EXAMPLES OF CAPACITY FOR SOME ELLIPTIC OPERATORS

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Abstract. We study $L$-capacities for uniformly elliptic operators of nondivergence form

$$L = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j a_j(x) \frac{\partial}{\partial x_j};$$

and construct examples of large sets having zero $L$-capacity for some $L$, and small sets having positive $L$-capacity. The relations between ellipticity constants of the coefficients and the sizes of these sets are also considered.

A compact set $S \subseteq \{|x| < 1\} \subseteq \mathbb{R}^n$, $n \geq 2$, has zero capacity for the Laplacian if and only if it is a removable set for the class of bounded subharmonic functions on $\{|x| < 1\}$; equivalently, there exists a positive superharmonic function $(\neq +\infty)$ on $\{|x| < 1\}\backslash S$ which approaches $+\infty$ continuously on $S$.

In this note, we study $L$-capacities for uniformly elliptic operators of nondivergence form

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and construct examples of large sets having zero $L$-capacity for some $L$, and small sets having positive $L$-capacity. We also study the relations between ellipticity constants of the coefficients and the sizes of these sets.

When $(a_{ij})$ are Dini continuous and $(a_j)$ are bounded, sets of $L$-capacity zero are precisely those of capacity zero for the Laplacian; this follows from the growth of the Green function for these operators. (See [1] and [7].) In general, there are elliptic operators $L$ with continuous coefficients for $n = 2$, bounded coefficients for $n \geq 3$, for which a single point has positive $L$-capacity; again this reflects the behavior of the Green function. (See [2, 4, 5 and 12].)

For uniformly elliptic operators of divergence form, the growth of Green function near its pole is comparable to that for the Laplacian [10]. Therefore sets of capacity zero are exactly those for the Laplacian.

Let $L$ be the above operator and coefficients of $L$ be continuous in a domain $\Omega \subseteq \mathbb{R}^n$. We consider strong solutions of $L = 0$ in $W^{2,\infty}_{\text{loc}}(\Omega)$ and call them $L$-solutions. The maximum principle, the existence and uniqueness of the solution to the Dirichlet problem and the Harnack principle are well known. A lower semicontinuous function $v$ is called an $L$-supersolution on $\Omega$, if for any closed ball $B \subseteq \Omega$ and any $L$-solution $u$ in $B$ with $u$ continuous on $\overline{B}$, the
inequality \( v \geq u \) on \( \partial B \) implies that \( v \geq u \) in \( B \). A function \( v \in C^2(\Omega) \) is an \( L \)-supersolution if and only if \( Lv \leq 0 \). A function \( v \) is called an \( L \)-subsolution if \( -v \) is an \( L \)-supersolution. (See [6, Chapter 9].)

Let \( D = \{ |x| < 1 \}, \quad \overline{x} = (\frac{1}{2}, 0, 0, \ldots, 0) \) and \( S \) be a compact set in \( \{ |x| \leq \frac{1}{4} \} \). We define the \( L \)-capacity of \( S \) as

\[
L \text{-cap } S = \inf \{ v(\overline{x}) : v \text{ is a positive } L \text{-supersolution on } D \text{ and } v \geq 1 \text{ on } S \},
\]

when the coefficients of \( L \) are bounded continuous in \( D \). When the coefficients of \( L \) are only known to be bounded continuous on \( D \setminus S \), we say \( L \)-cap \( S = 0 \) provided that

\[
\inf \left\{ v(\overline{x}) : v \text{ is a positive } L \text{-supersolution on } D \setminus S, \right. \\
\left. \quad \text{and } \liminf_{x \to x_0} v(x) \geq 1 \text{ at each } x_0 \in S \right\} = 0;
\]

otherwise we say \( L \)-cap \( S > 0 \). Both definitions of \( L \)-capacity zero agree when the coefficients of \( L \) are continuous on \( D \). We note that if there exists a positive \( L \)-supersolution on \( D \setminus S \) which approaches \( +\infty \) continuously on \( S \), then \( L \)-cap \( S = 0 \); and that if there exists a bounded positive \( L \)-subsolution on \( D \setminus S \) which approaches \( 0 \) continuously on \( \partial D \), then \( L \)-cap \( S > 0 \).

We recall that for the Laplacian, a set has positive capacity if it has positive \( h \)-Hausdorff measure for some \( h > 0 \) satisfying \( \int_0^1 h(r)/r^{n-1} \, dr < \infty \); and a set has zero capacity if it has finite \( (n-2) \)-dimensional Hausdorff measure when \( n \geq 3 \), or finite logarithmic measure when \( n = 2 \). Therefore \( n - 2 \) is the critical dimension for studying sets of capacity zero.

We shall prove the following:

**Theorem 1.** Let \( n \geq 2 \) and \( n - 2 < \alpha < n \). Then there exist a constant \( \Lambda_{n, n-\alpha} > 1 \), a compact set \( S \subseteq D \) of Hausdorff dimension \( \alpha \), an operator \( L = \sum a_{ij} \partial^2_j / \partial x_i \partial x_j \) with coefficients bounded smooth in \( D \setminus S \), satisfying

\[
|\xi|^2 \leq \sum a_{ij}(x) \xi_i \xi_j \leq \Lambda_{n, n-\alpha} |\xi|^2, \quad x, \xi \in \mathbb{R}^n,
\]

so that \( S \) has zero \( L \)-capacity in the sense (0.1). In fact, there is a positive \( L \)-supersolution \( v \) (\( \not\equiv +\infty \)) in \( D \setminus S \) approaching \( +\infty \) continuously on \( S \). Moreover,

\[
\Lambda_{n, n-\alpha} = 1 + O(1)(\alpha - n + 2) \quad \text{as } \alpha \to n - 2,
\]

and

\[
\Lambda_{n, n-\alpha} = O(1)(n - \alpha)^{-1} \quad \text{as } \alpha \to n,
\]

with the \( O(1) \) terms positive and independent of \( n \) and \( \alpha \).

We believe that (0.3) is sharp, and do not know whether (0.4) can be improved.

**Theorem 2.** Let \( n \geq 2, a > 0, \) and \( h(r) = r^{n-2} (\log \frac{1}{r})^{-1-a} \). Then there exist a constant \( \beta > 0 \), a compact set \( S \subseteq D \) of dimension \( n - 2 \), positive Hausdorff \( h \)-measure, an operator \( L = \sum a_{ij} \partial^2_j / \partial x_i \partial x_j \) with coefficients continuous in \( \mathbb{R}^n \), smooth off \( S \) satisfying

\[
|\xi|^2 \leq \sum a_{ij}(x) \xi_i \xi_j \leq \left\{ 1 + \beta \left( \log \frac{1}{\text{dist}(x, S)} \right)^{-1} \right\} |\xi|^2
\]
for all $x, \xi \in \mathbb{R}^n$, and a positive $L$-supersolution $v$ on $D$, which is an $L$-solution on $D \setminus S$ and approaches $+\infty$ continuously on $S$. In particular, $S$ has zero $L$-capacity, and positive capacity for the Laplacian.

**Theorem 3.** Let $n > 3$, $a > 0$, and $h(r) = r^{n-2}(\log ^{1/2})^a$. Then there exist a compact set $S \subseteq D$ of dimension $n - 2$, vanishing $h$-measure, a constant $\beta > 0$, an operator $L = \sum a_{ij} \partial^2 / \partial x_i \partial x_j$ with coefficients continuous in $\mathbb{R}^n$, smooth off $S$, and satisfying

$$\left\{ 1 - \beta \left( \log \frac{1}{\text{dist}(x, S)} \right) \right\} |\xi|^2 \leq \sum a_{ij}(x)\xi_i \xi_j \leq |\xi|^2$$

for $x, \xi \in \mathbb{R}^n$, and a bounded positive $L$-supersolution $w$ on $D$, which is an $L$-solution on $D \setminus S$. Thus $S$ has zero capacity for the Laplacian, and positive capacity for the operator $L$.

**Theorem 4.** Let $n > 3$ and $0 < \alpha < n - 2$. Then there exist a positive constant $\lambda_{n, \alpha} < 1$, a compact set $S \subseteq D$ of dimension $\alpha$, an operator $L = \sum a_{ij} \partial^2 / \partial x_i \partial x_j$ with $a_{ij}$ bounded smooth off $S$, satisfying

$$\lambda_{n, \alpha} |\xi|^2 \leq \sum a_{ij}(x)\xi_i \xi_j \leq |\xi|^2, \quad x, \xi \in \mathbb{R}^n,$$

so that $S$ has positive $L$-capacity. In fact, there exists a bounded positive $L$-subsolution $w$ on $D \setminus S$ which vanishes continuously on $\partial D$. Moreover

$$(0.6) \quad \lambda_{n, \alpha} = (1 - 2\alpha)(n - 1)^{-1}, \quad 0 < \alpha < 1/4,$$

and

$$(0.7) \quad \lambda_{n, \alpha} = 1 - O(1)(n - 2 - \alpha) \quad \text{as} \ \alpha \to n - 2,$$

with the $O(1)$ term positive and independent of $n$ and $\alpha$.

Since it is known that a point can have positive $L$-capacity, the only new part of Theorem 4 is the relation between the ellipticity constants and the dimension. In the proofs of all four theorems, we start with the Laplace operator, then modify the coefficients on a sequence of rings, accumulating on a Cantor set $S$, so that on the rings all eigenvalues are greater than 1 (or less than 1). When all are chosen properly, it will produce an $L$-supersolution which grows faster than (or slower than) the fundamental solution of the Laplacian near each point in $S$. This explains the relation between the normalization of the ellipticity constants and the size of the set $S$.

A related subject, the boundary regularity problem for the operator $L$, has been studied by many. A partial list includes [4, 7, 8, 9, 10, 11, 12, and 13].

1. Preliminary lemmas

Let $\Delta$ be the Laplace's operator and $r = |x|$ for $x \in \mathbb{R}^n$.

**Lemma 1.** Let $B(x)$ be positive continuous in a domain $\Omega$, and let

$$L = \frac{B(x)}{n - 1} \sum \frac{\partial^2}{\partial x_i^2} - \left( \frac{B(x)}{n - 1} - 1 \right) \sum \frac{x_i x_j}{|x|^2} \frac{\partial^2}{\partial x_i \partial x_j}$$
on \( \Omega \setminus \{0\} \). Then the coefficients \( a_{ij} \) of \( L \) are continuous, symmetric on \( \Omega \setminus \{0\} \), satisfying

\[
(1.1) \quad |\xi|^2 \leq \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leq \frac{B(x)}{n-1} |\xi|^2 \quad \text{when } B(x) \geq n-1,
\]

and

\[
(1.2) \quad \frac{B(x)}{n-1} |\xi|^2 \leq \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leq |\xi|^2 \quad \text{when } B(x) \leq n-1.
\]

The coefficients of \( L \) can be extended to be continuous on \( \Omega \) if \( B(0) = n-1 \). Moreover, \( |x|^{-B+1} \) is a solution of \( L = O(x \neq 0) \), when \( B(x) \equiv 0 \) a constant \( B \).

The characteristic values of \( (a_{ij}(x)) \) are \( 1, B(x)/(n-1), B(x)/(n-1), \ldots, B(x)/(n-1) \); and for \( x \neq 0 \),

\[
L = \frac{B(x)}{n-1} \Delta - \left( \frac{B(x)}{n-1} - 1 \right) \frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial r^2} + \frac{B(x)}{r} \frac{\partial}{\partial r} + \frac{B(x)}{n-1} r^{-2} \delta,
\]

where \( \delta \) is the Beltrami operator in the spherical coordinates. Whence the lemma follows.

Denote by \( D(x, a) \) the closed ball centered at \( x \) of radius \( a \), and recall that \( D = D(0, 1) \). When \( U \) is a ball, denote by \( cU \) the ball concentric to \( U \) of radius \( c \) times that of \( U \).

**Lemma 2.** Let \( 0 < \delta < \frac{1}{16} \) and \( D(a, r) \subseteq D(0, \delta) \). Then there exists a diffeomorphism \( y = Tx \) from \( \mathbb{R}^n \) onto \( \mathbb{R}^n \), which fixes every point in \( \mathbb{R}^n \setminus D(0, \frac{9}{16}) \), maps each point \( x \) in \( D(a, r) \) to \( x - a \), and satisfies on \( D(0, \delta) \):

\[
(1.3) \quad \frac{\partial y_i}{\partial x_j} - \delta_{ij} = c(x) a_i(x_j - a_j),
\]

\[
(1.4) \quad \left| \sum_i \frac{\partial^2 y_i}{\partial x_i \partial x_m} \xi_i \right| \leq 272 \delta |\xi|,
\]

where \( 0 < c(x) < 32 \), \( \delta_{ij} = 1 \) when \( i = j \) and \( \delta_{ij} = 0 \) when \( i \neq j \). Moreover if \( (a_{ij}(x)) \) is symmetric positive definite with all its eigenvalues bounded above by \( \Lambda \), and

\[
b_{ij}(x) = \sum_{l,m} a_{lm}(x) \frac{\partial y_i}{\partial x_l} \frac{\partial y_j}{\partial x_m},
\]

then

\[
(1.5) \quad \left| \sum_{i,j} b_{ij} \xi_i \xi_j - \sum_{i,j} a_{ij} \xi_i \xi_j \right| \leq 128 \delta \Lambda |\xi|^2,
\]

and

\[
(1.6) \quad \left| \sum b_{ii} - \sum a_{ii} \right| \leq 128 \delta \Lambda.
\]

**Proof.** Let

\[
\psi(s) = \begin{cases} 
10s^3 - 15s^4 + 6s^5, & 0 \leq s \leq 1, \\
0, & s < 0, \\
1, & s > 1;
\end{cases}
\]
and note that $\psi$ is $C^2$, $0 \leq \psi \leq 1$, $0 \leq \psi' \leq 15/8$ and $|\psi''| \leq 10/\sqrt{3}$. Let

$$\varphi(t) = 1 - \psi \left( \frac{t - \delta^2}{\frac{1}{4} - \delta^2} \right).$$

Thus $\varphi = 0$ for $t \geq \frac{1}{4}$, $\varphi = 1$ for $t \leq \delta^2$, $0 \leq \varphi \leq 1$, $-8 \leq \varphi' \leq 0$ and $|\varphi''| < 160$. Then $Tx = x - \varphi(|x - a|^2)a$ is a diffeomorphism on $\mathbb{R}^m$ that fixes every point in $\mathbb{R}^m \setminus D(0, \frac{9}{16})$ and maps $x \in D(a, r)$ to $x - a$. Moreover, on $D(0, \frac{9}{16})$, $T$ satisfies (1.3) and (1.4) with $c(x) = -2\varphi'(|x - a|^2)$.

To show (1.5), we let $x \in D(0, \frac{9}{16})$, and note that

$$b_{ij} - a_{ij} = \sum_{l,m} a_{lm}[\delta_{il} + c(x)a_i(x_l - a_l)][\delta_{jm} + c(x)a_j(x_m - a_m)] - a_{ij}$$

$$= c(x) \sum_{m} a_{lm}a_j(x_m - a_m) + c(x) \sum_{l} a_{lj}a_i(x_l - a_l)$$

$$+ c(x)^2 \sum_{l,m} a_{lm}a_i(x_l - a_l)(x_m - a_m).$$

Thus

$$\left| \sum_{i,j} b_{ij} \xi_i \xi_j - \sum_{i,j} a_{ij} \xi_i \xi_j \right|$$

$$\leq c(x) \left| \sum_{i,m} a_{im} \xi_i(x_m - a_m)a_j \xi_j \right| + c(x) \left| \sum_{j,l} a_{lj} \xi_j(x_l - a_l)a_i \xi_i \right|$$

$$+ c(x)^2 \left| \sum_{i,j,l,m} a_{lm}a_i(x_l - a_l)(x_m - a_m)a_j \xi_i \xi_j \right|.$$ 

Since $|a| < \delta$, $|x - a| < 1$ and eigenvalues of $(a_{ij})$ are bounded above by $\Lambda$, we conclude that

$$\left| \sum_{i,j} b_{ij} \xi_i \xi_j - \sum_{i,j} a_{ij} \xi_i \xi_j \right| \leq 2c(x)\Lambda \delta + c(x)^2 \Lambda \delta^2 \leq 128\Lambda \delta.$$ 

Similarly,

$$\left| \sum_{i} b_{ii} - \sum a_{ii} \right| \leq c(x) \left| \sum_{i,m} a_{im}a_i(x_m - a_m) \right| + c(x) \left| \sum_{i,l} a_{li}a_i(x_l - a_l) \right|$$

$$+ c(x)^2 \left| \sum_{l,m} a_{lm}^2 \sum_{l,m} a_{lm}(x_l - a_l)(x_m - a_m) \right| \leq 128\Lambda \delta.$$ 

### 2. The construction

Given $B^* \geq n - 1$, integer $k_0 > 0$, let $\{\delta_k\}$, $\{r_k\}$, and $\{N_k\}$ be sequences satisfying $0 < \delta_k < (2400B^*)^{-1}$, $0 < r_k < r_{k-1} < r_1 \leq \frac{1}{2}$ and $16\sqrt{n}/\delta_k < N_k < r_{k-1}/r_k$, for $k \geq k_0$. Then $r_{k+1} < \delta_{k+1}N_{k+1}r_{k+1} < N_{k+1}r_{k+1} < r_k$ for $k \geq k_0$. Let $[\ ]$ be the greatest integer function, $I_{k_0} = 1$ and $I_k = \prod_{j=k_0}^k [\delta_j N_j/16\sqrt{n}]^n$ for $k > k_0$. 
Denote by $D_{k_0,1} = D(0, r_{k_0})$. After $\{D_{k,l}: 1 \leq l \leq I_k\}$ are selected for some $k \geq k_0$ we let $D_{k,l}$ be the ball $\delta_{k+1}N_{k+1}r_{k+1}^{-1}D_{k,l}$ of radius $\delta_{k+1}N_{k+1}r_{k+1}$; and choose from each $D_{k,l}$ a number of $[\delta_{k+1}N_{k+1}/16\sqrt{n}]^n$ balls of radius $r_{k+1}$ to form the collection $\{D_{k+1,l}: 1 \leq l \leq I_{k+1}\}$. Moreover, we require their doublings $\{2D_{k+1,l}\}$ to be mutually disjoint and contained in $\cup D_{k,l}$.

Let $S$ be the Cantor set defined by $S = \bigcap_{k=k_0}^{\infty}(\bigcup_{l=1}^{I_k} D_{k,l})$. And let $\mu$ be the continuous measure on $S$, defined by $\mu(D_{k,l}) = I_k^{-1}$ for all $k \geq k_0$ and $1 \leq l \leq I_k$. For $k \geq k_0$, denote by $P_{k,l}$ the center of $D_{k,l}$, 

$$ R_{k,l} = \{N_{k+1}r_{k+1} \leq |x - P_{k,l}| \leq r_k\}, $$

and

$$ R'_{k,l} = \{\frac{3}{4}N_{k+1}r_{k+1} \leq |x - P_{k,l}| \leq \frac{5}{4}r_k\}, $$

and note that $\{R'_{k,l}: k \geq k_0, 1 \leq l \leq I_k\}$ are mutually disjoint.

Let $B(r)$ be a smooth function for $r > 0$, satisfying $n - 1 \leq B(r) \leq B^*$, with

\begin{align*}
B(r) &\equiv n - 1 & \text{on } \{r > \frac{3}{4}r_{k_0}\} \cup \bigcup_{k=k_0}^{\infty} \left[\frac{3}{4}r_k, \frac{5}{4}N_{k}r_k\right], \\
B(r) &> n - 1 & \text{on } \bigcup_{k \geq k_0} [N_kr_k, r_{k-1}],
\end{align*}

and $B(r)$ monotone in each of the remaining intervals. Define on $\mathbb{R}^n$ an elliptic operator

\begin{equation}
(2.2) \quad L = \begin{cases}
\Delta, & \text{on } \mathbb{R}^n \setminus \bigcup_{k,l} R'_{k,l}, \\
\frac{B(r)}{n-1} \Delta - \left(\frac{B(r)}{n-1} - 1\right) \frac{\partial^2}{\partial r^2}, & \text{at } x + P_{k,l} \in R'_{k,l},
\end{cases}
\end{equation}

where $r = |x|$. Rewrite $L$ in the standard form $\sum a_{ij} \partial^2/\partial x_i \partial x_j$. We note from Lemma 1 and properties of $B(r)$ that the coefficients $a_{ij}$ are symmetric and are smooth off $S$; and that $a_{ij}$ are continuous on $\mathbb{R}^n$ if $\lim_{r \to 0} B(r) = n - 1$. Let

\begin{equation}
(2.3) \quad B_k = \sup\{B(r): 0 < r \leq N_k r_k\},
\end{equation}

from (1.1) it follows that

\begin{equation}
(2.4) \quad |\xi|^2 \leq \sum a_{ij} \xi_i \xi_j \leq \frac{B_k}{n-1} |\xi|^2 & \text{on } R'_{k,l}.
\end{equation}

Next, we construct positive $L$-supersolutions.

Fix a point $x_0 \in S$ and rearrange the indices if necessary, we may assume that $x_0 \in \bigcap D_{k,1}$. Let

$$ D'_{k,1} = \{|x - P_{k,1}| \leq \frac{5}{4}r_k\}, $$

$$ D''_{k,1} = \{|x - P_{k,1}| \leq \frac{3}{4}N_{k+1}r_{k+1}\}, $$

$$ S_{k,1} = D''_{k-1,1} \setminus D'_{k,1}; $$
and note that \( D''_{k,1} \subseteq D_{k,1} \subseteq D'_{k,1} \subseteq D''_{k-1,1} \) and \( D'_{k,1} \subseteq D(P_{k-1,1}, \delta_k r_k) \).

Observe also that
\[
\mathbb{R}^n \setminus \{x_0\} = \bigcup_{k \geq k_0} R'_{k,1} \cup \bigcup_{k \geq k_0+1} S_{k,1} \cup \{|x| \geq \frac{5}{4} r_k\};
\]
and that \( \bigcup_k R'_{k,1} \) and \( \bigcup_k S_{k,1} \) meet on the boundaries only. Denote by
\[
\begin{align*}
a^k &= (a^1_k, a^2_k, \ldots, a^n_k) = P_{k-1} - P_{k-1,1}, \\
x^k &= (x^1_k, x^2_k, \ldots, x^n_k) = x - P_{k-1,1},
\end{align*}
\]
then \(|a^k| < \delta_k N_k r_k\) and \(|x^k - a^k| \leq N_k r_k\) if \( x^k \in S_{k,1} \).

Applying Lemma 2 to \( D''_{k-1,1} \) and \( D'_{k,1} \) instead of \( D(0, \frac{3}{4}) \) and \( D(a, r) \) for each \( k \geq 1 + k_0 \) in succession, we obtain, after a scale change, a diffeomorphism \( T \) from \( \mathbb{R}^n \setminus \{x_0\} \) onto \( \mathbb{R}^n \setminus \{0\} \) so that \( T \) fixes every point in \( \{|x| > \frac{3}{4} N_{k+1} r_{k+1}\} \), and is a translation on \( R'_{k,1} \) for each \( k \geq 1 + k_0 \) with
\[
T(R'_{k,1}) = \left\{ \frac{3}{4} N_{k+1} r_{k+1} \leq |y| \leq \frac{5}{4} r_k \right\};
\]
and that for \( x \in S_{k,1} \),
\[
(2.5) \quad \left| \frac{\partial y_j}{\partial x_j} - \delta_{ij} \right| \leq 32|a^k_j| |x^k_j - a^k_j| (N_k r_k)^{-2} \leq 32 \delta_k,
\]
\[
(2.6) \quad \left| \sum_j \frac{\partial^2 y_j}{\partial x_j \partial x_m} \xi_j \right| \leq 272 \delta_k |\xi| (N_k r_k)^{-1},
\]
and
\[
T(S_{k,1}) = \left\{ \frac{5}{4} r_k < |y| \leq \frac{3}{4} N_k r_k \right\}.
\]
Let \( T(x_0) = 0 \) and note that \( T \) is homeomorphic on \( \mathbb{R}^n \).

Let \( M \) be the operator on \( \mathbb{R}^n \setminus \{0\} \) defined by \( Mv(y) = L(v \circ T)(x) \) when \( y = Tx \); that is,
\[
M = \sum_{i,j} b_{ij} \frac{\partial^2}{\partial y_i \partial y_j} + \sum_j b_j \frac{\partial}{\partial y_j},
\]
with
\[
b_{ij} = \sum_{l,m} a_{lm} \frac{\partial y_i}{\partial x_l} \frac{\partial y_j}{\partial x_m} \quad \text{and} \quad b_j = \sum_{l,m} a_{lm} \frac{\partial^2 y_j}{\partial x_l \partial x_m}.
\]
Thus \( M \) is the Laplacian on \( \{|y| \geq \frac{3}{4} r_k\} \). Since \( T \) is a translation on \( R'_{k,1} \),
\[
M = \frac{B(\rho)}{n-1} \Delta - \left( \frac{B(\rho)}{n-1} - 1 \right) \frac{\partial^2}{\partial \rho^2} \quad \text{on } T(R'_{k,1})
\]
where \( \rho = |y| \). In view of (1.5) and (2.5), we obtain after a scale change that for \( x \in S_{k,1} \),
\[
\left| \sum_{i,j} b_{ij}(Tx)\xi_i \xi_j - \sum_{i,j} a_{ij}(x)\xi_i \xi_j \right| \leq 128 \delta_k \sup_{S_{k,1}} \sum_{i,j} a_{ij}(x)\xi_i \xi_j \leq \frac{128}{n-1} B_k \delta_k |\xi|^2.
\]
The last inequality follows from (2.3) and the fact that \( S_{k,1} \) contains rings from \( \{R'_{k,1}\} \) but none from the larger ones \( \{R'_{k-1,1}\} \). Similarly it follows from (1.6) and (2.5) that

\[
(2.9) \quad \left| \sum b_{ii}(Tx) - \sum a_{ii}(x) \right| \leq \frac{128}{n-1} B_k \delta_k \quad \text{on } S_{k,1}.
\]

Let \( f \) be a smooth function on \( r > 0 \), bounded above by \( B(r) \), with values \( f(r) = 0 \) for \( r \geq \frac{5}{4} r_{k_0} \), \( f(r) = B(r) \) on \( \bigcup_{k\geq k_0} [N_{k+1} r_{k+1}, r_k] \),

\[
(2.10) \quad f(r) = \left( n - 2 + \frac{n-1}{B_k} \right) (1 - 1200 B^* \delta_k) \quad \text{on } \left[ \frac{5}{4} r_k, \frac{3}{4} N_k r_k \right],
\]

for each \( k \geq 1 + k_0 \), and monotone in each of the remaining intervals. Define for \( \rho = |y| < 1 \),

\[
(2.11) \quad u(y) \equiv u(\rho) \equiv \int_\rho^1 \exp \int_t^1 \frac{f(s)}{s} \, ds \, dt,
\]

and claim that

\[
(2.12) \quad Mu \leq 0 \quad \text{in } D\setminus\{0\}.
\]

The idea of defining a radial \( M \)-supersolution in the form (2.11) comes from Gilbarg and Serrin [5] and Bauman [4]. It follows from (2.7) and the fact that \( f(r) \leq B(r) \) that \( Mu \leq 0 \) on \( \{ \frac{5}{4} r_k < |y| < 1 \} \cup T(R'_{k,1}) \). On \( T(S_{k,1}) \), we note that

\[
Mu(y) = \frac{u'(\rho)}{\rho} \left[ - \sum_{i,j} b_{ij} \frac{y_i y_j}{\rho^2} f(\rho) + \sum_i b_{ii} - \sum_{i,j} b_{ij} \frac{y_i y_j}{\rho^2} + \sum_j b_j y_j \right].
\]

Eigenvalues of \( (a_{ij}(x)) \) are in the form \( 1, \Lambda(x), \Lambda(x), \ldots, \Lambda(x) \), with

\[
(2.13) \quad 1 \leq \Lambda(x) \leq \frac{B_k}{n-1} \quad \text{on } S_{k,1}.
\]

We obtain from (2.6) that

\[
\left| \sum b_{j} y_{j} \right| = \sum l,m a_{lm} \sum_j \frac{\partial^2 y_j}{\partial x_l \partial x_m} y_j \leq 272 \delta_k B_k \frac{n}{n-1}
\]

on \( T(S_{k,1}) \). For \( x \in S_{k,1}, y = Tx \), and \( \rho = |y| \), we obtain from (2.8), (2.9), (2.10), (2.13) and the assumptions \( B(r) \leq B^* \) and \( \delta_k < (2400 B^*)^{-1} \) that

\[
\sum b_{ii} + \sum b_{ij} y_j \rho^2 - 1 \geq \frac{\sum a_{ii} - \frac{128}{n-1} B_k \delta_k - 272 \frac{n}{n-1} B_k \delta_k}{\sum a_{ij} y_i y_j \rho^2 + \frac{128}{n-1} B_k \delta_k} - 1
\]

\[
(2.14) \quad \geq 1 + \frac{(n-1) \Lambda(x) - \left( \frac{128}{n-1} + 272 \frac{n}{n-1} \right) B_k \delta_k}{\Lambda(x) + \frac{128}{n-1} B_k \delta_k} - 1
\]

\[
\geq \left( n - 2 + \frac{n-1}{B_k} \right) (1 - 1200 B^* \delta_k) \geq f(|y|).
\]

Hence \( Mu \leq 0 \) on \( T(S_{k,1}) \), and (2.12) is proved.
Let $H_{x_0}(x_0) = +\infty$ and
\begin{equation}
H_{x_0}(x) = u(Tx) \quad \text{on } D\setminus\{x_0\}.
\end{equation}
Since $LH_{x_0} \leq 0$ on $D\setminus\{x_0\}$ and the coefficients of $L$ are smooth off $S$, $H_{x_0}$ is an $L$-supersolution in $D\setminus S$. We shall estimate the growth of $H_{x_0}(x)$ near $x_0$.

In the rest of the paper, $C$ denotes positive constants depending at most on $n$, $\alpha$ and $a$ in the theorems; its value may vary from line to line.

3. Proof of Theorem 1

Let
\begin{equation}
B = \frac{2(n-1) + 2n(\alpha + 2 - n)}{n - \alpha},
\end{equation}
and note that $B > \alpha + 1 > n - 1$, $B \to n - 1$ as $\alpha \to n - 2$, and
\begin{equation}
\frac{\alpha}{n} < \frac{B - (1 + \alpha)}{B - (n - 2) - (n - 1)/B} < 1.
\end{equation}
Choose $\alpha'$, $\alpha < \alpha' < n$, so that
\begin{equation}
\frac{\alpha}{n} < \frac{B - (1 + \alpha')}{B - (n - 2) - (n - 1)/B} < 1,
\end{equation}
and denote by
\begin{equation*}
A = \frac{B - (1 + \alpha')}{B - (n - 2) - (n - 1)/B} \quad \text{and} \quad E = A - \frac{\alpha}{n}.
\end{equation*}
Let
\begin{equation}
\delta_k = \frac{16\sqrt{n}}{k}, \quad N_k = k^{A/E} \quad \text{and} \quad r_k = (k!)^{-1/E}
\end{equation}
and note that $N_k \leq r_{k-1}/r_k$ for $k \geq 1$. To specify $B(r)$ in (2.1), we let
\begin{equation}
B(r) = B \quad \text{on} \quad \bigcup_{k \geq k_0} [N_{k+1}r_{k+1}, r_k];
\end{equation}
and note that $n - 1 \leq B(r) \leq B$. Choose and fix integer $k_0 \geq 10^5B\sqrt{n}$.

It is clear that
\begin{equation}
\lim_{k \to \infty} (k!)^{\alpha/E}(\delta_k N_k r_k)^\gamma = 0 \quad \text{if } \gamma > \alpha;
\end{equation}
and that
\begin{equation}
\lim_{k \to \infty} (2^{-k}(k-1)!)^{-\alpha/E}/(\delta_k N_k r_k)^\eta = 0, \quad \text{if } \eta < \alpha.
\end{equation}

Note that there are $I_k = \prod_{j=k_0}^k[j^{A/E-1}]^n$ balls in $\{D_{k,i}\}_l$ and that
\begin{equation}
\left( \frac{k!}{2^kk_0!} \right)^{\alpha/E} \leq I_k \leq k!^{\alpha/E}
\end{equation}
when $k$ is large. And recall that $\mu$ is the continuous measure on $S$ defined by
\begin{equation}
\mu(D_{k,i}) = I_k^{-1} \quad \text{for each } l \text{ and } k \geq k_0.
\end{equation}
The smallest balls that carry a \( \mu \)-measure \( I_{k-1}^{-1} \) have radii proportional to \( \delta_k N_k r_k \). Using (3.5) and (3.6) one may check that for each \( \eta < \alpha \), \( \mu(D(x, r)) \leq C_n r^n \) for all \( x \in \mathbb{R}^n \) and \( r > 0 \). Hence \( S \) has positive \( \eta \)-dimensional measure for all \( \eta < \alpha \). In view of (3.4), \( S \) has Hausdorff dimension \( \alpha \).

For \( 0 < t < r_{k_0} \), let \( K = K(t) \) be the largest integer so that \( r_K \geq t \). We deduce from (2.10), (2.11), and (3.3) that, for \( 0 < \rho < r_{k_0} \),

\[
\begin{align*}
    u(\rho) \geq & \int_\rho^{r_{k_0}} \exp \left\{ -\sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} B - \left(n - 2 + \frac{n - 1}{B}\right) \left(1 - 1200B\delta_k\right) \frac{ds}{s} \right\} dt.
\end{align*}
\]

And note from the choices of \( A \) and \( \delta_k \) that

\[
\begin{align*}
    \sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} B - \left(n - 2 + \frac{n - 1}{B}\right) \left(1 - 1200B\delta_k\right) \frac{ds}{s} & \leq \sum_{k=k_0}^{K} \left[ \left(B - n + 2 - \frac{n - 1}{B}\right) + 1200B\delta_k(n - 1) \right] \frac{A}{E} \log k \\
    & \leq \sum_{k=k_0}^{K} \frac{(B - \alpha' - 1)}{E} \log k + C \frac{\log k}{k} \\
    & = C + (B - \alpha' - 1) \log \frac{1}{t} + C \left(\log \log \frac{1}{t}\right)^2.
\end{align*}
\]

Therefore, for \( 0 < \rho < r_{k_0} \),

\[
\begin{align*}
    u(\rho) \geq & \int_\rho^{r_{k_0}} \exp \left\{ -C + (\alpha' + 1) \log \frac{1}{t} - C \left(\log \log \frac{1}{t}\right)^2 \right\} dt \geq C \rho^{-y}
\end{align*}
\]

for some \( \gamma \) satisfying \( \alpha < \gamma < \alpha' \). From the property (2.5) of the transformation \( T \), it follows that

\[
(3.8) \quad H_{x_0}(x) \geq C|x - x_0|^{-\gamma} \quad \text{when } |x| < r_{k_0}.
\]

Let \( v(x) = \int_S H_T(x) d\mu(z) \), where \( \mu \) is the measure defined in (3.7). Clearly \( v \) is an \( L \)-supersolution on \( D \setminus S \). In view of (3.4), (3.6), and (3.8), \( v \) approaches \( +\infty \) as \( x \to x_0 \) for every \( x_0 \in S \). Since \( f(r) \) is bounded, \( v < +\infty \) on \( D \setminus S \).

Clearly (0.2) holds with \( \Lambda_{n, \alpha} = B/(n - 1) \).

The number \( B \) was chosen so that among other properties, (3.1) holds. As a consequence, (0.3) and (0.4) follow.

4. Proof of Theorem 2 \( (n > 3) \)

Let

\[
(4.1) \quad \delta_k = \frac{16\sqrt{n}}{k^{3/2}}, \quad N_k = k^{2n-5/2}
\]

and

\[
r_k = (k!)^{-2n+(2+\alpha)/(n-2)}/k
\]
for $k \geq 1$. Choose an integer $k_0 \geq 10^5(1 + a)n$, so that $N_k \leq r_{k-1}/r_k$ for $k \geq k_0$. It is easy to check that

\begin{align}
\lim_{k \to \infty} (k!)^{2n(n-2)}r_k^{\gamma} = 0 & \quad \text{if } \gamma > n - 2, \\
\lim_{k \to \infty} (k - 1)!^{-2n(n-2)/(\delta_k N_k r_k)^{n-2}} \left( \log \frac{1}{\delta_k N_k r_k} \right)^{-1-a} = 0, \\
\lim_{k \to \infty} (k - 1)!^{-2n(n-2)/(\delta_k N_k r_k)^{n-2}} \left( \log \frac{1}{\delta_k N_k r_k} \right)^{-3-a} = \infty.
\end{align}

There are $(k!/k_0!)^{2n(n-2)}$ balls in $\{D_{k,i}\}$, and that

\begin{equation}
\mu(D_{k,i}) = (k_0!/k!)^{2n(n-2)}
\end{equation}

for $k \geq k_0$. In view of (4.2) and (4.3), $S$ has Hausdorff dimension $n - 2$ and positive $h$-measure, where $h(r) = r^{n-2}(\log \frac{1}{r})^{-1-a}$.

Let

\begin{equation}
\beta = 12n(1 + a),
\end{equation}

and $B(r)$ be the function described in §2, with

\begin{equation}
B(r) = n - 1 + \frac{\beta}{\log \frac{1}{r}} \quad \text{on } \bigcup_{k \geq k_0} [N_{k+1}r_{k+1}, r_k].
\end{equation}

Clearly $n - 1 \leq B(r) \leq n - 1 + \beta/k_0 \equiv B^*$, and $\delta_k < (2400B^*)^{-1}$ for $k \geq k_0$.

Stirling's formula shows that

\begin{equation}
\left( \log \frac{1}{r_{k-1}} \right)^{-1} \leq \frac{1}{2nk \log k} \left( 1 + \frac{C}{\log k} \right),
\end{equation}

when $k$ is sufficiently large. Since $(\log \frac{1}{r})^{-1}$ is an increasing function of $s$ ($0 < s < 1$), and $r_{k-1} \geq N_k r_k$, we obtain from (4.7) that

\begin{equation}
1 - \frac{n - 1}{B_k} = \frac{B_k - (n - 1)}{B_k} \leq \frac{\beta}{(n - 1) \log 1/r_{k-1}} \leq \frac{\beta}{2n(n-1)k \log k} \left( 1 + \frac{C}{\log k} \right)
\end{equation}

for large $k$, here $B_k$ is the number defined in (2.3).

Thus, for $0 < \rho < r_{k_0}$,

\begin{equation}
u(\rho) \geq \int_{\rho}^{r_{k_0}} \exp \left\{ \int_{t}^{r_{k_0}} \frac{n - 1}{S} + \frac{\beta}{s \log \frac{1}{s}} \, ds - \sum_{k = k_0}^{K} \int_{r_k}^{N_k r_k} \frac{\beta}{s \log \frac{1}{s}} \, ds \right. \\
- \sum_{k = k_0}^{K} \int_{r_k}^{N_k r_k} (n - 1) - \left( n - 2 + \frac{n - 1}{B_k} \right) (1 - 1200B^* \delta_k) \frac{ds}{s} \right\} \, dt,
\end{equation}

where $K = k(t)$ is the largest integer satisfying $r_K \geq t$. We deduce from (4.8)
that

\[
\sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} (n-1) - \left( n-2 + \frac{n-1}{B_k} \right) \left( 1 - 1200B^* \delta_k \right) \frac{ds}{s} \leq \sum_{k=k_0}^{K} \left[ \frac{\beta}{2n(n-1)k \log k} \left( 1 + \frac{C}{\log k} \right) + Ck^{-3/2} \right] \log N_k
\]

(4.10)

\[
\leq C + \frac{\beta(2n-\frac{5}{2})}{2n(n-1)} \log K + C \log \log K
\]

\[
\leq C + \frac{\beta(2n-\frac{5}{2})}{2n(n-1)} \log \log \frac{1}{t} + C \log \log \log \frac{1}{t}.
\]

Again, from (4.7) and monotonicity of \((\log \frac{1}{s})^{-1}\), it follows that

\[
\sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} \frac{\beta}{s \log \frac{1}{s}} \frac{ds}{s} \leq \frac{\beta(2n-\frac{5}{2})}{2n} \sum_{k=k_0}^{K} \left( 1 + \frac{C}{\log k} \right) \frac{1}{k}
\]

(4.11)

\[
\leq C + \frac{\beta(2n-\frac{5}{2})}{2n} \log \log \frac{1}{t} + C \log \log \log \frac{1}{t}.
\]

We conclude from (4.6), (4.9), (4.10) and (4.11) that

\[
u(p) \geq \int_{\rho}^{r_k} \exp \left\{ -C + (n-1) \log \frac{1}{t} + \right. \]

\[
+ \beta \left[ 1 - \frac{2n-\frac{5}{2}}{2n} \left( 1 + \frac{1}{n-1} \right) \right] \log \log \frac{1}{t}
\]

(4.12)

\[
\left. -C \log \log \log \frac{1}{t} \right\} dt
\]

\[
\geq C \int_{\rho}^{r_k} t^{-n+1} \left( \log \frac{1}{t} \right)^{\beta/4(n-1)} \left( \log \log \frac{1}{t} \right)^{-C} dt
\]

\[
\geq C \rho^{-n+2} \left( \log \frac{1}{\rho} \right)^{3+a}
\]

for \(0 < \rho < r_k\). Therefore,

(4.13) \(H_{x_0}(x) \geq C |x - x_0|^{-n+2} \left( \log \frac{1}{|x - x_0|} \right)^{3+a}\) for \(|x| < r_k\).

The relation

\[
\frac{2n-\frac{5}{2}}{2n} \left( 1 + \frac{1}{n-1} \right) < 1
\]

used in (4.12) is prepared in the choices of \(r_k\) and \(N_k\).

The ellipticity of \(a_{ij}\), (0.5), follows from the choice of \(B(r)\); and the continuity of \(a_{ij}\) in \(\mathbb{R}^n\) follows from \(\lim_{r \to 0} B(r) = n - 1\). Recall that \(LH_{x_0} \leq 0\) on \(D \setminus \{x_0\}\); it follows from the maximum principle and the solvability of the Dirichlet problem for operators with continuous coefficients [5, pp. 220 and 252] and \(H_{x_0}\) is \(L\)-supersolution in \(D\).

Again, because \(a_{ij}\) are continuous, Green functions \(G(x, x_0)\) exist in \(D\) (see Bauman [3, 4]). In fact, for each \(x_0 \in D\), \(G(\cdot, x_0)\) is a positive \(L\)-solution.
in $D\setminus\{x_0\}$ with boundary value vanishing continuously on $|x| = 1$. Let $\bar{x} = (\frac{1}{4}, 0, 0, 0, \ldots 0)$ and assume that $G$ is normalized so that $G(\bar{x}, x_0) = 1$. We claim that for each $x_0 \in S$,

$$G(x_0) \geq C|x-x_0|^{-n+2} \left(\log \frac{1}{|x-x_0|}\right)^{3+a}$$

whenever $0 < |x-x_0| < r_k$.

Let $g(r) = \sup\{G(x_0): |x-x_0| = r\}$ for $0 < r < r_k$. Applying (4.13) and the maximum principle to the region $D\{|x-x_0| \leq r\}$, we obtain

$$1 = G(\bar{x}, x_0) \leq Cg(r)(\log \frac{1}{r})^{3+a}H_{x_0}(\bar{x}).$$

Because $g(r)$ is bounded and $|x_0 - \bar{x}| > \frac{1}{4}$, $H_{x_0}(\bar{x}) < C < \infty$ for all $x_0 \in S$. Hence $g(r) \geq Cr^{-n+2}(\log \frac{1}{r})^{3+a}$ for $0 < r < r_k$. Thus (4.14) follows from the Harnack principle.

In view of (4.14), the maximum principle and the solvability of the Dirichlet problem, $G(\cdot, x_0)$ is actually an $L$-supersolution on $D$. The function $v(x) = \int_S G(x, z) d\mu(z)$ approaches $+\infty$ on $S$ in view of (4.4) and (4.14), and it is the function desired.

5. Proof of Theorem 2 ($n = 2$)

Let $\delta_k \equiv \delta \equiv [50000(1 + a)^2]^{-1}$, $N_k \equiv N \equiv 1600000(1 + a)^2$, $r_k \equiv e^{-4k/(1+a)}/32\sqrt{2}$ for $k \geq 1$. Choose integers $k_0 \geq 1 + a$ so that

$$r_{k-1}/r_k \geq N^4 \quad \text{when} \quad k \geq k_0.$$ 

Note that

$$4^{-k} \left(\frac{1}{N\delta r_k}\right)^{1+a} = 1.$$ 

We note that there are $4^{k-k_0}$ disks in $\{D_{k,1}\}$, and that $\mu(D_{k,1}) = 4^{-k+k_0}$. In view of (5.2), $S$ has positive finite $h$-measure for $h(r) = (\log \frac{1}{r})^{-1-a}$.

Choose

$$B(r) = 1 + 20(1 + a)^2 \left(\log \frac{1}{r}\right)^{-1} \quad \text{on} \quad \bigcup_{k \geq k_0} [N_{k+1}r_{k+1}, r_k].$$

Clearly $1 \leq B(r) \leq 6(1 + a)^2$. Let $f(r)$ be the function in §2, satisfying all the properties there except (2.10); instead, let $f(r) \equiv 0$ on $\bigcup [\frac{2}{3}r_k, \frac{3}{4}Nkr_k]$. The fact that $Mu \leq 0$ in $D\setminus\{0\}$ is not affected by the change of $f(r)$ due to the estimate (2.14).

For $0 < \rho < r_k$,

$$u(\rho) \geq \int_\rho^{r_k} \exp \left\{ \int_t^{r_k} \frac{1}{s} + \frac{20(1 + a)^2}{s \log \frac{1}{s}} ds \right\} dt,$$

$$- \sum_{k=k_0}^K \int_{r_k}^{N_kr_k} \frac{1}{s} + \frac{20(1 + a)^2}{s \log \frac{1}{s}} ds \right\} dt,$$
where $K = K(t)$ is the largest integer so that $r_K \geq t$. We deduce from (5.1) that
$$
\sum_{k=k_0}^{K} \int_{r_k}^{r_{k+1}} \frac{1}{s \log \frac{1}{s}} \, ds \leq C + \frac{1}{4} \int_t^{r_0} \frac{1}{s \log \frac{1}{s}} \, ds \leq C + \frac{1}{4} \log \log \frac{1}{t},
$$
and that
$$
\sum_{k=k_0}^{K} \int_{r_k}^{r_{k+1}} \frac{1}{s \log \frac{1}{s}} \, ds = (K - K_0) \log N \leq C + 14(1 + a)^2 \log \log \frac{1}{t}.
$$
Combining the above estimates, we obtain
$$
u(\rho) \geq \int_0^{\rho} \exp \left\{ C + \log \frac{1}{\tau} + (1 + a)^2 \log \log \frac{1}{\tau} \right\} \, d\tau \geq C \left( \log \frac{1}{\rho} \right)^{1+(1+a)^2}
$$
for $0 < \rho < r_{k_0}$.

In view of (5.3), $a_{ij}$ are continuous in $D$. Thus the normalized Green function exists on $D$ and satisfies
$$
G(x, x_0) \geq C \left( \log \frac{1}{|x - x_0|} \right)^{1+(1+a)^2} \text{ for } |x - x_0| < r_{k_0}.
$$
The function $v(x) = \int_S G(x, y) \, d\mu(y)$ has all the properties in the theorem.

### 6. Proofs of Theorems 3 and 4

We follow the constructions in §2 and indicate the necessary changes.

Given $B^* = n - 1$, $k_0$, $\{\delta_k\}$, $\{r_k\}$ and $\{N_k\}$, let $S$ be the Cantor set and $\mu$ be the measure on $S$ defined in §2.

Let $B(r)$ be a new function, smooth on $r > 0$, with values $\frac{1}{2} < B(r) \leq n - 1$, satisfying (2.1),
$$
B(r) < n - 1 \text{ on } \bigcup_{k \geq k_0} [N_{k+1}r_{k+1}, r_k],
$$
and monotone in each of the remaining intervals. Define an operator $L$ associated with this $B(r)$ as in (2.2). Let
$$\beta_k = \inf\{B(r): 0 < r \leq N_k r_k\},$$
then
$$
\beta_k \left( \frac{1}{n - 1} |\xi|^2 \right) \leq \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leq |\xi|^2 \text{ on } R_{k,l}.
$$
Fix $x_0 \in S$, let $y = Tx$ be the diffeomorphism and $M$ be the operator defined before. Clearly (2.5) ~ (2.7) are retained; and (2.8) and (2.9) can be replaced respectively by
$$
\sum_{i,j} b_{ij}(Tx) \xi_i \xi_j - \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leq 128 \delta_k |\xi|^2 \text{ on } S_{k,l},
$$
and
$$
\sum_{i,j} b_{ii}(Tx) \xi_i \xi_j - \sum_{i,j} a_{ii}(x) \xi_i \xi_j \leq 128 \delta_k \text{ on } S_{k,l}.
$$
Suppose that $F$ is a smooth function on $r > 0$, with values $F(r) \geq B(r)$, $F(r) \equiv n - 1$ for $r \geq \frac{1}{2} r_{k_0}$, $F(r) = B(r)$ on $\bigcup_{k \geq k_0} [N_{k+1}r_{k+1}, r_k]$, $F(r) = \left( n - 2 + \frac{n - 1}{\beta_k} \right) \left( 1 + 5000 \delta_k \right) \text{ on } [\frac{1}{4} r_k, \frac{1}{4} N_k r_k]$. 

for each \( k \geq 1 + k_0 \), and that \( F \) is monotone in each of the remaining intervals. Define for \( \rho = |y| < 1 \),

\[
U(y) = U(\rho) = \int_{\rho}^{1} \exp \left( \int_{t}^{1} \frac{F(s)}{s} \, ds \, dt \right).
\]

Arguing as in §2, we conclude that for \( x \in S_{k,1} \), and \( y = Tx \),

\[
\sum b_{ii} + \sum b_{ij} y_j - 1 \leq f(|y|).
\]

From this, we may deduce that \( MU(y) > 0 \) on \( \{|y| < 1\} \setminus \{0\} \). Thus, \( Q_{x_0}(x) \equiv U(Tx) \) on \( D \setminus \{x_0\} \) is an \( L \)-subsolution in \( D \setminus S \).

To complete the proof of Theorem 3, we let \( \delta_k \) and \( N_k \) be the numbers defined in (4.1), let \( \tau > a/(n-2) \) and \( r_k = (k!)^{-2n-\tau/k} \). Fix an integer \( k_0 \geq 20(n^2 + \tau^2) \), so that \( N_k \leq r_{k-1}/r_k \) and \( \delta_k \leq (2400n)^{-1} \) for \( k \geq k_0 \). It is ready to check that

\[
\lim_{k \to \infty} (k!)^{2n(n-2)r^{-2}} \left( \log \frac{1}{r_k} \right)^a = 0,
\]

\[
\lim_{k \to \infty} ((k-1)!)^{-2n(n-2)}(\delta_k N_k r_k)^{-\eta} = 0 \quad \text{if} \ \eta < n-2,
\]

and

\[
\sum_{k \geq k_0} ((k-1)!)^{-2n(n-2)} r_k^{-n+2} \left( \log \frac{1}{r_k} \right)^{-2n(n+\tau)} < \infty.
\]

There are \( (k!/k_0!)^{2n(n-2)} \) balls in \( \{D_{k,i}\}_l \) for each \( k \geq k_0 \), and \( \mu(D_{k,i}) = (k_0!/k!)^{2n(n-2)} \). From (6.2) and (6.3) it follows that \( S \) has Hausdorff dimension \( n-2 \), and zero \( h \)-measure for \( h(r) = r^{n-2}(\log \frac{1}{r})^a \).

Let

\[
\beta = 16n^2(n+\tau),
\]

and

\[
B(r) = n-1 - \frac{\beta}{\log \frac{1}{r}} \quad \text{on} \quad \bigcup_{k \geq k_0} [N_k r_{k+1}, r_k].
\]

Thus, for \( 0 < \rho < r_{k_0} \),

\[
U(\rho) \leq \int_{\rho}^{1} \exp \left\{ \int_{t}^{1} \frac{n-1}{s} - \frac{\beta}{s \log \frac{1}{s}} \, ds + C + \sum_{k=k_0}^{K} \frac{\beta}{s \log \frac{1}{s}} \, ds \right\} \left[ -(n-1) + \left( n-2 + \frac{n-1}{\beta_k} \right) \left( 1 + 5000 \delta_k \right) \right] \frac{ds}{s} \right\} dt,
\]

where \( K = K(t) \) is the largest integer satisfying \( r_K \geq t \). Note that for large \( k \), inequality (4.7) still holds and

\[
\frac{n-1}{\beta_k} - 1 \leq \frac{9\beta}{16n(n-1)k \log k}.
\]

Thus, \( U(\rho) \leq C \rho^{-n+2}(\log \frac{1}{\rho})^{-\beta/8n} \).
Let $G(\cdot, x_0)$ be the normalized Green function on $D$ with
\[ G((\frac{1}{2}, 0, 0, \ldots), x_0) = 1. \]

Arguing as in §4, we obtain
\[
G(x, x_0) \leq C |x - x_0|^{-n + 2\left(\log \frac{1}{|x - x_0|}\right)^{-\beta/\delta n}}
\]
for all $x_0 \in S$ and $|x - x_0| < r_0$. We may also prove that
\[ G(x, x_0) \to +\infty \quad \text{as} \quad x \to x_0 \]
for each $x_0 \in S$, by constructing a positive $L$-supersolution approaching $+\infty$ at $x_0$. Thus $G(\cdot, x_0)$ is an $L$-supersolution on $D$. In view of (6.4), (6.5), and (6.6),
\[ w(x) = \int_S G(x, y) d\mu(y) \]
has all the properties stated in Theorem 3.

To prove Theorem 4, we need only to verify that the coefficients of $L$ can be chosen so that (0.6) and (0.7) are fulfilled. Let
\[
\beta = \begin{cases} 
1 - 2\alpha, & \text{if } 0 < \alpha < \frac{1}{4}, \\
(n-1) \left(1 - \frac{n-2-\alpha}{n-\frac{3}{2}-\alpha}\right), & \text{if } n - \frac{9}{4} < \alpha < n - 2,
\end{cases}
\]
and note that $\frac{1}{2} < b < 1 + \alpha < n - 1$ and that
\[ \frac{\alpha}{n} < \frac{1 + \alpha - b}{n - 2 + (n-1)/b - b} < 1. \]

Choose $\alpha' < \alpha$ so that
\[ \frac{\alpha}{n} < \frac{1 + \alpha' - b}{n - 2 + (n-1)/b - b} < 1, \]
and denote by
\[ A = \frac{1 + \alpha' - b}{n - 2 + (n-1)/b - b}, \quad E = A - \frac{\alpha}{n}. \]

Let $\delta_k$, $N_k$ and $r_k$ be defined according to (3.2), associated with the current choices of $A$ and $E$; and let the function $B(r)$ in (6.1) be chosen so that
\[ B(r) \equiv b \quad \text{on} \quad \bigcup_{k \geq k_0} [N_k r_k + r_k, r_k]. \]

It is ready to check that $S$ has dimension $\alpha$ and that the $L$-subsolution $Q_{x_0}(x)$ satisfies
\[ Q_{x_0}(x) \leq C |x - x_0|^{-\gamma}, \quad \text{when} \quad |x - x_0| < r_0 \]
for some $\gamma$ with $\alpha' < \gamma < \alpha$. The rest of the proof is routine and follows from the observation
\[ \sum_{k \geq k_0} (k - 1)!^{-\alpha/E} (\delta_k N_k r_k)^{-\gamma} < \infty \quad \text{if} \quad \gamma < \alpha. \]
References


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