|f\|_{\mathscr{B}}$ is defined by $\inf(\|\operatorname{Re} g\|_\infty + \|\operatorname{Im} h\|_\infty)$ where g and h vary over all pairs as above. Here $\|u\|_\infty = \sup_{|z|<1} |u(z)|$ for any function u defined in Δ.

Lemma 3.1. *Let f be holomorphic in Δ and suppose that there is a holomorphic function g and a constant $M > 0$ such that*

$$|f(z) + g(\overline{z})| \leq M \quad \text{for } |z| < 1. \tag{14}$$

Then $f \in \mathscr{B}$ and $\|f\|_{\mathscr{B}} \leq M$.

Proof. Let $s = \operatorname{Re} f$, $t = \operatorname{Im} f$, $u = \operatorname{Re} g$, and $v = \operatorname{Im} g$. The function G defined by $G(z) = \frac{1}{2}[f(z) + \overline{g(\overline{z})}]$ is holomorphic in Δ and $\operatorname{Re} G(z) = \frac{1}{2}[s(z) + u(\overline{z})]$. Hence (14) implies that $|\operatorname{Re} G(z)| \leq \frac{1}{2}M$ for $|z| < 1$. The function H defined by $H(z) = \frac{1}{2}[f(z) - \overline{g(\overline{z})}]$ is holomorphic in Δ and $\operatorname{Im} H(z) = \frac{1}{2}[t(z) + v(\overline{z})]$. Hence (14) implies $|\operatorname{Im} H(z)| \leq \frac{1}{2}M$ for $|z| < 1$. Since $f = G + H$ this yields $f \in \mathscr{B}$. Moreover $\|f\|_{\mathscr{B}} \leq \|\operatorname{Re} G\|_\infty + \|\operatorname{Im} H\|_\infty \leq M$.

Lemma 3.2. *Let $f \in H^\infty$ and let g be defined by*

$$g(z) = \frac{1}{z} \int_0^z \frac{f(w)}{1-w}\, dw \tag{15}$$

for $|z| < 1$. Then $|g'(z)| \leq B\|f\|_{H^\infty}/|1-z|$ for $|z| < 1$, where B is an absolute constant.

Proof. We first show that if $|z| < 1$ and α is the line segment from $w = 0$ to $w = z$ then

$$\int_\alpha \frac{1}{|1-w|^2} |dw| \leq \frac{\pi}{2} \frac{|z|}{|1-z|}. \tag{16}$$

This is clear if $z = 0$. Also if z is real and $z \neq 0$ then we have

$$\int_\alpha \frac{1}{|1-w|^2} |dw| = |z| \int_0^1 \frac{1}{(1-tz)^2}\, dt = \frac{|z|}{1-z},$$

and hence (16) follows. Henceforth assume that $|z| < 1$ and z is not real. Then

$$\begin{aligned}
\int_\alpha \frac{1}{|1-w|^2} |dw| &= |z| \int_0^1 \frac{1}{(1-tz)(1-t\overline{z})}\, dt \\
&= \frac{|z|}{z - \overline{z}} \left\{ \log \frac{1}{1-z} - \log \frac{1}{1-\overline{z}} \right\} \\
&= \frac{|z|}{z - \overline{z}} \int_\beta \frac{1}{1-w}\, dw
\end{aligned}$$

where β is the arc on the circle that is centered at $w = 1$ and goes from $\overline{z}$ to z. Let θ denote the angle subtended by the arc β and let L denote the length of β. Then $|z - \overline{z}| = 2|1 - z| \sin(\theta/2)$ and $L = |1 - z|\theta$. Therefore

$$\int_\alpha \frac{1}{|1 - w|^2} |dw| \leq \frac{|z|}{|z - \overline{z}|} \frac{1}{|1 - z|} L = \frac{\theta/2}{\sin(\theta/2)} \frac{|z|}{|1 - z|} \leq \frac{\pi}{2} \frac{|z|}{|1 - z|},$$

since $0 < \theta/2 \leq \pi/2$. This proves (16).

From (15) we obtain $zg'(z) + g(z) = f(z)/(1 - z)$. Hence an integration by parts yields

$$\begin{aligned} z^2 g'(z) &= \frac{zf(z)}{1 - z} - zg(z) = \frac{zf(z)}{1 - z} - \int_0^z \frac{f(w)}{1 - w}\, dw \\ &= \frac{zf(z)}{1 - z} - \frac{h(z)}{1 - z} + \int_0^z \frac{h(w)}{(1 - w)^2}\, dw, \end{aligned}$$

where

$$h(z) = \int_0^z f(w)\, dw \tag{17}$$

for $|z| < 1$. Clearly (17) implies $\|h\|_{H^\infty} \leq \|f\|_{H^\infty} \equiv M$. It follows that

$$|z^2 g'(z)| \leq \frac{M}{|1 - z|} + \frac{M}{|1 - z|} + M \int_\alpha \frac{1}{|1 - w|^2} |dw|.$$

Therefore (16) implies that

$$|z^2 g'(z)| \leq \left(2 + \frac{\pi}{2}\right) M \frac{1}{|1 - z|} \quad \text{for } |z| < 1.$$

The function G defined by $G(z) = (1 - z)z^2 g'(z)$ is analytic in Δ, has at least a second order zero at $z = 0$ and satisfies $|G(z)| \leq BM$ for $|z| < 1$ where $B = 2 + \pi/2$. Hence $|G(z)| \leq BM|z|^2$ for $|z| < 1$ and therefore $|g'(z)| \leq BM/|1 - z|$. □

Lemma 3.3. *Suppose that $f \in H^\infty$ and g is defined by*

$$g(z) = \frac{1}{z} \int_0^z \frac{f(w)}{1 - w}\, dw \tag{18}$$

for $|z| < 1$. Then $g \in \mathscr{B}$ and $\|g\|_{\mathscr{B}} \leq A\|f\|_{H^\infty}$ where A is an absolute constant.

Proof. By equation (18) and Lemma 3.2, there is an absolute constant B such that

$$|g'(z)| \leq \frac{B\|f\|_{H^\infty}}{|1 - z|} \quad \text{for } |z| < 1. \tag{19}$$

Let $|z| < 1$ and let γ denote the circle centered at 1 which passes through z and has radius $r = |1 - z|$. Let δ denote the subarc of γ from $\overline{z}$ to z. Then

$$g(z) - g(\overline{z}) = \int_\delta g'(w)\, dw$$

and hence (19) implies that

$$|g(z) - g(\overline{z})| \leq \frac{B\|f\|_{H^\infty}}{r} (\text{length of } \delta) \leq \frac{\pi}{2} B\|f\|_{H^\infty}.$$

An application of Lemma 3.1 in the special case where the functions there are related by $g = -f$ implies that $g \in \mathscr{B}$ and $\|g\|_{\mathscr{B}} \leq A\|f\|_{H^\infty}$ where $A = \pi B/2$. □

Lemma 3.4. *Suppose that* f *and* g *are functions holomorphic in* $\overline{\Delta}$ *and let* F *and* G *be defined by*

$$F(z) = \frac{1}{1-z}\int_z^1 f(w)\,dw \tag{20}$$

and

$$G(z) = \frac{1}{z}\int_0^z \frac{1}{1-w}g(w)\,dw\,. \tag{21}$$

Then

$$\int_0^{2\pi} f(e^{i\theta})G(e^{-i\theta})\,d\theta = \int_0^{2\pi} F(e^{i\theta})g(e^{-i\theta})\,d\theta\,. \tag{22}$$

Proof. There is a number $R > 1$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ for $|z| < R$. Then F also is holomorphic in $\{z : |z| < R\}$ and G is holomorphic in $\overline{\Delta}$ except possibly for a logarithmic singularity at $z = 1$. In particular, $G \in H^1$ (in fact, $G \in H^p$ for all $p > 0$). For $|z| < R$ we have

$$\begin{aligned} F(z) &= \frac{1}{1-z}\int_z^1 \left(\sum_{n=0}^{\infty} a_n w^n\right) dw = \frac{1}{1-z}\sum_{n=0}^{\infty}\frac{a_n}{n+1}(1-z^{n+1}) \\ &= \sum_{n=0}^{\infty}\left\{\frac{a_n}{n+1}\sum_{k=0}^{n} z^k\right\} = \sum_{n=0}^{\infty}\left\{\sum_{k=n}^{\infty}\frac{a_k}{k+1}\right\} z^n\,. \end{aligned}$$

Therefore

$$\int_0^{2\pi} F(e^{i\theta})g(e^{-i\theta})d\theta = 2\pi\sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\frac{a_k}{k+1}\right) b_n = 2\pi\sum_{n=0}^{\infty}\left\{\frac{a_n}{n+1}\sum_{k=0}^{n} b_k\right\}. \tag{23}$$

For $|z| < 1$, we have

$$\begin{aligned} G(z) &= \frac{1}{z}\int_0^z \left(\sum_{n=0}^{\infty} w^n\right)\left(\sum_{n=0}^{\infty} b_n w^n\right) dw \\ &= \frac{1}{z}\int_0^z \left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} b_k\right) w^n\right) dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} b_k\right) z^n\,. \end{aligned}$$

If $0 < r < 1$ then

$$\int_0^{2\pi} f(e^{i\theta})G(re^{-i\theta})d\theta = 2\pi\sum_{n=0}^{\infty}\left(\frac{a_n}{n+1}\sum_{k=0}^{n} b_k\right) r^n \equiv H(r)\,.$$

Since the series defining H converges at $r = 1$, Abel's theorem gives

$$\lim_{r\to 1-}\int_0^{2\pi} f(e^{i\theta})G(re^{-i\theta})\,d\theta = \lim_{r\to 1-} H(r) = 2\pi\sum_{n=0}^{\infty}\left(\frac{a_n}{n+1}\sum_{k=0}^{n} b_k\right). \tag{24}$$

Also, because $f(e^{i\theta})$ is bounded and $G \in H^1$ it follows that

$$(25) \qquad \lim_{r \to 1-} \int_0^{2\pi} f(e^{i\theta}) G(re^{-i\theta}) d\theta = \int_0^{2\pi} f(e^{i\theta}) G(e^{-i\theta}) d\theta .$$

Therefore by (23), (24), and (25),

$$\int_0^{2\pi} F(e^{i\theta}) g(e^{-i\theta}) d\theta = 2\pi \sum_{n=0}^{\infty} \left(\frac{a_n}{n+1} \sum_{k=0}^{n} b_k \right) = \int_0^{2\pi} f(e^{i\theta}) G(e^{-i\theta}) d\theta . \quad \square$$

We thank D. J. Hallenbeck for pointing out and rectifying an error in our initial proof of Lemma 3.4.

Theorem C (Vinogradov). *If* $f' \in H^1$ *, then* $f \in \mathscr{M}_1$.

Proof. Suppose that $f' \in H^1$ and $|\xi| = 1$. We first note that

$$\frac{f(z)}{\xi - z} = \frac{1}{\xi} \frac{f(z)}{1 - \overline{\xi} z} .$$

Therefore by Lemma 2.1, it is enough to show that $f(z)/(\xi - z) \in \mathscr{F}_1$, and that there is a constant $M > 0$ such that $\|f(z)/(\xi - z)\|_{\mathscr{F}_1} \le M$ for all $|\xi| = 1$. Also note that

$$\frac{f(z)}{\xi - z} = \frac{1}{\xi - z} \int_0^z f'(w)\, dw + \frac{f(0)}{\xi - z} .$$

Since $f(0)/(\xi - z) \in \mathscr{F}_1$ and since $\|f(0)/(\xi - z)\|_{\mathscr{F}_1} = |f(0)|$, it suffices to show that the function $(\xi - z)^{-1} \int_0^z f'(w)\, dw$ belongs to $\mathscr{F}_1$ and that for some $M > 0$, $\|(\xi - z)^{-1} \int_0^z f'(w)\, dw\|_{\mathscr{F}_1} \le M$ for all $|\xi| = 1$. The argument is carried out with $\xi = 1$ and a similar argument serves for all ξ providing the same bound on the norm.

In our formulation we replace f' by f . In other words, assume that $f \in H^1$ and let

$$(26) \qquad g(z) = \frac{1}{1 - z} \int_0^z f(w)\, dw \quad \text{for } |z| < 1 .$$

Then $g(z) = b/(1 - z) - (1 - z)^{-1} \int_z^1 f(w)\, dw$, where $b = \int_0^1 f(w)\, dw$.

First note that

$$\begin{aligned} |b| &\le \int_0^1 |f(w)|\, |dw| \le \int_{-1}^1 |f(w)| |dw| \\ &\le \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|\, d\theta = \pi \|f\|_{H^1} \quad [4, \text{p. } 46]. \end{aligned}$$

It follows that

$$(27) \qquad \left\| \frac{b}{1 - z} \right\|_{\mathscr{F}_1} \le \pi \|f\|_{H^1}$$

Next let $k(z) = (1 - z)^{-1} \int_z^1 f(w)\, dw$. Let A denote the space of functions holomorphic in Δ and continuous in $\overline{\Delta}$. To show that $k \in \mathscr{F}_1$ it suffices to prove that there is a constant $A > 0$ such that

$$(28) \qquad \left| \int_0^{2\pi} k(re^{i\theta}) h(e^{-i\theta})\, d\theta \right| \le A \|h\|_{H^\infty}$$

for $0 < r < 1$ and for all $h \in A$. This inequality will be obtained where $A = B\|f\|_{H^1}$ and B is an absolute constant. This will imply that

$$\|k\|_{\mathscr{F}_1} \leq B\|f\|_{H^1} \tag{29}$$

and it then follows from (26), (27), and (29) that $\|g\|_{\mathscr{F}_1} \leq (\pi + B)\|f\|_{H^1}$.

By first making the change of variables $z \to \rho z$ where $0 < \rho < 1$ and then letting $\rho \to 1$, we may assume that f and h are holomorphic in $\overline{\Delta}$. Then k is holomorphic in $\overline{\Delta}$. We now show that it suffices to prove that

$$\left| \int_0^{2\pi} k(e^{i\theta})h(e^{-i\theta})\,d\theta \right| \leq C\|f\|_{H^1}\|h\|_{H^\infty}, \tag{30}$$

where C is an absolute constant. For $0 \leq r \leq 1$ let $F(r) = \int_0^{2\pi} k(re^{i\theta})h(e^{i\theta})\,d\theta$. Assuming(30) we get $|F(1)| \leq C\|f\|_{H^1}\|h\|_{H^\infty}$. Since F is continuous in $[0, 1]$, there exists r_0 $(0 < r_0 < 1)$ such that $|F(r)| \leq 2|F(1)|$ for $r_0 \leq r \leq 1$. Therefore

$$|F(r)| \leq 2C\|f\|_{H^1}\|h\|_{H^\infty} \quad \text{for } r_0 \leq r < 1. \tag{31}$$

Suppose now that $0 \leq r \leq r_0$. Then

$$|F(r)| \leq \int_0^{2\pi} |k(re^{i\theta})|\,|h(e^{-i\theta})|d\theta \leq \|h\|_{H^\infty} \int_0^{2\pi} |k(r_0e^{i\theta})|d\theta.$$

Without loss of generality we may assume that $f \neq 0$. Then $k \neq 0$, $\|f\|_{H^1} > 0$, and $\int_0^{2\pi} |k(r_0e^{i\theta})|d\theta > 0$. Therefore for some $D > 0$, $\int_0^{2\pi} |k(r_0e^{i\theta})|d\theta = D\|f\|_{H^1}$. It follows that

$$|F(r)| \leq D\|f\|_{H^1}\|h\|_{H^\infty} \quad \text{for } 0 \leq r \leq r_0. \tag{32}$$

Letting $B = \max(2C, D)$, relations (31) and (32) imply that

$$|F(r)| \leq B\|f\|_{H^1}\|h\|_{H^\infty} \quad \text{for } 0 \leq r \leq 1.$$

This proves (28).

It remains to prove the assertion (30). Let $m(z) = z^{-1}\int_0^z (1-w)^{-1}h(w)\,dw$. Lemma 3.3 implies that $m \in \mathscr{B}$ and $\|m\|_{\mathscr{B}} \leq C\|h\|_{H^\infty}$ for an absolute constant C. We have $m = p + q$ where p and q are holomorphic in Δ and $u = \operatorname{Re} p$ and $v = \operatorname{Im} q$ are bounded and $\|u\|_\infty + \|v\|_\infty \leq C\|h\|_{H^\infty}$. Now

$$\int_0^{2\pi} f(e^{i\theta})m(e^{-i\theta})\,d\theta = \int_0^{2\pi} f(e^{i\theta})p(e^{-i\theta})\,d\theta + \int_0^{2\pi} f(e^{i\theta})q(e^{-i\theta})\,d\theta.$$

Using power series and the orthonormal relations for the trigonometric functions, this equals

$$\int_0^{2\pi} f(e^{i\theta})u(e^{-i\theta})\,d\theta + i\int_0^{2\pi} f(e^{i\theta})v(e^{-i\theta})\,d\theta.$$

Hence

$$\begin{aligned} \left| \int_0^{2\pi} f(e^{i\theta})m(e^{-i\theta})\,d\theta \right| &\leq \|u\|_\infty\|f\|_{H^1} + \|v\|_\infty\|f\|_{H^1} \\ &= (\|u\|_\infty + \|v\|_\infty)\|f\|_{H^1} \\ &\leq C\|f\|_{H^1}\|h\|_{H^\infty}. \end{aligned}$$

Because of Lemma 3.4, this yields

$$\left| \int_0^{2\pi} k(e^{i\theta})h(e^{-i\theta})\, d\theta \right| \leq C\|f\|_{H^1}\|h\|_{H^\infty},$$

which is the required inequality. □

The argument used to prove Theorem C does not depend on the duality theorem about H^1 and BMO proved by C. Fefferman [5, p. 245]. It is interesting to note that the function g defined in Lemma 3.3 can be shown to have bounded mean oscillation by a fairly direct argument.

The essential ideas for proving Theorem C as developed above are due to Boris Korenblum [12]. The authors would like to thank Korenblum for several helpful conversations about multipliers.

Theorem 3.5. *If* $f' \in H^1$, *then* $f \in \mathscr{M}_\alpha$ *for all* $\alpha > 0$.

Proof. Let $f' \in H^1$. By Theorem C, $f \in \mathscr{M}_1$, and by Theorem 2.2 it follows that $f \in \mathscr{M}_\alpha$ for every $\alpha > 1$.

In the case $0 < \alpha < 1$, let $g \in \mathscr{F}_\alpha$, and let $h = fg$. By Theorem A, it suffices to show that $h' \in \mathscr{F}_{\alpha+1}$.

Since $g \in \mathscr{F}_\alpha$, Theorem A implies that $g' \in \mathscr{F}_{\alpha+1}$. By the previous part of the proof, $f \in \mathscr{M}_{\alpha+1}$, and therefore

$$fg' \in \mathscr{F}_{\alpha+1}. \tag{33}$$

Because $f' \in H^1$, it follows that $f' \in \mathscr{F}_1$ [4, p. 34]. By assumption, $g \in \mathscr{F}_\alpha$ and so Theorem B implies that

$$f'g \in \mathscr{F}_{\alpha+1}. \tag{34}$$

Since $h' = fg' + f'g$, (33) and (34) show that $h' \in \mathscr{F}_{\alpha+1}$, or equivalently, $h \in \mathscr{F}_\alpha$. This proves that $f \in \mathscr{M}_\alpha$ for $0 < \alpha < 1$. □

Theorem 3.5 is sharp, since there are functions f such that $f' \in H^p$ $(0 < p < 1)$ and f is not bounded. By Theorem 2.3, such functions are not multipliers.

One example where Theorem 3.5 applies concerns bounded convex maps. Suppose that f is holomorphic in Δ and that f maps Δ one-to-one onto a bounded convex region. Since the boundary C of such a region is rectifiable and since C is a Jordan curve, it follows that $f' \in H^1$ [4, p. 44]. Therefore, $f \in \mathscr{M}_\alpha$, for $\alpha > 0$.

4

Suppose that $f(z) = \sum_{n=0}^\infty a_n z^n$ is holomorphic in Δ. Let

$$s_n(z) = \sum_{j=0}^n a_j z^j$$

and

$$\sigma_n(z) = \frac{1}{n+1} \sum_{j=0}^n s_j(z).$$

By a classical result [3, p. 439], the function f is bounded if and only if the sequence $\sigma_n(z)$ is uniformly bounded for $n = 0, 1, \ldots$ and for $|z| \leq 1$, and in this case, $\|f\|_{H^\infty} = \sup\{\|\sigma_n\|_{H^\infty} : n = 0, 1, \ldots\}$. This result is generalized in this section, in terms of polynomials which are generated in the study of the multiplier problem.

Definition. For $f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < 1)$, let

$$P_n(z; \alpha) = \frac{1}{A_n(\alpha)}\{A_n(\alpha)a_0 + A_{n-1}(\alpha)a_1 z + \cdots + A_1(\alpha)a_{n-1}z^{n-1} + A_0(\alpha)a_n z^n\}$$

where $\alpha > 0$, $n = 0, 1, \ldots$, and $z \in \mathbf{C}$.

Theorem 4.1. *If $f \in \mathcal{M}_\alpha$, then $\|P_n(z; \alpha)\|_{H^\infty} \leq \|f\|_{\mathcal{M}_\alpha}$ for $n = 0, 1, \ldots$.*

Proof. Let $f \in \mathcal{M}_\alpha$ and suppose that $M > \|f\|_{\mathcal{M}_\alpha}$. If $|x| = 1$ then we have $f(z)/(1 - xz)^\alpha \in \mathcal{F}_\alpha$. Also,

$$\left\| f(z)\frac{1}{(1 - xz)^\alpha} \right\|_{\mathcal{F}_\alpha} \leq M \quad \text{for all } |x| = 1.$$

Therefore for each x $(|x| = 1)$ there is a measure $\mu_x \in \mathcal{M}$ such that

$$f(z)\frac{1}{(1 - xz)^\alpha} = \int_\Gamma \frac{1}{(1 - yz)^\alpha}\, d\mu_x(y), \tag{35}$$

and $\|\mu_x\| \leq M$ for $|x| = 1$.

If $f(z) = \sum_{n=0}^{\alpha} a_n z^n$, then $f(z)/(1 - xz)^\alpha = \sum_{n=0}^{\infty} b_n z^n$ where

$$b_n = A_0(\alpha)a_n + A_1(\alpha)a_{n-1}x + \cdots + A_{n-1}(\alpha)a_1 x^{n-1} + A_n(\alpha)a_0 x^n.$$

If

$$\int_\Gamma \frac{1}{(1 - yz)^\alpha}\, d\mu_x(y) = \sum_{n=0}^{\infty} c_n z^n,$$

then

$$c_n = A_n(\alpha)\int_\Gamma y^n\, d\mu_x(y).$$

Because of (35), $b_n = c_n$, or

$$x^n P_n\left(\frac{1}{x}; \alpha\right) = \int_\Gamma y^n\, d\mu_x(y). \tag{36}$$

Since $\|\mu_x\| \leq M$ for $|x| = 1$, (36) implies that $|P_n(1/x; \alpha)| \leq M$ for $|x| = 1$ and $n = 0, 1, \ldots$. Equivalently $|P_n(z; \alpha)| \leq M$ for $|z| = 1$ and hence $\|P_n(z; \alpha)\|_{H^\infty} \leq M$. Since this holds for every $M > \|f\|_{\mathcal{M}_\alpha}$, this proves the theorem. □

The next results generalize the statement made previously concerning the Cesàro sums $\sigma_n(z)$ for a function holomorphic in Δ. Note that $\sigma_n(z) = P_n(z; 2)$ since the binomial coefficient $A_n(2) = n + 1$ for $n = 0, 1, \ldots$.

Theorem 4.2. *Suppose that f is holomorphic in Δ and that $|P_n(z; \alpha)| \leq M$ for $|z| \leq 1$ and $n = 0, 1, \ldots$. Then $f \in H^\infty$ and $\|f\|_{H^\infty} \leq M$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. Assume that $0 \le r < 1$ and $|x| = 1$. Then

$$\begin{aligned}\frac{1}{(1-r)^\alpha} f(rx) &= \left\{\sum_{n=0}^{\infty} A_n(\alpha) r^n\right\}\left\{\sum_{n=0}^{\infty} a_n r^n x^n\right\} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} A_{n-k}(\alpha) a_k x^k\right) r^n \\ &= \sum_{n=0}^{\infty} A_n(\alpha) P_n(x;\alpha) r^n .\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{(1-r)^\alpha}|f(rx)| &\le \sum_{n=0}^{\infty} A_n(\alpha)|P_n(x;\alpha)| r^n \\ &\le M\sum_{n=0}^{\infty} A_n(\alpha) r^n = M\frac{1}{(1-r)^\alpha},\end{aligned}$$

and so $|f(rx)| \le M$. Since this holds for all r and x, it follows that $|f(z)| \le M$ for $|z| < 1$. □

The following lemma will be used to establish a partial converse to Theorem 4.2. The kernels $T_n(\theta;\alpha)$ introduced in the lemma are well known, and are studied in [18].

Lemma 4.3. *Let* $\mu_0 = \frac{1}{2}$ *and for* $k = 1, 2, \ldots$ *let* $\mu_k(\theta) = \cos k\theta$. *Also let*

$$T_n(\theta;\alpha) = \frac{1}{A_n(\alpha)}\sum_{k=0}^{n} A_{n-k}(\alpha)\mu_k(\theta).$$

(a) If $\alpha \ge 2$ then $T_n(\theta;\alpha) \ge 0$ for $0 \le \theta \le 2\pi$ and $n = 0, 1, \ldots$.
(b) If $1 < \alpha < 2$ there is a constant $B(\alpha)$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} |T_n(\theta;\alpha)|\, d\theta \le B(\alpha) \quad \text{for } n = 0, 1, \ldots .$$

Proof. First consider the case $\alpha = 2$. Then (a) is a known fact and the argument for it is as follows. Since $A_n(2) = n+1$ for $n = 0, 1, \ldots$,

$$\begin{aligned}T_n(\theta;2) &= \frac{1}{n+1}\left\{\frac{n+1}{2} + \sum_{k=1}^{n}(n-k+1)\cos k\theta\right\} \\ &= \frac{1}{2}\sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ikt} = \frac{1}{2}\frac{1}{n+1}\left\{\frac{\sin\frac{n+1}{2}\theta}{\sin\frac{1}{2}\theta}\right\}^2 \ge 0.\end{aligned}$$

This proves (a) when $\alpha = 2$.

Suppose that $\alpha > 0$ and $\beta > 0$. Then

$$\sum_{n=0}^{\infty} A_n(\alpha+\beta)z^n = \frac{1}{(1-z)^{\alpha+\beta}} = \frac{1}{(1-z)^{\alpha}}\frac{1}{(1-z)^{\beta}}$$
$$= \sum_{n=0}^{\infty} A_n(\alpha)z^n \sum_{n=0}^{\infty} A_n(\beta)z^n$$
$$= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} A_{n-k}(\alpha)A_k(\beta)\right\} z^n .$$

This shows that

$$A_n(\alpha+\beta) = \sum_{k=0}^{n} A_{n-k}(\alpha)A_k(\beta) . \tag{37}$$

Now assume that $\alpha > 2$. From (37), it follows that

$$A_n(\alpha)T_n(\theta\,;\alpha) = \sum_{k=0}^{n} A_{n-k}(\alpha)\mu_k(\theta)$$
$$= \sum_{k=0}^{n}\left\{\sum_{j=0}^{n-k} A_{n-k-j}(2)A_j(\alpha-2)\right\}\mu_k(\theta)$$
$$= \sum_{j=0}^{n}\left\{\sum_{k=0}^{n-j} A_{n-j-k}(2)\mu_k(\theta)\right\}A_j(\alpha-2)$$
$$= \sum_{j=0}^{n} T_{n-j}(\theta\,;2)A_{n-j}(2)A_j(\alpha-2) .$$

Because $A_{n-j}(2) > 0$, $A_j(\alpha-2) > 0$, and $T_{n-j}(\theta\,;2) \geq 0$, this implies that $A_n(\alpha)T_n(\theta\,;\alpha) \geq 0$. This proves (a) for $\alpha > 2$.

A proof of (b) is contained in [18, Vol. 1, p. 94], where it is shown that the kernel

$$K_n^{\beta}(\theta) = \frac{1}{A_n(\beta+1)}\sum_{k=0}^{n} A_{n-k}(\beta)D_k(\theta)$$

is "quasipositive" for $0 < \beta < 1$. Here $D_k(\theta)$ denotes the Dirichlet kernel $\frac{1}{2}\sum_{j=-k}^{k} e^{ij\theta}$. Note that $K_n^{\alpha-1}(\theta) = T_n(\theta\,;\alpha)$, and since $1 < \alpha < 2$ by assumption, this establishes (b). □

The authors would like to thank B. Muckenhoupt, who provided the proof of (a) for $\alpha > 2$, and who pointed out that this fact is known.

Theorem 4.4. *For each* $\alpha > 1$ *there is a constant* $C(\alpha)$ *such that if* $f \in H^{\infty}$, *then*

$$\|P_n(z\,;\alpha)\|_{H^{\infty}} \leq C(\alpha)\|f\|_{H^{\infty}} , \tag{38}$$

for $n = 0, 1, \ldots$. *When* $\alpha \geq 2$, (38) *holds with* $C(\alpha) = 1$.

Proof. The orthonormal relations for the trigonometric functions imply that

$$\frac{1}{2\pi}\int_0^{2\pi} f(ze^{i\theta})T_n(\theta\,;\alpha)\,d\theta = \frac{1}{2}P_n(z\,;\alpha) \tag{39}$$

for $|z| < 1$.

Suppose that $\alpha \geq 2$, $|z| < 1$, and $f \in H^\infty$. Then (39) and (a) in Lemma 4.3 imply that

$$\frac{1}{2}|P_n(z;\alpha)| \leq \frac{1}{2\pi}\int_0^{2\pi} \|f\|_{H^\infty} T_n(\theta;\alpha)\,d\theta = \frac{1}{2}\|f\|_{H^\infty}.$$

This proves the theorem in the case $\alpha \geq 2$.

Now suppose that $1 < \alpha < 2$, $|z| < 1$, and $f \in H^\infty$. Then (39) and (b) in Lemma 4.3 imply that

$$\frac{1}{2}|P_n(z;\alpha)| \leq \|f\|_{H^\infty} \frac{1}{2\pi}\int_0^{2\pi} |T_n(\theta;\alpha)|d\theta \leq B(\alpha)\|f\|_{H^\infty}.$$

This proves the theorem where $C(\alpha) = 2B(\alpha)$. □

The assertion in Theorem 4.4 does not hold for $\alpha = 1$. This is because there are functions bounded and holomorphic in Δ such that the sequence of partial sums s_n is not uniformly bounded in Δ [3, p. 444]. Also note that $P_n(z;1) = s_n(z)$.

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