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ON LOCAL STRUCTURES OF THE SINGULARITIES A_k, D_k AND E_k OF SMOOTH MAPS

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Dedicated to Professor Haruo Suzuki on his 60th *birthday*

ABSTRACT. In studying the singularities of type A_k of smooth maps between manifolds N and P the Boardman manifold $\Sigma^{i,1,\dots,10}$ in $J^\infty(N, P)$ has been very useful. We will construct the submanifolds ΣD_k and ΣE_k in $J^\infty(N, P)$ playing the similar role for singularities D_k and E_k and study their properties in its process.

0. INTRODUCTION

Let A_k, D_k and E_k denote the types of the singularities of function germs studied in [4]. When a C^∞ stable map germ f is C^∞ equivalent to an unfolding of a function germ with singularity A_k, D_k or E_k, we say that f has a singularity A_k, D_k or E_k at the origin respectively. Let N and P be smooth (C^∞) manifolds with $\dim N = n$ and $\dim P = p$.

In his article [5], J. M. Boardman has constructed the submanifold Σ^I, $I = (\max(n-p+1, 1), 1, \dots, 1, 0)$ in $J^\infty(N, P)$ for A_k and introduced useful tools such as the total tangent bundle and the higher intrinsic derivatives. They have been important to study the topological properties of the singularities A_k of smooth maps in [1 and 2].

In this paper we shall study the local structures of the singularities D_k and E_k and construct the submanifolds ΣD_k and ΣE_k in $J^\infty(N, P)$ (see (3.8) and (3.10)) playing the similar role as Σ^I for A_k. Even though ΣE_k is easy to construct as seen in §4, ΣD_k needs some elaborate arguments using the higher intrinsic derivatives in §3 as Thom-Boardman submanifolds have done in [5]. Let X_k be one of D_k and E_k, then ΣX_k has the property that if $j^\infty f$ of a smooth map germ f is transverse to ΣX_k at x, then f is C^∞ stable and has a singularity X_k at x. The results of the paper are useful, for example, for the calculation of their Thom polynomials of smooth maps or singular foliations having only these singularities and for the elimination of those of the highest order by their vanishing property and h-principle (see [3 and 7]).

We shall review the definition of singularities A_k, D_k and E_k and the results of [5] in §1 and prepare some lemmas in §2. In §5 the connected components of ΣX_k will be described.

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1. Preliminaries

Throughout the paper N and P denote paracompact and Hausdorff C^∞ (simply smooth) manifolds of dimensions n and p respectively. We say that a smooth map germ $f:(N, x_0) \to (P, y_0)$ has the singularity of type A_k, D_k or E_k at x_0 is f is written as below respectively when we take suitable local coordinates $(y_1, \dots, y_p)$ of P near y_0 and ones of N near x_0 as

$$(x_1, \dots, x_{p-k}, x_p, \dots, x_{n-1}, s_0, \dots, s_{k-2}, u) \qquad \text{for } A_k,$$
$$(x_1, \dots, x_{p-k}, x_p, \dots, x_{n-2}, s_0, \dots, s_{k-2}, u, v) \qquad \text{for } D_k \text{ and } E_k.$$

Let $n > p$ and we set (see [4, 6, and 9]),

$$y_i \circ f(x) = x_i \qquad (1 \le i \le p-k),$$
$$y_i \circ f(x) = s_{i-p+k-1} \qquad (p-k < i \le p-1),$$

and $y_p \circ f(x)$ is as follows for A_k, D_k or E_k respectively.

$$(\mathrm{A}_k) \qquad \pm x_p^2 \pm \cdots \pm x_{n-1}^2 + \sum_{t=1}^{k-1} s_{t-1}u^t \pm u^{k+1} \qquad (k \ge 1),$$

$$(\mathrm{D}_k) \qquad Q + u^2v \pm v^{k-1} + \sum_{t=2}^{k-2} s_t v^t \qquad (k \ge 4),$$

$$(\mathrm{E}_6) \qquad Q + C \pm v^4 + s_4uv^2,$$

$$(\mathrm{E}_7) \qquad Q + C + uv^3 + s_4v^3 + s_5v^4,$$

$$(\mathrm{E}_8) \qquad Q + C + v^5 + s_4uv^2 + s_5v^3 + s_6uv^3,$$

where $Q = \pm x_p^2 \pm \cdots \pm x_{n-2}^2 + s_0u + s_1v$ and $C = u^3 + s_2uv + s_3v^2$.

Let $J^m(N, P)$ denote the m jet space $(0 \le m \le \infty)$. Next we shall review the fundamental properties of the Boardman submanifolds $\Sigma^I(N, P)$ (simply Σ^I) of $J^\infty(N, P)$ and the useful tools introduced in [5] for the symbols I related to A_k, D_k and E_k (see also [9, 10 and 11]). We shall utilize them to study the local structures of these singularities in the sequel.

Let $\pi_t^s : J^s(N, P) \to J^t(N, P)$ be the forgetting map for $s \ge t$ and π_N^m and π_P^m, the canonical projections of $J^m(N, P)$ onto N and P mapping an m-jet onto its source and target respectively. Let $\mathbf{D}$ denote the total tangent bundle over $J^\infty(N, P)$ (see the details of [5, Definition 1.9]). This notation is related to the derivative of smooth functions on $J^\infty(N, P)$. A real valued function ϕ defined on an open set U of $J^\infty(N, P)$ is called smooth if there is a smooth function ψ on some open subset of $J^m(N, P)$ for a finite number m such that $\phi = \psi \circ \pi_m^\infty$ on U. As seen below $\mathbf{D}$ is isomorphic to $\pi_N^*(TN)$ and any smooth section d of $\mathbf{D}$ over U determines a smooth function $d\phi$ on U characterized as follows. Any vector field d' on an open set of N induces a smooth section d of $\mathbf{D}$ such that $d\phi$ is determined by

$$d\phi(j_x^\infty f) = d'(\phi \circ j^\infty f)_x \tag{1.1}$$

for any smooth map germ $f:(N, x) \to (P, f(x))$ with $j_x^\infty f \in U$.

For a system of local coordinates $(x_1, \dots, x_n)$ near $x \in N$, we obtain smooth sections d_{x_i} of $\mathbf{D}$ coming from $\partial/\partial x_i$ by (1.1) (they are denoted by the symbol D_i in [5]. But we must use them for singularities D_k). Therefore $\mathbf{D}$ is defined as the vector bundle such that its any smooth section of $\mathbf{D}$ is locally represented as a linear combination of d_{x_i} over smooth function on $J^\infty(N, P)$, say $\sum \phi_i d_{x_i}$. For any smooth section d of $\mathbf{D}$ it follows from (1.1) that

$$d\phi \circ j^\infty f = ((j^\infty f)^* d)(\phi \circ j^\infty f). \tag{1.2}$$

Next we see the results about the higher intrinsic derivatives defined over Σ^I for $(n-p+1, 1\cdots 10)$, $(n-p+1, 2, 0)$ and $(n-p+1, 2, 1\cdots 10)$. We will formulate them over $J^m(N, P)$ for a sufficiently large number m (see [5, Lemma 1.12, 2.20 and p. 412]), even though they first have been done over $J^\infty(N, P)$. Let $\mathbf{D}' = (\pi_N)^*(TN)$ and $\mathbf{P} = (\pi_P)^*(TP)$. Then we have the homomorphism $d_1 : \mathbf{D}' \to \mathbf{P}$ over $J^m(N, P)$ and let Σ^{n-p+1} denote the subspace of all m jets z with $\dim(\mathrm{Ker}(d_{1,z})) = n-p+1$ where $d_{1,z} : \mathbf{D}'_z \to \mathbf{P}_z$ is the restriction of d_1 to the fibers over z (throughout the paper we shall use the similar notation). Set $\mathbf{K}_1 = \mathrm{Ker}(d_1)$ and $\mathbf{Q} = \mathrm{Cok}(d_1)$ over Σ^{n-p+1} (note that $\dim \mathbf{Q} = 1$). The second intrinsic derivative

$$d_2 : \mathbf{K}_1 \to \mathrm{Hom}(\mathbf{K}_1, \mathbf{Q}) \quad \text{over } \Sigma^{n-p+1}$$

defines $\Sigma^{n-p+1,j}$ as the subset of all $z \in \Sigma^{n-p+1}$ with $\dim(\mathrm{Ker}(d_{2,z})) = j$. Set $\mathbf{K}_2 = \mathrm{Ker}(d_2)$ over $\Sigma^{n-p+1,j}$. Then if $j = 1$, $\mathrm{Cok}(d_2)$ is isomorphic to $\mathrm{Hom}(\mathbf{K}_2, \mathbf{Q})$ over $\Sigma^{n-p+1,j}$. For a sequence $I_k = (n-p+1, 1\cdots 1)$ of k integers the definition of Σ^{I_k} proceeds by induction on k. In this process it is important to construct the $(k+1)$th intrinsic derivative $(k \geq 2)$,

$$d_{k+1} : \mathbf{K}_2 \to \mathrm{Hom}(\textstyle\bigotimes^k \mathbf{K}_2, \mathbf{Q}) \quad \text{over } \Sigma^{I_k}$$

and then $\Sigma^{I_{k+1}}$ is defined to be the set of all $z \in \Sigma^{I_k}$ such that $d_{k+1,z}$ vanishes. We define $\Sigma^{I_k,0} = \Sigma^{I_k} \backslash \Sigma^{I_{k+1}}$ as sets.

If $n > p$ and $j = 2$, the third intrinsic derivative

$$d_3 : \mathbf{K}_2 \to \mathrm{Hom}(\bigcirc^2 \mathbf{K}_2, \mathbf{Q}) \quad \text{over } \Sigma^{n-p+1,2}$$

define $\Sigma^{n-p+1,2,1}$ to be the set of all jets $z \in \Sigma^{n-p+1,2}$ with $\dim(\mathrm{Ker}(d_{3,z})) = 1$ (throughout the paper $V_1 \bigcirc \cdots \bigcirc V_t$ denote the symmetric product of subbundles $V_1, \dots, V_t$ of a vector bundle V in the tth symmetric product $\bigcirc^t V$). Set $\mathbf{K}_3 = \mathrm{Ker}(d_3)$ and then $\mathrm{Cok}(d_3)$ is isomorphic to $\mathrm{Hom}(\mathbf{K}_3 \bigcirc \mathbf{K}_2, Q)$ over $\Sigma^{n-p+1,2,1}$. The 4th intrinsic derivative

$$d_4 : \mathbf{K}_3 \to \mathrm{Hom}(\mathbf{K}_3 \bigcirc \mathbf{K}_3 \bigcirc \mathbf{K}_2, \mathbf{Q}) \quad \text{over } \Sigma^{n-p+1,2,1}$$

defines $\Sigma^{n-p+1,2,1,1}$ as the set of all jets $z \in \Sigma^{n-p+1,2,1}$ such that $d_{4,z}$ vanishes. Let $\mathbf{K}_4 = \mathbf{K}_3$ over $\Sigma^{n-p+1,2,1,1}$ and then $\mathrm{Cok}(d_4)$ is isomorphic to $\mathrm{Hom}(\bigcirc^3 \mathbf{K}_3 \bigcirc \mathbf{K}_2, \mathbf{Q})$. Finally we have the 5th intrinsic derivative

$$d_5 : \mathbf{K}_3 \to \mathrm{Hom}(\bigcirc^3 \mathbf{K}_3 \bigcirc \mathbf{K}_2, \mathbf{Q}) \quad \text{over } \Sigma^{n-p+1,2,1,1}.$$

We set $\Sigma^{n-p+1,2,1,0} = \Sigma^{n-p+1,2,1} \backslash \Sigma^{n-p+1,2,1,1}$ and $\Sigma^{n-p+1,2,1,1,0}$ as the subset of all jets $z \in \Sigma^{n-p+1,2,1,1}$ such that $d_{5,z}$ is injective.

There are important facts about the intrinsic derivatives. For every symbol $I_k = (i_1, \dots, i_k)$ appeared above we let $\mathbf{K}_{k+1} = \mathrm{Ker}(d_{k+1})$ and $\mathbf{P}_k$ denote the target bundle of d_{k+1}. Then d_{k+1} is extended to the surjective homomorphism (denoted by the same letter)

$$d_{k+1} : T\Sigma^{I_{k-1}}|\Sigma^{I_k} \to \mathbf{P}_k \quad \text{over } \Sigma^{I_k}$$

such that $\operatorname{Ker}(d_{k+1}) = T\Sigma^{I_k}$. By [5, (7.7)] we have that $\mathbf{K}_1 \cap T\Sigma^{I_{k-1}} = \mathbf{K}_k$ over Σ^{I_k}.

Here we briefly sketch how to define d_{k+1} for $I_k = (i_1, \dots, i_k)$ since it will be used in the construction of ΣD_k in §3. Take local coordinates system $(x_1, \dots, x_n)$ of N near x and $(y_1, \dots, y_p)$ of P near y. For any z of Σ^{I_k} with $\pi_N(z) = x$ and $\pi_P(z) = y$ we can choose a special I_k-flag (not necessarily unique) which is a series of subbundles $\mathbf{K}'_k$ of $\mathbf{D}|U$,

$$\mathbf{D}|U = \mathbf{K}_0 \supset \mathbf{K}'_1 \supset \mathbf{K}'_2 \supset \cdots \supset \mathbf{K}'_k$$

with $\dim \mathbf{K}'_t = i_t$ and $\mathbf{K}'_t|(\Sigma^{I_t} \cap U) = \mathbf{K}_t|(\Sigma^{I_t} \cap U)$ and a set of $n - i_k$ smooth functions c_j on U $(j = 1, \dots, n - i_k)$ satisfying

(a) $\operatorname{rk}(d_{x_i}(c_j))_{1 \le i \le n,\ 1 \le j \le n - i_k} = n - i_k$ on U (then we say that $c_1, \dots, c_{n-i_k}$ are totally independent on U).

(b) For any smooth section d' of $\mathbf{K}'_t$, $d'(c_j)$ is identically zero for $n - i_{t-1} < j \le n - i_t$.

(c) $c_1, \dots, c_{n-i_1}$ factor through π_P and for $n - i_{t-1} < j \le n - i_t$, $c_j \in \Gamma_{t-1}\Gamma_{t-2}\cdots\Gamma_2\Gamma_1\mathscr{F}(\pi_P U)$ where Γ_t denotes the module of all smooth sections of $\mathbf{K}'_t$ on U and $\mathscr{F}(\pi_P U)$, the module of all smooth functions on $\pi_P U$.

Then the $\mathbf{R}$-linear map

$$\Gamma_t \otimes \Gamma_{t-1} \otimes \cdots \otimes \Gamma_1 \otimes \mathscr{m}_y \to \mathbf{R}$$

is defined by mapping $d^t \otimes \cdots \otimes d^1 \otimes \phi$ onto $((d^t \cdots d^1)\phi)(z)$. It turns out that it vanishes when $\phi \in \mathscr{m}_y^2$ or $d^j \in \mathscr{m}_z\Gamma_j$ for some j and that it is symmetric. Hence it induces the homomorphism

$$\mathbf{K}_{t,z} \bigcirc \cdots \bigcirc \mathbf{K}_{1,z} \otimes \mathscr{m}_y/\mathscr{m}_y^2 \to \mathbf{R}$$

since $\mathbf{K}_{t,z} \cong \Gamma_t/\mathscr{m}_z\Gamma_t$. By identifying $\operatorname{Hom}(\mathscr{m}_y/\mathscr{m}_y^2, \mathbf{R})$ with TP_y we obtain the homomorphism

$$h_t : \mathbf{K}_t \bigcirc \cdots \bigcirc \mathbf{K}_1 \to \mathbf{P} \quad \text{over } \Sigma^{I_t} \cap U$$

although it may depend on the choice of a special flag. However it has been shown in [5] by using the intrinsic derivatives due to I. R. Porteous that the composition of h_t and the projection $\pi : \mathbf{P} \to \mathbf{Q}$ is invariantly defined when restricted as the following homomorphisms (we will use the notations below).

$$\begin{aligned}
d'_2 &= \pi \circ h_2 | \bigcirc^2 \mathbf{K}_1 \quad \text{for } t = 2, \\
d'_t &= \pi \circ h_t | \bigcirc^t \mathbf{K}_2 \quad \text{for } I = (n - p + 1, 1 \cdots 1), \\
d'_3 &= \pi \circ h_3 | \bigcirc^3 \mathbf{K}_2 \\
d'_4 &= \pi \circ h_4 | \bigcirc^3 \mathbf{K}_3 \bigcirc \mathbf{K}_2 \quad \text{for } I = (n - p + 1, 2, 1 \cdots 1). \\
d'_5 &= \pi \circ h_5 | \bigcirc^4 \mathbf{K}_3 \bigcirc \mathbf{K}_2
\end{aligned}$$

Moreover even though $\mathbf{K}_t$, $\mathbf{Q}$ and h_t are defined over $J^\infty(N, P)$, they uniquely factor through $J^m(N, P)$. Therefore we shall use the same notation for these notions over both of $J^m(N, P)$ and $J^\infty(N, P)$ throughout the paper. Thus d_t is defined to be the homomorphism induced from d'_t.

2. Lemmas

We shall prepare two lemmas about $\mathrm{Hom}(\bigcirc^3\mathbf{R}^2, \mathbf{R})$ to study d_3. Let u and v be the dual basis of $(1, 0)$ and $(0, 1)$ of $\mathbf{R}^2$. We always identify $\mathrm{Hom}(\bigcirc^3\mathbf{R}^2, \mathbf{R})$ with $\bigcirc^3 \mathrm{Hom}(\mathbf{R}^2, \mathbf{R})$ and then its element φ is written as $\varphi = au^3 + bu^2v + cuv^2 + dv^3$ under the symmetric product. By the isomorphism $\mathrm{Hom}(\mathbf{R}^2, \mathrm{Hom}(\bigcirc^2\mathbf{R}^2, \mathbf{R})) \cong \mathrm{Hom}(\mathbf{R}^2 \otimes (\bigcirc^2\mathbf{R}^2), \mathbf{R})$, φ induces a homomorphism $\varphi' : \mathbf{R}^2 \to \mathrm{Hom}(\bigcirc^2\mathbf{R}^2, \mathbf{R}) \cong \bigcirc^2 \mathrm{Hom}(\mathbf{R}^2, \mathbf{R})$. It is easy to see that φ' is written as

$$\varphi'(s, t) = (1/3)(s\partial/\partial u + t\partial/\partial v)(\varphi).$$

As shown in [4] $\mathrm{Hom}(\bigcirc^3\mathbf{R}^2, \mathbf{R})$ is decomposed into the five orbit manifolds of the action by $GL(2)$ through $u^2v \pm v^3$, u^2v, u^3 and 0. We denote them by $S_4^\pm$, S_5, S_E and 0 respectively. Let H be the set of all quadratic forms of rank 1 or 0 in $\mathrm{Hom}(\bigcirc^2\mathbf{R}^2, \mathbf{R})$. The first lemma is as follows.

Lemma 2.1. *For any element φ of S_5, $(\varphi')^{-1}(H)$ is a one-dimensional subspace of $\mathbf{R}^2$.*

Proof. As S_5 is the orbit through u^2v, φ is written as $\varphi = (u \circ A)^2(v \circ A)$ where $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in GL(2)$. It is easily seen that

$$3\varphi'(s, t) = (u \circ A, v \circ A) \begin{pmatrix} cs + dt & as + bt \\ as + bt & 0 \end{pmatrix} \begin{pmatrix} u \circ A \\ v \circ A \end{pmatrix}.$$

Therefore $(\varphi')^{-1}(H)$ is the kernel $as + bt = 0$ of $u \circ A$. Q.E.D.

Let $L \subset S_5 \times \mathbf{R}^2$ be the set of all pairs (φ, s, t) such that $\varphi'(s, t) \in H$. Let $\pi : L \to S_5$ be defined by $\pi(\varphi, s, t) = \varphi$.

Lemma 2.2. (i) *$\pi : L \to S_5$ is a vector subbundle of the two-dimensional trivial bundle over S_5.*

(ii) *The normal bundle of S_5 in $\mathrm{Hom}(\bigcirc^3\mathbf{R}^2, \mathbf{R})$ is isomorphic to the vector bundle $\mathrm{Hom}(\bigcirc^3 L, \mathbf{R})$.*

(iii) *Let L_φ be a fiber of L over $\varphi \in S_5$. Then $\varphi | L_\varphi \bigcirc L_\varphi \bigcirc \mathbf{R}^2$ is a null homomorphism.*

Proof. (i) Consider the action

$$\mu : GL(2) \times S_5 \times \mathbf{R}^2 \to S_5 \times \mathbf{R}^2$$

defined by $\mu(A, \varphi(u, v), (s, t)) = (\varphi(u \circ A, v \circ A), (s, t)A)$. Then $L \to S_5$ is the orbit through a fiber over u^2v and thus is a vector bundle.

(ii) Any element of the vector bundle $\mathrm{Hom}(\bigcirc^3 L, \mathbf{R})$ over S_5 is written as a pair (φ, c) for $\varphi \in S_5$ and a linear map $c : \bigcirc^3 L_\varphi \to \mathbf{R}$. It induces the linear map $c_\varphi = c \circ \pi_\varphi$ where π_φ is the orthogonal projection of $\mathbf{R}^2$ onto L_φ. This defines a smooth map

$$h : \mathrm{Hom}(\bigcirc^3 L, \mathbf{R}) \to \mathrm{Hom}(\bigcirc^3\mathbf{R}^2, \mathbf{R})$$

by $h(\varphi, c) = \varphi + c_\varphi$. To prove (ii) we shall show that h is an embedding near S_5. It is enough to prove it near u^2v by the action since h is an embedding on the zero section. Any element $\varphi \in S_5$ close to u^2v is written uniquely as $(u+a_1v)^2\{(1+a_2)v+a_3u\}$. Then L_φ is the line $s+a_1t=0$ and

$$c_\varphi = a_4(-a_1u+v)^3 .$$

So (a_1, a_2, a_3, a_4) is the local coordinate system of $\mathrm{Hom}(\bigcirc^3 L, \mathbf{R})$ near u^2v. Therefore we obtain

$$\begin{aligned}\varphi + c_\varphi &= (u+a_1v)^2\{(1+a_2)v+a_3u\} + a_4(-a_1u+v)^3 \\ &\equiv (u^2+2a_1uv)\{(1+a_2)v+a_3u\} + a_4v^3 \quad \text{modulo } (a_1, a_2, a_3, a_4)^2, \\ &\equiv (1+a_2)u^2v + 2a_1uv^2 + a_3u^3 + a_4v^3 \quad \text{modulo } (a_1, a_2, a_3, a_4)^2.\end{aligned}$$

Hence the Jacobian of h at u^2v equals $\partial(a_2, 2a_1, a_3, a_4)/\partial(a_1, a_2, a_3, a_4) = -2$. This means that h is a diffeomorphism near u^2v.

(iii) Again it is enough to show (iii) for $\varphi = u^2v$. Then L_φ is the line $s=0$. Under the identification $\mathrm{Hom}(\bigcirc^3\mathbf{R}^2, \mathbf{R}) \cong \bigcirc^3 \mathrm{Hom}(\mathbf{R}^2, \mathbf{R})$, $\varphi | L_\varphi \bigcirc L_\varphi \bigcirc \mathbf{R}^2$ is identified with the null homomorphism since $u|L_\varphi = 0$. Q.E.D.

By considering the branches of solutions of $\varphi = 0$, L is uniquely extended to the line bundle (denoted by the same letter) over the image $U(S_5)$ of a small neighborhood of the zero section of S_5 by h in proof of Lemma 2.2 where h is a diffeomorphism. In fact every element of $U(S_5)$ is written as $\varphi + c_\varphi = (u\circ A)^2(v\circ A) + \varepsilon(v\circ A)^3$ for a sufficiently small number ε. Then the fiber $L_{\varphi+c_\varphi}$ is defined as the line annihilated by $u\circ A$. Moreover it is clear that $\varphi + c_\varphi \in S_5$ if and only if $\varphi + c_\varphi | \bigcirc^3 L_{\varphi+c_\varphi}$ is a null homomorphism. The decomposition of $\mathrm{Hom}(\bigcirc^3\mathbf{R}^2, \mathbf{R})$ into $S_4^\pm$, S_5, S_E and 0 induces that of $\mathrm{Hom}(\bigcirc^3\mathbf{K}_2, \mathbf{Q})$ into five manifolds. We denote them by the same notation together with $U(S_5)$ and L over $U(S_5)$.

3. Singularities A_k and D_k

Let $I_k = (n-p+1, 1\cdots 1)$ be a sequence of k integers. It is well known that a smooth germ $f: \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ is C^∞ stable and has a singularity A_k at the origin if and only if j^mf is transverse to $\Sigma^{I_k,0}$ and $j^mf(0) \in \Sigma^{I_k,0}$. In studying global topological properties of singularities A_k in [1 and 2] Σ^{I_k} together with tools reviewed in §1 has played an important role. In order to study singularities D_k and E_k we shall construct the submanifolds ΣD_k and ΣE_k in $J^m(N, P)$ with the similar properties as those above of Σ^{I_k} in this section. In the following definition consider the vector bundle map (§1)

$$d_3' : \bigcirc^3\mathbf{K}_2 \to \mathbf{Q} \quad \text{over } \Sigma^{n-p+1,2,0}$$

as the smooth section of $\mathrm{Hom}(\bigcirc^3\mathbf{K}_2, \mathbf{Q})$ over $\Sigma^{n-p+1,2,0}$.

Definition 3.1. We define the subsets $\Sigma D_4^\pm(N, P)$ as $(d_3')^{-1}(S_4^\pm)$ and the subsets $\Sigma \overline{D}_5(N, P)$ as $(d_3')^{-1}(S_5)$ in $\Sigma^{n-p+1,2,0}(N, P)$ (we usually neglect (N, P) in this paper).

Let $U(D_5) = (d_3')^{-1}(U(S_5))$ and $\mathbf{L}$ be the induced subbundle $(d_3')^*L$ of $\mathbf{K}_2$. The restriction of d_3' to $\bigcirc^3\mathbf{L}$ over $U(D_5)$ is denoted by

$$r_3 : \bigcirc^3\mathbf{L} \to \mathbf{Q} \quad \text{over } U(D_5) \tag{3.2}$$

and considered as the section of $\mathrm{Hom}(\bigcirc^3\mathbf{L}, \mathbf{Q})$. It follows from the above remark that $z \in U(D_5)$ lies in $\Sigma\overline{D}_5$ if and only if $r_{3,z}$ is a null homomorphism. For an element $z \in \Sigma^{n-p+1,2,0}$ with $\pi_N(z) = x$ and $\pi_P(z) = y$, we can choose local coordinates $(x_1, \dots, x_{n-2}, u, v)$ of N near x and $(y_1, \dots, y_p)$ of P near y such that z is represented as $j^m f(0)$ with

$$\tag{3.3} \begin{aligned} y_i \circ f(x) &= x_i \qquad (1 \le i \le p-1), \\ y_p \circ f(x) &= \pm x_p^2 \pm \cdots \pm x_{n-2}^2 + \overline{f}(x_1, \dots, x_{p-1}, u, v), \end{aligned}$$

where $\overline{f}$ lies in the ideal $(x_1, \dots, x_{p-1}, u, v)^2$. It is easily checked that $\mathbf{K}_{1,z}$ is spanned by $d_{x_p}, \dots, d_{x_{n-2}}, d_u, d_v$, $\mathbf{K}_{2,z}$ by d_u, d_v and $\mathbf{Q}_z$ by the image of $\partial/\partial y_p$, say $\mathbf{e}$.

Lemma 3.4. *With the notation above let $d^i = a^i d_u + b^i d_v$ in $\mathbf{K}_{2,z}$ with constants a^i and b^i $(i = 1, 2, 3)$. Then*

$$d'_{3,z}(d^1 \bigcirc d^2 \bigcirc d^3) = \left[\left(\prod_{i=1}^{3}(a^i\partial/\partial u + b^i\partial/\partial v)\overline{f}\right)(0)\right]\mathbf{e}.$$

Proof. Recall a special $(n-p+1, 2, 0)$-flag defined near z, $\mathbf{D} \supset \mathbf{K}'_1 \supset \mathbf{K}'_2$ introduced in §1. Let $\mathbf{K}'_1$ be the set of all vectors annihilating functions y_i $(i = 1, \dots, p-1)$ considered as functions on a neighborhood of z through π_P. Therefore $\mathbf{K}'_1 | j^m f(U_x)$ is spanned by $d_{x_p}, \dots, d_{x_{n-2}}, d_u, d_v$ where U_x is a neighborhood of x. Since $j^m f(U_x)$ is locally a submanifold of $J^m(N, P)$, $\mathbf{K}'_1$ is spanned by $n-p+1$ vector fields v_j $(p \le j \le n)$ such that $(j^m f)^*(v_j) = \partial/\partial x_j$ $(j \le n-2)$, $(j^m f)^*(v_{n-1}) = \partial/\partial u$ and $(j^m f)^*(v_n) = \partial/\partial v$. In the set consisting of all smooth functions $d(y_p)$ where d is any smooth vector field of $\mathbf{K}'_1$ near z (note that as d kills $y_1, \dots, y_{p-1}$, this set is well defined), the functions $v_j(y_p)$ $(p \le j \le n-2)$ constitute $n-2$ totally independent functions together with $y_1, \dots, y_{p-1}$ near z (see §1). For, $v_j(y_p)$ $(j \le n-2)$ equals $\pm 2x_j$ over $j^m f(U)$ by (1.2) and

$$\begin{aligned} v_j(y_p) \circ j^m f &= (j^m f)^*(v_j)(y_p \circ \pi_P \circ j^m f) \\ &= \partial/\partial x_j(y_p \circ f) \\ &= \pm 2x_j. \end{aligned}$$

Then $\mathbf{K}'_2$ is the set of all vectors killing $y_1, \dots, y_{p-1}$ and $v_j(y_p)$ $(p \le j \le n-2)$ in $\mathbf{K}'_1$ near z. It is easily checked that $\mathbf{K}'_2 | j^m f(U)$ is spanned by d_u and d_v and $\mathbf{K}'_2$, by the extended vector fields v_{n-1} and v_n. Hence it follows from (1.1) and (1.2) that

$$\begin{aligned} d'_{3,z}(d^1 \bigcirc d^2 \bigcirc d^3) &= (h_3(d^1 \otimes d^2 \otimes d^3 \otimes y_p)|_z)\mathbf{e} \\ &= ((d^1 d^2 d^3(y_p)) \circ j^m f(0))\mathbf{e} \end{aligned}$$

and

$$\begin{aligned}(d^1d^2d^3(y_p))\circ j^m f &= ((j^m f)^*d^1)(d^2d^3(y_p)\circ j^m f)\\ &= (a^1\partial/\partial u + b^1\partial/\partial v)(((j^m f)^*d^2)((d^3y_p)\circ j^m f))\\ &= \prod_{i=1}^{2}(a^i\partial/\partial u + b^i\partial/\partial v)((j^m f)^*d^3)(y_p\circ j^m f)\\ &= \left(\prod_{i=1}^{3}(a^i\partial/\partial u + b^i\partial/\partial v)\right)(y_p\circ j^m f)\\ &= \left(\prod_{i=1}^{3}(a^i\partial/\partial u + b^i\partial/\partial v)\right)(\overline{f}).\end{aligned}$$

This proves the lemma. Q.E.D.

Proposition 3.5. *$d'_3 : \Sigma^{n-p+1,2,0} \to \mathrm{Hom}(\bigcirc^3\mathbf{K}_2, \mathbf{Q})$ is transverse to S_5 and so $\Sigma\overline{D}_5$ is a submanifold of $\Sigma^{n-p+1,2,0}$ with normal bundle $\mathrm{Hom}(\bigcirc^3\mathbf{L}, \mathbf{Q})$.*

Proof. Let $\Sigma^{n-p+1,2,0}_{x,y}$ be the fiber of $\Sigma^{n-p+1,2,0}$ over (x, y). We consider the submanifold S in $\Sigma^{n-p+1,2,0}_{x,y}$ which consists of all m jets $j^m f(0)$ where f is represented as in (2.5) and $\overline{f}$ varies in $(x_1, \dots, x_{p-1}, u, v)^2$ so that $\overline{f}_{uu}$, $\overline{f}_{uv}$ and $\overline{f}_{vv}$ generate $(x_1, \dots, x_{p-1}, u, v)^2$ together with $x_1, \dots, x_{n-2}$. The vector fields d_u, d_v and $\mathbf{e}$ determine the trivializations $\mathbf{K}_2|S \cong S\times\mathbf{R}^2$ and $\mathbf{Q}|S \cong S\times\mathbf{R}$ and $d'_3|S\colon S \to \mathrm{Hom}(\bigcirc^3\mathbf{K}_2, \mathbf{Q})|S \cong S\times\mathrm{Hom}(\bigcirc^3\mathbf{R}^2, \mathbf{R})$ is calculated by Lemma 3.4. Since $(d'_3|S)^{-1}(S_5) = S\cap\Sigma\overline{D}_5$ and $(d'_3|S)$ is surjective onto the second component, d'_3 is transverse to S_5. Lemma 2.2 shows that the normal bundle of $\Sigma\overline{D}_5$ is isomorphic to $\mathrm{Hom}(\bigcirc^3\mathbf{L}, \mathbf{Q})$ on $S\cap\Sigma\overline{D}_5$. Therefore the assertion follows all over $\Sigma\overline{D}_5$. Q.E.D.

Let $\mathbf{D}\supset\mathbf{K}'_1\supset\mathbf{K}'_2$ be a special $(n-p+1, 2, 0)$-flag on a small neighborhood U of $z\in U\cap U(D_5)$ and $\mathbf{L}'$, an extended line bundle of $\mathbf{L}$ in $\mathbf{K}'_2$ over U. Let Γ be the set of all smooth sections of $\mathbf{L}'$. For z with $\pi_N(z) = x$ and $\pi_P(z) = y$ we define

$$h_{t,z} : \bigotimes\Gamma\otimes\mathfrak{m}_y \to \mathbf{R}$$

by $h_{t,z}(d^t\otimes\cdots\otimes d^1\otimes\phi) = d^t(\cdots(d^1\phi))(z)$ where $d^j\in\Gamma$ and $\phi\in\mathfrak{m}_y$. When $z = j^m f(0)$, let a_y denote the set of germs $\alpha\in\mathfrak{m}_y$ such that $\alpha\circ f\in\mathfrak{m}_x^2$. We prove the following.

Lemma 3.6. *Let $z\in U\cap U(D_5)$. If $h_{j,z}$ vanishes for every j with $1\le j<t$, then $h_{t,z}$ induces $\overline{h}_{t,z} : \bigcirc^t\mathbf{L}_z\otimes\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathbf{R}$ such that the following diagram commutes.*

$$\begin{array}{ccccc} & & \bigotimes^t\Gamma\otimes\mathfrak{m}_y & \xrightarrow{h_{t,z}} & \mathbf{R}\\ & & \big\downarrow{\scriptstyle pr} & & \big\|\\ \bigcirc^t\mathbf{L}_z\otimes(a_y/a_y\cap\mathfrak{m}_y^2) & \xrightarrow{\ i\ } & \bigcirc^t\mathbf{L}_z\otimes\mathfrak{m}_y/\mathfrak{m}_y^2 & \xrightarrow{\overline{h}_{t,z}} & \mathbf{R}\end{array}$$

where pr and i are induced from the projections $\mathfrak{m}_y\to\mathfrak{m}_y/\mathfrak{m}_y^2$, $\Gamma\to\Gamma/\mathfrak{m}_z\Gamma\cong\mathbf{L}_z$ and the inclusion $a_y/a_y\cap\mathfrak{m}_y^2\to\mathfrak{m}_y/\mathfrak{m}_y^2$ respectively.

Proof. Notice that the null homomorphisms $d_1|\mathbf{L}_z$ and $d_2'|\mathbf{L}_z \bigcirc \mathbf{L}_z$ over $U(D_5)$ are induced from $h_{t,z}$ for $t = 1$ and 2 with similar commutative diagram. Since $\mathbf{L}_z \cong \Gamma/\mathscr{m}_z\Gamma$, we need to show for the proof that $h_{t,z}$ vanishes on $d^t \otimes d^{t-1} \otimes \cdots \otimes d^1 \otimes \alpha$ when either one of d^j lies in $\mathscr{m}_z\Gamma$ or α in $\mathscr{m}_y^2$. It is very like the proof of [5, Theorem 4.1] and so we omit it. Q.E.D.

Lemma 3.7. *The homomorphism $\overline{h}_{t,z}$ does not depend on the choice of a special flag $\mathbf{D} \supset \mathbf{K}_1' \supset \mathbf{K}_2'$ and $\mathbf{L}'$.*

Proof. We notice that $\mathbf{L}$ over a smooth submanifold $U(D_5)$ of $\Sigma^{n-p+1,2,0}$ is a subbundle of the bundle $\mathbf{K}_2|(U(D_5))$. Let $\Gamma_{U(D_5)}$ be the set of all smooth sections of $\mathbf{L}$. Then we can define the following homomorphism $k_{t,z}$ and $\overline{k}_{t,z}$ similarly as $h_{t,z}$ and $\overline{h}_{t,z}$ such that the following diagram commutes,

$$\begin{tikzcd}
\bigcirc^t\Gamma \otimes a_y \arrow[r] \arrow[d] & \bigcirc^t\mathbf{L}_z \otimes a_y/a_y \cap \mathscr{m}_y^2 \arrow[r, "\overline{h}_{t,z}"] \arrow[d] & \mathbf{R} \\
\bigcirc^t\Gamma_{(D_5)} \otimes (\pi_P|U(D_5))^* a_y \arrow[r] \arrow[urr, "k_{t,z}"] & \bigcirc^t\mathbf{L}_z \otimes (\pi_P|U(D_5))^*(a_y/a_y \cap \mathscr{m}_y^2) \arrow[ur, "\overline{k}_{t,z}"'] &
\end{tikzcd}$$

where the derivative of a smooth function on $U(D_5)$ by a vector field of $\Gamma_{U(D_5)}$ in the definition of $k_{t,z}$ is the usual one on a smooth manifold of finite dimension and the vertical maps are induced from the restrictions of Γ to $\Gamma_{U(D_5)}$ and $\pi_P|U(D_5)$. Then the lemma follows from the fact that $k_{t,z}$ is independent of the choice of a special flag and $\mathbf{L}'$. Q.E.D.

Definition 3.8. We inductively define the set $\Sigma\overline{D}_{t+1}(N, P)$ as the set of all jets $z \in U(D_5)$ such that $\overline{h}_{j,z}$ is defined and vanishes for all $j < t$ and set $\Sigma D_{t+1}(N, P) = \Sigma\overline{D}_{t+1}(N, P) \backslash \Sigma D_{t+2}(N, P)$ (we usually neglect (N, P) in this paper). We define the homomorphism

$$r_t : \bigcirc^t\mathbf{L}|\Sigma\overline{D}_{t+1} \longrightarrow \mathbf{Q}|\Sigma\overline{D}_{t+1}$$

so that $r_{t,z}$ is the induced one from $\overline{h}_{t,z}$ (or $\overline{k}_{t,z}$) by the identification of $\mathrm{Hom}(a_y/a_y \cap \mathscr{m}_y^2, \mathbf{R})$ with $\mathbf{Q}_z$.

Remark 3.9. Notice that r_3 in (3.8) coincides with r_3 in (3.2). It will be seen that $h_{t,z}$ is useful in the calculation of $r_{t,z}$ although $k_{t,z}$ is not so (see proof of Theorem 3.10).

Theorem 3.10. *The set $\Sigma\overline{D}_{k+1}$ is a submanifold of $J^\infty(N, P)$ of codimension $n-p+k+1$ with $U(D_5) \supset \Sigma\overline{D}_5 \supset \Sigma\overline{D}_6 \supset \cdots \supset \Sigma\overline{D}_{k+1} \supset \cdots$ satisfying that the intrinsic derivative of r_k,*

$$d(r_k) : T(\Sigma\overline{D}_{k+1})|\Sigma\overline{D}_{k+2} \to \mathrm{Hom}(\bigcirc^k\mathbf{K}, \mathbf{Q})|\Sigma\overline{D}_{k+2}$$

is surjective, that is, r_k is transverse to the zero section when considered as the section of $\mathrm{Hom}(\bigcirc^k\mathbf{L}, \mathbf{Q})|\Sigma\overline{D}_{k+1}$.

Proof. Notice that for $k = 3$ ($\Sigma\overline{D}_4$ means $U(D_5)$) r_3 is, by definition, nothing but $(d_3'|\bigcirc^3\mathbf{L})|U(D_5)$ and the assertion follows from Proposition 3.5. For any $z \in \Sigma\overline{D}_{k+1}$, we can choose suitable local coordinates $(x_1, \dots, x_{n-2}, u, l)$ near

$x = \pi_N(z)$ and $(y_1, \dots, y_p)$ near $y = \pi_P(z)$ such that $z = j^m f(0)$ and

$$(*) \qquad \begin{aligned} y_i \circ f &= x_i \qquad (1 \le i \le p-1), \\ y_p \circ f &= \pm x_p^2 \pm \cdots \pm x_{n-2}^2 + u^2 l + \overline{f}(x_1, \dots, x_{p-1}, u, l), \end{aligned}$$

where $\overline{f} \in m_x^2$ and $\overline{f}(0, \dots, 0, u, l) \in \{u, l\}^4$. Then it is easily checked by the similar arguments as in Lemma 3.4 that $\mathbf{L}'$ is spanned by a vector field d with $(j^m f)^* d = \partial/\partial l$ near x. Consider the submanifold S in $\Sigma\overline{D}_{k+1}$ consisting of all $j^m f(0)$ where f is as in $(*)$ and $\overline{f}$ varies so that $j^m f(0) \in \Sigma\overline{D}_{k+1}$. Then $\mathbf{L}|S$ is spanned by d_l and r_t $(t \le k)$ is calculated as follows. Let $d^t \in \Gamma$ with $(j^m f)^* d^t = a^t d_l$ for a constant a^t. Then

$$\begin{aligned} h_{t,z}(d^t \otimes \cdots \otimes d^1 \otimes y_p) &= (d^t \cdots d^1(y_p))(z) \\ &= (d^t \cdots d^1(y_p)) \circ j^m f|_{x=0} \\ &= ((j^m f)^* d^t)((d^{t-1} \cdots d^1(y_p)) \circ j^m f)|_{x=0} \\ &= (a^t \partial/\partial l)((d^{t-1} \cdots d^1(y_p)) \circ j^m f)|_{x=0} \\ &\cdots \\ &= \prod_{j=1}^{t} (a^j \partial/\partial l)(y_p \circ j^m f)|_{x=0} \\ &= a^t a^{t-1} \cdots a^1 \partial^t \overline{f}/\partial l^t(0). \end{aligned}$$

Hence it follows that

$$(3.11) \qquad r_{t,z}(a^t d_l \bigcirc \cdots \bigcirc a^1 d_l) = (a^t a^{t-1} \cdots a^1 \partial^t \overline{f}/\partial l^t(0))\mathbf{e}.$$

Thus for every k, $S \cap \Sigma\overline{D}_{k+1}$ is determined by the equations $\partial^t \overline{f}/\partial l^t(0) = 0$ for $1 \le t \le k-1$. Since $z \in S \cap \Sigma\overline{D}_{k+1}$ lies in $\Sigma\overline{D}_{k+2}$ if and only if $\partial^k \overline{f}/\partial l^k(0) = 0$, it is easily checked that r_k is transverse to the zero-section at any point $z \in \Sigma\overline{D}_{k+2}$. Q.E.D.

Corollary 3.12. *The normal bundle of* $\Sigma\overline{D}_{k+1}$ *is isomorphic to*

$$\operatorname{Hom}\left(\mathbf{K}_1 \oplus (\bigcirc^2 \mathbf{K}_2) \oplus \left(\bigoplus_{t=3}^{k-1} \bigcirc^t \mathbf{L}\right), \mathbf{Q}\right) \Bigg| \Sigma\overline{D}_{k+1}.$$

Remark 3.13. Let $(x_1, \dots, x_{n-2}, u, l)$ be the coordinates of $\mathbf{R}^n$ and $(y_1, \dots, y_p)$, that of $\mathbf{R}^p$. Let $f : \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ be a smooth map germ such that $y_i \circ f = x_i$ $(1 \le i \le p-1)$ and $y_p \circ f \in m_x^2$ and that the rank of $n-p-1$ matrix $(\partial^2 y_p \circ f/\partial x_s \partial x_t)_{p \le s, t \le n-2}$ is $n-p-1$ at the origin. Then $d'_{3,z}$ is calculated by the formula of Lemma 3.4 for $z = j^m f(0)$. Furthermore if $d'_{3,z} \in \operatorname{Hom}(\bigcirc^3 \mathbf{K}_{2,z}, \mathbf{Q}_z)$ is represented by the cubic form $u^2 l$ and $\partial^j y_p \circ \overline{f}/\partial l^j(0) = 0$ for $1 \le j \le t-1$, then $r_{t,z}$ is also calculated by the formula of (3.11). This is easily checked by proving the following fact. If we represent as $y_p \circ f = y_p \circ f - \overline{f} + \overline{f}$ where $\overline{f} = y_p \circ f(x_1, \dots, x_{p-1}, 0 \cdots 0, u, l)$, then there exists a local diffeomorphism: $(x_1, \dots, x_{n-2}, u, l) \to (x_1, \dots, x_{p-1}, x'_p, \dots, x'_{n-2}, u, l)$ such that $y_p \circ f = \pm x_p'^2 \pm \cdots \pm x_{n-2}'^2 + \overline{f}(x_1, \dots, x_{p-1}, u, l)$.

4. Singularities E_k

We shall define the submanifold ΣE_k by using the notation and the reviewed results in §1. First we have the following exact sequence and $d_4 : \mathbf{K}_3 \to \mathrm{Hom}(\bigcirc^2\mathbf{K}_3 \bigcirc \mathbf{K}_2, \mathbf{Q})$ over $\Sigma^{n-p+1,2,1}$.

$$0 \to \mathbf{K}_3 \to \mathbf{K}_2 \xrightarrow{d_3} \mathrm{Hom}(\mathbf{K}_2 \bigcirc \mathbf{K}_2, \mathbf{Q}) \to \mathrm{Hom}(\mathbf{K}_3 \bigcirc \mathbf{K}_2, \mathbf{Q}) \to 0.$$

The normal bundle of $\Sigma^{n-p+1,2,1}$ in $\Sigma^{n-p+1,2}$ is isomorphic to the vector bundle $\mathrm{Hom}(\bigcirc^2\mathbf{K}_3 \bigcirc \mathbf{K}_2, \mathbf{Q})$. By definition $\Sigma^{n-p+1,2,1,0}$ is the set of all jets $z \in \Sigma^{n-p+1,2,1}$ such that $d'_{4,z}$ does not vanish.

We define ΣE_6 as the set of all $z \in \Sigma^{n-p+1,2,1,0}$ such that $d'_{4,z}| \bigcirc^4 \mathbf{K}_3$ does not vanish and set $\Sigma E_7 = \Sigma^{n-p+1,2,1,0} \backslash \Sigma E_6$. It is easily checked that if we consider $d'_{4,z}| \bigcirc^4 \mathbf{K}_3$ as the section of the bundle $\mathrm{Hom}(\bigcirc^4\mathbf{K}_3, \mathbf{Q})$ over $\Sigma^{n-p+1,2,1,0}$, then it is transverse to the zero section. Thus ΣE_7 is a submanifold and the intrinsic derivative

$$d(d'_{4,z}) : T\Sigma^{n-p+1,2,1,0}|\Sigma E_7 \to \mathrm{Hom}(\bigcirc^4\mathbf{K}_3, \mathbf{Q})$$

is surjective.

Lemma 4.1. *The normal bundle of ΣE_7 in $\Sigma^{n-p+1,2,1,0}$ is isomorphic to* $\mathrm{Hom}(\bigcirc^4\mathbf{K}_3, \mathbf{Q})|\Sigma E_7$.

We define ΣE_8 as the set of all $z \in \Sigma^{n-p+1,2,1,1,0}$ such that $d'_{5,z}| \bigcirc^5 \mathbf{K}_4$ does not vanish. It is reasonable to set $\Sigma \overline{E}_6 = \Sigma^{n-p+1,2,1}$ and $\Sigma \overline{E}_7 = \Sigma E_7 \cup \Sigma^{n-p+1,2,1,1}$. For by the similar argument above $d'_{4,z}| \bigcirc^4 \mathbf{K}_3$ over $\Sigma^{n-p+1,2,1}$ is transverse to the zero section and its inverse image of the zero section, which is nothing but $\Sigma \overline{E}_7$, becomes a submanifold.

Proposition 4.2. *The normal bundle ν_k of ΣE_k in $J^\infty(N, P)$ are as follows.*

$$\nu_6 \cong \mathrm{Hom}(\mathbf{K}_1 \oplus \bigcirc^2\mathbf{K}_2 \oplus \bigcirc^2\mathbf{K}_3 \bigcirc \mathbf{K}_2, \mathbf{Q})|\Sigma E_6,$$
$$\nu_7 \cong \mathrm{Hom}(\mathbf{K}_1 \oplus \bigcirc^2\mathbf{K}_2 \oplus \bigcirc^2\mathbf{K}_3 \bigcirc \mathbf{K}_2 \oplus \bigcirc^4\mathbf{K}_3, \mathbf{Q})|\Sigma E_7,$$

and

$$\nu_8 \cong \mathrm{Hom}(\mathbf{K}_1 \oplus \bigcirc^2\mathbf{K}_2 \oplus \bigcirc^2\mathbf{K}_3 \bigcirc \mathbf{K}_2 \oplus \bigcirc^3\mathbf{K}_3 \bigcirc \mathbf{K}_2, \mathbf{Q})|\Sigma E_8.$$

The following theorem is an expected one and we omit its proof (see [4 and 8, Proposition 7.4]).

Theorem 4.3. *Let X_k denote D_k or E_k. Suppose that for a smooth map germ $f: \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$, $j^m f$ is transverse to ΣX_k at the origin and $j^m f(0) \in \Sigma X_k$. Then f is C^∞ stable and has a singularity X_k at the origin.*

5. Connected components

In this section ΣX_k denote the fiber of $\Sigma X_k(\mathbf{R}^n, \mathbf{R}^p)$ over the origin $(0, 0)$. We shall interpret the connected components of ΣX_k in our terminology and the result will be proved by the canonical forms of singularities X_k in [4] and known essentially (see, for example, [6]).

Let $z \in \Sigma X_k$. Then $d_{2,z}$ induces the nondegenerate quadratic form denoted by $q_z : (\mathbf{K}_{1,z}/\mathbf{K}_{2,z}) \bigcirc (\mathbf{K}_{1,z}/\mathbf{K}_{2,z}) \to \mathbf{Q}_z$. By taking an orientation of $\mathbf{Q}_z$, we can consider the index of q_z denoted by $i(z)$ and so we define the semi-index

$s(z)$ as $\min(i(z), \dim(\mathbf{K}_{1,z}/\mathbf{K}_{2,z}) - i(z))$. Let $\Sigma X_{k,s}$ be the set of all jets $z \in \Sigma X_k$ with semi-index s.

Now we first deal with $\Sigma A_k = \Sigma^{n-p+1,1\cdots 10}$. If $s \neq 1/2(n-p)$, we can determine the orientation of $\mathbf{Q}_z$ denoted by $o_1(z)$ so that q_z is represented as $x_p^2 + \cdots + x_{p+s-1}^2 - x_{p+s}^2 - \cdots - x_{n-1}^2$ under a basis of $\mathbf{K}_{1,z}/\mathbf{K}_{2,z}$. On the other hand if $k+1$ is even, the orientation of $\bigcirc^{k+1}\mathbf{K}_{2,z}$ induced from every one of $\mathbf{K}_{2,z}$ does not depend on its choice $(\dim \mathbf{K}_{2,z} = 1)$. Therefore the isomorphism $d'_{k+1,z} : \bigcirc^{k+1}\mathbf{K}_{2,z} \to \mathbf{Q}_z$ in §1 determines the second orientation $o_2(z)$ of $\mathbf{Q}_z$ coming from the above orientation of $\bigcirc^{k+1}\mathbf{K}_{2,z}$. So when $s \neq 1/2(n-p)$ and $k+1$ is even, we say that $z \in \Sigma A_k$ has plus or minus sign depending on whether $o_1(z)$ coincides with $o_2(z)$ or not.

Definition 5.1. When $s \neq 1/2(n-p)$ and $k+1$ is even, we define $\sum_{s,\pm}^{I_k,0}$ to be the set of all jets $z \in \sum_s^{I_k,0}$ with sign $\pm$.

Next we consider ΣD_k and let $n > p \geq 2$. If $s \neq 1/2(n-p-1)$ and $k+1$ is even, then we can determine the orientation $o_1(z)$ of Q_z for $z \in \Sigma D_k$ so that q_z is written as $x_p^2 + \cdots + x_{p+s-1}^2 - x_{p+s}^2 - \cdots - x_{n-2}^2$ under a suitable basis of $\mathbf{K}_{1,z}/\mathbf{K}_{2,z}$. The uniquely determined orientation of $\bigcirc^{k+1}\mathbf{L}_z$ induced from every one of $\mathbf{L}_z$ induces the second orientation $o_2(z)$ of $\mathbf{Q}_z$ by the isomorphism $r_{k+1,z} : \bigcirc^{k+1}\mathbf{L}_z \to \mathbf{Q}_z$. Then we define the sign of z similarly. If $k+1$ is odd, then there exist two isomorphisms

$$r_{k+1} : \bigcirc^{k+1}\mathbf{L} \to \mathbf{Q} \quad \text{and} \quad d''_3 : \mathbf{L} \otimes \mathbf{K}_2/\mathbf{L} \otimes \mathbf{K}_2/\mathbf{L} \to \mathbf{Q}$$

induced from d'_3 by (iii) of Lemma 2.2. Since $\xi \otimes \xi$ is trivial for a line bundle ξ, they induce the isomorphism

$$\mathbf{L} \cong \bigcirc^{k+1}\mathbf{L} \xrightarrow{r_{k+1}} \mathbf{Q} \xrightarrow{d_3''^{-1}} \mathbf{L} \otimes \mathbf{K}_2/\mathbf{L} \otimes \mathbf{K}_2/\mathbf{L} \cong \mathbf{L}.$$

We say that z has plus or minus sign depending on whether it preserves the orientation of $\mathbf{L}_z$ or not.

Definition 5.2. When (i) $s \neq \frac{1}{2}(n-p-1)$ and $k+1$ is even or (ii) $k+1$ is odd, we define $\Sigma D_{k,s}^{\pm}$ to be the set of all jets $z \in \Sigma D_{k,s}$ with sign $\pm$.

As for E_k, when $s \neq \frac{1}{2}(n-p-1)$, there exists the determined orientation of $\mathbf{Q}_{1,z}$ as above. When $k = 6$, $\bigcirc^4\mathbf{K}_3$ has the unique orientation. Since $d'_4|\bigcirc^4\mathbf{K}_3$ is isomorphism, we say that $z \in \Sigma E_6$ has plus or minus sign depending on whether $d'_{4,z}$ preserve the orientation or not.

Definition 5.3. When $s \neq \frac{1}{2}(n-p-1)$, we define $\Sigma E_{6,s}^{\pm}$ to be the set of all jets $z \in \Sigma E_{6,s}$ with sign $\pm$.

The following proposition is an expected one and will be proved by considering the normal forms of the unfoldings of smooth functions with singularity $X_{k,s}^{\pm}$ (see also [3]). So we omit its proof.

Proposition 5.4. *Let* $n > p \geq 2$. *Then*

(A) *If* $s \neq \frac{1}{2}(n-p)$ *and* $k+1$ *is even, then* $\sum_{s,\pm}^{I_k,0}$ *is connected. Otherwise* $\sum_s^{I_k,0}$ *is connected.*

(D) *If* (i) $s \neq \frac{1}{2}(n-p-1)$ *and* $k+1$ *is even or* (ii) $k+1$ *is odd, then* $\Sigma D^{\pm}_{k,s}$ *is connected. Otherwise* $\Sigma D_{k,s}$ *is connected.*

(E) $\Sigma E^{\pm}_{6,s}$ *is connected only for* $s \neq \frac{1}{2}(n-p-1)$. *Otherwise* $\Sigma E_{k,s}$ *is connected.*

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