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HOMOTOPY INVARIANTS OF NONORIENTABLE 4-MANIFOLDS

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ABSTRACT. We define a $\mathbf{Z}_4$-quadratic function on π_2 for nonorientable 4-manifolds and show that it is a homotopy invariant. We then use it to distinguish homotopy types of certain manifolds that arose from an analysis of toral action on nonorientable 4-manifolds.

1. INTRODUCTION

The oriented homotopy type or even the topological type of a closed simply-connected smooth 4-manifold is determined by the intersection pairing on H^2 [8, 1]. If the fundamental group is nontrivial, there are three immediate homotopy invariants supported on the 3-skeleton: π_1, π_2 as a $\mathbf{Z}[\pi_1]$-module, and the first k-invariant. There is also an immediate global homotopy invariant, the equivariant intersection pairing on π_2 with respect to the action of π_1. Surprisingly, no other invariants have garnered much attention so far. However it has been shown that these are actually enough to determine the homotopy types of orientable closed 4-manifolds (or just Poincaré complexes) with special finite fundamental groups [6, 2].

We will show here by examples that this fails for *nonorientable* smooth 4-manifolds at a very primitive stage by using two invariants: the $\mathbf{Z}_2$-intersection pairing on H^2, and a $\mathbf{Z}_4$-quadratic function q on $\pi_2 \otimes \mathbf{Z}_2$. Our invariants are not entirely new, but the independence of the first one to the above four invariants and a geometric proof of the homotopy invariance of the second one are new.

Our quadratic function can be derived from Wall's self-intersection μ [7]. However we use another definition which is more convenient for showing its homotopy invariance. We work this out in the next section. Then we discuss three nonorientable 4-manifolds doubly covered by $S^2 \times S^2$, and use our invariants to distinguish their homotopy types in the last section. In fact, the quadratic function was devised to distinguish these manifolds, which arose from an analysis by Myung Ho Kim of 2-dimensional toral actions on nonorientable 4-manifolds.

After completing this paper, Ian Hambleton pointed out to us that our invariant q, the $\mathbf{Z}_4$-quadratic function on $\pi_2 \otimes \mathbf{Z}_2$, can be evaluated as the difference between a self-intersection number and the Browder-Liversay invariant in the orientable double cover. The unobvious but key connection can be found in

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Lemma 4.6 of [10]. Hambleton and Milgram gave a homotopy theoretic definition of the Browder-Liversay invariant in [11]. Hence the homotopy invariance of q would already follow from [10, 11]. Our geometric proof of the homotopy invariance of the q-invariant seems to have intrinsic interest.

Ian Hambleton also kindly pointed out to us a mistake in our earlier version. We wish to express our thanks for Professor Hambleton for his interest and helpful comments. We also thank the referee for his suggestions.

2. A $\mathbf{Z}_4$-QUADRATIC FUNCTION

Let M be a closed smooth 4-manifold with a base point. The invariant we shall define does make sense for all smooth manifolds but contains something new only if M is nonorientable. Thus, to avoid unnecessary complications, we assume M is nonorientable throughout this section. Choose elements x and y of $\pi_2(M)$ and represent them by transversely immersed 2-spheres S_x and S_y which also mutually intersect transversely. We denote by $x \cdot y$ the number of mutual intersections of S_x and S_y modulo 2. The value is the usual $\mathbf{Z}_2$-intersection number of those cycles, however since our domain is π_2, the bilinear form is not quite the same as the homology intersection. For example, it may be singular. Another description of $\cdot$ is the $\mathbf{Z}_2$-reduction of the argumented equivariant intersection on π_2.

Let $\widetilde{M}$ be the orientable double cover of M with a base point which comes down to that of M. Then, fixing an orientation of $\widetilde{M}$, we define a function q with values in $\mathbf{Z}_4$ for $x \in \pi_2(M)$ by

$$q(x) = \chi(\nu(S_x)) + 2\,\#\,\mathrm{self}\, S_x \quad \text{modulo } 4.$$

Here $\nu(S_x)$ is the normal bundle of S_x (we regard S_x as a sphere by ignoring its self-intersection). Since we specified a homotopy class x and fixed the base point on $\widetilde{M}$, S_x has a unique lift $\widetilde{S}_x$ in $\widetilde{M}$. In particular, their normal bundles are canonically isomorphic with each other. We will identify S_x with $\widetilde{S}_x$ through this lifting when we are concerned with the normal bundle. $\chi(\ )$ is the Euler class evaluated with respect to the orientation of $\widetilde{M}$. More precisely, $\chi(\nu(S_x))$ is the Euler number evaluated with the local orientation induced from one on $\nu(\widetilde{S}_x)$, or equivalently the number to be identified with $\chi(\nu(\widetilde{S}_x))$. The Euler class makes sense as an integer. $\mathrm{self}\, S_x$ is the set of self-intersections of S_x and $\#$ stands for its cardinality.

Lemma 1. *q is a well-defined $\mathbf{Z}_4$-quadratic function with respect to $\cdot$, namely,*

(1) *$q(nx) = n^2 q(x)$ (in particular, $q(-x) = q(x)$), and*
(2) *$q(x+y) - q(x) - q(y) = 2\,x \cdot y$ (4).*

When we choose the other orientation of $\widetilde{M}$, q becomes $-q$.

Proof. By Whitney [9] (cf. Matsumoto [5]), homotopic spheres in this dimension are connected by a finite sequence of regular homotopies and homotopies of creating or killing small Whitney singularities. The regular homotopy obviously does not change the value of q. When we create or kill a Whitney singularity, we add or subtract one self-intersection and thereby change its Euler class by ± 2 according to the sign of intersection. Hence this operation does not change q either. Properties (1) and (2) are quite easy to verify. □

Remark. If M is orientable, q still makes sense by orienting M; then q is the homology self-intersection number of the cycle represented by S_x modulo 4 and hence there is nothing new.

Theorem 2. *The equivalence class of q $(q \sim -q)$ is a homotopy invariant. More precisely, if $f : M \to N$ is a homotopy equivalence, then*

$$q_M(x) = q_N(f_*(x)) \quad \textit{for all } x \in \pi_2(M)$$

with respect to relevant orientations of the orientable double covers $\widetilde{M}$ and $\widetilde{N}$.

The proof of Theorem 2 occupies the rest of this section. First of all, take the lift $\tilde{f} : \widetilde{M} \to \widetilde{N}$ of f which preserves the base points, and then orient $\widetilde{M}$ and $\widetilde{N}$ so that $\tilde{f}$ is of degree one. Choose a transversely immersed oriented 2-sphere S in M representing x. By property (1) of Lemma 1, $q(x)$ is independent of the orientation of S, but, nevertheless, we shall use an orientation to simplify our computations.

Homotop f so that the restriction of f to S is an immersion with transverse image and so that f extends to a bundle map: $\nu(S) \to \nu(f(S))$. This is attained by first perturbing f so that it becomes an immersion on S with transverse image. Then since $\tilde{f}$ is a degree one homotopy equivalence, the cycles $\widetilde{S}$ and $\tilde{f}(\widetilde{S})$ must have the same homological intersection number, which is equal modulo 2 to the Euler class of the normal bundle. In particular, $\chi(\nu(S)) \equiv \chi(\nu(f(S)))\ (2)$. Thus, by homotoping f by creating or killing small Whitney singularities, we can attain the identity $\chi(\nu(S)) = \chi(\nu(f(S)))$. It is then obvious how to homotop f to a bundle map on $\nu(S)$.

We adopt the following convention for orienting $f(S)$. Since S is oriented, $\nu(S)$ is compatibly oriented by the orientation of $\widetilde{M}$. Then since f extends to a bundle map, $\nu(f(S))$ admits an induced orientation. The orientation of $f(S)$ is to be the one that is compatible with the orientation of $\widetilde{N}$. We denote $f(S)$ with this orientation by S'.

The orientation of S' might be different from the orientation induced from one on S by f. The reason why we take this convention is to make the sign of the degree of f near S clear. The degree of $f|_S$ is 1 or -1 according to whether the orientation of S' agrees with the orientation induced by f on $f(S)$ or not.

Homotop f further fixing a neighborhood of S so that f is transverse regular to S'. This means, in part, that at the self-intersection points of S', f is a diffeomorphism near each of their preimages. For we may perturb f near the preimage of the self-intersection points and then perturb f near the preimage of $S' - \{\text{neighborhoods of self-intersections}\}$. Note that now there are no "manifold points" in the preimage of self-intersections by f since f is a diffeomorphism around them.

The inverse image $f^{-1}(S')$ then consists of transversely immersed *connected* surfaces C_j, $j = 0, 1, \ldots, n$, including S, with mutually transverse intersections. Since each C_j behaves much like a connected component, we call it a component of the inverse image. Namely $f^{-1}(S') = \bigcup_j C_j$, where each C_j is a component in our sense. We assume that $S = C_0$.

Lemma 3. *Each C_j lifts to $\widetilde{M}$. Moreover, each C_j is an orientable surface with transverse self-intersections.*

Proof. Since S' is a 2-sphere with self-intersections, it lifts to $\widetilde{N}$. In particular, the preimage of S' in $\widetilde{N}$ has two components in our sense. Think of the preimage of C_j in $\widetilde{M}$. Since it is mapped to the preimage of S' by $\tilde{f}$, it must contain at least two and hence exactly two components. This means that C_j is liftable.

By transverse regularity, $\nu(C_j)$ is a pullback of $\nu(S')$ by $f|_{C_j}$, and it is orientable since $\nu(S')$ is orientable. As C_j and hence $\nu(C_j)$ lifts to an orientable manifold $\widetilde{M}$, C_j itself must be an orientable surface. □

The component C_j has a unique lift which maps to $\widetilde{S}'$ by $\tilde{f}$. For our convenience, we denote it by $\widetilde{C}_j$. As we have been doing for S, we shall identify C_j and $\widetilde{C}_j$ when we are concerned with the normal bundle.

Orient C_j by the following procedure. Since S' was oriented, $\nu(S')$ has a compatible orientation with one on $\widetilde{N}$. Since f is transverse regular to S', $\nu(C_j)$ gets an induced orientation. We then orient C_j to be compatible with the orientation of $\widetilde{M}$. This is opposite to our previous procedure of orienting S' from the orientation of S, but we do recover the correct orientation for $C_0 = S$. From now on we use the notation C_j to represent an oriented surface.

Since each C_j is now oriented, the degree of $f|_{D_j}$, which we denote by d_j, is defined as an integer satisfying $f_*([C_j]) = d_j[S']$, where the notation $[*]$ denotes the integral homology class. For instance, $d_0 = \pm 1$. We should also note here that $[C_j] = d_j d_0[S]$ since f is homotopy equivalence.

We have two identities by the convention:

$$\chi(\nu(C_j)) = d_j\chi(\nu(S')) \quad \text{and} \quad \sum_j d_j = 1.$$

The Euler class for $\nu(C_j)$ in the first identity is evaluated with respect to our orientation convention. The second identity is the result of the fact that $\tilde{f}$ is of degree one in our orientation and verified by choosing a generic point on S' and checking how we count the degree of $\tilde{f}$ by d_j's.

We distinguish self-intersections according to their liftability to $\widetilde{M}$. Suppose C is a transversely immersed surface in M which has a specified lift $\widetilde{C}$ in $\widetilde{M}$. We let R_C be the set of self-intersections of C which lift to be self-intersections of $\widetilde{C}$, and let Q_C be those which do not. Obviously self C is a disjoint union of R_C and Q_C. The difference between these two is that R_C contributes to the homological self-intersection number of $\widetilde{C}$ but Q_C does not.

Lemma 4. *$f(Q_C)$ is contained in $Q_{S'}$ and $f(R_C)$ is contained in $R_{S'}$.*

Proof. Let $\omega_M : \pi_1(M) \to \mathbf{Z}_2$ be the corresponding homomorphism to the first Stiefel-Whitney class. Then to each self-intersection point of C, we assign the value of ω_M for a smooth path on C which starts from the point in question and comes back to the same point from the other branch. This is a well-defined function since C lifts, and the liftability of the point as a self-intersection point to $\widetilde{M}$ is classified by its value.

Since f is homotopy equivalent, it sends the orientable subgroup in $\pi_1(M)$ to that of $\pi_1(N)$ and hence $\omega_M = \omega_N \circ f$. Therefore the liftability of self-intersection points in C corresponds by means of f to the liftability of self-intersection points in S'. □

Lemma 5.

$$\begin{cases} \#R_{C_j} \equiv 0\ (2) & \textit{if} \quad d_j \equiv 0\ (4)\,, \\ \#R_{C_j} - \#R_{S'} \equiv 0\ (2) & \textit{if} \quad d_j \equiv 1\ (4)\,, \\ \#R_{C_j} \equiv \chi(\nu(S'))\ (2) & \textit{if} \quad d_j \equiv 2\ (4)\,, \\ \#R_{C_j} - \#R_{S'} \equiv \chi(\nu(S''))\ (2) & \textit{if} \quad d_j \equiv 3\ (4). \end{cases}$$

Proof. Recall the identity $f_*[C_j] = d_j[S']$ and hence $\tilde{f}_*[\widetilde{C}_j] = d_j[\widetilde{S}']$. Since $\tilde{f}$ is a degree one homotopy equivalence, the homological self-intersection number of corresponding elements must be the same. We thus get

$$[\widetilde{C}_j] \cdot [\widetilde{C}_j] = d_j^2\,[\widetilde{S}'] \cdot [\widetilde{S}'].$$

The left-hand side is equal to $\chi(\nu(C_j)) + 2\sum \operatorname{self} \widetilde{C}_j$, where the summation here involves the sign of self-intersections. On the other hand, the right hand side is equal to $d_j^2\,(\chi(\nu(S')) + 2\sum \operatorname{self} \widetilde{S}')$. Notice here that we have the identities: $\sum \operatorname{self} \widetilde{C}_j \equiv \#R_{C_j}\ (2)$, $\sum \operatorname{self} \widetilde{S}' \equiv \#R_{S'}\ (2)$, and $\chi(\nu(C_j)) = d_j \chi(\nu(S'))$. Thus, by substituting these identities into the first one, we have

$$d_j(d_j - 1)\chi(\nu(S')) \equiv 2(\#R_{C_j} - d_j^2\,\#R_{S'}) \quad (4).$$

The congruences in the statement of the lemma are consequences of this congruence as d_j varies modulo 4. □

Lemma 6. $\#Q_{S'} \equiv \sum_j \#Q_{C_j}\ (2)$.

Proof. Let A_j be the difference set $f|_{C_j}{}^{-1}(Q_{S'}) - Q_{C_j}$ and let B_j be the set $f|_{C_j}{}^{-1}(R_{S'}) - R_{C_j}$ corresponding to liftable intersections. In other words, A_j is the set of points in C_j which intersect with another component of $f^{-1}(S')$ and which map to unliftable self-interactions of S', and B_j is the corresponding set mapping to liftable self-intersections. The obvious identity obtained from Lemma 4 and transverse regularity of f to $\operatorname{self} S'$ yields

$$\#f^{-1}(Q_{S'}) = \sum_j \#Q_{C_j} + \frac{1}{2}\sum_j A_j.$$

The left-hand side is equal modulo 2 to $(\deg f)\#Q_{S'} \equiv \#Q_{S'}\ (2)$. Thus it is enough to show that the last term of the identity is even.

We first count the number of mutual intersections,

$$\begin{aligned} \frac{1}{2}\sum_j(\#A_j + \#B_j) &= \sum_{i<j} \#C_i \cap C_j \\ &\equiv \sum_{i<j} d_i d_0[S] \cdot d_j d_0[S] \quad (2) \\ &\equiv \sum_{i<j} d_i d_j(\chi(\nu(S)) + 2\#\operatorname{self} S) \quad (2) \\ &\equiv \sum_{i<j} d_i d_j\, \chi(\nu(S)) \quad (2). \end{aligned}$$

Another obvious identity obtained from Lemma 4 and transverse regularity of f on $\operatorname{self} S'$ is

$$\#f^{-1}(R_{S'}) = \sum_j \#R_{C_j} + \frac{1}{2}\sum_j \#B_j.$$

The left-hand side is equal modulo 2 to $(\deg f)\#R_{S'} \equiv \sum_j d_j\,\#R_{S'}\ (2)$. Hence we have

$$\begin{aligned}
\frac{1}{2}\sum_j \#B_j &\equiv \sum_j d_j\,\#R_{S'} - \sum_j \#R_{C_j} \quad (2)\\
&\equiv \sum_j (d_j\,\#R_{S'} - \#R_{C_j}) \quad (2)\\
&\equiv \left(\sum_{d_j\equiv 0\,(4)} + \sum_{d_j\equiv 1\,(4)} + \sum_{d_j\equiv 2\,(4)} + \sum_{d_j\equiv 3\,(4)}\right)(d_j\,\#R_{S'} - \#R_{C_j}) \quad (2)\\
&\equiv \sum_{d_j\equiv 2,3} \chi(\nu(S')) \quad (2)\\
&\equiv \sum_{d_j\equiv 2,3} d_0\,\chi(\nu(S)) \quad (2),
\end{aligned}$$

by Lemma 5. The term we are interested in now becomes

$$\frac{1}{2}\sum_j \#A_j \equiv \left(\sum_{i<j} d_i d_j + \sum_{d_i\equiv 2,3} d_0\right)\chi(\nu(S)) \quad (2).$$

We conclude the lemma by showing that the coefficient of $\chi(\nu(S))$ is even. Let D_k be the number of d_j's whose residue modulo 4 is k. Since the sum of the d_j's is equal to 1, we obtain

$$0\cdot D_0 + 1\cdot D_1 + 2\cdot D_2 + 3\cdot D_3 \equiv 1 \quad (4).$$

Then we can easily obtain

$$\tfrac{1}{2}(D_1 + D_3)(D_1 + D_3 - 1) \equiv (D_2 + D_3) \quad (2).$$

The left-hand side is the first summation in the coefficient, while the right-hand side is the last one. □

Proof of Theorem 2. Since $q_N(f_*(x)) = q_N(-f_*(x))$ by Lemma 1, we are allowed to use our oriented 2-sphere S' to compute $q_N(f_*(x))$. The difference of their value is

$$q_M(x) - q_N(f_*(x)) = \chi(\nu(S)) - \chi(\nu(S')) + 2(\#R_S - \#R_{S'}) + 2(\#Q_S - \#Q_{S'}).$$

By Lemma 5, the first four terms cancel each other modulo 4. We will show that the last two terms also cancel by deriving a contradiction from the opposite supposition.

Hence assume that $\#Q_S \not\equiv \#Q_{S'}\ (2)$. We first claim that there is a component C of $f^{-1}(S')$ so that $\#Q_C \not\equiv \#Q_{dS}\ (2)$, where d is the degree of $f|_C$ times d_0, and dS is a cycle representing $d[S] = [C]$ obtained by taking d "parallel" cross-sections of $\nu(S)$ with mutual transverse intersections.

The self-intersections of dS either are produced due to nontriviality of the Euler class of $\nu(S)$ or inherit the self-intersections of S. The second one forms a lattice in a small neighborhood of each self-intersection of S. Since dS is contained in $\nu(S)$, it has a unique lift $d\widetilde{S}$ contained in $\nu(\widetilde{S})$. The self-intersections due to nontrivial Euler class all are liftable to $d\widetilde{S}$. The lattice intersections around R_S are also all liftable. But those around Q_S are certainly not, and each point in Q_S produces d^2 unliftable self-intersections. Thus we get a nice numerical property $\#Q_{dS} = d^2 \#Q_S$.

We take all congruences modulo 2 in the verification of our claim. Suppose $\#Q_{S'} \equiv 1$. Then $\#Q_S \equiv 0$ by the supposition. Hence there must be another component C with $\#Q_C \equiv 1$ by Lemma 6. Then, since $\#Q_{dS} = d^2 \#Q_S \equiv 0$ is not equal to $\#Q_C$, we are done.

Suppose $\#Q_{S'} \equiv 0$. Then $\#Q_S \equiv 1$ by the supposition. In this case, by Lemma 6, the number of C_j's with $\#Q_{C_j} \equiv 1$ must be even. If one such C_j has even degree, let it be C. Then $\#Q_{dS} = d^2 \#Q_S \equiv 0$ which is not equal to $\#Q_C$ and we are done. When all of the C_j's with $\#Q_{C_j} \equiv 1$ have odd degree, since there is an even number of such C_j's, there must be another component C_i with $\#Q_{C_i} \equiv 0$ and odd degree because the total degree is 1, which is odd. Thus let that component be C. Then $\#Q_{dS} = d^2 \#Q_S \equiv 1$ which is not equal to $\#Q_C$ and we have now established the claim.

We finish the proof of invariance by getting a contradiction. Recall that C is homologous to dS. Surger C and dS respectively in M by removing a small neighborhood of each liftable intersection in R_C and R_{dS} and then sewing back an annulus with the compatible orientation. Denote the resultant immersed surfaces by C^* and dS^*. The surgery does not change their homology class, and they are still homologous to each other. Since we surgered around the liftable self-intersections, the resultants lift to $\widetilde{M}$ also. Choose their unique lifts $\widetilde{C}^*$ and $d\widetilde{S}^*$ corresponding to $\widetilde{C}$ and $d\widetilde{S}$. These are homologous since $\tilde{f}$ is a homotopy equivalence and $\tilde{f}_*([\widetilde{C}^*]) = \tilde{f}_*([d\widetilde{S}^*])$. Also since we surgered on all the liftable self-intersections, they turn out to be embedded surfaces. Thus they are L-equivalent, that is, there is a proper orientable submanifold V in $\widetilde{M} \times [0, 1]$ so that $V \cap \widetilde{M} \times \{0\} = \widetilde{C}^*$ and $V \cap \widetilde{M} \times \{1\} = -d\widetilde{S}^*$. Then perturb V slightly without moving a neighborhood of the boundary; we may assume that V has a transverse image in $M \times [0, 1]$. Hence its self-intersectional singularity forms a 1-dimensional proper submanifold in $M \times [0, 1]$. Each nonclosed component of the singularity has two end points at the boundary, which are the members of the unliftable self-intersections $Q_C \cup Q_{dS}$. But this set was claimed and shown to have an odd number of elements. This is a contradiction. □

Remark. Our quadratic function can be derived from Wall's self-intersection μ [7]. It is defined in our case on π_2 of the associated Stiefel bundle over M with values in $\lambda = \mathbf{Z}[\pi_1]$ modulo some ambiguity. This homotopy group is identified with the set of regular homotopy classes of a 2-sphere in M. The ambiguity will disappear if we reduce its image to $\mathbf{Z}_2[\mathbf{Z}_s]$ by the homomorphism: $\pi_1(M) \to \mathbf{Z}_2$ associated to the first Stiefel-Whitney class and the $\mathbf{Z}_2$ reduction of the coefficients. Furthermore, if we compose the collapsing map of the constant factor: $\mathbf{Z}_2[\mathbf{Z}_2] \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to \mathbf{Z}_2 : a + bg \to b$, to the reduction, then μ comes down to the map from $\pi_2(M)$. Let us denote this map by $\mu^* : \pi_2(M) \to \mathbf{Z}_2$. Then since $\mu^*(x) \equiv \#Q_{S_x}$ (2), we have the identity

$$q(x) = \chi(\nu(S_x)) + 2\#R_{S_x} + 2\mu^*(x) = [\widetilde{S}_x] \cdot [\widetilde{S}_x] + 2\mu^*(x).$$

3. EXAMPLES

The invariants are practical tools to detect the homotopy type of manifolds. Hence it would be instructive to compute them from concrete examples. We present here three nonorientable manifolds with the same 3-skeleton and see how the invariants work. The computation shows the nontriviality of the $\mathbf{Z}_2$-intersection pairing and q-function, and their independence to the other invariants, and unfortunately leaves a few unanswered questions.

Start with a D^2-bundle over S^2 with Euler class $2n$. Denote it by E_n. The boundary of E_n is a circle bundle over S^2. It admits a free involution τ_0 rotating each fiber half, and another free involution τ_1 rotating each fiber half and simultaneously rotating the base half along some axis. On E_n, we identify the orbits of τ_j on the boundary of E_n with points. We shall denote this quotient space by $N_{nj} = E_n/\tau_j$, $j = 0, 1$.

Split the base S^2 into two disks D_+ and D_- along the equator perpendicular to the axis of rotation. Over D_+ and D_- the fibration E_n is split into E_n^+ and E_n^-. Both involutions leave each component invariant and we get the decomposition $N_{nj} = E_n^+/\tau_j \cup E_n^-/\tau_j$, where these two parts are diffeomorphic to each other. From this decomposition it can be seen that N_{n0} is diffeomorphic to $S^2 \times RP^2 \cong N_{00}$ and that N_{n1} is diffeomorphic to either N_{01} or N_{11} according to whether n is even or odd. For example, with τ_0, E_n^+/τ_0 and E_n^-/τ_0 are two copies of $D^2 \times RP^2$ glued together by a diffeomorphism f_n along $S^1 \times RP^2$. The diffeomorphism, which is isotopic to the identity, extends to a diffeomorphism from $N_{00} = D_+ \times RP^2 \cup_{\mathrm{id}} D_- \times RP^2$ to $N_{n0} = D_+ \times RP^2 \cup_{f_n} D_- \times RP^2$. On the other hand, $E_n/\tau_1 = E_n^+/\tau_1 \cup_{g_n} E_n^-/\tau_1$ is the union of two nontrivial 2-disk bundles over RP^2. The glueing mapping g_n is defined over the equator on the twisted S^2-bundle over S^1. This time g_n, because of τ_1, is isotopic to g_{n+2} and we get a diffeomorphism of N_{n1} to N_{01} if n is even and to N_{11} if n is odd.

Let us review the invariants of $N_{00} = S^2 \times RP^2$. It has $\pi_1 \cong \mathbf{Z}_2$ and $\pi_2 \cong \mathbf{Z} \oplus \mathbf{Z}$ generated by each factor, say x and y. Then the action of π_1 is given by $(x, y) \to (x, -y)$. Since π_1 twists the second factor of π_2, $\mathrm{H}^3(\pi_1; \pi_2) \cong \mathbf{Z}_2$. The first k-invariant is supported by an embedded RP^2, and hence is nontrivial.

If we let e be a 2-disk in E_n bounded by the invariant circle on ∂E_n, $e \cup \partial E_n/\tau_1$ forms a 3-skeleton of N_{n1}. It is homeomorphic to $e \cup \partial E_n/\tau_0$, which is a 3-skeleton of N_{00}. Hence each N_{n1} has a common 3-skeleton with N_{00} up to homotopy. In particular, they share π_1, π_2 as a Λ-module and the unique nontrivial first k-invariant in $\mathrm{H}^3(\pi_1; \pi_2) \cong \mathbf{Z}_2$ with N_{00}. Since τ_1 is isotopic to the identity, both N_{01} and N_{11} are doubly covered by the double of E_n, which is diffeomorphic to $S^2 \times S^2$. On the other hand, every orientation reversing involution induces the automorphism of $\pi_2(S^2 \times S^2)$ described by $(x, y) \to (x, -y)$ up to base change. Hence this fact forces N_{01} and N_{11} to have the same equivariant intersection paring on π_2 with N_{00}.

Homotopy invariants. N_{00} has the hyperbolic $\mathbf{Z}_2$-intersection pairing and the others have the standard one, while N_{01} has a trivial q-function and the others have a nontrivial one. In particular, they are not mutually homotopy equivalent to each other.

Computation. The $\mathbf{Z}_2$-intersection pairing of N_{00} is obvious. To compute it for the others, recall the decomposition $N_{n1} = E_n^+/\tau_1 \cup E_n^-/\tau_1$. Each factor, as we have observed, is a disk bundle over RP^2, and its boundary is a nonorientable bundle. The exact sequence for this pair shows that its spine, RP^2, is a self dual Poincaré dual class in $E_n^\pm/\tau_1$. Using this fact and the Mayer-Vietoris exact sequence, we can verify that $\mathrm{H}^2(N_{n1}; \mathbf{Z}_2)$ for all n is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and has the standard (odd) form.

To compute q-functions for N_{00} and N_{11}, look at E_1/τ_j $(j = 0, 1)$ and find an embedded S^2 with Euler class 2 as the zero section of E_1. In particular, their q-functions are nontrivial. To compute it for N_{01}, recall that the q-function is $\mathbf{Z}_4$-quadratic. Thus we only need to know values of two primitive independent elements of $\pi_2(N_{01}) \cong \mathbf{Z} \oplus \mathbf{Z}$. An element to compute easily is the zero section of E_0 whose value is zero. The other element to compute easily is the sphere lying on $\partial(E_0^+/\tau_1)$. Since this is an embedded sphere with trivial normal bundle, its value is also zero. It is then obvious by looking at the universal cover that these two form a primitive pair of $\pi_2(N_{01})$, and the q-function of N_{01} is trivial. □

Remark. Hambleton and Kreck suggested the connection between k-invariants and $\mathbf{Z}_2$-intersection pairings in [2]. However our examples show that they are rather independent, at least in the nonorientable case.

Finally, we review these examples from homotopy theoretic viewpoints. Take a 3-skeleton of $N_{00} = S^2 \times RP^2$ sitting as $K = S^2 \times RP^1 \cup * \times RP^2$. N_{00} is then a union of K and a single 4-cell e^4. The universal cover $\widetilde{K}$ of K is homeomorphic to $S^2 \times S^1 \cup e \cup e'$, where e and e' are 2-cells attached to $* \times S^1$ canonically. The covering transformation acts on $\widetilde{K}$ by the (identity) $\times$ (half-rotation) on the $S^2 \times S^1$ part and by the antipodal map on $e \cup e' = S^2$. On the other hand, $\widetilde{K}$ has the homotopy type $S^2 \vee S^2 \vee S^3$, where the first two S^2 correspond to $S^2 \times *$ and $e \cup e'$ respectively. Hence $\pi_3(K) \cong \pi_3(\widetilde{K})$ is isomorphic to $\mathbf{Z}^4$. The generators can be described by g_1 and g_2, the Hopf maps to the first two spheres, g_3, the Whitehead product to $S^2 \vee S^2$, and g_4, the standard degree one map to the S^3 factor.

Lemma 7. *The action of the covering transformation T on $\pi_3(\widetilde{K}) \cong \pi_3(K)$ is given by $T(g_j) = g_j$ for $j = 1, 2$, $T(g_3) = -g_3$ and $T(g_4) = g_3 + g_4$.*

Sketch of proof. The action for g_1 and g_3 should be obvious. The action for g_2 is identical because the reflection map on the base sphere of the Hopf fibration is covered by an orientation preserving map of S^3.

We roughly sketch how to get the last claim. Let us decompose S^3 into two solid tori U and V. Thicken two 2-cells in $\widetilde{K}$ by producing D^2 so that $e \times D^2$ attaches to $S^2 \times S^1$ in a tubular neighborhood of $* \times S^1$. The complement W of its attaching part in $S^2 \times S^1$ is also a solid torus. Then the map g_4 is the map sending U to the complement W and V to $e \cup D^2$, while $T(g_4)$ maps U to the same W but V to $e' \cup D^2$. Then it is not hard to find a homotopy of the map $T(g_4) - g_4$ to the Whitehead product g_3 which maps U to $S^2 \times *$ and V to $e \cup e'$. □

By attaching a 4-cell to K by a glueing map $\gamma \in \pi_3(K) \cong \mathbf{Z}^4$, we obtain a 4-complex $K_\gamma = K \cup_\gamma e^4$. Since the $N_{n,j}$'s have a common 3-skeleton, they all are so obtained. But not all γ's produces a manifold or even a Poincaré complex. Also, different γ's might produce homotopy equivalent complexes.

Poincaré complexes. If $\gamma = g_4$, $g_4 + g_1$, $g_4 + g_2$, or $g_4 + g_1 + g_2$, then K_γ is a Poincaré complex doubly covered by a homotopy $S^2 \times S^2$. They are not mutually homotopy equivalent.

Computation. Suppose that K_γ is a Poincaré complex. Then since $\mathrm{H}_3(K_\gamma)$ is trivial, the Hurewicz image of γ must generate $\mathrm{H}_3(\widetilde{K})$ as a $\mathbf{Z}[\pi_1]$-module. In our case, the only g_4 factor in $\pi_3(K)$ survives in the image. Also we must have a nonsingular symmetric bilinear form on $\pi_2(K) \cong \pi_2(K_\gamma)$, which corresponds to the intersection form on $\mathrm{H}_2(\widetilde{K}_\gamma)$. In our case, the direct summand of $\pi_3(K)$ generated by g_1, g_2, g_3 is identified with the set of symmetric bilinear forms on $\pi_2(K)$ (cf. [3]), and $(1-T)\gamma$ represents the form on $\pi_2(\widetilde{K})$. If both properties are satisfied, then K_γ is a Poincaré complex.

It is then quite easy by Lemma 7 to compute which γ produces a Poincaré complex. To rule out obvious homotopic examples, observe that K_γ is homotopy equivalent to $K_{-\gamma}$ and $K_{T(\gamma)}$. Hence one significant family of K_γ's having the form $-g_3$ on $\pi_2(K_\gamma)$ is obtained by the sum of g_4 with arbitrary linear combinations of g_1 and g_2. Since we chose $-g_3$ for the intersection form, each K_γ is doubly covered by a homotopy $S^2 \times S^2$.

On the other hand, suppose that there is a homotopy equivalence $f : K_{\gamma'} \to K_\gamma$ so that f is the identity on a 2-skeleton $K^{(2)}$. Since there certainly exists a continuous extension of $f|_{K^{(2)}}$ from $K_{\gamma'}$, the obstruction class defined by $f|_{K^{(3)}}$ in $\mathrm{H}^4(K_{\gamma'}; \pi_3(K_\gamma))$ must vanish. By Poincaré duality with Wall's twisted coefficients (see Wall [7, p. 25]), we have the isomorphisms

$$\mathrm{H}^4(K_{\gamma'}; \pi_3(K_\gamma)) \cong \mathrm{H}_0^t(K_{\gamma'}; \pi_3(K_\gamma)) \cong \pi_3(K_\gamma) \otimes_\Lambda \mathbf{Z},$$

and the obstruction element is represented by γ'. Thus consider an epimorphism $\pi_3(K) \to \pi_3(K_\gamma) \otimes_\Lambda \mathbf{Z}$. It descends to $\pi_3(K) \otimes_\Lambda \mathbf{Z} \to \pi_3(K_\gamma) \otimes_\Lambda \mathbf{Z}$, where the domain is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}$ generated by g_1, g_2, and g_4 in each summand. Hence, letting $\{\gamma, \gamma'\}$ be any pair in $\{g_4, g_4+g_1, g_4+g_2, g_4+g_1+g_2\}$, we can verify that the K_γ's are homotopically distinct from each other, at least fixing $K^{(2)}$.

Then it can be shown without too much difficulty that N_{00}, N_{01}, N_{11} actually correspond to the elements $\gamma = g_4$, $g_4 + g_1$, $g_4 + g_1 + g_2$, respectively. Also, Hambleton and Milgram showed in [11, §3] that the Poincaré complex defined by $\gamma = g_4 + g_2$ ($J + \eta_1$ in their notation) is not homotopy equivalent to a topological manifold. Hence they are actually homotopically distinct. $\square$

We would like to finish this paper by asking the following

Question. Does the set of six invariants appearing here determine the homotopy type of a nonorientable 4-dimensional Poincaré complex with $\pi_1 \cong \mathbf{Z}_2$?

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