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THE CENTER OF $\mathbb{Z}[S_{n+1}]$ IS THE SET OF SYMMETRIC POLYNOMIALS IN n COMMUTING TRANSPOSITION-SUMS

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ABSTRACT. Let S_{n+1} be the symmetric group on the $n+1$ symbols $0, 1, 2, \dots, n$. We show that the center of the group-ring $\mathbb{Z}[S_{n+1}]$ coincides with the set of symmetric polynomials with integral coefficients in the n elements $s_1, \dots, s_n \in \mathbb{Z}[S_{n+1}]$, where $s_k = \sum_{0 \le i < k}(i, k)$ is a sum of k transpositions, $k = 1, \dots, n$. In particular, every conjugacy-class-sum of S_{n+1} is a symmetric polynomial in $s_1, \dots, s_n$.

0. INTRODUCTION

It is well known that the symmetric group S_{n+1} on the $n+1$ symbols $0, \dots, n$ is generated by transpositions, and that it has sets of generating transpositions of size n, but not less. By analogy, the problem arises whether the center C^{n+1} of the group-algebra $\mathbb{C}[S_{n+1}]$ over the complex numbers is generated by its transposition-analogue, the sum s of all transpositions in S_{n+1}. More specifically, C^{n+1} is known to consist of all linear combinations over $\mathbb{C}$ of conjugacy-class-sums (CCSs) of S_{n+1}. It is a commutative subring of $\mathbb{C}[S_{n+1}]$. Denote by $\widetilde{C}^{n+1}$ the subring with identity of C^{n+1} generated by its member s consisting of all polynomials in s with complex coefficients. Does $\widetilde{C}^{n+1} = C^{n+1}$ hold for all n? Rather unexpectedly, the answer, which can be read from the character table of S_{n+1} (see e.g. [KM]) is negative. While $\widetilde{C}^{n+1} = C^{n+1}$ for $n = 1, 2, 3, 4, 6$, it fails first for $n = 5$ ($\widetilde{C}^6 \neq C^6$) and again for $n \geq 7$. In fact, for $n \geq 8$ no CCS x in C^n generates C^n. This is clear if the class is even. When the class is odd it is seen as follows. In general, $x \in C^n$ generates C^n as an algebra over $\mathbb{C}$ if and only if all the coefficients $\omega_D(x)$ in the linear expansion of x by the central idempotents in C^n are distinct, where the index D varies over all Young diagrams of size n. But if x is a CCS for some odd conjugacy class, then $\omega_D(x) = 0$ for every self-conjugate Young diagram D. As for $n \geq 8$ there always exist two distinct self-conjugate Young diagrams, no odd CCS x in C^n can generate C^n.

On the other hand, n transpositions that generate S_{n+1} certainly generate all the group algebra $\mathbb{C}[S_{n+1}]$. The more delicate problem of generating C^{n+1} by *commuting* transposition-sums is both natural and of importance in treating the n-body problem in theoretical physics [D, C].

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Recently [KM] it was noted that every CCS of S_{n+1} is a polynomial with integral coefficients in the transposition-sums $s_1, \dots, s_n$, where $s_j = \sum_{0\le i<j}(i, j)$. Denote by P the set of all these polynomials, and let C_k denote the center of $\mathbb{Z}[S_k]$, D_{k+1} the centralizer of S_k in $\mathbb{Z}[S_{k+1}]$. Then it is actually shown in [KM] that P is the subring generated by $D_1, D_2, \dots, D_{n+1}$ in $\mathbb{Z}[S_{n+1}]$—a commutative subring that properly contains the center C_{n+1} of $\mathbb{Z}[S_{n+1}]$.

In this paper we show that the subring of P formed by the *symmetric* polynomials in $s_1, \dots, s_n$ is, in fact, C_{n+1}. As C_{n+1} is known to be the set of integral linear combination of CCSs of S_{n+1}, we deduce that every CCS of S_{n+1} is actually a *symmetric* polynomial in $s_1, \dots, s_n$. Now $\langle 3^*\rangle_3 = (0, 1, 2) + (0, 2, 1)$, the class-sum of 3-cycles in $\mathbb{Z}[S_3]$, satisfies

$$\langle 3^*\rangle_3 = s_1^2 + s_2^2 - 3 = s_1 s_2 = s_2^2 - 2.$$

Thus, even if the total degree of the polynomial is restricted to minimum, symmetry of a polynomial representing a CCS is not guaranteed; and adding the symmetry restriction does not guarantee its uniqueness, either.

We explain in some detail the notation we use in §1, so as to render the paper as self-contained as possible. Section 2 is devoted to the proof that every symmetric polynomial in $s_1, \dots, s_n$ is in the center of $\mathbb{Z}[S_{n+1}]$. By the fundamental Theorem about symmetric polynomials (Proposition 1.1), it is enough to prove it for elementary symmetric polynomials. Indeed, we show that the rth elementary polynomial in $s_1, \dots, s_n$ is the sum of all permutations in S_{n+1} representable as a product of r transpositions, but not less (Theorem 2.1). The proof we chose to give in detail depends on the selection of a particular shortest representation of a permutation as a product of transpositions, which we call *monotone*. A shorter proof is sketched at the end of §2.

The other inclusion, namely, every element of the center C_{n+1} is a symmetric polynomial in $s_1, \dots, s_n$, follows from its validity for CCSs. This is proved in §3 (Theorem 3.1), and depends on another selection of shortest product of transpositions for a given permutation, which we call the *stellar* one. The proof proceeds by induction on a weight function that combines the length of shortest product of transpositions and the number of symbols moved by a member of the given conjugacy class.

1. Preliminaries and results

We let $|A|$ denote the cardinality of a finite set A, whose elements are referred to as *symbols*. S_A denotes the symmetric group over A, consisting of all permutations of A. We let $\mathbb{Z}$, $\mathbb{N}_0$, $\mathbb{N}$ denote the sets of integers, nonnegative integers and positive integers. It will be convenient to adopt the convention $n = \{0, \dots, n-1\}$ for $n \in \mathbb{N}_0$, thereby letting n denote a specific set of n symbols. Thus, $n+1 = \{0, \dots, n\}$ and S_{n+1} is the group of all permutations of the $n+1$ symbols $0, \dots, n$.

We briefly recall the construction of the group-ring $R[G]$ for a finite group G and a commutative ring with identity R. $R[G] = {}^G R$ is the set of all functions from G to R, with addition and product by a member of R defined pointwise:

$$(x+y)(g) = x(g) + y(g), \quad (rx)(g) = r \cdot x(g), \qquad x, y \in {}^G R, \ r \in R.$$

For $g \in G$ define $g^*\colon G \to R$ by $g^*(h) = \delta_{gh}$, $h \in G$. With this definition, every $x \in R[G]$ admits a unique representation $x = \sum_{g\in G} x(g) \cdot g^*$, and with little risk of ambivalence, this is conveniently shortened to

$$x = \sum_{g \in G} x(g) \cdot g.$$

We similarly use 1 to denote the unit element of G, of R and of $R[G]$ (as well as the natural number one), and write g for $1 \cdot g$, r for $r \cdot 1$ $(g \in G, r \in R)$ thereby considering R, G as subsets of $R[G]$.

The product operation in $R[G]$ is defined by requiring that the product is preserved in R and in G, and that $rg = gr$ for $r \in R$, $g \in G$. These requirements and distributivity yield the following definition of the product in $R[G]$:

$$\begin{aligned}\left(\sum_{g \in G} x(g)g\right)\left(\sum_{h \in G} y(h)h\right) &= \sum_{g,h \in G} x(g)y(h)gh \\ &= \sum_{k \in G}\left(\sum_{g \in G} x(g)y(g^{-1}k)\right)k.\end{aligned} \tag{1.0}$$

Let $H \subseteq G$. Then $\langle H\rangle = \sum_{h \in H} h \in R[G]$ denotes the sum of its members. Let $H_1, \dots, H_k \subset G$, and for $g \in G$ let

$$n_g^{H_1, \dots, H_k} = |\{(h_1, \dots, h_k) \in H_1 \times \cdots \times H_k : h_1 \cdots h_k = g\}|.$$

Thus $n_g^{H_1, \dots, H_k} \in \mathbb{N}_0$, and, whenever R has characteristic zero, i.e. $\mathbb{Z} \subseteq R$, we have by (1.0)

$$\langle H_1\rangle \cdots \langle H_k\rangle = \sum_{g \in G} n_g^{H_1, \dots, H_k} g. \tag{1.1}$$

(If $\operatorname{char} R = r > 0$ then $n_g^{H_1, \dots, H_k}$ should be replaced in (1.1) by its residue in $\mathbb{Z}_r$, the subring generated by 1_R in R.)

For $h \in G$ let $[h] = \{ghg^{-1} : g \in G\}$ denote its conjugacy class (COC) in G. We call $\langle [h]\rangle$ a *conjugacy-class-sum* (CCS) in $R[G]$. The center C of $R[G]$, consisting of all $x \in R[G]$ satisfying $xy = yx$ for every $y \in R[G]$, is spanned over R in $R[G]$ by the CCSs (see e.g. [KM]).

Let $R[z_1, \dots, z_n]$ denote the ring of polynomials in the commuting variables $z_1, \dots, z_n$ over R, spanned over R by the monomials $z_1^{k_1} \cdots z_n^{k_n}$, $k_1, \dots, k_n \in \mathbb{N}_0$. A permutation $\theta \in S_{\{z_1, \dots, z_n\}}$ acts on such a monomial by

$$\theta(z_1^{k_1} \cdots z_n^{k_n}) := \theta(z_1)^{k_1} \cdots \theta(z_n)^{k_n},$$

and this action is linearly extended to an action of $S_{\{z_1, \dots, z_n\}}$ on $R[z_1, \dots, z_n]$. A polynomial $q(z_1, \dots, z_n) \in R[z_1, \dots, z_n]$ is called a *symmetric polynomial* iff $\theta q = q$ for all $\theta \in S_{\langle z_1, \dots, z_n\rangle}$. We denote by $\operatorname{Syp}_R[z_1, \dots, z_n]$ the subring of $R[z_1, \dots, z_n]$ consisting of all symmetric polynomials in $z_1, \dots, z_n$.

For $n \in \mathbb{N}_0$, $0 \le h \le n$, the hth elementary polynomial in $z_1, \dots, z_n$ is defined by

$$\begin{aligned} e_0(z_1, \dots, z_n) &:= 1, \\ e_h(z_1, \dots, z_n) &:= \sum_{1 \le i_1 < i_2 < \cdots < i_h \le n} z_{i_1} z_{i_2} \cdots z_{i_n}. \end{aligned} \tag{1.2}$$

We shall make use of the fundamental theorem on symmetric polynomials [Mac].

Proposition 1.1. $\mathrm{Syp}_R[z_1, \dots, z_n]$ *is the ring generated by* $e_0(z_1, \dots, z_n), \dots, e_n(z_1, \dots, z_n)$ *in* $R[z_1, \dots, z_n]$. *Moreover, every symmetric polynomial* $q(z_1, \dots, z_n) \in \mathrm{Syp}_R[z_1, \dots, z_n]$ *has a unique polynomial of* n *variables* h *over* R, *such that* $q = h(e_1, \dots, e_n)$.

By a *type* we mean any function $t\colon \mathbb{N} \to \mathbb{N}_0$ satisfying $t(n) \neq 0$ for finitely many n's. Addition and the product by a nonnegative integer are defined for types naturally by $(t+t')(n) = t(n) + t'(n)$, $(kt)(n) = kt(n)$ $(n \in \mathbb{N}, k \in \mathbb{N}_0)$. For each $n \in \mathbb{N}$ let n^* be the type defined by $n^*(m) = \delta_{nm}$. Then every type t admits a unique representation

$$t = \sum_{n \in \mathbb{N}} t(n) n^* .$$

Let $\theta \in S_A$, $a \in A$. Then we denote by $\theta(a)$ the symbol into which θ maps a. Hence, the product $\xi\eta$ of ξ, $\eta \in S_A$ is the permutation obtained by acting first with η, then with ξ. Thus, for instance, $(a, b)(b, c) = (a, b, c)$ for distinct symbols $a, b, c \in A$, where for distinct $a_0, \dots, a_{k-1} \in A$, we let $\theta = (a_0, \dots, a_{k-1})$ denote the *k-cycle* $\theta \in S_A$ mapping a_i to a_{i+1} for $0 \le i < k$ (with $a_k = a_0$) and fixing every other symbol. A 2-cycle is called *transposition.*

For $\theta \in S_A$, $a \in A$ let $(a)_\theta$ denote the θ-orbit containing a:

$$(a)_\theta := \{\theta^m(a)\colon m \in \mathbb{Z}\} .$$

We let $\overline{\theta}(k)$ denote the number of θ-orbits of cardinality k:

$$\overline{\theta}(k) := |\{(a)_0\colon |(a)_\theta| = k\}| .$$

Thus, $\overline{\theta}$ is a type, determined by the partition of A into θ-orbits, and $\theta' \in S_A$ is conjugate to θ iff $\overline{\theta} = \overline{\theta}'$.

For a type t define

$$|t| := \sum_{n \in \mathbb{N}} t(n) \cdot n , \qquad m(t) := \sum_{\substack{n \in \mathbb{N} \\ 1<n}} t(n) \cdot n ,$$

$$c(t) := \sum_{n \in \mathbb{N}} t(n) , \qquad tr(t) := |t| - c(t) = \sum_{n \in \mathbb{N}} t(n)(n-1) .$$

If $\theta \in S_A$, let $|\theta| := |\overline{\theta}|$, $m(\theta) := m(\overline{\theta})$, $c(\theta) := c(\overline{\theta})$, $tr(\theta) := tr(\overline{\theta})$. Thus, $|\theta| = |A|$ is the cardinality of θ's domain. $m(\theta)$ is the cardinality of the set $M(\theta) := \{a \in A\colon \theta(a) \neq a\}$ of symbols actually moved by θ, $c(\theta) = |\{(a)_\theta\colon a \in A\}|$ is the number of θ-orbits, and $tr(\theta)$, *the transposition-number of* θ, is the smallest number of transpositions whose product equals θ. (See Proposition 2.2.)

We use types to denote COCs in S_n and CCSs in the group-ring $R[S_n]$ as follows. If $|t| \le n$, then

$$[t]_n = \{\theta \in S_n\colon \overline{\theta} = t + (n - |t|)1^*\} , \qquad \langle t \rangle_n = \langle [t]_n \rangle .$$

Thus, for instance, $[1^*]_n$ is the identity COC in S_n and $\langle 1^* \rangle_n$ is the identity of $R[S_n]$, $[2^*]_n$ is the tranposition COC in S_n and $\langle 2^* \rangle_n$ is the corresponding CCS in $R[S_n]$.

For arbitrary pairwise commuting $x_1, \ldots, x_n \in R[S_{n+1}]$ let $\mathrm{Syp}_R[x_1, \ldots, x_n]$ denote the set of all elements $q(x_1, \ldots, x_n) \in R[S_{n+1}]$ obtained by substituting $x_1, \ldots, x_n$ for the variables $z_1, \ldots, z_n$ in $q(z_1, \ldots, z_n)$.

Let C_{n+1}^R denote the center of $R[S_{n+1}]$. For $k = 1, \ldots, n$ let $s_k = \sum_{0 \leq i < k}(i, k) \in R[S_{n+1}]$. Our main result is

Theorem 1′. $C_{n+1}^R = \mathrm{Syp}_R[s_1, \ldots, s_n]$.

It is an immediate consequence of its special case

Theorem 1. $C_{n+1} = \mathrm{Syp}[s_1, \ldots, s_n]$.

Here C_{n+1} is the center of $\mathbb{Z}[S_{n+1}]$ and $\mathrm{Syp}[S_{n+1}] = \mathrm{Syp}_{\mathbb{Z}}[s_1, \ldots, s_n]$ is the set of all symmetric polynomials with integral coefficients in $s_1, \ldots, s_n$.

2. $\mathrm{Syp}(s_1, \ldots, s_n) \subseteq C_{n+1}$

For $r = 0, \ldots, n$ let $e_r = e_r(s_1, \ldots, s_n) \in \mathbb{Z}[S_{n+1}]$ denote the rth elementary symmetric polynomial in $s_1, \ldots, s_n$. Thus

$$e_0 = 1, \quad e_r = \sum_{1 \leq j_1 < \cdots < j_r \leq n} s_{j_1} \ldots s_{j_r}, \qquad r = 1, \ldots, n.$$

Also, let $u_r \in C_{n+1}$ be the sum of all permutations $\theta \in S_{n+1}$ satisfying $tr(\theta) = r$; thus

$$\begin{aligned} u_0 &= \langle 1^* \rangle_{n+1} = 1, \\ u_1 &= \langle 2^* \rangle_{n+1}, \\ u_2 &= \langle 3^* \rangle_{n+1} + \langle 2 \cdot 2^* \rangle_{n+1} \qquad (3 \leq n) \\ &\vdots \\ u_n &= \langle (n+1)^* \rangle_{n+1}. \end{aligned}$$

We shall prove

Theorem 2.1. $e_r = u_r$, $r = 0, \ldots, n$.

$\mathrm{Syp}[s_1, \ldots, s_n] \subseteq C_{n+1}$ follows by Proposition 1.1.

We start with some preliminaries. If A is a finite set, $\theta = (\theta_0, \ldots, \theta_{r-1}) \in {}^rS_A$ is a sequence of length r, then we write $l\theta = r$ and let $\pi\theta = \theta_0 \cdots \theta_{r-1} \in S_A$ denote the product of its elements in order. By definition, $\pi\theta = 1$ if $r = 0$, i.e. if θ is the empty sequence.

The following proposition is well known. We shall sketch its proof for the convenience of the reader.

Proposition 2.2. *Let A be a finite set and let $\xi \in S_A$. Let $\sigma = (\sigma_0, \ldots, \sigma_{r-1})$ be a shortest sequence of transpositions satisfying $\xi = \pi\sigma$. Then $r = tr(\xi)$, and $M(\xi) = \bigcup_{i=0}^{r-1} M(\sigma_i)$.*

Proof. First notice that $r \leq tr(\xi)$, as ξ is always a product of $tr(\xi)$ transpositions (a k-cycle is a product of $k-1$ transpositions, see Proposition 3.3). Also $M(\xi) \subseteq \bigcup_{i=0}^{r-1} M(\sigma_i)$ is obvious.

To prove the converse inclusion, notice first that for $0 \leq i < j < r$ we have $\sigma_i \neq \sigma_j$. Indeed, if $\sigma_i = \sigma_j$ then $\sigma_i\sigma_{i+1} \cdots \sigma_{j-1}\sigma_j = \sigma_i\sigma_{i+1} \cdots \sigma_{j-1}\sigma_i = \tilde{\sigma}_{i+1} \cdots \tilde{\sigma}_{j-1}$ where $\tilde{\sigma}_m = \sigma_i\sigma_m\sigma_i$ is again a transposition, and so $\tilde{\sigma} = (\sigma_0, \ldots,$

$\sigma_{i-1}, \tilde{\sigma}_{i+1}, \dots, \tilde{\sigma}_{j-1}, \sigma_{j+1}, \dots, \sigma_{r-1})$ is a shorter sequence of transpositions with $\pi\tilde{\sigma} = \xi$.

Now consider the simple graph with set of vertices A and with r edges $M(\sigma_i)$, $i = 0, \dots, r-1$. Every ξ-orbit is obviously contained in a connected component of this graph. As a connected component with k vertices has at least $k-1$ edges, we see that $tr(\xi) \leq r$. It follows that $tr(\xi) = r$ and our graph is a forest, each of whose components have a ξ-orbit as the set of vertices. $\bigcup_{i=0}^{r-1} M(\sigma_i) \subseteq M(\xi)$ follows. □

For $\xi \in S_{n+1}$, $\xi \neq 1$ let $j_\xi \in \{0, \dots, n\}$ be the largest symbol moved by ξ. A sequence of transpositions $\sigma = (\sigma_0, \dots, \sigma_{r-1}) \in S_{n+1}$ is called an increasing sequence iff $r = 0$, or $j_{\sigma_0} < j_{\sigma_1} < \cdots < j_{\sigma_{r-1}}$.

Proposition 2.3. *Every $\xi \in S_{n+1}$ with $tr(\xi) = r$ has a unique increasing sequence $\sigma^\xi = (\sigma_0^\xi, \dots, \sigma_{r-1}^\xi)$ of transpositions satisfying $\xi = \pi\sigma^\xi$.*

Proof. We proceed by induction on $r = tr(\xi)$. For $r = 0$, $\xi = 1$ and σ^ξ is the empty sequence.

Let now $tr(\xi) = r$ with $r > 0$, and let $\sigma = (\xi^{-1}(j_\xi), j_\xi)$, $\zeta = \xi\sigma$. Then $\zeta(j_\xi) = j_\xi$, and it is readily checked that a representation of ζ as a product of disjoint cycles (pdc) is obtained from a pdc of ξ by deleting the symbol j_ξ, which is fixed by ζ. Hence $c(\zeta) = c(\xi) + 1$ and so $tr(\zeta) = tr(\xi) - 1 = r - 1$.

Let $\sigma^\zeta = (\sigma_0^\zeta, \dots, \sigma_{r-2}^\zeta)$ be as provided by the induction hypothesis: An increasing sequence of transpositions of length $r-1$ satisfying $\pi\sigma^\zeta = \zeta$. As $r - 1 = tr(\zeta)$, Proposition 2.2 yields that every symbol moved by $\sigma_0^\zeta, \dots, \sigma_{r-2}^\zeta$ is also moved by ζ, hence smaller then j_ξ and so the sequence

$$\sigma^\xi = (\sigma_0^\zeta, \dots, \sigma_{r-2}^\zeta, \sigma) \overset{\text{def}}{=} (\sigma_0^\xi, \dots, \sigma_{r-1}^\xi)$$

is an increasing sequence of transpositions of length r, satisfying $\pi\sigma^\xi = \zeta\sigma = \xi$.

We now prove uniqueness. Let $\sigma = (\sigma_0, \dots, \sigma_{r-1})$ be any increasing sequence of transpositions satisfying $\pi\sigma = \xi$. Since $tr(\xi) = r$, we have by Proposition 2.2 $M(\xi) = \bigcup_{i=0}^{r-1} M(\sigma_i)$. Thus $j_\xi = j_{\sigma_i}$ for some $0 \leq i < r$, and $j_\xi \geq j_{\sigma_k}$ for all $0 \leq k < r$. Since σ is increasing, we actually have $j_\xi = j_{\sigma_{r-1}}$, and $j_{\sigma_k} < j_\xi$ for $0 \leq k < r-2$. It follows that $j_\xi = \xi(\xi^{-1}(j_\xi)) = \pi\sigma(\xi^{-1}(j_\xi)) = \sigma_{r-1}(\xi^{-1}(j_\xi))$ (recall that σ_{r-1} acts first in $\pi\sigma$). Thus, $\sigma_{r-1} = (j_\xi, \xi^{-1}(j_\xi)) = \sigma_{r-1}^\xi$ and so $\sigma_0, \dots, \sigma_{r-2} = \xi\sigma_{r-1} = \xi\sigma_{r-1}^\xi = \zeta$. But then $(\sigma_0, \dots, \sigma_{r-2})$ is an increasing sequence of transpositions of length $r - 1 = tr(\zeta)$ with product ζ, so by the induction hypothesis $(\sigma_0, \dots, \sigma_{r-2}) = \sigma^\zeta = (\sigma_0^\zeta, \dots, \sigma_{r-2}^\zeta) = (\sigma_0^\xi, \dots, \sigma_{r-2}^\xi)$ and so $\sigma = \sigma^\xi$. □

We notice that this proof provides us with an algorithm for deriving the sequence σ^ξ from a representation of ξ as a pdc, by repeatedly juxtaposing to the right of such a pdc the transposition exchanging the maximal symbol moved with its predecessor, and deleting this symbol from the pdc. For instance

$$\begin{aligned}(1, 7, 6)(2, 3, 5, 8)(0.4) &= (1, 7, 6)(2, 3, 5)(0, 4)|(5, 8)\\ &= (1, 6)(2, 4, 5)(0, 4)|(1, 7)(5, 8)\\ &= (2, 3, 5)(0, 4)|(1, 6)(1, 7)(5, 8)\\ &= (2, 3)(0, 4)|(3, 5)(1, 6)(1, 7)(5, 8)\end{aligned}$$

so for $\xi = (1, 7, 6)(2, 3, 5, 8)(0, 4)$, we have

$$\sigma^{\xi} = ((2, 3), (0, 4), (3, 5), (1.6), (1, 7), (5, 8)).$$

Proof of Theorem 2.1. We obviously have $e_0 = u_0 = 1$, so assume $r > 0$. Define $E_j \subseteq S_{n+1}$ by

$$E_j = \{(i, j): 0 \leq i < j\}, \qquad j = 1, \dots, n.$$

Thus, $s_j = \langle E_j \rangle$, $j = 1, \dots, n$. Given $1 \leq j_1 < j_2 < \cdots < j_r \leq n$, let

$$E_{j_1, \dots, j_r} = E_{j_1} \times \cdots \times E_{j_r}.$$

Thus, $E_{j_1, \dots, j_r}$ consists of all increasing sequences of transpositions $\sigma = (\sigma_0, \dots, \sigma_{r-1})$ satisfying $j_{\sigma_0} = j_1, \dots, j_{\sigma_{r-1}} = j_r$. By Proposition 2.3, we know that mapping π sending $\sigma \in E_{j_1, \dots, j_n}$ to $\pi\sigma \in S_{n+1}$ is one-to-one, and so letting

$$\pi E_{j_1, \dots, j_r} = \{\pi\sigma : \sigma \in E_{j_1, \dots, j_r}\}$$

we have

$$\langle \pi E_{j_1, \dots, j_r} \rangle = s_{j_1} \cdots s_{j_r}.$$

Now let

$$E^r = \bigcup_{1 \leq j_1 < \cdots < j_r \leq n} E_{j_1, \dots, j_r}. \tag{2.1}$$

Again, by Proposition 2.3, the mapping π maps E^r one to one onto the set

$$U_r = \{\xi \in S_{n+1} : tr(\xi) = r\}$$

satisfying $u_r = \langle U_r \rangle$.

Since the union on the right hand of (2.1) is a disjointed union we have by (2.0),

$$u_r = \langle U_r \rangle = \langle \pi E_r \rangle = \sum_{1 \leq j_1 < \cdots < j_r \leq n} \pi E_{j_1, \dots, j_r}$$

$$= \sum_{1 \leq j_1 < \cdots j_r \leq n} s_{j_1} s_{j_2} \cdots s_{j_r} = e_r. \quad \square$$

A second proof of Theorem 2.1. We sketch another proof of Theorem 2.1, stating the steps and leaving details to the reader:

1. $e_r(z_1, \dots, z_n) = e_r(z_1, \dots, z_{n-1}) + e_{r-1}(z_1, \dots, z_{n-1}) \cdot z_n$.

2. Let u_r^k denote the sum of all permutations θ in S_k satisfying $tr(\theta) = k$. Then $u_r^{n+1} = u_r^n + u_{r-1}^n s_n$.

To prove 2, use

3. Let $\theta \in S_n$ be represented as a product of disjoint cycles $\theta = \theta_1 \cdots \theta_c$—including 1-cycles—over the set of symbols n. Then θs_n is the sum of n permutations of S_{n+1}—all moving the symbol n—obtained by inserting n in the representation $\theta_1 \cdots \theta_c$ after a symbol $i \in n$ and summing for $i = 0, \dots, n-1$.

4. Theorem 2.1 follows by induction on n.

3. $C_{n+1} \subseteq \mathrm{Syp}[s_1, \dots, s_n]$

Recall that C_{n+1} is linearly spanned over $\mathbb{Z}$ by the CCSs of S_{n+1}. Let T_{n+1} be the set of types defined by

$$t \in T_{n+1} : \Leftrightarrow 0 < |t| \leq n + 1 \,\&\, (t = 1^* \text{ or } t(1) = 0).$$

Clearly, for each COC K in S_{n+1} there is a unique $t \in T_{n+1}$ with $K = [t]_{n+1}$, $\langle K\rangle = \langle t\rangle_{n+1}$. The inclusion $C_{n+1} \subseteq \mathrm{Syp}[s_1, \dots, s_n]$ follows therefore from

Theorem 3.1. *Each $t \in T_{n+1}$ has a symmetric polynomial $\tilde{q}_t(z_1, \dots, z_n) \in \mathrm{Syp}[z_1, \dots, z_n]$ of total degree $tr(t)$ satisfying*

$$\langle t\rangle_{n+1} = \tilde{q}_t(s_1, \dots, s_n). \tag{3.0}$$

Theorem 3.1 follows from Theorem 3.1′ that we formulate next.

Define a weight function w on S_{n+1} by $w(\theta) := (tr(\theta), m(\theta))$. Similarly, for a type t let $w(t) := (tr(t), m(t))$. As both $tr(t)$ and $m(t)$ are independent of $t(1)$, we see that $w(\theta) = w(t)$ whenever $\overline{\theta} = \overline{\theta}(1) \cdot 1^* + t$, hence whenever $t \in T_{n+1}$ and $[\theta] = [t]_{n+1}$.

Since $0 \le tr(\theta) \le n$, $0 \le m(\theta) \le n+1$, we have

$$w(\theta) \in (n+1) \times (n+2) = \{(r, m)\colon 0 \le r \le n, 0 \le m \le n+1\}.$$

We order $(n+1) \times (n+2)$ lexicographically

$$(r, m) \le (r', m')\colon \Leftrightarrow r < r', \quad \text{or} \quad r = r' \quad \text{and} \quad m \le m'.$$

For $x = \sum_{\theta \in S_{n+1}} x(\theta) \cdot \theta \in \mathbb{Z}[S_{n+1}]$, let $s(x) = \{\theta \in S_{n+1}\colon x(\theta) \ne 0\}$ denote the support of x. Define its weight $w(x)$ by $w(0) := (0, 0)$, and for $x \ne 0$:

$$w(x) := \max\{w(\theta)\colon x(\theta) \ne 0\}.$$

Theorem 3.1′. *Each $t \in T_{n+1}$ has a symmetric polynomial $q_t(z_1, \dots, z_n) \in \mathrm{Syp}[z_1, \dots, z_n]$ homogeneous of degree $tr(t)$ satisfying*

$$q_t(s_1, \dots, s_n) = \langle t\rangle_{n+1} + x \tag{3.1}$$

where $x = 0$ or $w(x) < w(t)$.

Proof of Theorem 3.1, *given Theorem* 3.1′. We proceed by induction on $w(t)$. If $t \in T_{n+1}$, $w(t) = (0, 0)$ then necessarily $t = 1^*$. Let $q_1^* = \tilde{q}_1^*$ be the symmetric polynomial 1.

Assume now $w(t) > (0, 0)$. By Theorem 3.1′ let

$$q_t(z_1, \dots, z_n) \in \mathrm{Syp}[z_1, \dots, z_n]$$

be a symmetric homogeneous polynomial satisfying (3.1). By Theorem 2.1 $q_t(s_1, \dots, s_n) \in C_{n+1}$, and hence so does $x = q_t(s_1, \dots, s_n) - \langle t\rangle_{n+1}$. As $w(x) < w(t)$ we have

$$x = \sum_{\substack{t' \in T_{n+1} \\ w(t') < w(t)}} x(t') \cdot \langle t'\rangle_{n+1}$$

where $x(t')$ is the common value $x(\theta)$ taken by x at $\theta \in [t']_{n+1}$. By induction, a symmetric polynomial $\tilde{q}_{t'}(z_1, \dots, z_n) \in \mathrm{Syp}[z_1, \dots, z_n]$ exists of total degree $tr(t') \le tr(t)$ with $\langle t'\rangle_{n+1} = \tilde{q}_t(s_1, \dots, s_n)$, so that for

$$\tilde{q}_t(z_1, \dots, z_n) = q_t(z_1, \dots, z_n) - \sum_{\substack{t' \in T_{n+1} \\ w(t') < w(t)}} x(t') q_t(z_1, \dots, z_n),$$

(3.0) holds

$$\tilde{q}_t(s_1, \dots, s_n) = \langle t\rangle_{n+1}$$

and $\tilde{q}_t(z_1, \dots, z_n)$ is of total degree $tr(t)$. □

The rest of this section is dedicated to the proof of Theorem 3.1.

Let us start by giving an explicit definition of the symmetric polynomial $q_t(z_1, \dots, z_n)$ satisfying (3.1). Fix a type $t \in T_{n+1}$, $t \neq 1^*$. Let $s(t) = \{k \in \mathbb{N}: t(k) > 0\}$, and assume $|s(t)| = p$, say $s(t) = \{k_1, \dots, k_p\}$ with $1 < k_1 < \cdots < k_p \leq n+1$. Define $r_i = k_i - 1$, $i = 1, \dots, p$. For nonnegative integers h, r put

$$e_h^r(z_1, \dots, z_n) = e_h(z_1^r, \dots, z_n^r)$$

where $e_h(z_1, \dots, z_n)$ is the hth elementary symmetric polynomial in $z_1, \dots, z_n$ (see (1.2)).

Let further $h_i = t(k_i)$, $i = 1, \dots, p$, so that

$$t = h_1 k_1^* + \cdots + h_p k_p^*, \qquad h_1, \dots, h_p > 0.$$

Finally, put

$$q_t(z_1, \dots, z_n) = e_{h_1}^{r_1}(z_1, \dots, z_n) \cdots e_{h_p}^{r_p}(z_1, \dots, z_n), \tag{3.2}$$

As $e_h^r(z_1, \dots, z_n)$ is homogeneous degree $h \cdot r$, we see that q_t is homogeneous of degree $tr(t)$, as $tr(t) = \sum_{k \in s(t)} t(k)(k-1) = \sum_{i=1}^p h_i r_i$. It follows that every $\theta \in s(q_t(s_1, \dots, s_n))$ is a product of $tr(t)$ transpositions and so θ is the same parity as t and $tr(\theta) \leq tr(t)$.

Given $y \in \mathbb{Z}[S_{n+1}]$ we say that y *is monic in* $\theta \in S_{n+1}$ iff $y(\theta) = 1$. By the previous remarks, Theorem 3.1′ is a consequence of

Proposition 3.2. *Let* $t \in T_{n+1}$, $t \neq 1^*$, *and let* $q_t = q_t(s_1, \dots, s_n)$, *where* $q_t(z_1, \dots, z_n)$ *is given in* (3.2). *Then*

(i) q_t *is monic in* θ *for each* $\theta \in [t]_{n+1}$,

(ii) *If* $w(\theta) = w(t)$ *and* $q_t(\theta) \neq 0$ *then* $\theta \in [t]_{n+1}$.

Proposition 3.2 will be the last consequence of the sequel.

Let $\binom{A}{h}$ denote the set of all subsets of cardinality h of the set A, and let $[n] = \{1, \dots, n\}$. For $L \subseteq [n]$, say $L = \{j_1, \dots, j_h\}$, $1 \leq j_1 < j_2 < \cdots < j_h \leq n$, define $v_L(z_1, \dots, z_n) = \prod_{j \in L} z_j = z_{j_1} \cdots z_{j_h} \in \mathbb{Z}[z_1, \dots, z_n]$ so that $v_L^r(z_1, \dots, z_n) = z_{j_1}^r \cdots z_{j_r}^r$.

By definition we have

$$e_h^r(z_1, \dots, z_n) = \sum_{L \in \binom{[n]}{h}} v_L^r(z_1, \dots, z_n).$$

Let now $t = h_1 k_1^* + \cdots + h_p k_p^*$, $1 < k_1 < \cdots < k_p$, $h_1, \dots, h_p > 0$ and put

$$\binom{[n]}{t} = \binom{[n]}{h_1} \times \cdots \times \binom{[n]}{h_p},$$

i.e.,

$$\binom{[n]}{t} = \left\{ L = (L_1, \dots, L_p) : L_i \in \binom{[n]}{h_i},\ i = 1, \dots, p \right\}.$$

Then from (3.2) we obtain, with $r_i = k_i - 1$, $i = 1, \dots, p$:

$$q_t(z_1, \dots, z_n) = \sum_{(L_1, \dots, L_p) \in \binom{[n]}{t}} \prod_{i=1}^p v_{L_i}^{r_i}(z_1, \dots, z_n).$$

Letting $v_L^r = v_L^r(s_1, \dots, s_n) \in \mathbb{Z}[S_{n+1}]$, we have

$$q_t = \sum_{(L_1, \dots, L_p) \in \binom{[n]}{t}} \prod_{i=1}^{p} v_{L_i}^{r_i}. \tag{3.3}$$

Let $\theta \in S_{n+1}$. Define j_θ to be the largest symbol moved by θ;

$$j_\theta := \max M(\theta).$$

For a sequence $\theta = (\theta_1, \dots, \theta_r) \in {}^rS_{n+1}$ let $l(\theta) = r$ denote as before the length of the sequence θ, and put

$$M(\theta) := \bigcup_{i=1}^{r} M(\theta_i) \quad \text{and} \quad j_\theta := \max\{j_{\theta_1}, \dots, j_{\theta_r}\}.$$

Let $r > 0$. A sequence of transpositions $\sigma = (\sigma_1, \dots, \sigma_r)$ will be called a *starlike sequence* if $j_\sigma = j_{\sigma_1} = \cdots = j_{\sigma_r}$. If in addition $\sigma_1, \dots, \sigma_r$ are all distinct, we call σ a *star sequence.*

Proposition 3.3. (i) *Let $\theta \in S_{n+1}$ be a cycle of length $r+1$. Then there is a unique star sequence σ^θ satisfying $\pi\sigma^\theta = \theta \cdot \sigma^\theta$ is of length r and satisfies $j_\sigma = j_\theta$.*

(ii) *If σ in any star sequence of length r then $\theta = \pi\sigma$ is an $(r+1)$-cycle satisfying $j_\sigma = j_\theta$.*

(iii) *If σ is a starlike sequence and $\theta = \pi\sigma$ is an $(r+1)$-cycle, then σ is a star sequence.*

Proof. Let $i_1, \dots, i_r, j$ be distinct symbols. Then

$$(i_r, j) \cdots (i_2, j)(i_1, j) = (i_1, i_2, \dots, i_r, j).$$

Thus, if $i_1, \dots, i_r < j$ then $\sigma = ((i_r, j), \dots, (i_1, j))$ is the only star sequence satisfying $\pi\sigma = \theta$. (i) and (ii) follow.

For (iii) notice that if σ is not a star sequence then $tr(\theta) < r$ (see proof of Proposition 2.2) and so by $m(\theta) \le r+1$ θ cannot be an $(r+1)$-cycle, whose transposition number is r. $\square$

Consider now the set of transpositions $E_j = \{(i, j) \colon i < j\} \subseteq S_{n+1}$. We have $s_j = \langle E_j \rangle$, and for $r \in [n]$, s_j^r is the sum of all $\pi\sigma$, where $\sigma = (\sigma_1, \dots, \sigma_r) \in {}^rE_j$. Now, ${}^rE_j$ is actually the set of all starlike sequences σ of length r satisfying $j_\sigma = j$, and $\sigma \in {}^rE_j$ is a star sequence iff $\sigma_1, \dots, \sigma_r$ are distinct. Thus, if σ is not a star sequence, then $\theta = \pi\sigma$ is a product of less than r transpositions (see proof of Propositions 2.2) hence θ is not an $(r+1)$-cycle. Combining these remarks with Proposition 3.3 we obtain

Proposition 3.4. *Let $j, r \in [n]$. Then*

(i) *If $\theta \in S_{n+1}$ is an $(r+1)$-cycle, then $s_j^r(\theta) = \delta_{j, j_\theta}$. Thus, s_j^r is monic at θ if $j_\theta = j$, and $s_j^r(\theta) = 0$ if $j_\theta \ne j$.*

(ii) *If θ is not an $(r+1)$-cycle, and $s_j^r(\theta) \ne 0$, then $tr(\theta) < r$.*

Let us denote by $*$ the concatenation of finite sequences. A nonempty sequence of transpositions σ will be called a *prestellar sequence* if it admits a representation

$$\sigma = \sigma_1 * \cdots * \sigma_c \tag{3.4}$$

where

(a) $\sigma_1, \ldots, \sigma_c$ are starlike sequences,

(b) $l\sigma_1 \leq l\sigma_1 \leq \cdots \leq l\sigma_c$,

(c) if $l\sigma_1 = l\sigma_{i+1}$ then $j_{\sigma_i} < j_{\sigma_{i+1}}$.

It is readily checked that a prestellar sequence σ admits a unique representation (3.4). We call it *the prestellar representation of* σ.

A prestellar sequence σ with prestellar representation (3.4) is called *prestellar sequence of type* t, when $t = \sum_{i=1}^{c}(l\sigma_i + 1)^*$. Notice that in this case t satisfies $t(1) = 0$, and so $|t| = m(t)$.

A prestellar sequence σ, with the prestellar representation (3.4) will be called *a stellar sequence* if it satisfies in addition:

(d) $M(\sigma_i) \cap M(\sigma_k) = \varnothing$, $1 \leq i < k \leq c$,

(e) σ_i is a star sequence, $1 \leq i \leq c$.

We then refer to (3.4) as *the stellar representation of* σ.

Let σ be a stellar sequence with stellar representation (3.4), and let $\theta_i = \pi\sigma_i$, $i = 1, \ldots, c$. By (e) and Proposition 3.3(ii), θ_i is an $(l\sigma_i + 1)$-cycle, and by (d), $M(\theta_i) \cap M(\theta_k) = \varnothing$ for $1 \leq i < k \leq c$. Thus, letting $\theta = \pi\sigma$, we conclude that

$$\theta = \theta_1 \cdots \theta_c \tag{3.5}$$

is a representation of θ as a product of disjoint nontrivial cycles, where the cycles appear in nondecreasing-length-order, and within a given length by the order of the largest symbol moved. Moreover, if σ is a stellar sequence of type t, then $\overline{\theta} = \overline{\theta}(1) \cdot 1^* + t$, i.e., $[\theta] = [t]_{n+1}$.

Conversely, let $\theta \in S_{n+1}$, $\theta \neq 1$, satisfy $\overline{\theta} = \theta(1) \cdot 1^* + t$, (so that $[\theta] = [t]_{n+1}$). Let (3.5) be the unique representation of θ as a product of c nontrivial cycles in nondecreasing-length-order first, and within a given length by the magnitude of the largest symbol moved. Then selecting for each θ_i its unique star sequence $\sigma_i = \sigma^{\theta_i}$ and setting $\sigma = \sigma_1 * \cdots * \sigma_c$ we obtain a stellar sequence σ^θ satisfying $\pi\sigma = \theta$. Moreover, if σ is any stellar sequence satisfying $\pi\sigma = \theta$, then necessarily $\sigma = \sigma^\theta$, as its stellar representation coincides with that of σ^θ. We summarize this discussion into

Proposition 3.5. *Let* $t \in T_{n+1}$, $t \neq 1^*$; *then*

(i) *If* $\theta \in S_{n+1}$, $\overline{\theta} = \overline{\theta}(1) \cdot 1^* + t$, *then there is a unique stellar sequence* σ^θ *of type* t *satisfying* $\pi\sigma^\theta = \theta$.

(ii) *If* σ *is a stellar sequence of type* t, *then* $\theta = \pi\sigma$ *satisfies* $\overline{\theta} = \overline{\theta}(1) \cdot 1^* + t$, *and* $\sigma = \sigma^\theta$.

(iii) *If* σ *is a prestellar sequence of type* t *which is not stellar,* $\theta = \pi\sigma$, *then* $tr(\theta) < tr(t)$ *or* $m(\theta) < m(t)$, *so* $w(\theta) < w(t)$.

Proof. (i) and (ii) are established in the remarks preceding the proposition. We establish (iii).

Let σ be a prestellar sequence of type t, which is not stellar, and let $\sigma = \sigma_1 * \cdots * \sigma_c$ be its prestellar representation. Since σ is not stellar, either condition (d) or condition (e) fail.

Assume first that (e) fails. Then for some $1 \leq i \leq c$, σ_i is a nonstar starlike sequence. Thus, by Proposition 3.3(iii), $tr(\pi\sigma_i) < l\sigma_i$ and so $tr(\theta) = tr(\pi\sigma) \leq tr(\pi\sigma_1) + \cdots + tr(\pi\sigma_c) < l\sigma_1 + \cdots + l\sigma_c$. But as $t = (l\sigma_1 + 1)^* + \cdots + (l\sigma_c + 1)^*$ we have $tr(t) = l\sigma_1 + \cdots + l\sigma_c$, and so $tr(\theta) < tr(t)$.

Assume now that (e) holds, but (d) fails. Thus, $\sigma_1, \ldots, \sigma_k$ are star sequences, and so $\theta_i = \pi\sigma_i$ is an $l\sigma_i + 1$ cycle for $i = 1, \ldots, c$, i.e. $m(\sigma_i) = l\sigma_i + 1$. Since (d) fails, $M(\theta_i) \cap M(\theta_k) \neq \varnothing$ for some $1 \leq i < k \leq c$ and so $m(\theta) < m(\theta_1) + \cdots + m(\theta_c) = (l\theta_1 + 1) + \cdots + (l\theta_c + 1) = |t| = m(t)$. $\square$

Proof of Proposition 3.2. Fix a type $t \in T_{n+1}$, $t \neq 1^*$. Then for some $p > 0$,

$$t = h_1 k_1^* + \cdots + h_p k_p^*, \qquad 1 < k_1 < \cdots < k_p \leq n+1, \ \ h_1 \cdots h_p > 0.$$

For $i = 1, \ldots, p$ let $r_i = k_i - 1$, and let σ be a prestellar sequence of type t. *Then*

$$\sigma = \varepsilon_1 * \cdots * \varepsilon_p$$

where

$$\varepsilon_i = \sigma_{i1} * \cdots * \sigma_{ih_i}$$

and $\sigma_{i\nu}$ is a starlike sequence of length r_i.

Letting $j_{i\nu} = j_{\sigma_{i\nu}}$ we have $1 \leq j_{i1} < j_{ik} < \cdots < j_{ih_i} \leq n$. Let

$$L_i^\sigma = \{j_{i1}, \ldots, j_{ih_i}\} \in \binom{[n]}{h_i},$$

and put

$$\overrightarrow{L}^\sigma := (L_1^\sigma, \ldots, L_p^\sigma) \in \binom{[n]}{h_1} \times \cdots \times \binom{[n]}{h_p} = \binom{[n]}{t}.$$

We shall say that σ is of *class* $(t, \overrightarrow{L})$.

For r, $h \in [n]$, $L \in \binom{[n]}{h}$, say $L = \{j_1, \ldots, j_h\}$ with $j_1 < \cdots < j_h$ put

$$E_L^r := {}^rE_{j_1} * {}^rE_{j_2} * \cdots * {}^rE_{j_h}$$

(where $E * E' := \{\sigma * \sigma' \colon \sigma \in E, \ \sigma' \in E'\}$ for sets of finite sequences E, E', and as before, $E_j = \{(i, j) \colon i < j\}$). For $\overrightarrow{L} = (L_1, \ldots, L_p) \in \binom{[n]}{t}$ define

$$E_{\overrightarrow{L}}^t = E_{L_1}^{r_1} * \cdots * E_{L_p}^{r_p}$$

Then obviously, $E_{\overrightarrow{L}}^t$ consists of all prestellar sequences σ of class $(t, \overrightarrow{L})$.

If E is a set of finite sequences of members of $\mathbb{Z}[S_{n+1}]$, put

$$\langle \pi E \rangle = \sum_{\sigma \in E} \pi\sigma.$$

Then by (1.1) we have for $j = 1, \ldots, n$, $r \in \mathbb{N}$ and $L = \{j_1, \ldots, j_h\} \subseteq [n]$, $1 \leq j_1 < \cdots < j_h \leq n$,

$$\langle \pi E_j \rangle = s_j, \quad \langle \pi\, {}^rE_j \rangle = s_j^r, \quad \langle \pi E_L^r \rangle = s_{j_1}^r \cdots s_{j_h}^r.$$

Recalling that for $L \subseteq [n]$ we denote $v_L = \prod_{j \in L} s_j$, we see that for $\overrightarrow{L} = (L_1, \ldots, L_p) \in \binom{[n]}{t}$ we have

$$\langle \pi E_{\overrightarrow{L}}^t \rangle = \prod_{i=1} v_{L_i}^{r_i}.$$

Let us denote by E^t the set of all prestellar sequences of type t. Then

$$E^t = \bigcup_{\vec{L} \in \binom{[n]}{t}} E^t_{\vec{L}}$$

Since this union is disjoint, we have

$$\langle \pi E^t \rangle = \sum_{\vec{L} \in \binom{[n]}{t}} \langle \pi E^t_{\vec{L}} \rangle$$

and so, by (3.3),

$$\langle \pi E^t \rangle = \sum_{(L_1, \dots, L_p) \in \binom{[n]}{t}} \prod_{i=1}^{p} v_{L_i}^{r_i} = q_t .$$

Now let $\theta \in S_{n+1}$ satisfy $\overline{\theta} = \theta(1) \cdot 1^* + t$. We show that q_t is monic at θ. Let σ^θ be the unique stellar sequence of θ, as provided by Proposition 3.5(i). Then $\pi\sigma^\theta = \theta$, and for any $\sigma \in E^t$, $\pi\sigma = \theta$ implies $\sigma = \sigma^\theta$. Thus, $\langle \pi E^t \rangle$ is monic and θ, so by $\langle \pi E^t \rangle = q_t$, q_t is monic at θ, and (i) is proved.

Assume now that $q_t(\theta) \neq 0$ and $w(\theta) = w(t)$. Again by $q_t = \langle \pi E^t \rangle$, some prestellar sequence σ of type t exists with $\pi\sigma = \theta$. By Proposition 3.5(iii), σ is actually a stellar sequence, and so by Proposition 3.5(ii), $\theta = \pi\sigma$ satisfies $\overline{\theta} = \theta(1) \cdot 1^* + t$, i.e. $\theta \in [t]_{n+1}$, and (ii) is proved. □

The proof of Theorem 3.1′ is complete.

Remark. Let us say that $p(z_1, \dots, z_n) \in \mathbb{Z}_{[z_1, \dots, z_n]}$ is *even* (*odd*) iff it is a linear integral combination of monomials of even (odd) total degree. It is readily checked that one may add in Theorem 3.1 the statement that $q_t(z_1, \dots, z_n)$ is even (odd) according as $tr(t)$ is even (odd), i.e., according as $[t]_{n+1}$ is an even (odd) COC in S_{n+1}.

Added in Proof

While this paper was in press, we have learned of earlier occurrences of Theorem 1. It was probably first noted in the early seventies by Jucys [J]. Jucys proved Theorem 2.1, using the argument given here as second proof. He then combined it with an earlier result of Farahat and Higman [FH] that states that C_{n+1} is generated by $u_1, \dots, u_n$ to obtain Theorem 1 (Farahat-Higman's result in turn is given here an independent proof, namely Theorem 2.1 followed by Theorem 3.1). In 1979 G. E. Murphy, unaware of Jucys' work, rediscovered Theorem 1, and published it later [M].

We are gateful to A. Kerber, G. E. Murphy and R. P. Stanley for their communication.

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