ON TWISTOR SPACES OF ANTI-SELF-DUAL HERMITIAN SURFACES

MASSIMILIANO PONTECORVO

Abstract. We consider a complex surface $M$ with anti-self-dual hermitian metric $h$ and study the holomorphic properties of its twistor space $Z$. We show that the naturally defined divisor line bundle $[X]$ is isomorphic to the $-\frac{1}{2}$ power of the canonical bundle of $Z$, if and only if there is a Kähler metric of zero scalar curvature in the conformal class of $h$. This has strong consequences on the geometry of $M$, which were also found by C. Boyer [3] using completely different methods. We also prove the existence of a very close relation between holomorphic vector fields on $M$ and $Z$ in the case that $M$ is compact and Kähler.

1. Introduction

The aim of this section is to give the basic definitions and results which will be used later.

In this work $(M, h)$ will denote a complex surface $M$ together with a hermitian metric $h$ whose Weyl tensor $W$ is anti-self-dual. We write $h = g - 2i\omega$ where $g$ and $\omega$ are the associated Riemannian metric and fundamental 2-form, respectively. This is equivalent to having a Riemannian 4-dimensional manifold $(M, g)$ with an integrable almost complex structure $J$ satisfying $g(JX, JY) = g(X, Y)$ for all tangent vectors $X, Y$ on $M$.

As the real dimension of $M$ is four we have a famous splitting of the bundle of 2-forms: $\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M)$ into the eigenspaces of the Hodge star operator $*: \Lambda^2(M) \to \Lambda^2(M)$, because $*^2 = 1$. Looking at the curvature operator $\mathcal{R}: \Lambda^2(M) \to \Lambda^2(M)$ we can then ask that its conformally invariant part $W = W_+ + W_-$ be anti-self-dual, that is $W_+ = 0$ with respect to the orientation of $M$ given by $J$.

This conformal property of $h$ has an important consequence [1] on the twistor space $Z$ of $M$ which we are briefly going to describe. The real 6-dimensional manifold $Z$ can be defined to be the bundle of almost complex structures on $M$ which are compatible with the metric $g$ and the orientation given by $J$:

$$Z = \{ I \in \mathfrak{O}(TM) | I^2 = -\text{id}, I > 0 \}.$$  

We denote by $t: Z \to M$ the twistor fibration and notice that the fiber is $SO(4)/U(2) \cong S^2$. The important point is that $Z$ has a natural almost complex
structure $J'$ given as follows: at each point $z \in Z$ we can use the Levi-Civita connection of $M$ to split the tangent space to $Z$ into vertical and horizontal components: $T_z Z = V_z \oplus H_z$. The vector space $V_z$ is tangent to the fiber which is an oriented metric 2-sphere and then has a natural almost complex structure $J_1$, namely rotation by $+90^\circ$. On the other hand, $H_z$ can be identified with $T_{t(z)} M$ and given the tautological almost complex structure $J_2$ defined by $z$ itself. Finally the almost complex structure of $Z$ is $J' = J_1 \oplus J_2$. The important theorem [1] is that

\[(1) \quad J' \text{ is integrable if and only if } W_+ = 0 \text{ on } W.\]

In this case the fibers of $t$ become complex submanifolds of $Z$, isomorphic to $\mathbb{CP}_1$ and called twistor lines; the antipodal map on each line induces an anti-holomorphic involution $\sigma$ of $Z$ called the real structure.

The reason why the integrability of $J'$ only depends on the conformal class of $g$, is because the whole construction is indeed conformally invariant [1].

In fact there is an important interplay, called the Penrose correspondence, between holomorphic properties of $Z$ and conformal properties of $M$. An instance of this is the following: as the Lie algebra $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, the locally defined spin bundle of $M$ splits into two complex subbundles, denoted by $S_+$ and $S_-$, which have rank 2 and satisfy $S_+ \oplus S_- = CT^* M$. By $S^m_+$ and $S^m_-$ we indicate their symmetric $m$th powers and notice that for $m$ even these bundles are globally defined even when $M$ is not spin.

One then considers covariant differentiation

\[\nabla : \Gamma(S^m_+) \rightarrow \Gamma(S^m_+ \otimes CT^* M) = \Gamma(S^m_+ \otimes S_+ \otimes S_-)\]

which, together with the orthogonal decomposition [1],

\[S^m_+ \otimes S_+ \otimes S_- = (S^{m-1}_+ \otimes S_-) \oplus (S^{m-1}_+ \otimes S_-)\]

gives, by projection, the Dirac operator

\[D_m : \Gamma(S^m_+) \rightarrow \Gamma(S^{m-1}_+ \otimes S_-)\]

and the twistor operator

\[\overline{D}_m : \Gamma(S^m_+) \rightarrow \Gamma(S^{m+1}_+ \otimes S_-).\]

Finally recall that the canonical line bundle $K$ of $Z$ always admits a preferred square root $K^{1/2}$. And in fact a fourth root exactly when $M$ is spin.

Having said all this we can state a result of Hitchin [6] which says that global holomorphic sections of $K^{-m/4}$ exactly corresponds to solutions of the twistor equation:

\[(2) \quad H^0(Z, K^{-m/4}) \cong \text{Ker} \overline{D}_m \text{ for } m \geq 0.\]

We will be mainly interested in the case $m = 2$, in the case in fact $s^2_+ \cong \Lambda^2_+(M)$ and the Dirac operator $D_2$ is just exterior differentiation $d$ restricted to self-dual 2-forms [6]. Now the fundamental 2-form $\omega$ of $(M, h)$ is self-dual and it is very well known that $\omega$ is closed if and only if is parallel, $h$ is called a Kähler metric in this case. In the same spirit we have the following result which will be needed later:
Lemma 1.1. Let $\omega$ be the fundamental 2-form of a hermitian surface, then
\[ \overline{D}_2 \omega = 0 \iff \nabla \omega = d \omega = 0. \]

Proof. In terms of spinor indices the formula relating covariant differentiation
to the Dirac and twistor equations on $S^2 \cong \Lambda^2_+$ is
\[ \nabla^A_\omega^{BC} = \nabla^{(A}_\omega^{BC)} + \frac{2}{3} \epsilon^{A(B} \nabla^{D(C)} \omega_{D)}. \]
This says that for a self-dual 2-form $\omega$,
\[ \overline{D}_2 \omega = 0 \text{ if and only if } \nabla \omega = d \omega. \]
But when $\omega$ is the fundamental 2-form of a hermitian metric one has [8, p. 148] that for all vector fields $X, Y, Z, \omega$ on $M$,
\[ (\nabla_X \omega)(Y, Z) = \frac{1}{2} d \omega(X, JY, JZ) - \frac{3}{2} d \omega(X, Y, Z) \]
therefore $\overline{D}_2 \omega = 0$ if and only if
\[ d \omega(X, Y, Z) = 3 d \omega(X, JY, JZ). \]
And using that $J^2 = -1$ we get $d \omega = 0$; then $\nabla \omega = 0$ also. \qed

2. The twistor space

Let $t: Z \rightarrow M$ denote the twistor fibration and suppose $M$ is hermitian and
anti-self-dual. Two things are clear from the definition of the almost complex
structure of $Z$: first, $t$ is never a holomorphic map; second, the complex
structure $J$ of $M$ defines a cross section $J: M \rightarrow Z$, whose image we denote
by $\Sigma$. By the integrability of $J$, $\Sigma$ is indeed a complex hypersurface of $Z$ biholomorphic to $M$ [4]. Similarly $-J: M \rightarrow Z$ defines hypersurface $\overline{\Sigma}$. The 
"real structure" $\sigma$ of $Z$ switches the two hypersurfaces identifying one with
the other in an antiholomorphic fashion. If $X$ denotes the divisor $\Sigma + \overline{\Sigma}$ in
$Z$, we can consider the holomorphic line bundle $[X]$; since $\sigma(X) = \sigma(\Sigma + \overline{\Sigma}) = \Sigma + \overline{\Sigma} = X$, $[X]$ is called a "real" bundle.

We then investigate the relation between the holomorphic line bundle $[X]$ and
the complex structure of $Z$.

First, when $M$ is compact, one has the following topological remark [13]:
\[ c_1([X]) = c_1(K^{-1/2}_Z) \]
where $K_Z$ denotes the canonical bundle.

It is then natural to ask when is $[X]$ isomorphic to $K^{-1/2}_Z$.

Now if $H^1(Z, \mathcal{O}) = 0$, the Chern class map $c_1: H^1(Z, \mathcal{O}) \rightarrow H^2(Z, \mathcal{Z})$ is injective and the above implies $[X] \cong K^{-1/2}_Z$, however by the Ward correspondence [1, Theorem 5.2], $H^1(M, \mathcal{R})$ has to be zero in this case.

The general philosophy of the Twistor Program of R. Penrose is to relate the
conformal geometry of $M$ to the holomorphic properties of $Z$. In this context,
whether $M$ is compact or not, we have

Theorem 2.1.
\[ [X] \cong K^{-1/2}_Z \]
if and only if $h$ is conformal to a Kähler metric.
Proof. We start by assuming that \( h \) is a Kähler metric and prove that \([X] \cong K_Z^{-1/2}\), in two steps. We first define a holomorphic section \( \omega \in H^0(Z, K^{-1/2}) \) by using the Kähler form \( \omega \) of \( M \); then we show that \( X = \{\omega = 0\} \). In the course of this proof we will often use the following (see [1]): \( Z = P(S_+) \); the symplectic form of \( S_+ \) defines a linear isomorphism \( \varepsilon: S_+ \to S_+^* \) and the hermitian form an antilinear isomorphism \( h: S_+ \to \overline{S}_+ \) so that if \( \eta \in S_+ \), \( \overline{\eta} \) will denote its image and we will write \( \eta \otimes \overline{\eta} \in S_+^2 \).

Step 1. Recall that \( \bigwedge_+^2 (M) = S_+^2 \), then the Kähler form \( \omega \) of \( M \) is a section of \( S_+^2 \). Now according to [6, §2] any section \( \psi \in S_+^2 \) tautologically defines a complex valued function on \( S_+ \setminus 0 \) which is a homogeneous polynomial of degree 2 on each fiber; this in turns gives a section \( \bar{\psi} \in \Gamma(Z, \mathcal{O}(2)) = \Gamma(Z, K^{-1/2}) \). And furthermore \( \bar{\psi} \) is a holomorphic section, i.e. \( \bar{\psi} \in H^0(Z, \mathcal{O}(K^{-1/2})) \), if and only if \( \psi \) satisfies the twistor equation \( D_2 \psi = 0 \). It is clear from the definition of the operators \( D_m \) and \( \overline{D}_m \) that in general every parallel section of \( S_+^2 \) is a solution to both the Dirac and twistor equations (in fact, by the Weitzenböck formulas, these are the only solutions when \( M \) is compact and \( R = 0 \)). Therefore since \( \omega \) is parallel, \( \omega \in H^0(Z, K^{-1/2}) \) is holomorphic.

Step 2. Since \( M \) is hermitian we have two sections \( \phi \) and \( \overline{\phi}: M \to Z \) representing the almost complex structures \( J \) and \( \overline{J} \). Let \( \omega \in \bigwedge_+^2 (M) = S_+^2 \) be the Kähler form. According to [1, §1], at each point \( p \in M \), \( \omega = \phi \otimes \overline{\phi} \) where \( \phi \in S_+ \) and \( \overline{\phi} \in S_+ \) represent \( \phi \) and \( \overline{\phi} \) respectively. Now let \( \alpha \in Z = P(S_+) \) be a twistor at \( p \). By using the isomorphism \( \varepsilon: S_+ \to S_+^* \) it makes sense to solve the equation \( \alpha = 0 \). Since \( \varepsilon \) is given by the symplectic form and \( S_+ \) has complex dimension 2, the only solution is \( \alpha = \phi \). Similarly for \( \overline{\phi} \) and we have shown that \( \omega(\alpha) = 0 \) if and only if \( \alpha = \phi \) or \( \alpha = \overline{\phi} \) that is \( X = \{\omega = 0\} \). This proves one direction of the statement.

To complete the proof we assume now that \([X] \cong K_Z^{-1/2}\), and show that there is a Kähler metric in the conformal class of \( h \). By hypothesis we have a holomorphic section \( \rho \) of \( K_Z^{-1/2} \) vanishing exactly on \( X \). Furthermore since \( H^0(Z, K_Z^{-1/2}) \) has a “real” structure, we can choose \( \rho \) to be invariant under the anti-holomorphic involution \( \sigma \) of \( Z \). The corresponding self-dual 2-form \( \rho \) is then real and satisfies the twistor equation: \( \overline{D}_2 \rho = 0 \) [6]. By Lemma 1.1 is then enough to prove that \( \rho \) is the fundamental 2-form of a hermitian metric in the conformal class of \( h \).

Now if \( \omega \) is the fundamental 2-form of \( h \), we have already shown that \( \omega \) also vanishes exactly on \( X \), but is not necessarily holomorphic. However, on each twistor line \( P(S_+) \), \( \rho \) and \( \omega \) are homogeneous polynomial of degree two vanishing on the same two antipodal points and therefore they differ by a nonzero multiplicative constant \( f(x) \) which is real. It follows that \( \rho = f \omega \) for a never-zero real function \( f \) on \( M \). Assume that \( M \) is connected, this means that either \( \rho \) or \( -\rho \) is the fundamental 2-form of the hermitian metric \( |f|h \). ☐

Remark 2.2. It was proved for example in [9] that a metric is anti-self-dual and Kähler if and only if it is Kähler of zero scalar curvature. So that the hermitian
anti-self-dual surfaces for which \([X] \cong K^{−1/2}_Z\) are precisely the Kähler surfaces of zero scalar curvature. The problem of their classification was posed in [16]. The above theorem also gives a “twistor proof” of a result of C. Boyer:

**Corollary 2.3** [3]. Let \((m, h)\) be a compact anti-self-dual surface then:
- If \(b_1(M)\) is even, 
  
h is globally conformal to a Kähler metric of zero scalar curvature.
- If \(b_1(M)\) is odd, 
  
h is locally conformal to a Kähler metric of zero scalar curvature.

**Proof.** By (3), \([X] = K^{−1/2}_Z\) where \(F\) is a holomorphic line bundle of zero Chern class on \(Z\). By the Ward correspondence then, \(F = t^*E\) where \(E\) is a hermitian line bundle over \(M\), with anti-self-dual connection and zero Chern class. In particular the curvature of the connection is harmonic and therefore zero, by Hodge theory. Now we consider the following commutative diagram:

\[
\begin{array}{ccc}
\Z & \overset{q}{\longrightarrow} & \Z \\
\downarrow i & & \downarrow i \\
\tilde{M} & \overset{p}{\longrightarrow} & M
\end{array}
\]

where \(p: \tilde{M} \rightarrow M\) and \(q: \tilde{Z} \rightarrow \Z\) are universal coverings, and \(i : \tilde{Z} \rightarrow \tilde{M}\) is the twistor fibration. The pulled-back connection on the line bundle \(p^*E\) over \(\tilde{M}\) is trivial, because it is flat and \(\tilde{M}\) is simply connected. As a consequence \(\tilde{F} := i(p^*E) = q^*(t^*E) = q^*F\) is trivial [1]. And therefore \([\tilde{X}] = K^{−1/2}_{\tilde{Z}}\) on \(\tilde{Z}\), where \(\tilde{X}\) is the universal covering of \(X\). Then \(\tilde{M}\) is globally conformally Kähler by 2.1.

It follows that \(M\) is locally conformally Kähler (l.c.k. in the notation of Vaisman). But any compact l.c.k. surface is globally conformally Kähler exactly when \(b_1(M)\) is even [3], because the Hodge decomposition \(H^1(M, \mathbb{C}) = H^{1,0}(M) \oplus H^{0,1}(M)\) holds in this case. \(\square\)

### 3. Compact Kähler Surfaces

From now on we will assume that \(M\) is compact and \(b_1(M)\) is even. All known examples of anti-self-dual compact complex surfaces of this type are the following:

- Flat tori and K3 surfaces with a Yau metric. These are the hyperkahler surfaces and are the universal coverings of:
- The other Ricci-flat Kähler surfaces, i.e. the hyperelliptic and the Enriques surfaces.
- \(S_g \times \mathbb{C}P_1\), where \(S_g\) is a compact Riemann surface of genus \(g \geq 2\) with a metric of constant scalar curvature \(-1\), and \(\mathbb{C}P_1\) is the Riemann sphere with constant curvature \(+1\), or, more generally, ruled surfaces which are flat \(S^2\)-bundle over \(S_g\), \(g \geq 2\).
- Recently LeBrun has constructed zero scalar curvature Kähler metrics on ruled surfaces blown up at two or more points [10].
The reason why these are hermitian anti-self-dual manifolds is that they have a Kähler metric of zero scalar curvature.

Notice that the complex projective plane \( \mathbb{CP}_2 \) with its standard orientation and metric is self-dual and Kähler, while the same manifold with orientation reversed, \( \overline{\mathbb{CP}_2} \), does not even admit an almost complex structure; otherwise \( c_1^2 \) would be equal to \( 2\chi + 3\tau = 3 \), which implies that the first Chern class \( c_1 \) cannot be represented by an integral 2-form.

The above theorem also gives precise informations on the normal bundle of \( X \) in \( Z \), denoted by \( \nu_{X/Z} \).

**Corollary 3.1.** When \( M \) is compact and \( b_1(M) \) is even, the normal bundle of \( X \) in \( Z \) is isomorphic to the anticanonical bundle: \( \nu_{X/Z} \cong K_X^{-1} \), similarly \( \nu_{\Sigma/Z} \cong K_{\Sigma}^{-1} \) and \( \nu_{\Sigma/Z} \cong K_{\Sigma}^{-1} \).

**Proof** [13]. The adjunction formulas [5] state that \( \nu_{X/Z} \cong [X]_X \) and \( K_X \cong (K_Z \otimes [X])_X \) therefore \( \nu_{X/Z} \cong K_{Z/X}^{-1/2} \) and \( K_X \cong (K_Z \otimes K_Z^{-1/2})_X \cong K_Z^{1/2} \) as wanted. The rest clearly follows from \( X = \Sigma \cup \Sigma \). □

Theorem 2.1 says that the line bundle \( K^{-1/2} \) has global holomorphic sections and this easily implies that \( K^{-m/2} \) has global holomorphic sections for each \( m \geq 0 \). In fact using the ideas of [13] one can show that these are the only line bundles, within their Chern class, to have global holomorphic sections.

**Proposition 3.2.** If \( c_1(L) = c_1(K^{-m/2}) \) then \( H^0(Z, L) \neq 0 \) if and only if \( L \cong K^{-m/2} \) for each \( m \geq 0 \).

When one considers a compact twistor space \( t: Z \to M \) as a complex manifold, there is a theorem of Hitchin [7] which states that \( Z \) is Kähler (in fact algebraic) if and only if \( M \) is either \( S^4 \) or \( \mathbb{CP}_2 \), with its standard conformal structure. It is then interesting to investigate "how far is a twistor space from being algebraic," for example by looking at its algebraic dimension \( a(Z) \). To this respect Poon has found some very interesting relations between \( a(Z) \) and the geometry of \( M \) [12, 13].

The methods of [12] show that the algebraic dimension of a compact twistor space \( Z \) is achieved by the Kodaira dimension \( k(Z, F) \) of some "real" holomorphic line bundle \( F \to Z \), see [14] for definitions. Therefore

**Remark 3.3.**

\[
a(Z) = k(Z, [X]) = k(Z, K^{-1/2})
\]

when \( Z \) is the twistor space of a compact Kähler surface of zero scalar curvature.

The above discussion can then be used as in [11], to give a more direct proof of a theorem of Poon [13] which states that

\[
a(Z) \leq 1 \quad \text{for the twistor space of a compact Kähler surface of zero scalar curvature. Furthermore equality holds precisely when } M \text{ is Ricci-flat.}
\]

The situation is different when \( b_1(M) \) is odd and in [11] we gave the first example of a twistor space with algebraic dimension equal to two. It is the twistor space of a Hopf surface.
4. Holomorphic vector fields

In this section we still assume that \( M \) is compact and Kähler; we show that there is a close relation between the Lie algebras of holomorphic vector fields of \( M \) and \( Z \), which we denote by \( H^0(M, \Theta) \) and \( H^0(Z, \Theta) \). Again, as in Poon’s theorem the results reflect whether or not \( M \) is Ricci-flat.

For a holomorphic vector bundle \( E \) over a compact complex manifold \( N \), \( h^0(N, E) \) will denote the complex dimension of \( H^0(N, E) \). We will prove

**Theorem 4.1.** If \( M \) is Ricci-flat

\[
H^0(Z, \Theta) \cong H^0(M, \Theta) \oplus H^0(M, \Theta)
\]

which is also isomorphic to the complexification of the Lie algebra of real parallel vector fields on \( M \); so that

\[
h^0(Z, \Theta) = b_1(M) = 2h^0(M, \Theta).
\]

**Theorem 4.2.** If \( M \) is not Ricci-flat

\[
H^0(Z, \Theta) \cong H^0(M, \Theta).
\]

To explain this, recall that in the general case, by the Penrose correspondence, \( H^0(Z, \Theta) \) is the complexification of the Lie algebra of conformal Killing vector fields on \( M \). This in turn is closely related to \( H^0(M, \Theta) \) when \( M \) is Kähler.

To prove the above theorems we will use the following [B]:

**Theorem 4.3 (Bochner).** On a compact riemannian manifold \((N, g)\) with Ric \(\leq 0\), every Killing vector field is parallel.

Similarly if \( g \) is Kähler, then every holomorphic vector field is parallel.

**Theorem 4.4 (Lichnerowicz).** On a compact Kähler manifold of constant scalar curvature

\[
H^0(M, \Theta) \cong a \oplus h
\]

where \( a \) is the abelian Lie algebra of all parallel holomorphic vector fields and \( h \) is the complexification of a Lie algebra consisting of Killing vector fields.

Another result of Lichnerowicz states that: on any compact Kähler manifold of dimension at least 2 a conformally Killing vector field is automatically Killing. In complex dimension 2 we also have an elementary proof of this fact:

**Lemma 4.5.** If \( M \) is a compact Kähler surface every conformal vector field is real holomorphic and in fact Killing.

**Proof.** Suppose \( \mathcal{L}_V g = fg \) for some function \( f \); we start by showing that \( \mathcal{L}_V \omega = 0 \) where \( \omega \) denotes the Kähler form. In fact let \( \phi_t \) be the flow of \( V \). For each \( t \), \( \phi_t \) is a conformal isometry homotopic to the identity. Since \( \omega \) is a self-dual closed 2-form, it is also harmonic, and it is easy to check that the Hodge-star operator \( *: \Lambda^n \to \Lambda^n \), on a manifold of real dimension \( 2n \), is invariant under a conformal rescaling of the metric; so that \( \phi_t^* \omega \) is again harmonic. But \( [\phi_t^* \omega] = [\omega] \in H^2_{dR}(M) \) and so by Hodge theory, \( \phi_t^* \omega = \omega \), i.e. \( \mathcal{L}_V \omega = 0 \).
Now the complex structure $J = g^{-1} \circ \omega$ as an endormorphism of the tangent bundle, therefore
\[
\mathcal{L}_\nu J = (\mathcal{L}_\nu g^{-1}) \circ \omega + g^{-1} \circ (\mathcal{L}_\nu \omega) = fg^{-1} \circ \omega = fJ,
\]
on the other hand $J^2 = -\text{id}$ implies that
\[
0 = \mathcal{L}_\nu (-\text{id}) = \mathcal{L}_\nu J^2 = J(\mathcal{L}_\nu J) + (\mathcal{L}_\nu J)J = -2f
\]
i.e. $f = 0$, $\mathcal{L}_\nu g = 0$ and $\mathcal{L}_\nu J = 0$. \(\Box\)

It is also straightforward to check that

**Lemma 4.6.** On any Kähler manifold if $V$ and $JV$ are both Killing vector fields, they are also parallel with respect to the Levi-Civita connection.

**Proof of 4.1.** By the Bochner theorem and 4.5 we have that $H^0(Z, \Theta)$ is the complexification of the Lie algebra of parallel vector fields. Now recall the Weitzenböck decomposition of 1-forms:
\[
\Delta = dd^* + d^*d = \nabla^* \nabla + \text{Ric}
\]
it says that on a Ricci-flat riemannian manifold a 1-form is harmonic if and only if is parallel with respect to the Levi-Civita connection. Using the metric to pass from 1-forms to vector fields we have:
\[
h^0(Z, \Theta) = \dim \mathbb{R} (\text{Lie algebra of parallel vector fields}) = b_1(M)
\]
and we are left to prove that $2h^0(M, \Theta) = b_1(M)$. By the Bochner theorem every holomorphic vector field is parallel, so the dual \((0,1)\)-form is parallel; since $M$ is Kähler, $\Delta = 2\Theta = \overline{\partial} \partial^* = \overline{\partial}^* \partial$ and a \((0,1)\)-form is parallel if and only if is $\overline{\partial}$-harmonic; we conclude that
\[
h^0(M, \Theta) = h^0(M, \Omega^1) = \frac{1}{2} b_1(M). \quad \Box
\]

**Proof of 4.2.** Suppose $M$ has no parallel vector fields, then by the Lichnerowicz theorem and 4.5, $H^0(M, \Theta)$ is the complexification of the Lie algebra of all conformal Killing vector fields on $M$ and therefore isomorphic to $H^0(Z, \Theta)$, and we have proved the result. Then it is enough to show that $M$ admits no parallel holomorphic vector fields.

To show this is true, we first reduce to the case of a minimal surface: suppose $M$ is not minimal (i.e. it contains a holomorphically embedded, irreducible rational curve $C$ with self-intersection = $-1$). Then if $\Theta_{M,C}$ denotes the sheaf of holomorphic vector fields on $M$ which are tangent to $C$, along $C$, we have an exact sequence $0 \to \Theta_{M,C} \to \Theta_m \to \nu_{C/M} \to 0$. As $H^0(C, \nu_{C/M}) \cong H^0(\mathbb{CP}^1, \Theta(-1)) = 0$, it follows that every holomorphic tangent vector on $M$ is tangent to $C$, along $C$. Since $C \cong \mathbb{CP}^1$, every holomorphic vector field vanishes somewhere. (In fact a direct image argument shows that it has to vanish identically, along $C$.)

However, if $M$ is minimal and the total scalar curvature is nonnegative, Yau [15] has shown that $M \cong \mathbb{CP}^2$ or else is a $\mathbb{CP}^1$-bundle over a Riemann surface $S_g$. This says that $\chi(M) \neq 0$, and therefore $M$ has no parallel vector fields unless it is a $\mathbb{CP}^1$-bundle over a torus; in this case however $\chi(M) = \tau(M) = 0$.

\footnote{Warning: Proposition 4 in [15] is false, counterexample: $\mathbb{CP}^1 \times S_g$.}
On the other hand, by Chern-Weil theory [3], when \( M \) is anti-self-dual with zero scalar curvature, this implies that \( M \) is actually flat, which is absurd. \( \square \)

Notice that the result of 4.2 holds for any half-conformally flat compact Kähler surface with no parallel holomorphic vector field, e.g. \( \mathbb{CP}_2 \); or trivially, for any such surface of negative Ricci curvature.

**Acknowledgments**

Most of the material presented here is part of the author's Ph.D. thesis. I would like to thank my advisor Claude LeBrun for his guidance, help and encouragement, and Y. S. Poon for informing me of his work.

**References**


SISSA, STRADA COSTIERA 11, 34014 TRIESTE, ITALY