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# ON TWISTOR SPACES OF ANTI-SELF-DUAL HERMITIAN SURFACES

MASSIMILIANO PONTECORVO

ABSTRACT. We consider a complex surface $M$ with anti-self-dual hermitian metric $h$ and study the holomorphic properties of its twistor space $Z$. We show that the naturally defined divisor line bundle $[X]$ is isomorphic to the $-\frac{1}{2}$ power of the canonical bundle of $Z$, if and only if there is a Kähler metric of zero scalar curvature in the conformal class of $h$. This has strong consequences on the geometry of $M$, which were also found by C. Boyer [3] using completely different methods. We also prove the existence of a very close relation between holomorphic vector fields on $M$ and $Z$ in the case that $M$ is compact and Kähler.

## 1. INTRODUCTION

The aim of this section is to give the basic definitions and results which will be used later.

In this work $(M, h)$ will denote a complex surface $M$ together with a hermitian metric $h$ whose Weyl tensor $W$ is anti-self-dual. We write $h = g - 2i\omega$ where $g$ and $\omega$ are the associated Riemannian metric and fundamental 2-form, respectively. This is equivalent to having a Riemannian 4-dimensional manifold $(M, g)$ with an integrable almost complex structure $J$ satisfying $g(JX, JY) = g(X, Y)$ for all tangent vectors $X$, $Y$ on $M$.

As the real dimension of $M$ is four we have a famous splitting of the bundle of 2-forms: $\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M)$ into the eigenspaces of the Hodge star operator $*: \Lambda^2(M) \to \Lambda^2(M)$, because $*^2 = 1$. Looking at the curvature operator $\mathscr{R}: \Lambda^2(M) \to \Lambda^2(M)$ we can then ask that its conformally invariant part $W = W_+ + W_-$ be anti-self-dual, that is $W_+ = 0$ with respect to the orientation of $M$ given by $J$.

This conformal property of $h$ has an important consequence [1] on the twistor space $Z$ of $M$ which we are briefly going to describe. The real 6-dimensional manifold $Z$ can be defined to be the bundle of almost complex structures on $M$ which are compatible with the metric $g$ and the orientation given by $J$:

$$Z = \{I \in \mathscr{O}(TM) | I^2 = -\mathrm{id}, I > 0\}.$$

We denote by $t: Z \to M$ the twistor fibration and notice that the fiber is $SO(4)/U(2) \cong S^2$. The important point is that $Z$ has a natural almost complex

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structure $J'$ given as follows: at each point $z \in Z$ we can use the Levi-Civita connection of $M$ to split the tangent space to $Z$ into vertical and horizontal components: $T_z Z = V_z \oplus H_z$. The vector space $V_z$ is tangent to the fiber which is an oriented metric 2-sphere and then has a natural almost complex structure $J_1$, namely rotation by $+90°$. On the other hand, $H_z$ can be identified with $T_{t(z)}M$ and given the tautological almost complex structure $J_2$ defined by $z$ itself. Finally the almost complex structure of $Z$ is $J' = J_1 \oplus J_2$. The important theorem [1] is that

$$(1) \qquad J' \text{ is integrable if and only if } W_+ = 0 \text{ on } W.$$

In this case the fibers of $t$ become complex submanifolds of $Z$, isomorphic to $\mathbf{CP}_1$ and called twistor lines; the antipodal map on each line induces an anti-holomorphic involution $\sigma$ of $Z$ called the real structure.

The reason why the integrability of $J'$ only depends on the conformal class of $g$, is because the whole construction is indeed conformally invariant [1].

In fact there is an important interplay, called the Penrose correspondence, between holomorphic properties of $Z$ and conformal properties of $M$. An instance of this is the following: as the Lie algebra $\mathrm{so}(4) = \mathrm{su}(2) \oplus \mathrm{su}(2)$, the locally defined spin bundle of $M$ splits into two complex subbundles, denoted by $\mathbf{S}_+$ and $\mathbf{S}_-$, which have rank 2 and satisfy $\mathbf{S}_+ \oplus \mathbf{S}_- = \mathbf{C}T^*M$. By $\mathbf{S}_+^m$ and $\mathbf{S}_-^m$ we indicate their symmetric $m$th powers and notice that for $m$ even these bundles are globally defined even when $M$ is not spin.

One then considers covariant differentiation

$$\nabla\colon \Gamma(\mathbf{S}_+^m) \longrightarrow \Gamma(\mathbf{S}_+^m \otimes \mathbf{C}T^*M) = \Gamma(\mathbf{S}_+^m \otimes \mathbf{S}_+ \otimes \mathbf{S}_-)$$

which, together with the orthogonal decomposition [1],

$$\mathbf{S}_+^m \otimes \mathbf{S}_+ \otimes \mathbf{S}_- = (\mathbf{S}_+^{m-1} \otimes \mathbf{S}_-) \oplus (\mathbf{S}_+^{m-1} \otimes \mathbf{S}_-)$$

gives, by projection, the Dirac operator

$$D_m\colon \Gamma(\mathbf{S}_+^m) \longrightarrow \Gamma(\mathbf{S}_+^{m-1} \otimes \mathbf{S}_-)$$

and the twistor operator

$$\overline{D}_m\colon \Gamma(\mathbf{S}_+^m) \longrightarrow \Gamma(\mathbf{S}_+^{m+1} \otimes \mathbf{S}_-).$$

Finally recall that the canonical line bundle $K$ of $Z$ always admits a preferred square root $K^{1/2}$. And in fact a fourth root exactly when $M$ is spin.

Having said all this we can state a result of Hitchin [6] which says that global holomorphic sections of $K^{-m/4}$ exactly corresponds to solutions of the twistor equation:

$$(2) \qquad H^0(Z, K^{-m/4}) \cong \operatorname{Ker} \overline{D}_m \quad \text{for } m \geq 0.$$

We will be mainly interested in the case $m = 2$, in the case in fact $\mathbf{s}_+^2 \cong \Lambda_+^2(M)$ and the Dirac operator $D_2$ is just exterior differentiation $d$ restricted to self-dual 2-forms [6]. Now the fundamental 2-form $\omega$ of $(M, h)$ is self-dual and it is very well known that $\omega$ is closed if and only if is parallel, $h$ is called a Kähler metric in this case. In the same spirit we have the following result which will be needed later:

**Lemma 1.1.** *Let $\omega$ be the fundamental 2-form of a hermitian surface, then*

$$\overline{D}_2\omega = 0 \Leftrightarrow \nabla\omega = d\omega = 0.$$

*Proof.* In terms of spinor indices the formula relating covariant differentiation to the Dirac and twistor equations on $\mathbf{S}^2_+ \cong \Lambda^2_+$ is

$$\nabla^A_{A'}\omega^{BC} = \nabla^{(A}_{A'}\omega^{BC)} + \tfrac{2}{3}\varepsilon^{A(B}\nabla_{A'D}\omega^{C)D}.$$

This says that for a self-dual 2-form $\omega$,

$$\overline{D}_2\omega = 0 \quad \text{if and only if} \quad \nabla\omega = d\omega.$$

But when $\omega$ is the fundamental 2-form of a hermitian metric one has [8, p. 148] that for all vector fields $X$, $Y$, $Z$, on $M$,

$$(\nabla_X\omega)(Z, Y) = \tfrac{3}{2}\, d\omega(X, JY, JZ) - \tfrac{3}{2}\, d\omega(X, Y, Z)$$

therefore $\overline{D}_2\omega = 0$ if and only if

$$d\omega(X, Y, Z) = 3\, d\omega(X, JY, JZ).$$

And using that $J^2 = -1$ we get $d\omega = 0$; then $\nabla\omega = 0$ also. □

## 2. The twistor space

Let $t\colon Z \to M$ denote the twistor fibration and suppose $M$ is hermitian and anti-self-dual. Two things are clear from the definition of the almost complex structure of $Z$: first, $t$ is never a holomorphic map; second, the complex structure $J$ of $M$ defines a cross section $J\colon M \to Z$, whose image we denote by $\Sigma$. By the integrability of $J$, $\Sigma$ is indeed a complex hypersurface of $Z$ biholomorphic to $M$ [4]. Similarly $-J\colon M \to Z$ defines hypersurface $\overline{\Sigma}$. The "real structure" $\sigma$ of $Z$ switches the two hypersurfaces identifying one with the other in an antiholomorphic fashion. If $X$ denotes the divisor $\Sigma + \overline{\Sigma}$ in $Z$, we can consider the holomorphic line bundle $[X]$; since $\sigma(X) = \sigma(\Sigma + \overline{\Sigma}) = \overline{\Sigma} + \Sigma = X$, $[X]$ is called a "real" bundle.

We then investigate the relation between the holomorphic line bundle $[X]$ and the complex structure of $Z$.

First, when $M$ is compact, one has the following topological remark [13]:

$$c_1([X]) = c_1(K_Z^{-1/2}) \tag{3}$$

where $K_Z$ denotes the canonical bundle.

It is then natural to ask when is $[X]$ isomorphic to $K_Z^{-1/2}$.

Now if $H^1(Z, \mathscr{O}) = 0$, the Chern class map $c_1\colon H^1(Z, \mathscr{O}^*) \to H^2(Z, \mathbf{Z})$ is injective and the above implies $[X] \cong K_Z^{-1/2}$, however by the Ward correspondence [1, Theorem 5.2], $H^1(M, \mathbf{R})$ has to be zero in this case.

The general philosophy of the Twistor Program of R. Penrose is to relate the conformal geometry of $M$ to the holomorphic properties of $Z$. In this context, whether $M$ is compact or not, we have

**Theorem 2.1.**

$$[X] \cong K_Z^{-1/2}$$

*if and only if $h$ is conformal to a Kähler metric.*

*Proof*. We start by assuming that $h$ is a Kähler metric and prove that $[X] \cong K_Z^{-1/2}$, in two steps. We first define a holomorphic section $\check{\omega} \in H^0(Z, K^{-1/2})$ by using the Kähler form $\omega$ of $M$; then we show that $X = \{\check{\omega} = 0\}$. In the course of this proof we will often use the following (see [1]): $Z = \mathbf{P}(\mathbf{S}_+)$; the symplectic form of $\mathbf{S}_+$ defines a linear isomorphism $\varepsilon\colon \mathbf{S}_+ \to \mathbf{S}_+^*$ and the hermitian form an antilinear isomorphism $h\colon \mathbf{S}_+ \to \overline{\mathbf{S}}_+$ so that if $\eta \in \mathbf{S}_+$, $\overline{\eta}$ will denote its image and we will write $\eta \otimes \overline{\eta} \in \mathbf{S}_+^2$.

*Step* 1. Recall that $\bigwedge_+^2(M) = \mathbf{S}_+^2$, then the Kähler form $\omega$ of $M$ is a section of $\mathbf{S}_+^2$. Now according to [6, §2] any section $\psi \in \mathbf{S}_+^2$ tautologically defines a complex valued function on $\mathbf{S}_+ \setminus 0$ which is a homogeneous polynomial of degree 2 on each fiber; this in turns gives a section $\check{\psi} \in \Gamma(Z, \mathscr{O}(2)) = \Gamma(Z, K^{-1/2})$. And furthermore $\check{\psi}$ is a holomorphic section, i.e. $\check{\psi} \in H^0(Z, \mathscr{O}(K^{-1/2}))$, if and only if $\psi$ satisfies the twistor equation $\overline{D}_2\psi = 0$. It is clear from the definition of the operators $D_m$ and $\overline{D}_m$ that in general every parallel section of $\mathbf{S}_+^m$ is a solution to both the Dirac and twistor equations (in fact, by the Weitzenböck formulas, these are the only solutions when $M$ is compact and $R = 0$). Therefore since $\omega$ is parallel, $\check{\omega} \in H^0(Z, K^{-1/2})$ is holomorphic.

*Step* 2. Since $M$ is hermitian we have two sections $\varphi$ and $\overline{\varphi}\colon M \to Z$ representing the almost complex structures $J$ and $\overline{J}$. Let $\omega \in \bigwedge_+^2(M) = \mathbf{S}_+^2$ be the Kähler form. According to [1, §1], at each point $p \in M$, $\omega = \phi \otimes \overline{\phi}$ where $\phi \in \mathbf{S}_+$ and $\overline{\phi} \in \mathbf{S}_+$ represents $\varphi$ and $\overline{\varphi}$ respectively. Now let $\alpha \in Z = \mathbf{P}(\mathbf{S}_+)$ be a twistor at $p$. By using the isomorphism $\varepsilon\colon \mathbf{S}_+ \to \mathbf{S}_+^*$ it makes sense to solve the equation $\phi(\alpha) = 0$. Since $\varepsilon$ is given by the symplectic form and $\mathbf{S}_+$ has complex dimension 2, the only solution is $\alpha = \varphi$. Similarly for $\overline{\phi}$ and we have shown that $\check{\omega}(\alpha) = 0$ if and only if $\alpha = \varphi$ or $\alpha = \overline{\varphi}$ that is $X = \{\check{\omega} = 0\}$. This proves one direction of the statement.

To complete the proof we assume now that $[X] \cong K_Z^{-1/2}$, and show that there is a Kähler metric in the conformal class of $h$. By hypothesis we have a holomorphic section $\check{\rho}$ of $K_Z^{-1/2}$ vanishing exactly on $X$. Furthermore since $H^0(Z, K_Z^{-1/2})$ has a "real" structure, we can choose $\check{\rho}$ to be invariant under the anti-holomorphic involution $\sigma$ of $Z$. The corresponding self-dual 2-form $\rho$ is then real and satisfies the twistor equation: $\overline{D}_2\rho = 0$ [6]. By Lemma 1.1 is then enough to prove that $\rho$ is the fundamental 2-form of a hermitian metric in the conformal class of $h$.

Now if $\omega$ is the fundamental 2-form of $h$, we have already shown that $\check{\omega}$ also vanishes exactly on $X$, but is not necessarily holomorphic. However, on each twistor line $\mathbf{P}(\mathbf{S}_{+_x})$, $\check{\rho}$ and $\check{\omega}$ are homogeneous polynomial of degree two vanishing on the same two antipodal points and therefore they differ by a nonzero multiplicative constant $f(x)$ which is real. It follows that $\rho = f\omega$ for a never-zero real function $f$ on $M$. Assume that $M$ is connected, this means that either $\rho$ or $-\rho$ is the fundamental 2-form of the hermitian metric $|f|h$. □

*Remark* 2.2. It was proved for example in [9] that a metric is anti-self-dual and Kähler if and only if it is Kähler of zero scalar curvature. So that the hermitian

anti-self-dual surfaces for which $[X] \cong K_Z^{-1/2}$ are precisely the Kähler surfaces of zero scalar curvature. The problem of their classification was posed in [16].

The above theorem also gives a "twistor proof" of a result of C. Boyer:

**Corollary 2.3** [3]. *Let $(m, h)$ be a compact anti-self-dual surface then:*
*If $b_1(M)$ is even,*
*$h$ is globally conformal to a Kähler metric of zero scalar curvature.*
*If $b_1(M)$ is odd,*
*$h$ is locally conformal to a Kähler metric of zero scalar curvature.*

*Proof*. By (3), $[X] = K_Z^{-1/2}F$ where $F$ is a holomorphic line bundle of zero Chern class on $Z$. By the Ward correspondence then, $F = t^*E$ where $E$ is a hermitian line bundle over $M$, with anti-self-dual connection and zero Chern class. In particular the curvature of the connection is harmonic and therefore zero, by Hodge theory. Now we consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{q} & Z \\ {\scriptstyle \tilde{t}}\downarrow & & {\scriptstyle t}\downarrow \\ \widetilde{M} & \xrightarrow{p} & M \end{array}$$

where $p\colon \widetilde{M} \longrightarrow M$ and $q\colon \widetilde{Z} \to Z$ are universal coverings, and $\tilde{t}\colon \widetilde{Z} \to \widetilde{M}$ is the twistor fibration. The pulled-back connection on the line bundle $p^*E$ over $\widetilde{M}$ is trivial, because it is flat and $\widetilde{M}$ is simply connected. As a consequence $\widetilde{F} := \tilde{t}(p^*E) = q^*(t^*E) = q^*F$ is trivial [1]. And therefore $[\widetilde{X}] = K_{\widetilde{Z}}^{-1/2}$ on $\widetilde{Z}$, where $\widetilde{X}$ is the universal covering of $X$. Then $\widetilde{M}$ is globally conformally Kähler by 2.1.

It follows that $M$ is locally conformally Kähler (l.c.k. in the notation of Vaisman). But any compact l.c.k. surface is globally conformally Kähler exactly when $b_1(M)$ is even [3], because the Hodge decomposition $H^1(M, \mathbf{C}) = H^{1,0}(M) \oplus H^{0,1}(M)$ holds in this case. □

## 3. Compact Kähler surfaces

From now on we will assume that $M$ is compact and $b_1(M)$ is even. All known examples of anti-self-dual compact complex surfaces of this type are the following:

- Flat tori and $K3$ surfaces with a Yau metric. These are the hyperkähler surfaces and are the universal coverings of:
- The other Ricci-flat Kähler surfaces, i.e. the hyperelliptic and the Enriques surfaces.
- $S_g \times \mathbf{CP}_1$, where $S_g$ is a compact Riemann surface of genus $g \geq 2$ with a metric of constant scalar curvature $-1$, and $\mathbf{CP}_1$ is the Riemann sphere with constant curvature $+1$, or, more generally, ruled surfaces which are flat $S^2$-bundle over $S_g$, $g \geq 2$.
- Recently LeBrun has constructed zero scalar curvature Kähler metrics on ruled surfaces blown up at two or more points [10].

The reason why these are hermitian anti-self-dual manifolds is that they have a Kähler metric of zero scalar curvature.

Notice that the complex projective plane $\mathbf{CP}_2$ with its standard orientation and metric is self-dual and Kähler, while the same manifold with orientation reversed, $\overline{\mathbf{CP}}_2$, does not even admit an almost complex structure; otherwise $c_1^2$ would be equal to $2\chi + 3\tau = 3$, which implies that the first Chern class $c_1$ cannot be represented by an integral 2-form.

The above theorem also gives precise informations on the normal bundle of $X$ in $Z$, denoted by $\nu_{X/Z}$.

**Corollary 3.1.** *When $M$ is compact and $b_1(M)$ is even, the normal bundle of $X$ in $Z$ is isomorphic to the anticanonical bundle: $\nu_{X/Z} \cong K_X^{-1}$, similarly $\nu_{\Sigma/Z} \cong K_\Sigma^{-1}$ and $\nu_{\overline{\Sigma}/Z} \cong K_{\overline{\Sigma}}^{-1}$.*

*Proof* [13]. The adjunction formulas [5] state that $\nu_{X/Z} \cong [X]_{|X}$ and $K_X \cong (K_Z \otimes [X])_{|X}$ therefore $\nu_{X/Z} \cong K_{Z|X}^{-1/2}$ and $K_X \cong (K_z \otimes K_Z^{-1/2})_{|X} \cong K_{Z|X}^{1/2}$ as wanted. The rest clearly follows from $X = \Sigma \amalg \overline{\Sigma}$ $\square$

Theorem 2.1 says that the line bundle $K^{-1/2}$ has global holomorphic sections and this easily implies that $K^{-m/2}$ has global holomorphic sections for each $m \geq 0$. In fact using the ideas of [13] one can show that these are the only line bundles, within their Chern class, to have global holomorphic sections:

**Proposition 3.2.** *If $c_1(L) = c_1(K^{-m/2})$ then $H^0(Z, L) \neq 0 \Leftrightarrow L \cong K^{-m/2}$ and $m \geq 0$.*

When one considers a compact twistor space $t\colon Z \to M$ as a complex manifold, there is a theorem of Hitchin [7] which states that $Z$ is Kähler (in fact algebraic) if and only if $M$ is either $S^4$ or $\mathbf{CP}_2$, with its standard conformal structure. It is then interesting to investigate "how far is a twistor space from being algebraic," for example by looking at its algebraic dimension $a(Z)$. To this respect Poon has found some very interesting relations between $a(Z)$ and the geometry of $M$ [12, 13].

The methods of [12] show that the algebraic dimension of a compact twistor space $Z$ is achieved by the Kodaira dimension $k(Z, F)$ of some "real" holomorphic line bundle $F \to Z$, see [14] for definitions. Therefore

*Remark* 3.3.

$$a(Z) = k[Z, [X]) = k(Z, K^{-1/2})$$

when $Z$ is the twistor space of a compact Kähler surface of zero scalar curvature.

The above discussion can then be used as in [11], to give a more direct proof of a theorem of Poon [13] which states that

> *$a(Z) \leq 1$ for the twistor space of a compact Kähler surface of zero scalar curvature. Furthermore equality holds precisely when $M$ is Ricci-flat.*

The situation is different when $b_1(M)$ is odd and in [11] we gave the first example of a twistor space with algebraic dimension equal to two. It is the twistor space of a Hopf surface.

## 4. Holomorphic vector fields

In this section we still assume that $M$ is compact and Kähler; we show that there is a close relation between the Lie algebras of holomorphic vector fields of $M$ and $Z$, which we denote by $H^0(M, \Theta)$ and $H^0(Z, \Theta)$. Again, as in Poon's theorem the results reflect whether or not $M$ is Ricci-flat.

For a holomorphic vector bundle $E$ over a compact complex manifold $N$, $h^0(N, E)$ will denote the complex dimension of $H^0(N, E)$. We will prove

**Theorem 4.1.** *If $M$ is Ricci-flat*

$$H^0(Z, \Theta) \cong H^0(M, \Theta) \oplus H^0(M, \Theta)$$

*which is also isomorphic to the complexification of the Lie algebra of real parallel vector fields on $M$; so that*

$$h^0(Z, \Theta) = b_1(M) = 2h^0(M, \Theta).$$

**Theorem 4.2.** *If $M$ is not Ricci-flat*

$$H^0(Z, \Theta) \cong H^0(M, \Theta).$$

To explain this, recall that in the general case, by the Penrose correspondence, $H^0(Z, \Theta)$ is the complexification of the Lie algebra of conformal Killing vector fields on $M$. This in turn is closely related to $H^0(M, \Theta)$ when $M$ is Kähler.

To prove the above theorems we will use the following [B]:

**Theorem 4.3** (Bochner). *On a compact riemannian manifold $(N, g)$ with* Ric $\leq 0$, *every Killing vector field is parallel.*

*Similarly if $g$ is Kähler, then every holomorphic vector field is parallel.*

**Theorem 4.4** (Lichnerowicz). *On a compact Kähler manifold of constant scalar curvature*

$$H^0(M, \Theta) \cong \mathbf{a} \oplus \mathbf{h}$$

*where* $\mathbf{a}$ *is the abelian Lie algebra of all parallel holomorphic vector fields and* $\mathbf{h}$ *is the complexification of a Lie algebra consisting of Killing vector fields.*

Another result of Lichnerowicz states that: on any compact Kähler manifold of dimension at least 2 a conformally Killing vector field is automatically Killing. In complex dimension 2 we also have an elementary proof of this fact:

**Lemma 4.5.** *If $M$ is a compact Kähler surface every conformal vector field is real holomorphic and in fact Killing.*

*Proof.* Suppose $\mathscr{L}_V g = fg$ for some function $f$; we start by showing that $\mathscr{L}_V \omega = 0$ where $\omega$ denotes the Kähler form. In fact let $\varphi_t$ be the flow of $V$. For each $t$, $\varphi_t$ is a conformal isometry homotopic to the identity. Since $\omega$ is a self-dual closed 2-form, it is also harmonic, and it is easy to check that the Hodge-star operator $*: \bigwedge^n \to \bigwedge^n$, on a manifold of real dimension $2n$, is invariant under a conformal rescaling of the metric; so that $\varphi_t^* \omega$ is again harmonic. But $[\varphi_t^* \omega] = [\omega] \in H^2_{dR}(M)$ and so by Hodge theory, $\varphi_t^* \omega = \omega$, i.e. $\mathscr{L}_V \omega = 0$.

Now the complex structure $J = g^{-1} \circ \omega$ as an endormorphism of the tangent bundle, therefore

$$\mathscr{L}_V J = (\mathscr{L}_V g^{-1}) \circ \omega + g^{-1} \circ (\mathscr{L}_V \omega) = f g^{-1} \circ \omega = f J ,$$

on the other hand $J^2 = -\mathrm{id}$ implies that

$$0 = \mathscr{L}_V(-\mathrm{id}) = \mathscr{L}_V J^2 = J(\mathscr{L}_V J) + (\mathscr{L}_V J) J = -2f$$

i.e. $f = 0$, $\mathscr{L}_V g = 0$ and $\mathscr{L}_V J = 0$. □

It is also straightforward to check that

**Lemma 4.6.** *On any Kähler manifold if $V$ and $JV$ are both Killing vector fields, they are also parallel with respect to the Levi-Civita connection.*

*Proof of* 4.1. By the Bochner theorem and 4.5 we have that $H^0(Z, \Theta)$ is the complexification of the Lie algebra of parallel vector fields. Now recall the Weitzenböch decomposition of 1-forms:

$$\Delta = dd^* + d^*d = \nabla^*\nabla + \mathrm{Ric}$$

it says that on a Ricci-flat riemannian manifold a 1-form is harmonic if and only if is parallel with respect to the Levi-Civita connection. Using the metric to pass from 1-forms to vector fields we have:

$$h^0(Z, \Theta) = \dim_{\mathbf{R}} (\text{Lie algebra of parallel vector fields}) = b_1(M)$$

and we are left to prove that $2h^0(M, \Theta) = b_1(M)$. By the Bochner theorem every holomorphic vector field is parallel, so the dual $(0, 1)$-form is parallel; since $M$ is Kähler, $\Delta = 2\Box = \overline{\partial}\overline{\partial}^* = \overline{\partial}^*\overline{\partial}$ and a $(0, 1)$-form is parallel if and only if is $\overline{\partial}$-harmonic; we conclude that

$$h^0(M, \Theta) = h^0(M, \Omega^1) = \tfrac{1}{2} b_1(M). \quad \Box$$

*Proof of* 4.2. Suppose $M$ has no parallel vector fields, then by the Lichnerowicz theorem and 4.5, $H^0(M, \Theta)$ is the complexification of the Lie algebra of all conformal Killing vector fields on $M$ and therefore isomorphic to $H^0(Z, \Theta)$, and we have proved the result. Then it is enough to show that $M$ admits no parallel holomorphic vector fields.

To show this is true, we first reduce to the case of a minimal surface: suppose $M$ is not minimal (i.e. it contains a holomorphically embedded, irreducible rational curve $C$ with self-intersection $= -1$). Then if $\Theta_{M,C}$ denotes the sheaf of holomorphic vector fields on $M$ which are tangent to $C$, along $C$, we have an exact sequence $0 \to \Theta_{M,C} \to \Theta_m \to \nu_{C/M} \to 0$. As $H^0(C, \nu_{C/M}) \cong H^0(\mathbf{CP}_1, \mathscr{O}(-1)) = 0$, it follows that every holomorphic tangent vector on $M$ is tangent to $C$, along $C$. Since $C \cong \mathbf{CP}_1$, every holomorphic vector field vanishes somewhere. (In fact a direct image argument shows that it has to vanish identically, along $C$.)

However, if $M$ is minimal and the total scalar curvature is nonnegative, Yau [15][1] has shown that $M \cong \mathbf{CP}_2$ or else is a $\mathbf{CP}_1$-bundle over a Riemann surface $S_g$. This says that $\chi(M) \neq 0$, and therefore $M$ has no parallel vector fields unless it is a $\mathbf{CP}_1$-bundle over a torus; in this case however $\chi(M) = \tau(M) = 0$.

[1]Warning: Proposition 4 in [15] is false, counterexample: $\mathbf{CP}_1 \times S_g$.

On the other hand, by Chern-Weil theory [3], when $M$ is anti-self-dual with zero scalar curvature, this implies that $M$ is actually flat, which is absurd. □

Notice that the result of 4.2 holds for any half-conformally flat compact Kähler surface with no parallel holomorphic vector field, e.g. $\mathbf{CP}_2$; or trivially, for any such surface of negative Ricci curvature.

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SISSA, Strada Costiera 11, 34014 Trieste, Italy