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NONCOMMUTATIVE MATRIX JORDAN ALGEBRAS

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ABSTRACT. We consider noncommutative degree two Jordan algebras $\mathcal{J}$ of two by two matrices whose off diagonal entries are from an anticommutative algebra $\mathcal{S}$. We give generators and relations for the automorphism group of $\mathcal{J}$ and determine the derivation algebra $\operatorname{Der}\mathcal{J}$ in terms of mappings on $\mathcal{S}$. We also give an explicit construction of all $\mathcal{S}$ for which $\operatorname{Der}\mathcal{J}$ does not kill the diagonal idempotents and give conditions for nonisomorphic $\mathcal{S}$'s to give isomorphic $\mathcal{J}$'s.

1. INTRODUCTION

All of the algebras and vector spaces under discussion will be over a field k, $\operatorname{char} k \neq 2$. Note, however, that we are not assuming finite dimensionality and we are not assuming that the algebras are associative.

Suppose $\mathcal{S}$ is an anticommutative algebra having a nondegenerate symmetric bilinear form B which is associative, that is,

$$B(\alpha\beta, \gamma) = B(\alpha, \beta\gamma) \tag{1.1}$$

for all $\alpha, \beta, \gamma \in \mathcal{S}$. Suppose further that $s, t \in k$ with $st \neq 0$. We define a new algebra $\mathcal{J} = \mathcal{J}(\mathcal{S}, B, s, t)$ as

$$\mathcal{J} := \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} : a, b \in k, \alpha, \beta \in \mathcal{S} \right\}$$

with addition and scalar multiplication defined componentwise and multiplication defined by

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} := \begin{pmatrix} ac + B(\alpha, \delta) & a\gamma + d\alpha + t\beta\delta \\ c\beta + b\delta + s\alpha\gamma & B(\beta, \gamma) + bd \end{pmatrix}. \tag{1.2}$$

It is easy to check that with these operations $\mathcal{J}$ is a noncommutative Jordan algebra, that is, for all $x, y \in \mathcal{J}$

$$x(yx) = (xy)x, \tag{1.3}$$

$$x^2(yx) = (x^2y)x. \tag{1.4}$$

These algebras were introduced in [4] as a generalization of the Zorn matrix construction of the octonions O.

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Example 1.5. Take $\mathcal{S} = k^3$, three-dimensional vectors over k, with the vector cross product as the algebra multiplication. Define B on $\mathcal{S}$ by $B(\alpha, \beta) = -\alpha \cdot \beta$, where $\alpha \cdot \beta$ denotes the dot product of α and β. Then $\mathcal{J} = \mathcal{J}(\mathcal{S}, B, 1, 1)$ is the Zorn matrix construction of the split octonions O (see [2]).

These generalizations of O were futher studied in [5]. Note that the Zorn matrix realization of O in Example 1.5 makes the computation of G_2 as $\operatorname{Der} O$ easy (see [2]), where the *derivation algebra* $\operatorname{Der}\mathcal{J}$ of $\mathcal{J}$ is the Lie algebra of all $D \in \operatorname{End}_k \mathcal{J}$ satisfying

$$(xy)D = (xD)y + x(yD) \tag{1.6}$$

for all $x, y \in \mathcal{J}$. Our goal in this paper is to compute $\operatorname{Der}\mathcal{J}$ for all $\mathcal{J}$ whose product is given by (1.2) and to also compute the *automorphism group*

$$\operatorname{Aut}\mathcal{J} := \{A \in GL(\mathcal{J}) \mid (xy)A = (xA)(yA) \text{ for all } x, y \in \mathcal{J}\}$$

of $\mathcal{J}$.

The assumption that B is nondegenerate is motivated by the following result from [4]:

Theorem 1.7. *$\mathcal{J}$ is simple if and only if B is nondegenerate.*

Since the nondegeneracy of B will be preserved by field extensions, we get the following corollary:

Corollary 1.8. *$\mathcal{J}$ is central simple.*

If $x = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}$, then x satisfies the quadratic equation

$$x^2 - (a+b)x + (ab - B(\alpha, \beta))I = 0$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $\mathcal{J}$ is a degree two algebra (see [3] for the definition). We also note that if $y = \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix}$, the bilinear form $C(\ ,\ )$ on $\mathcal{J}$ defined by

$$C(x, y) := \operatorname{trace}(xy) = ac + bd + B(\alpha, \delta) + B(\beta, \gamma) \tag{1.9}$$

is nondegenerate, symmetric, and associative. Hence if we let $\mathcal{S}$ be $\mathcal{J}$ with the product $[x, y] := xy - yx$ we can construct a new Jordan algebra $\mathcal{J}' = \mathcal{J}(\mathcal{S}, C, s', t')$.

The organization of the paper is as follows: In §2 we give a basic decomposition of $\operatorname{Der}\mathcal{J}$ in terms of

$$\mathcal{G} = \left\{ D \in \operatorname{Der}\mathcal{J} \,\middle|\, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

(Theorem 2.9) and characterize $\mathcal{G}$ in terms of mappings on $\mathcal{S}$ having certain properties ((2.1), (2.2), and Lemma 2.4). In §3 we show that the codimension of $\mathcal{G}$ in $\operatorname{Der}\mathcal{J}$ is 0, 2, or 6 (Theorem 3.8) and in fact, if $\dim \mathcal{S} \geq 4$, this codimension, codim $\mathcal{G}$, is either 0 or 2 (Theorem 3.4). We give instances in Example 3.6 of algebras $\mathcal{S}$ for which $\dim \mathcal{S} \geq 4$ and $\operatorname{codim}\mathcal{G} = 2$. In §4 we show that if $\dim\mathcal{S} \geq 4$ and $\operatorname{codim}\mathcal{G} = 2$, then $\mathcal{S}$ is one of the algebras constructed in Example 3.6 (Theorem 4.2) and we compute $\operatorname{Der}\mathcal{J}$ for these algebras (Corollary 4.6). In §5 we give generators (Theorem 5.7 and Corollary 5.8) and relations (Proposition 5.9 and Proposition 5.13) for $\operatorname{Aut}\mathcal{J}$. We explicitly calculate $\operatorname{Aut}\mathcal{J}$ when $\mathcal{S}$ is one of the algebras constructed in

Example 3.6 (Theorem 5.12 and Proposition 5.13). §6 concerns conditions for two of these matrix Jordan algebras to be isomorphic and §7 concerns results on $\operatorname{Der}\mathscr{J}$ and $\operatorname{Aut}\mathscr{J}$ when $s=0$ or $t=0$.

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2. Derivations

Let $\mathscr{G}$ be the set of $(D_1, D_2) \in \operatorname{End}_k\mathscr{S} \oplus \operatorname{End}_k\mathscr{S}$ satisfying

$$B(\gamma D_i, \delta) = -B(\gamma, \delta D_j), \tag{2.1}$$

$$(\gamma\delta)D_i = (\gamma D_j)\delta + \gamma(\delta D_j) \tag{2.2}$$

for all $\gamma, \delta \in \mathscr{S}$, where $i, j = 1, 2$ and $i \neq j$. $\mathscr{G}$ is a Lie algebra with the Lie bracket given by $[(D_1, D_2), (E_1, E_2)] = ([D_1, E_1], [D_2, E_2])$ and has a faithful representation on $\mathscr{J} = \mathscr{J}(\mathscr{S}, B, s, t)$ defined by

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}(D_1, D_2) := \begin{pmatrix} 0 & \alpha D_1 \\ \beta D_2 & 0 \end{pmatrix} \tag{2.3}$$

for all $\left(\begin{smallmatrix} a & \alpha \\ \beta & b \end{smallmatrix}\right) \in \mathscr{J}$. The following lemma is a straightforward verification that (D_1, D_2) acts as a derivation on $\mathscr{J}$.

Lemma 2.4. *$\mathscr{G} \subseteq \operatorname{Der}\mathscr{J}$ where the action of $\mathscr{G}$ on $\mathscr{J}$ is given by* (2.3).

For $\mathscr{S}$ and B as in Example 1.5 $sl(k, 3) \cong \mathscr{G}$ [2], where $A \mapsto (A, -A^t)$, so $\dim\mathscr{G} = 8$. Since $\dim\operatorname{Der} O = 14$, clearly there are derivations of O not in $\mathscr{G}$, i.e., $\mathscr{G} \neq \operatorname{Der}\mathscr{J}$. For $a \in k$ let V_a be the linear subspace of all $\alpha \in \mathscr{S}$ such that for all $\gamma, \delta \in \mathscr{S}$

$$a\alpha(\gamma\delta) = B(\alpha, \gamma)\delta - B(\alpha, \delta)\gamma. \tag{2.5}$$

Lemma 2.6. *$\alpha \in V_a$ if and only if for all $\gamma, \delta \in \mathscr{S}$*

$$a(\alpha\gamma)\delta = B(\gamma, \delta)\alpha - B(\alpha, \delta)\gamma. \tag{2.6.1}$$

Proof. Suppose α satisfies (2.6.1) for all $\gamma, \delta \in \mathscr{S}$. Then for all $\gamma, \delta, \varepsilon \in \mathscr{S}$

$$\begin{aligned} -aB(\gamma, \alpha(\delta\varepsilon)) &= -aB(\gamma\alpha, \delta\varepsilon) = aB(\alpha\gamma, \delta\varepsilon) = aB((\alpha\gamma)\delta, \varepsilon) \\ &= B(\gamma, \delta)B(\alpha, \varepsilon) - B(\alpha, \delta)B(\gamma, \varepsilon) \\ &= -B(\gamma, B(\alpha, \delta)\varepsilon - B(\alpha, \varepsilon)\delta) \end{aligned}$$

since B is associative and $\mathscr{S}$ is anticommutative. Hence, by the nondegeneracy of B, $a\alpha(\delta\varepsilon) = B(\alpha, \delta)\varepsilon - B(\alpha, \varepsilon)\delta$ which is (2.5). The converse is similar.

Another straightforward verification using Lemma 2.6 gives the following result.

Lemma 2.7. *Suppose $\alpha, \beta \in V_{st}$. Define $E_{\alpha,\beta} \in \operatorname{End}\mathscr{J}$ by*

$$\begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} E_{\alpha,\beta} := \begin{pmatrix} -B(\beta, \gamma) - B(\alpha, \delta) & (c-d)\alpha - t\beta\delta \\ (c-d)\beta + s\alpha\gamma & B(\beta, \gamma) + B(\alpha, \delta) \end{pmatrix}. \tag{2.7.1}$$

Then $E_{\alpha,\beta} \in \operatorname{Der}\mathscr{J}$.

Clearly $E_{\alpha,\beta} \notin \mathscr{G}$ if $\alpha \neq 0$ or $\beta \neq 0$ since

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} E_{\alpha,\beta} = 2\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}. \tag{2.8}$$

Also by (2.8) $\dim \mathscr{H} = 2\dim V_{st}$ for $\mathscr{H} := \{E_{\alpha,\beta} \mid \alpha, \beta \in V_{st}\}$. For $\mathscr{S}$ and B as in Example 1.5, $V_1 = \mathscr{S}$, giving $\dim \mathscr{H} = 6$ and so $\operatorname{Der} O = \mathscr{G} \oplus \mathscr{H}$ as a vector space. Note that $\mathscr{H}$ is not a subalgebra in this case. The same decomposition of $\operatorname{Der} \mathscr{J}$ holds in the general case:

Theorem 2.9. $\operatorname{Der} \mathscr{J} = \mathscr{G} \oplus \mathscr{H}$ *as a vector space, where* $\mathscr{H} := \{E_{\alpha,\beta} \mid \alpha, \beta \in V_{st}\}$.

Proof. Lemmas 2.4 and 2.7 establish $\mathscr{H} \oplus \mathscr{H} \subseteq \operatorname{Der} \mathscr{J}$. To finish the proof we must show that if $D \in \operatorname{Der} \mathscr{J}$, then $D \in \mathscr{G} \oplus \mathscr{H}$. Therefore, suppose $D \in \operatorname{Der} \mathscr{J}$. $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) D = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the identity of $\mathscr{J}$. Since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, if $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) D = \left(\begin{smallmatrix} a & \alpha \\ \beta & b \end{smallmatrix}\right)$, then the derivation condition (1.6) gives

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix},$$

so $2a = 0$ and $-2b = 0$. Hence

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \tag{2.9.1}$$

for some $\alpha, \beta \in \mathscr{S}$. Now define mappings $D_{11}, D_{22}, E_{11}, E_{22}$ from $\mathscr{S}$ to k and mappings $D_{12}, D_{21}, E_{12}, E_{21}$ from $\mathscr{S}$ to $\mathscr{S}$ by

$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} D = \begin{pmatrix} \gamma D_{11} & \gamma D_{12} \\ \gamma D_{21} & \gamma D_{22} \end{pmatrix} \tag{2.9.2}$$

and

$$\begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} D = \begin{pmatrix} \delta E_{11} & \delta E_{12} \\ \delta E_{21} & \delta E_{22} \end{pmatrix} \tag{2.9.3}$$

for all $\gamma, \delta \in \mathscr{S}$. We will use the derivation condition (1.6) to show $\alpha, \beta \in V_{st}$, $(D_{12}, E_{21}) \in \mathscr{G}$, and $D = (D_{12}, E_{21}) + \frac{1}{2}E_{\alpha,\beta}$, thus showing $D \in \mathscr{G} \oplus \mathscr{H}$.

Using (1.6) and definitions (2.9.2) and (2.9.3) on $\left(\begin{smallmatrix} 0 & 0 \\ s\gamma\delta & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & \delta \\ 0 & 0 \end{smallmatrix}\right)$ yields for all $\gamma, \delta \in \mathscr{S}$

$$s(\gamma\delta)E_{11} = B(\gamma, \delta D_{21}), \tag{2.9.4}$$

$$s(\gamma\delta)E_{22} = B(\gamma D_{21}, \delta), \tag{2.9.5}$$

$$s(\gamma\delta)E_{12} = (\gamma D_{11})\delta + (\delta D_{22})\gamma, \tag{2.9.6}$$

$$s(\gamma\delta)E_{21} = s(\gamma D_{12})\delta + s\gamma(\delta D_{12}). \tag{2.9.7}$$

A similar computation using $\left(\begin{smallmatrix} 0 & t\gamma\delta \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ \gamma & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)$ yields for all $\gamma, \delta \in \mathscr{S}$

$$t(\gamma\delta)D_{11} = B(\gamma E_{12}, \delta), \tag{2.9.8}$$

$$t(\gamma\delta)D_{22} = B(\gamma, \delta E_{12}), \tag{2.9.9}$$

$$t(\gamma\delta)D_{21} = (\gamma E_{22})\delta + (\delta E_{11})\gamma, \tag{2.9.10}$$

$$t(\gamma\delta)D_{12} = t(\gamma E_{21})\delta + t\gamma(\delta E_{21}). \tag{2.9.11}$$

Now since $\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$, we see by (1.6) that

(2.9.12) $$\gamma D_{21} = \tfrac{1}{2} s\alpha\gamma,$$

(2.9.13) $$\gamma D_{22} = \tfrac{1}{2} B(\beta, \gamma),$$

and since $\begin{pmatrix} 0 & -\gamma \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we see that

(2.9.14) $$\gamma D_{11} = -\tfrac{1}{2} B(\beta, \gamma).$$

Similarly $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\gamma & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$ give

(2.9.15) $$\gamma E_{11} = -\tfrac{1}{2} B(\alpha, \gamma),$$

(2.9.16) $$\gamma E_{12} = -\tfrac{1}{2} t\beta\gamma,$$

(2.9.17) $$\gamma E_{22} = \tfrac{1}{2} B(\alpha, \gamma).$$

Hence

$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} D = \begin{pmatrix} -\frac{1}{2}B(\beta, \gamma) & \gamma D_{12} \\ \frac{1}{2}s\alpha\gamma & \frac{1}{2}B(\beta, \gamma) \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} D = \begin{pmatrix} -\frac{1}{2}B(\alpha, \delta) & -\frac{1}{2}t\beta\delta \\ \delta E_{21} & \frac{1}{2}B(\alpha, \delta) \end{pmatrix}.$$

Thus, since

$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} = \frac{1}{2}B(\gamma, \delta)\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right],$$

we get

(2.9.18) $$B(\gamma D_{12}, \delta) + B(\gamma, \delta E_{21}) = 0,$$

and (2.6.1) for α and β with $a = st$. Hence $\alpha, \beta \in V_{st}$. Also (2.9.7) and (2.9.11) show that (D_{12}, E_{21}) satisfies (2.2), and (2.9.18) is (2.1) for (D_{12}, E_{21}) so $(D_{12}, E_{21}) \in \mathscr{G}$.

Finally, let $\widetilde{D} = D - (D_{12}, E_{21}) \in \operatorname{Der} \mathscr{J}$, i.e.,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\widetilde{D} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\widetilde{D} = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}\widetilde{D} = \begin{pmatrix} -\frac{1}{2}B(\beta, \gamma) & 0 \\ \frac{1}{2}s\alpha\gamma & \frac{1}{2}B(\beta, \gamma) \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix}\widetilde{D} = \begin{pmatrix} -\frac{1}{2}B(\alpha, \delta) & -\frac{1}{2}t\beta\delta \\ 0 & \frac{1}{2}B(\alpha, \delta) \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix}\widetilde{D} &= \frac{1}{2}\begin{pmatrix} -B(\beta, \gamma) - B(\alpha, \delta) & (c-d)\alpha - t\beta\delta \\ (c-d)\beta + s\alpha\gamma & B(\beta, \gamma) + B(\alpha, \delta) \end{pmatrix} \\ &= \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix}\frac{1}{2}E_{\alpha, \beta} \end{aligned}$$

so $\widetilde{D} = \frac{1}{2}E_{\alpha, \beta} \in \mathscr{H}$. Hence $D = (D_{12}, E_{21}) + \frac{1}{2}E_{\alpha, \beta} \in \mathscr{G} \oplus \mathscr{H}$ and we are done.

Corollary 2.10. $\operatorname{Der}\mathcal{J} = \mathcal{G}$ *iff* $V_{st} = \{0\}$.

Corollary 2.11. *If* $D \in \operatorname{Der}\mathcal{J}$ *and* $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)D = \left(\begin{smallmatrix}0 & \alpha\\ \beta & 0\end{smallmatrix}\right)$, *then* $\alpha, \beta \in V_{st}$ *and* $D = \frac{1}{2}E_{\alpha,\beta} + F$ *for some* $F \in \mathcal{G}$.

The following result is another easy verification using Theorem 2.9. Recall that $C(x, y)$ is the nondegenerate symmetric bilinear form on $\mathcal{J}$ defined by (1.9).

Proposition 2.12. *Let* $\mathcal{L}_C = \{D \in \operatorname{End}\mathcal{J} \mid C(xD, y) = -C(x, yD)$ *for all* $x, y \in \mathcal{J}\}$. *Then* $\operatorname{Der}\mathcal{J} \subseteq \mathcal{L}_C$.

3. Dimension of V_{st}

Lemma 3.1. *Suppose* $\dim\mathcal{S} > 1$ *and* $V_a \neq \{0\}$. *Then* $a \neq 0$ *and* $V_b = \{0\}$ *for all* $b \neq a$.

Proof. Suppose $\alpha \in V_a$ with $\alpha \neq 0$. Then, by Lemma 2.6, $a(\alpha\gamma)\delta = B(\gamma, \delta)\alpha - B(\alpha, \delta)\gamma$ for all $\gamma, \delta \in \mathcal{S}$. If $a = 0$, then $B(\gamma, \delta)\alpha = B(\alpha, \delta)\gamma$ for all $\gamma, \delta \in \mathcal{S}$, which is impossible since $\dim\mathcal{S} > 1$. Now suppose $\beta \in V_b$ where $b \neq a$. So, by Lemma 2.6, $b(\beta\gamma)\delta = B(\gamma, \delta)\beta - B(\beta, \delta)\gamma$ for all $\gamma, \delta \in \mathcal{S}$. Hence using (2.5) for α yields

$$\begin{aligned} &bB(\alpha, \beta)\delta - bB(\alpha, \delta)\beta \\ &\quad = ab\alpha(\beta\delta) = -ab(\beta\delta)\alpha = -aB(\delta, \alpha)\beta + aB(\beta, \alpha)\delta, \end{aligned}$$

so $(b-a)B(\alpha, \beta)\delta = (b-a)B(\alpha, \delta)\beta$ for all $\delta \in \mathcal{S}$. Since $b \neq a$, we get $B(\alpha, \beta)\delta = B(\alpha, \delta)\beta$ for all $\delta \in \mathcal{S}$. If $\beta \neq 0$, choosing δ linearly independent of β gives $B(\alpha, \beta) = 0 = B(\alpha, \delta)$, so $B(\alpha, \delta) = 0$ for all $\delta \in \mathcal{S}$, contradicting the nondegeneracy of B. Thus $\beta = 0$ and $V_b = \{0\}$.

Remark 3.2. Lemma 3.1 explains why $\mathcal{J}_1 = \mathcal{J}(\mathcal{S}, B, 1, 1)$ is not isomorphic to $\mathcal{J}_2 = \mathcal{J}(\mathcal{S}, B, 1, -1)$ for $\mathcal{S}$ and B as in Example 1.5, as noted in [4], since $\operatorname{Der}\mathcal{J}_1 = \mathcal{G} \oplus \mathcal{H}$ with $\dim\mathcal{H} = 6$ and $\operatorname{Der}\mathcal{J}_2 = \mathcal{G}$ by Theorem 2.9, Corollary 2.10, and Lemma 3.1.

Lemma 3.3. V_a *is a subalgebra of* $\mathcal{S}$.

Proof. If $\dim\mathcal{S} = 1$ or $V_a = \{0\}$, the result is obvious, so suppose $\dim\mathcal{S} > 1$ and $V_a \neq \{0\}$. By Lemma 3.1, $a \neq 0$. Choose $s, t \in k$ with $st = a \neq 0$. Then if $\alpha, \beta \in V_a$, we have $E_{\alpha,\beta}, E_{\beta,\alpha} \in \operatorname{Der}\mathcal{J}$, $\mathcal{J} = \mathcal{J}(\mathcal{S}, B, s, t)$, and hence $[E_{\alpha,\beta}, E_{\beta,\alpha}] \in \operatorname{Der}\mathcal{J}$. It is easy to check that $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)[E_{\alpha,\beta}, E_{\beta,\alpha}] = 2\left(\begin{smallmatrix}0 & 2t\beta\alpha\\ 2s\beta\alpha & 0\end{smallmatrix}\right)$, so, by Corollary 2.11, $\beta\alpha \in V_{st} = V_a$. Hence V_a is a subalgebra.

Theorem 3.4. *If* $\dim\mathcal{S} \geq 4$, *then* $\dim V_a \leq 1$.

Proof. By Lemma 3.3 if $\alpha, \beta \in V_a$, $\alpha\beta \in V_a$, so by (2.5)

$$a(\alpha\beta)(\gamma\delta) = B(\alpha\beta, \gamma)\delta - B(\alpha\beta, \delta)\gamma \tag{3.4.1}$$

for all $\gamma, \delta \in \mathcal{S}$. Since $\alpha \in V_a$, Lemma 2.6 give $a(\alpha\beta)(\gamma\delta) = B(\beta, \gamma\delta)\alpha - B(\alpha, \gamma\delta)\beta$ so this and (3.4.1) give

$$B(\alpha\beta, \gamma)\delta - B(\alpha\beta, \delta)\gamma = B(\beta, \gamma\delta)\alpha - B(\alpha, \gamma\delta)\beta \tag{3.4.2}$$

for all $\alpha, \beta \in V_a$ and all $\gamma, \delta \in \mathcal{S}$.

Now suppose by way of contradiction that $\alpha, \beta \in V_a$ are linearly independent. Since $\dim \mathscr{S} \geq 4$, there are $\gamma, \delta \in \mathscr{S}$ so that $\{\alpha, \beta, \gamma, \delta\}$ is linearly independent. Hence $B(\alpha\beta, \gamma) = B(\alpha\beta, \delta) = 0$ and since $B(\alpha\beta, \alpha) = B(\beta, \alpha^2) = 0 = B(\alpha, \beta^2) = B(\alpha\beta, \beta)$, we see $\alpha\beta = 0$ by the nondegeneracy of B and so (3.4.2) becomes $0 = B(\beta, \gamma\delta)\alpha - B(\alpha, \gamma\delta)\beta$ for all $\gamma, \delta \in \mathscr{S}$. Since α and β are linearly independent, we get $B(\alpha, \gamma\delta) = B(\alpha\gamma, \delta) = 0$ for all $\gamma, \delta \in \mathscr{S}$, so again using the nondegeneracy of B gives $\alpha\gamma = 0$ for all $\gamma \in \mathscr{S}$. But then $\alpha(\gamma\delta) = 0$ for all $\gamma, \delta \in \mathscr{S}$, so by (2.5) $0 = B(\alpha, \gamma)\delta - B(\alpha, \delta)\gamma$ and choosing δ linearly independent of γ shows $B(\alpha, \gamma) = 0$ for all $\gamma \in \mathscr{S}$. Thus $\alpha = 0$, contradicting the choice of α and β. Therefore, $\dim V_a \leq 1$.

Remark 3.5. If $\dim \mathscr{S} \geq 4$, Theorem 3.4 implies Lemma 3.3, and in fact Theorem 3.4 can be proven without using Lemma 3.3, as follows: If $\alpha \neq 0$ is in V_a, let $\alpha^\perp := \{\gamma \mid B(\alpha, \gamma) = 0\}$. If $\gamma \in \alpha^\perp$ and $\delta \in \mathscr{S}$ with $B(\alpha, \delta) \neq 0$, then, by (2.5) $a\alpha(\gamma\delta) = -B(\alpha, \delta)\gamma$. This, and the fact that $B(\alpha, \alpha\sigma) = B(\alpha^2, \sigma) = 0$ for all $\sigma \in \mathscr{S}$, proves that $\alpha^\perp = \{\alpha\sigma \mid \sigma \in \mathscr{S}\}$.

Now suppose by way of contradiction that $\dim V_a > 1$ and choose $\beta \in V_a$, $\beta \neq 0$, with $B(\alpha, \beta) = 0$. Then $\alpha \in \beta^\perp$ and $\alpha = \beta\sigma$ for some $\sigma \in \mathscr{S}$. If γ is such that $B(\beta, \gamma) = B(\sigma, \gamma) = 0$ then, by Lemma 2.6, $a\alpha\gamma = a(\beta\sigma)\gamma = B(\sigma, \gamma)\beta - B(\beta, \gamma)\sigma = 0$, so $\operatorname{codim}\{\gamma \mid \alpha\gamma = 0\} \leq 2$ (where codim is the codimension). Hence $\dim \alpha^\perp \leq 2$ and so $\dim \mathscr{S} \leq 3$.

The next example shows that there are numerous $(\mathscr{S}, B)$'s for which $\dim \mathscr{S} \geq 4$ and $\dim V_a = 1$.

Example 3.6. Let W be a vector space of dimension at least three and having a nondegenerate symmetric bilinear form B for which there is an $A \in GL(W)$ such that $A^2 = cI$ for some nonzero scalar c and $B(\gamma A, \delta) = -B(\gamma, \delta A)$ for all $\gamma, \delta \in W$. Note that the existence of such an A forces $\dim W$ to be even if $\dim W < \infty$, since $\langle \gamma, \delta \rangle := B(\gamma A, \delta)$ is a nondegenerate skew symmetric bilinear form on W, and nondegenerate skew symmetric forms only exist on even dimensional spaces.

Let $\mathscr{S} = k\alpha \oplus W$, fix $d \in k$, $d \neq 0$, and extend B to $\mathscr{S}$ by

$$B(a\alpha + \gamma, b\alpha + \delta) := abcd + B(\gamma, \delta)$$

for $\gamma, \delta \in W$. Define a multiplicaton on $\mathscr{S}$ by

$$(3.6.1) \qquad (a\alpha + \gamma)(b\alpha + \delta) := B(\alpha, \alpha)^{-1} B(\gamma A, \delta)\alpha + (a\delta - b\gamma)A$$

for $\gamma, \delta \in W$. With this multiplication $\mathscr{S}$ is an anticommutative algebra, B is associative, and $k\alpha = V_d$.

Finally we consider the possibilities when $\dim \mathscr{S} \leq 3$.

Proposition 3.7. (i) *If* $\dim \mathscr{S} = 1$, *then* $\mathscr{S}$ *is abelian and* $\mathscr{S} = V_1$, *so* $\dim V_1 = 1$.

(ii) *If* $\dim \mathscr{S} = 2$, *then* $\mathscr{S}$ *is abelian and* $V_a = \{0\}$ *for all* $a \in k$.

(iii) *If* $\dim \mathscr{S} = 3$, *then either* $\mathscr{S}$ *is abelian and* $V_a = \{0\}$ *for all* $a \in k$, *or* $\mathscr{S}$ *is simple and* $\mathscr{S} = V_d$ *for some* $d \in k$, *in which case* $\mathscr{S}$ *is a Lie algebra of type* A_1 *and* B *is a multiple of the Killing form. Thus* $\dim V_a = 0$ *or* 3.

Proof. Since $\mathscr{S}$ is anticommutative, $\alpha^2 = 0$ for all $\alpha \in \mathscr{S}$. If $\alpha, \beta \in \mathscr{S}$, then $B(\alpha\beta, \alpha) = B(\beta, \alpha^2) = 0 = B(\alpha, \beta^2) = B(\alpha\beta, \beta)$ since B is associative, so $\alpha\beta$ is orthogonal to both α and β relative to B. This shows that $\mathscr{S}$ is abelian if $\dim\mathcal{S} = 1$ or $\dim\mathcal{S} = 2$. Then if $\dim\mathcal{S} = 1$, (2.5) is trivially satisfied with $a = 1$, so $\mathcal{S} = V_1$.

If $\mathcal{S}$ is abelian and $\dim\mathcal{S} > 1$, then (2.5) gives $B(\alpha, \gamma)\delta = B(\alpha, \delta)\gamma$ for all $\alpha \in V_a$, $\gamma, \delta \in \mathcal{S}$, so choosing γ and δ linearly independent of each other gives $B(\alpha, \gamma) = 0$ for all $\gamma \in \mathcal{S}$. Thus $\alpha = 0$ and $V_a = \{0\}$.

This only leaves the case $\dim\mathcal{S} = 3$, $\mathcal{S}$ not abelian, to be considered. Let $\{\alpha, \beta, \gamma\}$ be an orthogonal basis of $\mathcal{S}$. Then $\alpha\beta = a\gamma$, $\beta\gamma = b\alpha$, $\gamma\alpha = c\beta$ for some $a, b, c \in k$. Now using the associativity of B gives $aB(\gamma, \gamma) = B(\alpha\beta, \gamma) = B(\alpha, \beta\gamma) = bB(\alpha, \alpha) = B(\beta, \gamma\alpha) = cB(\beta, \beta)$. A straightforward verification shows that $\mathcal{S} = V_d$ with $d = -B(\alpha, \alpha)a^{-1}c^{-1}$. But then by (2.5) $\mathcal{S}$ satisfies the Jacobi identity and hence must be a Lie algebra of type A_1 since $\mathcal{S}$ is not solvable. Since B is associative and $\mathcal{S}$ is simple, B must be a multiple of the Killing form.

Theorem 3.8. *For all* $a \in k$ $\dim V_a = 0, 1,$ *or* 3, *and* $\dim V_a = 3$ *for some* $a \in k$ *iff* $\mathcal{S}$ *is a Lie algebra of type* A_1.

Proof. This follows from Theorem 3.4 and Proposition 3.7.

4. The $\dim V_{st} = 1$ case

In this section we will show that if $\dim\mathcal{S} \geq 4$ and $\dim V_a = 1$ for some $a \in k$, then $\mathcal{S}$ is one of the algebras constructed in Example 3.6. We will also compute $\mathrm{Der}\mathcal{J}(\mathcal{S}, B, s, t)$ in this case.

Lemma 4.1. *Suppose* $\dim\mathcal{S} \geq 4$ *and* $\dim V_a = 1$. *Then* $\mathcal{S}$ *is simple.*

Proof. Suppose $I \lhd \mathcal{S}$ with $I \neq \{0\}$ and suppose $0 \neq \gamma \in I$, $0 \neq \alpha \in V_a$, and $\delta \in \mathcal{S}$ with $B(\gamma, \delta) \neq 0$. Then $(\alpha\gamma)\delta \in I$ and $B(\alpha, \delta)\gamma \in I$, so by Lemma 2.6 $B(\gamma, \delta)\alpha \in I$, giving $\alpha \in I$. Now choose $\delta \in \mathcal{S}$ with $B(\alpha, \delta) \neq 0$. Then for all $\eta \in \mathcal{S}$, $a(\alpha\eta)\delta - B(\eta, \delta)\alpha = -B(\alpha, \delta)\eta \in I$, again by Lemma 2.6, so $\eta \in I$ and $I = \mathcal{S}$.

Theorem 4.2. *Suppose* $\dim\mathcal{S} \geq 4$ *and* $\dim V_a = 1$. *Then* $\mathcal{S}$ *is the algebra constructed in Example* 3.6. *Hence if* $\dim\mathcal{S} < \infty$, $\dim\mathcal{S}$ *is odd.*

Proof. Suppose $V_a = k\alpha$. First suppose $B(\alpha, \alpha) = 0$. Then there is a $\beta \in \mathcal{S}$ with $B(\alpha, \beta) = 1 = B(\beta, \alpha)$ and $B(\beta, \beta) = 0$. We can decompose $\mathcal{S}$ relative to B as $\mathcal{S} = (k\alpha \oplus k\beta) \perp W$. Then for $\gamma, \delta \in W$ we get the following from (2.5) and (2.6.1):

(4.2.1) $$(\alpha\gamma)\delta = a^{-1}B(\gamma, \delta)\alpha,$$

(4.2.2) $$\alpha(\alpha\beta) = -a^{-1}\alpha,$$

(4.2.3) $$\beta(\alpha\gamma) = a^{-1}\gamma.$$

Now suppose $\alpha\beta = c\alpha + d\beta + \eta$ for some $\eta \in W$. Then $c = B(\alpha\beta, \beta) = B(\alpha, \beta^2) = 0$ and $d = B(\alpha\beta, \alpha) = B(\beta, \alpha^2) = 0$, so $\alpha\beta \in W$. By (4.2.2) and (4.2.1) for $\delta \in W$

$$\alpha\delta = -a[\alpha(\alpha\beta)]\delta = -aa^{-1}B(\alpha\beta, \delta)\alpha = -B(\alpha\beta, \delta)\alpha,$$

so by (4.2.3) $\beta(\alpha\delta) = a^{-1}\delta = \beta[-B(\alpha\beta, \delta)\alpha] = -B(\alpha\beta, \delta)\beta\alpha$. Thus $\dim W = 1$ and $\dim\mathcal{S} = 3$, contradicting the hypothesis $\dim\mathcal{S} \geq 4$. Hence no such $\mathcal{S}$ exists.

Thus $B(\alpha, \alpha) \neq 0$. Again we decompose $\mathscr{S}$ relative to B as $\mathscr{S} = k\alpha \perp W$ and get the following from (2.5) and (2.6.1) for $\gamma, \delta \in W$:

(4.2.4) $$\alpha(\gamma\delta) = 0,$$

(4.2.5) $$(\alpha\gamma)\delta = a^{-1}B(\gamma, \delta)\alpha,$$

(4.2.6) $$\alpha(\alpha\gamma) = a^{-1}B(\alpha, \alpha)\gamma.$$

Now suppose $\gamma, \delta \in W$ and $\gamma\delta = b\alpha + \eta$ for some $\eta \in W$. Then by (4.2.4), $0 = \alpha(\gamma\delta) = \alpha(b\alpha + \eta) = \alpha\eta$. Hence using (4.2.6) gives $0 = \alpha 0 = \alpha(\alpha\eta) = a^{-1}B(\alpha, \alpha)\eta$, so $\eta = 0$ and $\gamma\delta = b\alpha$ for some $b \in k$. In fact, $bB(\alpha, \alpha) = B(\gamma\delta, \alpha) = B(\delta, \alpha\gamma)$, so $b = B(\alpha, \alpha)^{-1}B(\alpha\gamma, \delta)$. Thus for all $\gamma, \delta \in W$

(4.2.7) $$\gamma\delta = B(\alpha, \alpha)^{-1}B(\alpha\gamma, \delta)\alpha.$$

If $\gamma \in W$ and $\alpha\gamma = c\alpha + \eta$ for some $\eta \in W$, by (4.2.5)

$$\begin{aligned} a^{-1}B(\gamma, \gamma)\alpha = (\alpha\gamma)\gamma &= (c\alpha + \eta)\gamma = c\alpha\gamma + \eta\gamma \\ &= c(c\alpha + \eta) + B(\alpha, \alpha)^{-1}B(\alpha\eta, \gamma)\alpha \\ &= [c^2 + B(\alpha, \alpha)^{-1}B(\alpha\eta, \gamma)]\alpha + c\eta \end{aligned}$$

by (4.2.7), so $a^{-1}B(\gamma, \gamma) = c^2 + B(\alpha, \alpha)^{-1}B(\alpha\eta, \gamma)$ and $c\eta = 0$. Hence either $c = 0$ or $\eta = 0$. But if $\eta = 0$, $\alpha\gamma = c\alpha$, so by (4.2.6) $\gamma = aB(\alpha, \alpha)^{-1}\alpha(\alpha\gamma) = aB(\alpha, \alpha)^{-1}c\alpha^2 = 0$. Thus $\eta \neq 0$ so $c = 0$ and $\alpha\gamma = \eta \in W$ and $B(\alpha\eta, \gamma) = a^{-1}B(\alpha, \alpha)B(\gamma, \gamma)$. Hence for $\gamma \in W$,

(4.2.8) $\alpha\gamma = \eta$ for some $\eta \in W$, where $B(\alpha\gamma, \eta) = -a^{-1}B(\alpha, \alpha)B(\gamma, \gamma)$.

Note that since $\eta = \alpha\gamma$, $B(\alpha\gamma, \alpha\gamma) = -a^{-1}B(\alpha, \alpha)B(\gamma, \gamma)$.

$\mathscr{S}$ is simple by Lemma 4.1, so $W = \alpha W$ since $W^2 \subseteq k\alpha$ by (4.2.7). Let $A: W \to W$ be defined by $\gamma A = \alpha\gamma$, so we have $B(\gamma A, \gamma A) = -a^{-1}B(\alpha, \alpha)B(\gamma, \gamma)$ and $A^2 = a^{-1}B(\alpha, \alpha)I$ by (4.2.6), and we get from (4.2.7) and (4.2.8) that

(4.2.9) $$(c\alpha + \gamma)(d\alpha + \delta) = B(\alpha, \alpha)^{-1}B(\gamma A, \delta)\alpha + (c\delta - d\gamma)A,$$

which is (3.6.1). As noted in Example 3.6, $\dim W$ is even if $\dim W < \infty$, so $\dim \mathscr{S}$ is odd.

We need two lemmas for the computation of $\operatorname{Der}\mathscr{J}(\mathscr{S}, B, s, t)$.

Lemma 4.3. *If $\alpha \in V_a$ and $(D_1, D_2) \in \mathscr{G}$, then $\alpha D_1, \alpha D_2 \in V_a$.*

Proof. This is a straightforward verificaton done by applying D_1 and D_2 to (2.5).

Lemma 4.4. *Suppose $\dim \mathscr{S} \geq 4$ and $\dim V_a = 1$. Define $\pi : \mathscr{G} \to \operatorname{End}\mathscr{S}$ by $(D_1, D_2)\pi := D_1$. Then $\mathscr{G} \cong \mathscr{G}\pi \cong \mathscr{L} \oplus kc$, where $\mathscr{L}$ is a symplectic Lie algebra with $\alpha l = 0$ for all $l \in \mathscr{L}$, $\alpha \in V_a$, and c is a central element of $\mathscr{G}$ defined by*

(4.4.1) $$(b\alpha + \gamma)c := b\alpha - \tfrac{1}{2}\gamma$$

for all $\gamma \in \mathscr{S}$ such that $B(\alpha, \gamma) = 0$, $\alpha \in V_a$.

Proof. By Theorem 4.2 if $V_a = k\alpha$, then $\mathscr{S} = k\alpha \perp W$ relative to B and the multiplication in $\mathscr{S}$ is given by (4.2.9), where $A \in GL(W)$ is defined by $\gamma A := \alpha\gamma$ and satisfies $A^2 = a^{-1}B(\alpha, \alpha)I$ and $B(\gamma A, \gamma A) = -a^{-1}B(\alpha, \alpha)B(\gamma, \gamma)$ for all $\gamma \in W$. Then $\langle \gamma, \delta \rangle := B(\gamma A, \delta)$ is a nondegenerate skew-symmetric bilinear form on W. Let $\mathscr{L} := \{D \in \operatorname{End} W | \langle \gamma D, \delta \rangle = -\langle \gamma, \delta D \rangle$ for all $\gamma, \delta \in W\}$ and define $\overline{D} \in \operatorname{End}\mathscr{S}$ for $D \in \operatorname{End} W$ by $(a\alpha + \gamma)\overline{D} := \gamma D$ for all $\gamma \in W$. Then if $D \in \mathscr{L}$, it is easy to check that $(\overline{D}, \overline{A^{-1}DA}) \in \mathscr{G}$ and $D \mapsto (\overline{D}, \overline{A^{-1}DA})$ is a homomorphism of Lie algebras. Also $(c, -c) \in \mathscr{G}$ so $\mathscr{L} \oplus kc$ is isomorphic to a subalgebra of $\mathscr{G}$.

Conversely, suppose $(D_1, D_2) \in \mathscr{G}$. By Lemma 4.3, $\alpha D_i = a_i\alpha$ for some $a_i \in k$. Since $B(\alpha D_i, \alpha) = -B(\alpha, \alpha D_j)$ we see $a_1 = -a_2$. Moreover, if $\gamma \in W$ and $\gamma D_i = b_i\alpha + \gamma_i$, then $B(\gamma D_i, \alpha) = -B(\gamma, \alpha D_j)$ implies $b_i = 0$, so $WD_i \subseteq W$. Let $(E_1, E_2) = (D_1, D_2) - a_1(c, -c)$, so $(E_1, E_2) \in \mathscr{G}$ and $\alpha E_i = 0$ for $i = 1, 2$. Then for $\gamma \in W$, $(\alpha\gamma)E_i = (\alpha E_j)\gamma + \alpha(\gamma E_j) = \alpha(\gamma E_j)$ implies $\gamma A E_i = \gamma E_j A$, i.e., $E_2|_W = A^{-1}DA$ for $D := E_1|_W$, so $(E_1, E_2) = (\overline{D}, \overline{A^{-1}DA})$. Also for $\gamma, \delta \in W$, $0 = B(\alpha, \alpha)^{-1}B(\gamma A, \delta)\alpha E_2 = (\gamma\delta)E_2 = (\gamma E_1)\delta + \gamma(\delta E_1) = B(\alpha, \alpha)^{-1}[B(\gamma E_1 A, \delta) + B(\gamma A, \delta E_1)]\alpha$, so $D \in \mathscr{L}$ and we are done.

Theorem 4.5. *Suppose* $\dim \mathscr{S} \geq 4$, $\dim V_a = 1$, $\mathscr{J} = \mathscr{J}(\mathscr{S}, B, s, t)$, $st = a$, *and* $c \in \operatorname{End}\mathscr{S}$ *is defined by* (4.4.1). *Then* $[\mathscr{H} \oplus k(c, -c)] \triangleleft \operatorname{Der}\mathscr{J}$ *is isomorphic to* $sl(2, k)$, *where* $\mathscr{H} = \{E_{\alpha,\beta} | \alpha, \beta \in V_a\}$, $E_{\alpha,\beta}$ *defined by* (2.7.1). *Moreover,* $\mathscr{L} \triangleleft \operatorname{Der}\mathscr{J}$ *so* $\operatorname{Der}\mathscr{J}$ *is the direct sum of an* $sl(2, k)$ *ideal and a symplectic ideal.*

Proof. A straightforward verification shows that if $\alpha_i, \beta_i \in V_{st}$ for $i = 1, 2$, then $[E_{\alpha_1,\beta_1}, E_{\alpha_2,\beta_2}] = E_{\beta,\alpha} + (D_1, D_2)$, where $\beta = 2t\beta_1\beta_2$, $\alpha = 2s\alpha_1\alpha_2$, and $(D_1, D_2) \in \mathscr{G}$ is defined by $\gamma D_1 := 2B(\beta_2, \gamma)\alpha_1 - st\beta_2(\alpha_1\gamma) - 2B(\beta_1, \gamma)\alpha_2 + st\beta_1(\alpha_2\gamma)$ and $\gamma D_2 := 2B(\alpha_2, \gamma)\beta_1 - st\alpha_2(\beta_1\gamma) - 2B(\alpha_1, \gamma)\beta_2 + st\alpha_1(\beta_2\gamma)$ for all $\gamma \in \mathscr{S}$, and it is easy to check that if $\alpha, \beta \in V_{st}$ and $(D_1, D_2) \in \mathscr{G}$, then $[E_{\alpha,\beta}, (D_1, D_2)] = E_{\alpha',\beta'}$, where $\alpha' = \alpha D_1$ and $\beta' = \beta D_2$, which are both in V_{st} by Lemma 4.3. Since $\alpha l = 0$ for all $l \in \mathscr{L} \subseteq \mathscr{G}\pi$ and $(c, -c)$ is a central element of $\mathscr{G}$, we see that $\mathscr{L} \triangleleft \operatorname{Der}\mathscr{J}$. Since $\dim V_{st} = 1$, $\dim \mathscr{H} = 2$ and $\mathscr{H}$ is spanned by $\{E_{\alpha,0}, E_{0,\beta}\}$, where α, β are nonzero elements of V_{st} with $\beta = B(\alpha, \alpha)^{-1}\alpha$. Hence $\mathscr{H} \oplus k(c, -c)$ is three dimensional and by the previous formulas $[E_{\alpha,0}, E_{0,\beta}] = 2(c, -c)$, $[E_{\alpha,0}, 2(c, -c)] = 2E_{\alpha,0}$, and $[E_{0,\beta}, 2(c, -c)] = -2E_{0,\beta}$, so $\mathscr{H} \oplus k(c, -c)$ is isomorphic to $sl(2, k)$. That $\mathscr{H} \oplus k(c, -c)$ is an ideal follows from Lemma 4.4 and Theorem 2.9.

Corollary 4.6. *Suppose* $\dim \mathscr{S} \geq 4$ *and* $\dim V_a = 1$.

(i) *If* $st = a$, *then* $\operatorname{Der}\mathscr{J}(\mathscr{S}, B, s, t) \cong \mathscr{L} \oplus sl(2, k)$, *where* $\mathscr{L}$ *is a symplectic Lie algebra,* $\mathscr{L} \triangleleft \operatorname{Der}\mathscr{J}(\mathscr{S}, B, s, t)$, *and* $sl(2, k) \triangleleft \operatorname{Der}\mathscr{J}(\mathscr{S}, B, s, t)$.

(ii) *If* $st \neq a$, *then* $\operatorname{Der}\mathscr{J}(\mathscr{S}, B, s, t) \cong \mathscr{L} \oplus kc$, *where* $\mathscr{L}$ *is a symplectic Lie algebra,* $\mathscr{L} \triangleleft \operatorname{Der}\mathscr{J}(\mathscr{S}, B, s, t)$, *and* c *is a central element of* $\operatorname{Der}\mathscr{J}$ *acting semisimply on* $\mathscr{S}$.

Proof. For (ii) use Corollary 2.10 and Lemma 3.1.

5. Automorphisms

In this section we will determine the automorphism group of $\mathscr{J}(\mathscr{S}, B, s, t)$. The following proposition is a straightforward verification.

Proposition 5.1. *Suppose* $\mathscr{J} = \mathscr{J}(\mathscr{S}, B, s, t)$.

(i) *Suppose* $R = (R_1, R_2) \in GL(\mathscr{S}) \times GL(\mathscr{S})$ *such that for all* $\gamma, \delta \in \mathscr{S}$, $B(\gamma R_i, \delta R_j) = B(\gamma, \delta)$ *and* $(\gamma\delta)R_i = (\gamma R_j)(\delta R_j)$ *for* $i, j = 1, 2, i \neq j$.

Then $A_R \in \operatorname{Aut}\mathcal{J}$, *where* A_R *is defined by*

$$(5.1.1)\qquad \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} A_R := \begin{pmatrix} c & \gamma R_1 \\ \delta R_2 & d \end{pmatrix}.$$

(ii) *Suppose* $S = (S_1, S_2) \in GL(\mathcal{S}) \times GL(\mathcal{S})$ *such that* $B(\gamma S_1, \delta S_2) = B(\gamma, \delta)$, $s(\gamma\delta)S_1 = t(\gamma S_2)(\delta S_2)$, *and* $t(\gamma\delta)S_2 = s(\gamma S_1)(\delta S_1)$ *for all* $\gamma, \delta \in \mathcal{S}$. *Then* $B_S \in \operatorname{Aut}\mathcal{J}$, *where* B_S *is defined by*

$$(5.1.2)\qquad \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} B_S := \begin{pmatrix} d & \delta S_1 \\ \gamma S_2 & c \end{pmatrix}.$$

(iii) *Suppose* $\alpha, \beta \in V_{st}$. *Then* $E_\alpha, F_\beta \in \operatorname{Aut}\mathcal{J}$, *where* E_α *and* F_β *are defined by*

$$(5.1.3)\qquad \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} E_\alpha := \begin{pmatrix} c - B(\alpha, \delta) & \gamma + (c-d)\alpha - B(\alpha, \delta)\alpha \\ \delta + s\alpha\gamma & d + B(\alpha, \delta) \end{pmatrix},$$

$$(5.1.4)\qquad \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} F_\beta := \begin{pmatrix} c + B(\beta, \gamma) & \gamma + t\beta\delta \\ \delta + (d-c)\beta - B(\beta, \gamma)\beta & d - B(\beta, \gamma) \end{pmatrix}.$$

Note that E_α is the exponential of the derivation $E_{\alpha,0}$ and F_β is the exponential of the derivation $E_{0,-\beta}$.

We will now show that $\operatorname{Aut}\mathcal{J}$ is generated by the A_R's, B_S's, E_α's, and F_β's. For this suppose $E \in \operatorname{Aut}\mathcal{J}$. Then $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ so

$$(5.2)\qquad \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} E = \begin{pmatrix} c + a(d-c) & (c-d)\alpha \\ (d-c)\beta & d + b(c-d) \end{pmatrix}$$

for some $a, b \in k$ and some $\alpha, \beta \in \mathcal{S}$. Also, there are linear mappings $E_{11}, E_{22}, F_{11}, F_{22} : \mathcal{S} \to k$ and $E_{12}, E_{21}, F_{12}, F_{21} : \mathcal{S} \to \mathcal{S}$ such that

$$(5.3)\qquad \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} E = \begin{pmatrix} \gamma E_{11} & \gamma E_{12} \\ \gamma E_{21} & \gamma E_{22} \end{pmatrix}$$

and

$$(5.4)\qquad \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} E = \begin{pmatrix} \delta F_{11} & \delta F_{12} \\ \delta F_{21} & \delta F_{22} \end{pmatrix}$$

for all $\gamma, \delta \in \mathcal{S}$.

Lemma 5.5. (i) $a(a-1) = b(b-1) = B(\alpha, \beta)$.

(ii) $(b-a)\alpha = 0 = (a-b)\beta$.

(iii) $E_{22} = -E_{11}$ *and* $F_{22} = -F_{11}$.

(iv) *If* $\alpha \neq 0$ *or* $\beta \neq 0$, *then* $a = b$.

Proof. Since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 1-a & \alpha \\ -\beta & b \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, the computation of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)E$ gives (i) and (ii). For (iii), computation of

$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} E \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

yields

$$(5.5.1)\qquad (\gamma E_{11})^2 + B(\gamma E_{12}, \gamma E_{21}) = 0 = B(\gamma E_{21}, \gamma E_{12}) + (\gamma E_{22})^2$$

and

$$(5.5.2)\qquad (\gamma E_{11})(\gamma E_{12}) + (\gamma E_{22})(\gamma E_{12}) = 0 = (\gamma E_{11})(\gamma E_{21}) + (\gamma E_{22})(\gamma E_{21}).$$

By (5.5.1) $(\gamma E_{11})^2 = (\gamma E_{22})^2$, so $\gamma E_{22} = \varepsilon\gamma E_{11}$ for $\varepsilon = \pm 1$. Therefore (5.5.2) becomes $(\gamma E_{11} + \varepsilon\gamma E_{11})(\gamma E_{12}) = 0 = (\gamma E_{11} + \varepsilon\gamma E_{11})(\gamma E_{21})$ so either $\gamma E_{11} + \varepsilon\gamma E_{11} = 0$ or $\gamma E_{12} = 0 = \gamma E_{21}$. But if $\gamma E_{12} = 0 = \gamma E_{21}$, (5.5.1) gives $(\gamma E_{11})^2 = 0 = (\gamma E_{22})^2$ so $\left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, contradicting the nonsingularity of E. Hence $\gamma E_{11} + \varepsilon\gamma E_{11} = 0$, so $\gamma E_{22} = -\gamma E_{11}$. A similar argument with $\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ shows $\delta F_{22} = -\delta F_{11}$. (iv) follows directly from (ii).

Lemma 5.6. *Suppose* $\alpha = 0 = \beta$ *in* (5.2). *Then* $E = A_R$, *where* $R = (E_{12}, F_{21})$, *or* $E = B_S$, *where* $S = (F_{12}, E_{21})$.

Proof. By Lemma 5.5(i) a and b can have only 0 and 1 as values. If $a = 0$ and $b = 1$, then $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and if $a = 1$ and $b = 0$, then $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, either case being a contradiction of the nonsingularity of E. Hence $a = b = 0$ or $a = b = 1$.

Suppose $a = b = 0$. Then $\left(\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right)E = \left(\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right)$ so $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)E$ yields $\gamma E_{21} = 0$ and $\gamma E_{11} = 0$ by Lemma 5.5(iii). Hence

$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} E = \begin{pmatrix} 0 & \gamma E_{12} \\ 0 & 0 \end{pmatrix}, \tag{5.6.1}$$

so E_{12} must be nonsingular since E is nonsingular. A similar argument with $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E$ yields $\delta F_{11} = 0$ and $\delta F_{12} = 0$, so

$$\begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} E = \begin{pmatrix} 0 & 0 \\ \delta F_{21} & 0 \end{pmatrix} \tag{5.6.2}$$

and F_{21} is nonsingular. $\left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} B(\gamma,\delta) & 0 \\ 0 & 0 \end{smallmatrix}\right)E$ gives $B(\gamma E_{12}, \delta F_{21}) = B(\gamma, \delta)$ for all $\gamma, \delta \in \mathscr{S}$, and $\left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 0 & \eta \\ 0 & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & 0 \\ s\gamma\eta & 0 \end{smallmatrix}\right)E$ and $\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 0 & 0 \\ \tau & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & t\delta\tau \\ 0 & 0 \end{smallmatrix}\right)E$ yield $(\gamma\eta)F_{21} = (\gamma E_{12})(\eta E_{12})$ and $(\delta\tau)E_{12} = (\delta F_{21})(\tau F_{21})$. Thus

$$\begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} E = \begin{pmatrix} c & \gamma E_{12} \\ \delta F_{21} & d \end{pmatrix} = \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} A_R,$$

where $R = (E_{12}, F_{21})$.

If $a = b = 1$, then $\left(\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right)E = \left(\begin{smallmatrix} d & 0 \\ 0 & c \end{smallmatrix}\right)$ and calculations similar to the ones above show $E = B_S$, where $S = (F_{12}, E_{21})$.

We now return to the general case.

Theorem 5.7. Aut$\mathscr{J}$ *is generated by the* A_R's, B_S's, E_α's, *and* F_β's *of Proposition* 5.1.

Proof. By Lemma 5.6 it only remains to deal with the case $\alpha \neq 0$ or $\beta \neq 0$. Assuming $\alpha \neq 0$, we get $a = b$ by Lemma 5.5(iv) so

$$\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} E = \begin{pmatrix} c + a(d-c) & (c-d)\alpha \\ (d-c)\beta & d + a(c-d) \end{pmatrix}. \tag{5.7.1}$$

By Lemma 5.5(iii)

$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} E = \begin{pmatrix} \gamma E_{11} & \gamma E_{12} \\ \gamma E_{21} & -\gamma E_{11} \end{pmatrix} \tag{5.7.2}$$

and

$$\begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} E = \begin{pmatrix} \delta F_{11} & \delta F_{12} \\ \delta F_{21} & -\delta F_{11} \end{pmatrix}. \tag{5.7.3}$$

Once we have shown $\alpha, \beta \in V_{st}$, then $E_\alpha, F_\beta \in \operatorname{Aut}\mathcal{J}$, where E_α and F_β are defined by (5.1.3) and (5.1.4) respectively. If $a = 1$, then $\left(\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right)EE_\alpha F_\beta = \left(\begin{smallmatrix} d & 0 \\ 0 & c \end{smallmatrix}\right)$ since $B(\alpha, \beta) = 0$ by Lemma 5.5(i), so $EE_\alpha F_\beta \in \operatorname{Aut}\mathcal{J}$ is of the form B_S by Lemma 5.6. If $a \neq 1$ and $\alpha' = (a-1)^{-1}\alpha$, then $\left(\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right)EE_{\alpha'}F_{-\beta} = \left(\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right)$ so $EE_{\alpha'}F_{-\beta} \in \operatorname{Aut}\mathcal{J}$ is of the form A_R by Lemma 5.6. Thus, the proof will be complete once we have shown $\alpha, \beta \in V_{st}$.

To see that $\alpha \in V_{st}$ we compute the diagonal terms of $\left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} B(\gamma,\delta) & 0 \\ 0 & 0 \end{smallmatrix}\right)E$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)E$, and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E$ to get the following identities:

(5.7.4) $$(1-a)B(\gamma, \delta) = (\gamma E_{11})(\delta F_{11}) + B(\gamma E_{12}, \delta F_{21}),$$

(5.7.5) $$aB(\gamma, \delta) = B(\gamma E_{21}, \delta F_{12}) + (\gamma E_{11})(\delta F_{11}),$$

(5.7.6) $$a\gamma E_{11} = B(\alpha, \gamma E_{21}),$$

(5.7.7) $$(1-a)\gamma E_{11} = B(\beta, \gamma E_{12}),$$

((5.7.8) $$(1-a)\delta F_{11} = -B(\alpha, \delta F_{21}),$$

(5.7.9)) $$a\delta F_{11} = -B(\beta, \delta F_{12}).$$

Now if $a \neq 1$, E_{12} must be nonsingular, for if $\gamma E_{12} = 0$ for some $\gamma \neq 0$, then $\gamma E_{11} = 0$ by (5.7.7), so $B(\gamma, \delta) = 0$ for all $\delta \in \mathcal{S}$ by (5.7.4), contradicting the nondegeneracy of B. Similarly, if $a \neq 1$, F_{21} is nonsingular by (5.7.8) and (5.7.4). If $a = 1$, then E_{21} and F_{12} must both be nonsingular by (5.7.5), (5.7.6), and (5.7.9). Computing the off-diagonal terms of $\left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} B(\gamma,\delta) & 0 \\ 0 & 0 \end{smallmatrix}\right)E$, $\left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)E$, and $\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)E = \left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)E$ yields the following identities:

(5.7.10) $$B(\gamma, \delta)\alpha = (\gamma E_{11})(\delta F_{12}) - (\delta F_{11})(\gamma E_{12}) + t(\gamma E_{21})(\delta F_{21}),$$

(5.7.11) $$-B(\gamma, \delta)\beta = (\delta F_{11})(\gamma E_{21}) - (\gamma E_{11})(\delta F_{21}) + s(\gamma E_{12})(\delta F_{12}),$$

(5.7.12) $$(1-a)\gamma E_{21} = -(\gamma E_{11})\beta - s(\gamma E_{12})\alpha,$$

(5.7.13) $$a\gamma E_{12} = -(\gamma E_{11})\alpha + t(\gamma E_{21})\beta,$$

(5.7.14) $$(1-a)\delta F_{12} = (\delta F_{11})\alpha - t(\delta F_{21})\beta,$$

(5.7.15) $$a\delta F_{21} = (\delta F_{11})\beta + s(\delta F_{12})\alpha.$$

Suppose $a \neq 1$. Solving (5.7.8) for δF_{11} and (5.7.7) for γE_{11} and substituting into (5.7.4) yields

(5.7.16) $$(1-a)B(\gamma, \delta) = -(1-a)^{-2}B(\beta, \gamma E_{12})B(\alpha, \delta F_{21}) + B(\gamma E_{12}, \delta F_{21}).$$

Solving for γE_{11} in (5.7.7), δF_{11} in (5.7.8), γE_{21} in (5.7.12), and δF_{12} in (5.7.14) and eliminating these quantities from (5.7.10) gives

(5.7.17) $$\begin{aligned} B(\gamma, \delta)\alpha = &- (1-a)^{-3}B(\gamma E_{12}, \beta)B(\delta F_{21}, \alpha)\alpha \\ &+ (1-a)^{-1}B(\alpha, \delta F_{21})(\gamma E_{12}) \\ &+ (1-a)^{-1}st[\alpha(\gamma E_{12})](\delta F_{21}). \end{aligned}$$

We now multiply (5.7.17) by $(1-a)$ and (5.7.16) by α to obtain

(5.7.18) $$0 = B(\alpha, \delta F_{21})(\gamma E_{12}) + st[\alpha(\gamma E_{12})](\delta F_{21}) - B(\gamma E_{12}, \delta F_{21})\alpha.$$

Hence α satisfies (2.6.1) since E_{12} and F_{21} are nonsingular when $a \neq 1$, so $\alpha \in V_{st}$.

If $a=1$, a similar argument using (5.7.5), (5.7.6), (5.7.9), (5.7.11), (5.7.13), and (5.7.15) proves $\beta \in V_{st}$. The computations showing $\beta \in V_{st}$ if $a \neq 1$ and $\alpha \in V_{st}$ if $a=1$ are also similar to the one above.

Corollary 5.8. *Suppose* $\sqrt[3]{ts^{-1}} \in k$. *Then* $T \in \operatorname{Aut}\mathscr{J}$, *where* T *is defined by*

$$\begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} T := \begin{pmatrix} d & \sqrt[3]{ts^{-1}}\,\delta \\ \sqrt[3]{st^{-1}}\,\gamma & c \end{pmatrix}. \tag{5.8.1}$$

Moreover $\operatorname{Aut}\mathscr{J}$ *is generated by the* A_R*'s,* E_α*'s,* F_β*'s, and* T.

Proof. That (5.8.1) defines an automorphism of $\mathscr{J}$ is an easy check. The rest follows from Theorem 5.7 and the fact that $TB_S = A_R$ for all $B_S \in \operatorname{Aut}\mathscr{J}$, $S=(S_1, S_2)$, where $R=(\sqrt[3]{st^{-1}}S_1,\ \sqrt[3]{ts^{-1}}S_2)$.

The following relations among the generators of $\operatorname{Aut}\mathscr{J}$ are easy verifications.

Proposition 5.9. *Let* $\theta : GL(\mathscr{S}) \times GL(\mathscr{S}) \to GL(\mathscr{S}) \times GL(\mathscr{S})$ *be defined by* $(R_1, R_2)\theta := (R_2, R_1)$.

(i) $A_R A_S = A_{RS}$, *where* $RS=(R_1S_1, R_2S_2)$ *for* $R=(R_1, R_2)$ *and* $S=(S_1, S_2)$.
(ii) $A_R B_S = B_{(R\theta)S}$ *and* $B_S A_R = B_{SR}$.
(iii) $B_R B_S = A_{(R\theta)S}$.
(iv) *If* $\alpha, \beta \in V_{st}$, *then* $E_\alpha A_R = A_R E_{\alpha R_1}$ *and* $F_\beta A_R = A_R F_{\beta R_2}$ *for* $R=(R_1, R_2)$.
(v) *If* $\alpha, \beta \in V_{st}$, *then* $E_\alpha B_S = B_S F_{\alpha S_2}$ *and* $F_\beta B_S = B_S E_{\beta S_1}$ *for* $S=(S_1, S_2)$.
(vi) *If* $\alpha, \beta \in V_{st}$, *then* $E_\alpha^{-1} = E_{-\alpha}$ *and* $F_\beta^{-1} = F_{-\beta}$.
(vii) *Suppose* $\sqrt[3]{ts^{-1}} \in k$. *Then* $T^{-1} = T$.

Let $G := \{(R_1, R_2) \in GL(\mathscr{S}) \times GL(\mathscr{S}) \mid B(\gamma R_i, \delta R_j) = B(\gamma, \delta)$ and $(\gamma\delta)R_i = (\gamma R_j)(\delta R_j)$ for all $\gamma, \delta \in \mathscr{S}\}$ and let C_n be the cyclic group of order n. $R=(R_1, R_2) \in G$ if and only if $A_R \in \operatorname{Aut}\mathscr{J}$, so we may regard G as a subgroup of $\operatorname{Aut}\mathscr{J}$ via the identification $R \leftrightarrow A_R$.

Corollary 5.10. *Suppose* $\mathscr{J} = \mathscr{J}(\mathscr{S}, B, s, t)$ *and* $\dim V_{st} = 0$. *Then either* $\operatorname{Aut}\mathscr{J} \cong G$, *or* $G \lhd \operatorname{Aut}\mathscr{J}$ *and* $\operatorname{Aut}\mathscr{J}/G \cong C_2$. *If* $\sqrt[3]{st^{-1}} \in k$, *then* $\operatorname{Aut}\mathscr{J} \cong G \rtimes C_2$.

Recall that $\mathscr{G}_0 := \{(D_1, D_2) \in \mathscr{G} \mid D_1 = D_2\}$.

Lemma 5.11. *Suppose* k *is algebraically closed,* $\dim\mathscr{S} < \infty$, $\mathscr{S}$ *is not abelian,* $\operatorname{Der}\mathscr{J} = \mathscr{G}_0$, *and* $\mathscr{S}$ *is an irreducible* $\mathscr{G}_0$*-module. Then* $\operatorname{Aut}\mathscr{J} \cong (\operatorname{Aut}_B\mathscr{S} \times C_3) \rtimes C_2$, *where* $\operatorname{Aut}_B\mathscr{S} := \{R \in \operatorname{Aut}\mathscr{S} \mid B(\gamma R, \delta R) = B(\gamma, \delta)$ *for all* $\gamma, \delta \in \mathscr{S}\}$.

Proof. Since $\operatorname{Der}\mathscr{J} = \mathscr{G}_0$, where $\mathscr{J} = \mathscr{J}(\mathscr{S}, B, s, t)$, $\dim V_{st} = 0$ by Theorem 2.9. Hence by Corollary 5.10 $\operatorname{Aut}\mathscr{J} \cong G \rtimes C$, since k is algebraically closed. Thus it is only necessary to show that $G \cong \operatorname{Aut}_B\mathscr{S} \times C_3$. Suppose $(R_1, R_2) \in G$ and $(D, D) \in \operatorname{Der}\mathscr{J}$. Then $(R_1, R_2)^{-1}(D, D)(R_1, R_2) = (R_1^{-1}DR_1, R_2^{-1}DR_2) \in \operatorname{Der}\mathscr{J}$. Since $\operatorname{Der}\mathscr{J} = \mathscr{G}_0$, we have $R_1^{-1}DR_1 = R_2^{-1}DR_2$, i.e., $DR_1R_2^{-1} = R_1R_2^{-1}D$ for all $(D, D) \in \operatorname{Der}\mathscr{J}$. Since $\mathscr{S}$ is an irreducible $\mathscr{G}_0$-module (where the action is defined by $\gamma\cdot(D, D) := \gamma D$), we have $R_1R_2^{-1} = cI$ for some $c \in k$, $c \neq 0$, by Schur's Lemma. Hence $R_1 = cR_2$.

Since $\mathcal{S}$ is not abelian and $c(\gamma\delta) = (\gamma\delta)R_1R_2^{-1} = (\gamma R_2R_1^{-1})(\delta R_2R_1^{-1}) = c^{-2}(\gamma\delta)$, we get $c^3 = 1$. Thus $(cR_1, c) \in \mathrm{Aut}_B\mathcal{S} \times C_3$. Conversely if $(R, c) \in \mathrm{Aut}_B\mathcal{S} \times C_3$, it is easy to check that $(c^{-1}R, c^{-2}R) \in G$ and we are done.

We now consider the $\dim V_{st} = 1$ case.

Theorem 5.12. *Suppose k is algebraically closed, $\dim\mathcal{S} \geq 4$, and $\dim V_a = 1$. Then $G \cong H \times k^*$, where H is a symplectic group and k^* is the multiplicative group of k. Also the copy of H in G is a normal subgroup of $\mathrm{Aut}\mathcal{J}$.*

Proof. By Theorem 4.2, $\mathcal{S} = k\alpha \perp W$ relative to B, where $V_a = k\alpha$, and the multiplication in $\mathcal{S}$ is given by (4.2.9), where $A \in GL(W)$ is defined by $\gamma A := \alpha\gamma$. Moreover, for all $\gamma \in W$, $B(\gamma A, \gamma A) = -a^{-1}B(\alpha, \alpha)B(\gamma, \gamma)$ and $A^2 = a^{-1}B(\alpha, \alpha)I$. Hence $\langle\gamma, \delta\rangle := B(\gamma A, \delta)$ is a nondegenerate skew-symmetric bilinear form on W and so $H := \{R \in GL(W) \mid \langle\gamma R, \delta R\rangle = \langle\gamma, \delta\rangle$ for all $\gamma, \delta \in W\}$ is a symplectic group. For $R \in GL(W)$ define $\overline{R} \in GL(\mathcal{S})$ by $(a\alpha + \gamma)\overline{R} = a\alpha + (\gamma R)$ for $\gamma \in W$. It is easy to check using (4.2.9) that if $R \in H$, then $(\overline{R}, \overline{A^{-1}RA}) \in G$. Also, if $f \in k^*$, then $(R_f, R_{f^{-1}}) \in G$, where $R_f \in GL(\mathcal{S})$ is defined by

$$(5.12.1) \qquad (a\alpha + \gamma)R_f := af^{-2}\alpha + f\gamma$$

for $\gamma \in W$. Thus $H \times k^*$ is isomorphic to a subgroup of G.

Conversely, suppose $(R_1, R_2) \in G$. Then αR_i satisfies (2.5) for $i = 1, 2$ so $\alpha R_i = e_i\alpha$ for some $e_i \in k^*$, $i = 1, 2$, and since $B(\alpha R_1, \alpha R_2) = B(\alpha, \alpha)$, $e_1 = e_2^{-1} = e$. Moreover if $\gamma \in W$ and $\gamma R_i = a_i\alpha + \delta_i$, then $0 = B(\alpha, \gamma) = B(\alpha R_j, \gamma R_i) = B(e_j\alpha, a_i\alpha + \delta_i) = e_ja_iB(\alpha, \alpha)$ implies $a_i = 0$, i.e., $WR_i \subseteq W$. Thus from $(\alpha\gamma)R_j = (\alpha R_i)(\gamma R_i)$ we get $\gamma R_i = e_j\gamma A^{-1}R_jA$ for $\gamma \in W$ and from $(\gamma\delta)R_i = (\gamma R_j)(\delta R_j)$ we get $e_iB(\gamma A, \delta) = B(\gamma R_jA, \delta R_j)$ for all $\gamma, \delta \in W$. Let $f \in k^*$ such that $e = f^2$ and define $S \in GL(W)$ by $S = (R_1R_f)|_W$, where R_f is defined by (5.12.1). Then $S \in H$ and $(R_1, R_2) = (\overline{S}, \overline{A^{-1}SA})(R_{f^{-1}}, R_f)$, so $G \cong H \times k^*$. Clearly $H \triangleleft G$. Using the identification of G with a subgroup of $\mathrm{Aut}\mathcal{J}$ given above, we can regard H as a subgroup of $\mathrm{Aut}\mathcal{J}$. The relations in (iv) of Proposition 5.9 show that H is normalized by the E_α's and F_β's. H is also normalized by the mapping T defined by (5.8.1), so $H \triangleleft \mathrm{Aut}\mathcal{J}$ by Corollary 5.8.

The rest of the relations in the $\dim V_{st} = 1$ case are given by the following proposition.

Proposition 5.13. *Suppose $\dim\mathcal{S} \geq 4$, $\dim V_a = 1$, $\mathcal{J} = \mathcal{J}(\mathcal{S}, B, s, t)$, where $st = a$, and $\alpha, \beta \in V_a$. Then*

(i) *$E_\alpha E_\beta = E_{\alpha+\beta}$ and $F_\alpha F_\beta = F_{\alpha+\beta}$.*

(ii) *If $B(\alpha, \beta) = -1$, then $E_\alpha F_\beta = B_S E_{-\alpha}$, where $S = (S_1, S_2)$ is defined by*

$$(5.13.1) \qquad \delta S_1 := t\beta\delta - B(\alpha, \delta)\alpha,$$

$$(5.13.2) \qquad \gamma S_2 := s\alpha\gamma - B(\beta, \gamma)\beta.$$

(iii) *If $B(\alpha, \beta) \neq -1$, let $a = -(B(\alpha, \beta) + 1)$, $\alpha' = a^{-1}\alpha$, and $\beta' = a\beta$. Then $E_\alpha F_\beta = A_R F_{-\beta'} E_{-\alpha'}$, where $R = (R_1, R_2)$ is defined by*

$$(5.13.3) \qquad \gamma R_1 := -a\gamma + B(\beta, \gamma)[-1 + a^{-1} - a^{-2}]\alpha,$$

(5.13.4) $$\delta R_2 := -a^{-1}\delta + B(\alpha, \delta)[1 - a - a^{-1}]\beta.$$

Recall that $C(x, y) := \text{trace}(xy)$ for all $x, y \in \mathcal{J}$. The proof of the following proposition is straightforward.

Proposition 5.14. *$C(xE, yE) = C(x, y)$ for all $E \in \text{Aut}\,\mathcal{J}$ and all $x, y \in \mathcal{J}$.*

6. Isomorphisms

Turning now to isomorphisms between $\mathcal{J}_1 = \mathcal{J}(\mathcal{S}_1, B_1, s_1, t_1)$ and $\mathcal{J}_2 = \mathcal{J}(\mathcal{S}_2, B_2, s_2, t_2)$, we have the following theorem, whose proof is quite similar to the proof of Theorem 5.7 and so is omitted.

Theorem 6.1. *Suppose $E : \mathcal{J}_1 \to \mathcal{J}_2$ is an isomorphism of Jordan algebras. Then there is a $C \in \text{Aut}\,\mathcal{J}_2$ such that EC has one of the following forms:*

(1) $\left(\begin{smallmatrix} c & \gamma \\ \delta & d \end{smallmatrix}\right)EC = \left(\begin{smallmatrix} c & \gamma R_1 \\ \delta R_2 & d \end{smallmatrix}\right)$, *where $R_1, R_2 : \mathcal{S}_1 \to \mathcal{S}_2$ satisfy*

(6.1.1) $$B_2(\gamma R_1, \delta R_2) = B_1(\gamma, \delta),$$

(6.1.2) $$t_1 t_2^{-1}(\gamma\delta)R_1 = (\gamma R_2)(\delta R_2),$$

(6.1.3) $$s_1 s_2^{-1}(\gamma\delta)R_2 = (\gamma R_1)(\delta R_1)$$

for all $\gamma, \delta \in \mathcal{S}_1$, or

(2) $\left(\begin{smallmatrix} c & \gamma \\ \delta & d \end{smallmatrix}\right)EC = \left(\begin{smallmatrix} d & \delta S_1 \\ \gamma S_2 & c \end{smallmatrix}\right)$ *where $S_1, S_2 : \mathcal{S}_1 \to \mathcal{S}_2$ satisfy*

(6.1.4) $$B_2(\gamma S_1, \delta S_2) = B_1(\gamma, \delta),$$

(6.1.5) $$s_1 t_2^{-1}(\gamma\delta)S_1 = (\gamma S_2)(\delta S_2),$$

(6.1.6) $$t_1 s_2^{-1}(\gamma\delta)S_2 = (\gamma S_1)(\delta S_1)$$

for all $\gamma, \delta \in \mathcal{S}_1$.

Conversely, if $R_1, R_2 : \mathcal{S}_1 \to \mathcal{S}_2$ satisfy (6.1.1), (6.1.2), *and* (6.1.3), *then $E : \mathcal{J}_1 \to \mathcal{J}_2$ defined by $\left(\begin{smallmatrix} c & \gamma \\ \delta & d \end{smallmatrix}\right)E := \left(\begin{smallmatrix} c & \gamma R_1 \\ \delta R_2 & d \end{smallmatrix}\right)$ is an isomorphism of Jordan algebras, and if $S_1, S_2 : \mathcal{S}_1 \to \mathcal{S}_2$ satisfy* (6.1.4), (6.1.5), *and* (6.1.6), *then $F : \mathcal{J}_1 \to \mathcal{J}_2$ defined by $\left(\begin{smallmatrix} c & \gamma \\ \delta & d \end{smallmatrix}\right)F := \left(\begin{smallmatrix} d & \delta S_1 \\ \gamma S_2 & c \end{smallmatrix}\right)$ is an isomorphism of Jordan algebras. Moreover, if $\sqrt[3]{t_2 s_2^{-1}} \in k$ and $E : \mathcal{J}_1 \to \mathcal{J}_2$ is an isomorphism of Jordan algebras, then it is possible to choose $C \in \text{Aut}\,\mathcal{J}_2$ so that EC has form* (1).

A bijective linear mapping $R : \mathcal{S}_1 \to \mathcal{S}_2$ is a *scalar isomorphism* if there is an $x_R \in k$, $x_R \neq 0$, such that for all $\gamma, \delta \in \mathcal{S}_1$

(6.2) $$(\gamma\delta)R = x_R(\gamma R)(\delta R).$$

If there is a scalar isomorphism from $\mathcal{S}_1$ to $\mathcal{S}_2$, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are *scalar isomorphic*. x_R is the *scaling factor* of R. Note that $x_R R$ is an isomorphism from $\mathcal{S}_1$ to $\mathcal{S}_2$.

Suppose $S \in GL(\mathcal{S}_1)$ such that for all $\gamma, \delta \in \mathcal{S}_1$

(6.3) $$B_1(\gamma S, \delta) = B_1(\gamma, \delta S),$$

and there is a $w_S \in k$, $w_S \neq 0$, such that for all $\gamma, \delta \in \mathcal{S}_1$

(6.4) $$((\gamma S)(\delta S))S = w_S \gamma\delta.$$

We can define a new product $\cdot_S$ on $\mathscr{S}_1$ by

(6.5) $$\gamma \cdot_S \delta := (\gamma\delta)S$$

and a new nondegenerate symmetric bilinear form B_S by

(6.6) $$B_S(\gamma, \delta) = B_1(\gamma S^{-1}, \delta).$$

It is easy to check that $\mathscr{S}_1$ with the product $\cdot_S$ is an anticommutative algebra and B_S is associative. We will refer to $\mathscr{S}_1$ with the product $\cdot_S$ as $\mathscr{S}_S$.

Theorem 6.7. *Suppose* $\sqrt[3]{t_2 s_2^{-1}} \in k$. *Then* $\mathscr{J}_1 \cong \mathscr{J}_2$ *as Jordan algebras iff there is an* $S \in GL(\mathscr{S}_1)$ *satisfying* (6.3) *and* (6.4) *with* $w_S = t_1^{-1} s_1 t_2 s_2^{-1}$ *and a scalar isomorphism* $R : \mathscr{S}_S \to \mathscr{S}_2$ *with* $x_R = t_1^{-1} t_2$ *such that* R *is an isometry from* B_S *to* B_2.

Proof. First suppose $S \in GL(\mathscr{S}_1)$ satisfies (6.3) and (6.4) with $w_S = t_1^{-1} s_1 t_2 s_2^{-1}$, and $R : \mathscr{S}_S \to \mathscr{S}_2$ is a scalar isomorphism with $x_R = t_1^{-1} t_2$ such that R is an isometry from B_S to B_2. Define $E : \mathscr{J}_1 \to \mathscr{J}_2$ by $\left(\begin{smallmatrix} c & \gamma \\ \delta & d \end{smallmatrix}\right)E := \left(\begin{smallmatrix} c & \gamma SR \\ \delta R & d \end{smallmatrix}\right)$. Then E is an isomorphism of Jordan algebras.

Conversely, suppose $\mathscr{J}_1 \cong \mathscr{J}_2$. By Theorem 6.1 we can find an isomorphism $E : \mathscr{J}_1 \to \mathscr{J}_2$ of Jordan algebras such that $\left(\begin{smallmatrix} c & \gamma \\ \delta & d \end{smallmatrix}\right)E = \left(\begin{smallmatrix} c & \gamma R_1 \\ \delta R_2 & d \end{smallmatrix}\right)$, where $R_1, R_2 : \mathscr{S}_1 \to \mathscr{S}_2$ satisfy (6.1.1), (6.1.2), and (6.1.3). If we let $S = R_1 R_2^{-1}$ and $R = R_2$, then S satisfies (6.3) and (6.4) with $w_S = t_1^{-1} s_1 t_2 s_2^{-1}$, $R : \mathscr{S}_S \to \mathscr{S}_2$ is a scalar isomorphism with $x_R = t_1^{-1} t_2$, and R is an isometry from B_S to B_2.

Suppose $\mathscr{S}_1, B_1, s_1, t_1$ are given and $S \in GL(\mathscr{S}_1)$ satisfies (6.3) and (6.4) for some $w_S \in k$, $w_S \neq 0$. For any $T \in GL(\mathscr{S}_1)$ and any $x \in k$, $x \neq 0$, we can define a new product $\cdot_T$ and a new bilinear form B_T on $\mathscr{S}_1$ by $\gamma \cdot_T \delta := [(\gamma T^{-1}) \cdot_S (\delta T^{-1})]T$, where $\cdot_S$ is defined by (6.5), and $B_T(\gamma, \delta) := x^2 B_S(\gamma T^{-1}, \delta T^{-1})$, where B_S is defined by (6.6). Now if we define $R := x^{-1}T$, then R is an isometric scalar isomorphism from $(\mathscr{S}_1, \cdot_S, B_S)$ to $(\mathscr{S}_1, \cdot_T, B_T)$ and $x_R = x$. By Theorem 6.7

$$\mathscr{J}(\mathscr{S}_1, B_1, s_1, t_1) \cong \mathscr{J}(\mathscr{S}_1, \cdot_T, B_T, s_2, t_2),$$

where $t_2 = x t_1$ and $s_2 = t_1^{-1} s_1 t_2 w_S^{-1}$ and every $(\mathscr{S}_2, B_2)$ such that $\mathscr{J}_1 \cong \mathscr{J}_2$ is isometrically isomorphic to $(\mathscr{S}_1, \cdot_T, B_T)$ for some choice of S, T, and x. Thus it is clear that the question of which $(\mathscr{S}_2, B_2)$ gives $\mathscr{J}_2$ isomorphic to $\mathscr{J}_1$ for given $\mathscr{S}_1, B_1, s_1, t_1$ reduces to the study of those $S \in GL(\mathscr{S}_1)$ which satisfy (6.3) and (6.4).

Finally we have the following proposition, whose proof is straightforward.

Proposition 6.8. *Suppose* $S, T \in GL(\mathscr{S}_1)$ *both satisfy* (6.3) *and* (6.4) *with* $w_S = w_T$. *Then* $\mathscr{S}_S \cong \mathscr{S}_T$ *under an isometry from* B_S *to* B_T *iff there are* $R_1, R_2 \in GL(\mathscr{S}_1)$ *such that for all* $\gamma, \delta \in \mathscr{S}_1$

(6.8.1) $$B_1(\gamma R_1, \delta R_2) = B_1(\gamma, \delta)$$

and

(6.8.2) $$(\gamma\delta)R_i = (\gamma R_j)(\delta R_j) \quad \textit{for } i, j = 1, 2, \ i \neq j,$$

and $T = R_1 S R_2^{-1}$. *In this case,* $R_2 : \mathscr{S}_T \to \mathscr{S}_S$ *is an isometric isomorphism. Also, if* $S \in GL(\mathscr{S}_1)$ *satisfies* (6.3) *and* (6.4) *and* $R_1, R_2 \in GL(\mathscr{S}_1)$ *satisfy* (6.8.1) *and* (6.8.2), *then* $T := R_1 S R_2^{-1}$ *satisfies* (6.3) *and* (6.4), *with* $w_T = w_S$.

Note that if $R_1, R_2 \in GL(\mathscr{S}_1)$ satisfy (6.8.1) and (6.8.2), then $A_R \in \operatorname{Aut}\mathscr{J}_1$ for $R = (R_1, R_2)$. The actual computation of $\{S \in GL(\mathscr{S}_1) \mid S$ satisfies (6.3) and (6.4)$\}$ may be quite difficult for particular $\mathscr{S}_1$, but we note that (6.3) and (6.4) together imply that S is in the structure group of $\mathscr{S}_1$, about which much is known, at least for commutative Jordan algebras.

7. Relaxing the $st \neq 0$ requirement

If we assume that s or t is zero, then results similar to the ones in the previous sections are true and are stated here without proof. Verification of these statements depends on calculations that are like those appearing elsewhere in this paper.

Theorem 7.1. *Suppose* $s = 0$ *or* $t = 0$ *and* $\mathscr{J} = \mathscr{J}(\mathscr{S}, B, s, t)$. *Then*

(i) $\mathscr{J}$ *is central simple.*

(ii) *Let* $\mathscr{G}$ *be the set of* $(D_1, D_2) \in \operatorname{End}_k \mathscr{S} \oplus \operatorname{End}_k \mathscr{S}$ *satisfying the following for all* $\gamma, \delta \in \mathscr{S}$:

(7.1.1) $$B(\gamma D_i, \delta) = -B(\gamma, \delta D_j) \quad \text{for } i, j = 1, 2, \ i \neq j,$$

and

(7.1.2) $$\text{if } s \neq 0 : (\gamma\delta)D_2 = (\gamma D_1)\delta + \gamma(\delta D_1),$$

or

(7.1.3) $$\text{if } t \neq 0 : (\gamma\delta)D_1 = (\gamma D_2)\delta + \gamma(\delta D_2).$$

Then $\operatorname{Der}\mathscr{J} = \mathscr{G}$ *where the action of* $(D_1, D_2) \in \mathscr{G}$ *is given by* $\begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix}(D_1, D_2) := \begin{pmatrix} 0 & \gamma D_1 \\ \delta D_2 & 0 \end{pmatrix}$.

(iii) *Let* $R = (R_1, R_2) \in GL(\mathscr{S}) \times GL(\mathscr{S})$ *satisfy*

(7.1.4) $$B(\gamma R_1, \delta R_2) = B(\gamma, \delta) \quad \text{for all } \gamma, \delta \in \mathscr{S}.$$

Define $A_R, B_R \in GL(\mathscr{J})$ *by*

$$\begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} A_R := \begin{pmatrix} c & \gamma R_1 \\ \delta R_2 & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} B_R := \begin{pmatrix} d & \delta R_1 \\ \gamma R_2 & c \end{pmatrix}.$$

Then:

(1) *If* $s = t = 0$ or $\mathscr{S}$ is abelian, $\operatorname{Aut}\mathscr{J} = \{A_R, B_R \mid R$ *satisfies* (7.1.4)$\}$.

(2) *If* $s \neq 0$, $t = 0$, *and* $\mathscr{S}$ *is not abelian,* $\operatorname{Aut}\mathscr{J} = \{A_R \mid R$ *satisfies* (7.1.4) *and* $(\gamma\delta)R_2 = (\gamma R_1)(\delta R_1)$ *for all* $\gamma, \delta \in \mathscr{S}\}$.

(3) *If* $s = 0$, $t \neq 0$, *and* $\mathscr{S}$ *is not abelian,* $\operatorname{Aut}\mathscr{J} = \{A_R \mid R$ *satisfies* (7.1.4) *and* $(\gamma\delta)R_1 = (\gamma R_2)(\delta R_2)$ *for all* $\gamma, \delta \in \mathscr{S}\}$.

Finally, we note that results similar to those of Theorem 7.1 are true for $\mathscr{J}$ when the multiplication in $\mathscr{J}$ is not defined by (1.2), but rather by the more general formula

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}\begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} := \begin{pmatrix} ac + z_1 B(\alpha, \delta) & \begin{matrix} x_1 a\gamma + (1 - x_1)b\gamma + y_2 d\alpha \\ +(1 - y_2)c\alpha + t\beta\delta \end{matrix} \\ \begin{matrix} x_2 b\delta + (1 - x_2)a\delta + y_1 c\beta \\ +(1 - y_1)d\beta + s\alpha\gamma \end{matrix} & bd + z_2 B(\beta, \gamma) \end{pmatrix}$$

where $x_1, x_2, y_1, y_2, z_1, z_2 \in k$ are arbitrary with $z_1 \neq 0, z_2 \neq 0$. We get (1.2) from (7.2) by choosing $x_1 = x_2 = y_1 = y_2 = z_1 = z_2 = 1$. In fact, the only choices for these scalars that give a noncommutative Jordan algebra are $x_1 = x_2 = y_1 = y_2 = 1$ and $z_1 = z_2$ and we have dealt with these in our previous results.

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