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A SPECTRAL SEQUENCE FOR PSEUDOGROUPS ON $\mathbb{R}$

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ABSTRACT. Consider a pseudogroup P of local homeomorphisms of $\mathbb{R}$ satisfying the following property: given points $x_0 < \cdots < x_p$ and $y_0 < \cdots < y_p$ in $\mathbb{R}$, there is an element of P, with domain an interval containing $[x_0, x_p]$, taking each x_i to y_i. The pseudogroup P^r of local C^r homeomorphisms, $0 \leq r \leq \infty$, is of this type as is the pseudogroup P^ω of local real-analytic homeomorphisms. Let Γ be the topological groupoid of germs of elements of P. We construct a spectral sequence which involves the homology of a sequence of discrete groups $\{G_p\}$. Consider the set $\{f \in P \mid f(i) = i, \ i = 0, 1, \dots, p\}$; define $f \sim g$ if f and g agree on a neighborhood of $[0, p] \subset \mathbb{R}$. The equivalence classes under composition form the group G_p. *Theorem*: There is a spectral sequence with $E^1_{p,q} = H_q(BG_p)$ which converges to $H_{p+q}(B\Gamma)$. Our spectral sequence can be considered to be a version which covers the real-analytic case of some well-known theorems of J. Mather and G. Segal. The article includes some observations about how the spectral sequence applies to $B\Gamma_1^\omega$. Further applications will appear separately.

INTRODUCTION

Underlying the structure of a manifold on a topological space, or of a flow or foliation on a differentiable manifold is an appropriate pseudogroup P of local transformations of Euclidean space. P carries the transition functions of the structure. The universal properties of P are captured in its classifying space. As Haefliger showed in 1970, each pseudogroup P has a universal space $B\Gamma(P)$ which classifies, up to homotopy, structures compatible with P in much the same way that each topological group G classifies, up to isomorphism, structures compatible with G (namely G-bundles). The simplicial structure of the $B\Gamma(P)$'s will be the focus of our attention in this paper. We will see from our main result that intrinsic to a pseudogroup P on $\mathbb{R}$ is a family of discrete groups $\{G_p\}$ of diffeomorphisms such that $B\Gamma(P)$ is built up in a natural way out of the G_p.

More specifically, consider a pseudogroup P of local homeomorphisms of $\mathbb{R}$ satisfying the following property: given points $x_0 < \cdots < x_p$ and $y_0 < \cdots < y_p$ in $\mathbb{R}$, there is an element of P, with domain an interval containing $[x_0, x_p]$, taking each x_i to y_i. The pseudogroup P^r of local C^r homeomorphisms, $0 \leq r \leq \infty$, is of this type as is the pseudogroup P^ω of local real-analytic homeomorphisms. Let Γ be the topological groupoid of germs of elements of

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P. The spectral sequence we construct involves the homology of a sequence of discrete groups $\{G_p\}$, $p = 0, 1, 2, \ldots$. Consider the set $\{f \in P \mid f(i) = i, i = 0, 1, \ldots, p\}$; define $f \sim g$ if f and g agree on a neighborhood of $[0, p] \subset \mathbb{R}$. The equivalence classes under composition form the group G_p.

Theorem 1. *There is a spectral sequence with* $E^1_{p,q} = H_q(BG_p)$ *which converges to* $H_{p+q}(B\Gamma)$.

The differentials will be defined in §2.

Our spectral sequence is closely related to some well-known results about the homology of $B\Gamma$. In 1974, J. Mather proved that for certain P (including the C^r pseudogroups, $0 \le r \le \infty$) there is an integer-homology equivalence $BG_c \to \Omega B\Gamma$ where G_c is the discrete group of homeomorphisms of $\mathbb{R}$ which are in P and have compact support. Mather's theorem does not apply to the real-analytic case; in fact $B\Gamma_1^\omega$ is a $K(\pi, 1)$ with π an uncountable perfect group, whereas $B\Gamma_1^r$, $0 \le r \le \infty$ is simply connected. Our spectral sequence is a version of Mather's theorem which covers the real-analytic pseudogroup on $\mathbb{R}$. In what sense our result is a generalization is explained in §4. Our spectral sequence also provides a version of a theorem of G. Segal. In 1978 Segal proved that $B\Gamma_1^r$, $0 \le r \le \infty$ has the same homology as the discrete group of C^r diffeomorphisms of $\mathbb{R}$. In §5 we show how to prove Segal's theorem in our setting.

We conclude the paper with some observations about how the spectral sequence applies to the C^ω case.

Applications to $B\Gamma_1^\omega$ will appear in separate articles.

1. Pseudogroups, groupoids, and classification

Let Z be a topological space and P a *pseudogroup of local homeomorphisms* of Z. There is a *topological category* Γ naturally associated to P: the objects of Γ, $O(\Gamma)$, is the space Z and the space of morphisms of Γ, $M(\Gamma)$ is the set of germs of elements of P at the points of Z with the sheaf topology.

The topological categories which will be of interest to us are *groupoids*, (all morphisms are isomorphisms), arising from pseudogroups of local homeomorphisms of $\mathbb{R}$.

Let P_1^r be the pseudogroup of local C^r homeomorphisms of $\mathbb{R}$ which are orientation preserving, and let Γ_1^r be the associated groupoid of germs of these homeomorphisms. Here $0 \le r \le \infty$ or $r = \omega =$ real analytic.

The difference between the C^r and C^ω pseudogroups is captured in the following definitions.

We call a pseudogroup P on $\mathbb{R}$ *weakly connected* if it satisfies properties (i) and (ii) below:

(i) orientation preserving: if $x < y \in \mathbb{R}$ then for all $f \in P$, with domain containing $[x, y]$, $f(x) < f(y)$.

(ii) If $x_0 < \cdots < x_p$ and $y_0 < \cdots < y_p$ are in $\mathbb{R}$ then there is an element $f \in P$ with domain containing $[x_0, x_p] \subset \mathbb{R}$ such that $f(x_i) = y_i$ for all i.

We call a pseudogroup P on $\mathbb{R}$ *strongly connected* if it is weakly connected and in addition satisfies property (δ) of Mather [12].

(δ) Let f_x, g_y be in $\Gamma(P)$ with $x < y$ and $f(x) < g(y)$. Then there exists $h \in P$ with domain containing $[x, y]$ such that $h_x = f_x$ and $h_y = g_y$.

Note P_1^r, $0 \le r \le \infty$, $r = \omega$, are all weakly connected, P_1^r, $0 \le r \le \infty$, are strongly connected, and P_1^ω is not strongly connected.

The sequence of constructions

$$P \to \Gamma \to N\Gamma \to B\Gamma$$

produces the classifying space for structures compatible with P. Here $N\Gamma$ is a simplicial space which is the nerve of Γ, and $B\Gamma$ is the thick realization of $N\Gamma$, $B\Gamma = \|N\Gamma\|$.

Basic references for this section are [2, 3, 16 and 20].

2. The intrinsic spectral sequence of $B\Gamma$

The first step in constructing the spectral sequence is to describe $B\Gamma$ up to homology as the realization of a bisimplicial set $S_L\Gamma_{**}$ which arises out of a simplification of the singular bicomplex $S\Gamma_{**}$. Theorem 1 then interprets the homology of the bicomplex $S_L\Gamma_{**}$.

Consider a pseudogroup P of local homeomorphisms of $\mathbb{R}$ which is weakly connected. Let Γ be the topological groupoid associated to P and let Γ_δ denote Γ with the discrete topology.

We construct a set $S_L\Gamma$ which is both a discrete groupoid and a discrete category in distinct but natural ways.

$S_L\Gamma$ as a groupoid. Let $S_L\mathbb{R}$ be the set of closed intervals $[a, b] \subset \mathbb{R}$, $a \le b$. For $[a, a']$, $[b, b']$ elements of $S_L\mathbb{R}$ consider the set of all $f \in P$ with connected domain containing $[a, a']$ satisfying $f(a) = b$, $f(a') = b'$. Define an equivalence relation on this set by $f \sim g$ if $f = g$ on some neighborhood of $[a', a']$. The morphisms of $S_L\Gamma$ from $[a, a']$ to $[b, b']$ are these equivalence classes.

$S_L\Gamma$ as a category. If $f : [a, a'] \to [b, b']$ is an element of $S_L\Gamma$ then we define the source of f to be the germ of f at a and the target of f to be the germ of f at a'. We define composition by juxtaposition. That is if $f : [a, a'] \to [b, b']$ and $g : [a', a''] \to [b', b'']$ are given and the germ of f at a' equals the germ of g at a'; then define the composition $f * g$ to be the unique element of $S_L\Gamma$ with $f * g \,|\, [a, a'] = f$, $f * g \,|\, [a', a''] = g$.

In summary we have the following diagram:

$$\begin{array}{ccc} \Gamma_\delta & \leftleftarrows & S_L\Gamma \\ \downdownarrows & & \downdownarrows \\ \mathbb{R}_\delta & \leftleftarrows & S_L\mathbb{R} \end{array}$$

The *horizontal maps*, are the source and target maps of $S_L\Gamma$ as a category with objects Γ_δ. The *vertical maps* are the source and target maps of $S_L\Gamma$ as a groupoid with objects $S_L\mathbb{R}$.

We will always identify the objects of a category as the subset of identity morphisms.

Now the above diagram extends to a bisimplicial set $S_L\Gamma_{**}$ by extending by nerves in the horizontal (h) and vertical (v) directions

$$S_L\Gamma_{p,q} = N_v^q N_h^p S_L\Gamma.$$

To see why this construction works note that the nerve of $S_L\Gamma$ considered as a category is itself a simplicial groupoid with the groupoid structure on $N_h^p S_L\Gamma$

induced in the obvious way from that of $S_L\Gamma$. The simplicial set $N_v^* N_h^p S_L\Gamma$ is then the nerve of the groupoid $N_h^p S_L\Gamma$. Equivalently $S_L\Gamma_{p,q} = N_h^p N_v^q S_L\Gamma$.

In any case we see that $S_L\Gamma_{p,1}$ is a groupoid for each p and $S_L\Gamma_{1,q}$ is a category for each q and their nerves fit together in a way that yields a bisimplicial set $S_L\Gamma_{**}$.

Lemma 2. *There is a map* $\|S_L\Gamma_{**}\| \to B\Gamma$ *inducing an isomorphism on all integer homology groups.*

We prove the lemma and the main theorem in the following section.

First we describe the differentials $d_{p,q}^1$ of the intrinsic spectral sequence.

Consider the groupoid $N_h^p S_L\Gamma$. Let G_p be the discrete group of all elements of $N_h^p S_L\Gamma$ keeping the object $[0, 1, \dots, p] = [0, 1] * \cdots * [p-1, p] \in N_h^p S_L\mathbb{R}$ fixed. For any given face of $[0, 1, \dots, p]$ there is, by weak connectedness, a morphism of $N_h^{p-1} S_L\Gamma$ with source the given face and target $[0, 1, \dots, p-1]$. For each face choose once and for all such a morphism. Then each face map $\partial_i : N_h^p S_L\Gamma \to N_h^{p-1} S_L\Gamma$ induces a homomorphism $\phi_i : G_p \to G_{p-1}$ obtained by taking the face in $N_h S_L\Gamma$ and appropriately conjugating it to an element of G_{p-1}.

The differentials of Theorem 1 are given by

$$d_{p,q}^1 = \sum_{i=0}^{p} (-1)^i (\phi_i)_*,$$

where the ∂_i are the face maps described above.

3. Proofs

Proof of Lemma 2. We begin with some general remarks. Let X_* be a simplicial space, and let $SX_{p,q}$ be a set of singular p-simplices on X_q. Then SX_{**} is a bisimplicial set whose realization $\|SX_{**}\|$ is weakly homotopy equivalent to $\|X_*\|$; see [14, 17]. As a result there are two spectral sequences converging to $H_{p+q}(\|SX_{**}\|)$, [15].

(1) $E_{p,q}^2 = H_p^h H_q^v (SX_{**}) \Rightarrow H_{p+q}(\|SX_{**}\|)$,

(2) $E_{p,q}^2 = H_q^v H_p^h (SX_{**}) \Rightarrow H_{p+q}(\|SX_{**}\|)$.

We apply these constructions with $X = N\Gamma$.

To prove Lemma 2 we define below two intermediate bisimplicial sets $OS_{\mathscr{U}}\Gamma$ and $S_{\mathscr{U}}\Gamma$ with bisimplicial maps as shown

$$S_L\Gamma \xleftarrow{C} OS_{\mathscr{U}} \xrightarrow{B} S_{\mathscr{U}}\Gamma \xrightarrow{A} S(N\Gamma).$$

We will see, essentially from acyclic models, that each of the maps A, B, and C induce chain equivalences on horizontal bisimplicial sets, and moreover that C induces a homotopy equivalence on the realization of each horizontal bisimplicial set. Then Lemma 2 follows immediately from the spectral sequence (2).

Construction of $S_{\mathscr{U}}\Gamma$. Let Y be a space and $\mathscr{U}$ an open cover of Y. Let $S_{\mathscr{U}}^p Y$ be the set of singular p-simplices on X subordinate to the open cover $\mathscr{U}$. The simplicial set $S_{\mathscr{U}}Y : p \to S_{\mathscr{U}}^p Y$ is a subsimplicial set of SY. It is well known that the inclusion $S_{\mathscr{U}}Y \to SY$ induces an equivalence on chains (see e.g. [19, p. 178]).

Let $\mathscr{U}_q$ be the cover of $N\Gamma(q)$ by basic open sets in the sheaf topology, and let $S_{\mathscr{U}}\Gamma_{**}$ be the bisimplicial set $S^p_{\mathscr{U}}N\Gamma(q)$. We have a bisimplicial map $A : S_{\mathscr{U}}\Gamma \to S\Gamma$ induced by the inclusion of horizontal bisimplicial sets and by our above remarks the map A is an equivalence on horizontal chain complexes.

Construction of $OS_{\mathscr{U}}\Gamma$. Consider the simplicial set $OS\mathbb{R}$ of order preserving simplices on $\mathbb{R}$: a simplex $m \in S^p\mathbb{R}$ is order preserving if $m(v_{i+1}) \geq m(v_i)$ where $v_0, v_1, \ldots, v_p$ are the ordered vertices of the standard p-simplex. Let $OS^p\Gamma$ be the set of singular p-simplices on Γ whose source (and hence target) is in $OS^p\mathbb{R}$. As before, $OS^p\Gamma$ becomes a discrete groupoid with objects $OS^p\mathbb{R}$ and we extend vertically by the nerve of $OS^p\Gamma$ for each p to obtain a bisimplicial set $OS\Gamma_{**}$. If we consider only simplices subordinate to the open covers $\mathscr{U}_q$ of $N\Gamma(q)$ we obtain a bisimplicial set $OS_{\mathscr{U}}\Gamma_{**}$ and a map $B : OS_{\mathscr{U}}\Gamma_{**} \to S_{\mathscr{U}}\Gamma_{**}$.

To see that B is an equivalence on horizontal chains one easily modifies the classical acyclic models argument used to show that the ordered chain complex of a simplicial complex is chain equivalent to the standard one (see e.g. [19, p. 171]).

The map C. Now consider $OS^p_{\mathscr{U}}\Gamma$, the ordered p-simplices on Γ subordinate to the cover $\mathscr{U}$ by basic open sets. For $m, n \in OS^p_{\mathscr{U}}\Gamma$ define $m \sim n$ if there is a basic open set $U \in \mathscr{U}$ such that $\text{image}(m) \subset U$, $\text{image}(n) \subset U$ and $m(v) = n(v)$ for each vertex v in the standard p-simplex. Then $N^p_h S_L\Gamma = OS^p\Gamma/R$ where R is the equivalence relation generated by $\sim$.

So there is a quotient simplicial map ρ from $OS^*_{\mathscr{U}} : p \to OS^p_{\mathscr{U}}\Gamma$ to $S^*_L\Gamma : p \to N^p_h S_L\Gamma$ and it is easy to see that ρ induces a homotopy equivalence on realizations: to find a homotopy inverse $\bar{\rho}$ choose a representative $OS^p_{\mathscr{U}}\Gamma$ for each element of $N^p_{\mathscr{U}}S_L\Gamma$. This can be done simplicially in the p direction. Now $\rho\bar{\rho} = \text{id}$ and because each U is contractible $\bar{\rho}\rho$ is homotopic to the identity.

ρ extends naturally to a bisimplicial map $OS_{\mathscr{U}}\Gamma_{**} \to S_L\Gamma_{**}$ and for each fixed q there is, in analogy to the above, a homotopy equivalence from $p \to OS^p_{\mathscr{U}}N\Gamma(q)$ to $p \to N^q_v N^p_h S_L\Gamma$. Hence there is a quotient bisimplicial map $OS_{\mathscr{U}}\Gamma_{**} \to S_L\Gamma_{**}$ inducing a homotopy equivalence $\|OS_{\mathscr{U}}\Gamma_{**}\| \to \|S_L\Gamma_{**}\|$. This completes the proof of Lemma 2. □

With a little more effort one can actually show that A and B are homotopy equivalences [6].

Proof of Theorem 1. To prove Theorem 1 we compute homology using the spectral sequence (1)

$$E^2_{p,q} = H^h_p H^v_q(S_L\Gamma_{**}) \Rightarrow H_{p+q}(\|S_L\Gamma_{**}\|).$$

To simplify the statement somewhat let us consider the spectral sequence as beginning with E^1; let us write $S^p_L\Gamma$ for the discrete groupoid $N^p_h S_L\Gamma$ whose nerve forms the pth column of $S_L\Gamma_{**}$, and let us use Lemma 2 to identify $\|S_L\Gamma_{**}\|$ with $B\Gamma$. This gives

$$E^1_{p,q} = H_q(BS^p_L\Gamma) \Rightarrow H_{p+q}(B\Gamma).$$

In order to obtain a satisfactory description of the groups in the E^1 term we will make use of some elementary facts about discrete groupoids [4].

Given G a discrete groupoid, define an equivalence relation $\approx$ on Objects (G) : $x \approx y$ if there is a morphism from x to y. Clearly $\pi_0(BG)$ is in one-to-one correspondence with Objects$(G)/\approx$. Choose one object α in each equivalence class. The set $\mathscr{A} = \{\alpha\}$ is called a set of base points for G. The isotropy group of the base point α is $\pi_\alpha = \{m \in G \mid \text{source}(m) = \text{target}(m) = \alpha\}$. It is well known (and easy to prove) that

$$BG = \coprod_{\alpha \in \mathscr{A}} K(\pi_\alpha, 1).$$

Next let $F : G \to G'$ be a functor of groupoids, and let $\mathscr{A}$ and $\mathscr{A}'$ be sets of base points for G and G' respectively. For each object x of G' pick once and for all a morphism $\rho(x)$ from the base point of the component containing x to x. Call $\{\rho(x)\}$ a set of base paths for G'. Then F induces a homomorphism $F_\# : \pi_\alpha \to \pi_{\alpha'}$, where α' is the base point of the component containing $F(\alpha)$. Namely, $F_\#(m) = \rho(F(m))^{-1} \circ F(m) \circ \rho(F(m))$, for $m \in \pi_\alpha$.

Now we choose *bases for the groupoids* $S^p_L\Gamma$. Let $[a_0, \dots, a_p] \in S^p_L\mathbb{R}$ be nondegenerate. Then we have $a_0 < \cdots < a_p$ and by weak connectedness there exists an $f \in S^p_L\Gamma$ with source $[0, 1, \dots, p]$ and target $[a_0, \dots, a_p]$. Moreover if $[b_0, \dots, b_p]$ is degenerate then there is at least one b_i so that $b_{i+1} = b_i$ and there is a $g \in S^p_L\Gamma$ with target $[b_0, \dots, b_p]$, and source $[0, \dots, q]$ where $q < p$ and all entries are integers.

As a set of base points of $S^p_L\Gamma$ we may take all distinct $[b_0, \dots, b_p] \in S^p_L\mathbb{R}$ so that $b_0 \le \cdots \le b_p$, $b_0 = 0$, and $b_{i+1} - b_i = \{0 \text{ or } 1\}$. Note there is one base point $[0, 1, \dots, p]$ for the unique component of $S^p_L\Gamma$ consisting of nondegenerate simplices.

To complete the construction we choose a morphism from a basis element $[0, \dots, p]$ to $[a_0, \dots, a_p]$ once and for all for each $[a_0, \dots, a_p] \in S^p_L\mathbb{R}$. Then the isotropy group of the base point $[0, 1, \dots, p]$ of the unique component of $S^p_L\Gamma$ containing the nondegenerate simplices is exactly the group G_p defined in the statement of Theorem 1, and each face map $S^p_L\Gamma \to S^{p-1}_L\Gamma$ induces a homomorphism $G_p \to G_{p-1}$ which depends on the choice of base points and base paths.

To complete the proof we factor out by degenerate simplices $D^1_{p,q} : p \to DH_q(BS^p_L\Gamma)$ and let $\widetilde{E}^1_{p,q} = E^1_{p,q}/D^1_{p,q} = H_q(BG_p)$. Moreover, $E^1_{p,q} \to \widetilde{E}^1_{p,q}$ induces an isomorphism on the homology of all the horizontal complexes [10] so that $\widetilde{E}^1_{p,q}$ is the required spectral sequence. □

4. Compactly supported diffeomorphisms of $\mathbb{R}$ and $B\Gamma$

Mather's theorem [12] states that for a strongly connected P there is a map $BG \to \Omega B\Gamma$ inducing an isomorphism of all integer homology groups. The proof of the theorem involves the construction of a space BBG and has two steps.

(i) There is a weak homotopy equivalence $BBG \to B\Gamma$;

(ii) There is a map $BG \to \Omega BBG$ inducing an isomorphism on all integer homology groups.

We indicate below how to construct BBG and deduce (i) from the main constructions of §3. The proof of step (ii) is due to Quillen and is given by Mather in [12].

Consider $S_L\mathbb{R} = \{[a, b] | a \leq b \in \mathbb{R}\}$, as before. We construct a discrete groupoid $S_C\Gamma$ with objects $S_L\mathbb{R}$ as follows. There is a morphism from $[a, b]$ to $[a', b']$ if and only if $a = a'$, $b = b'$ in which case the set of all morphisms from $[a, b]$ to $[a, b]$ is the group of all compactly supported P-homeomorphisms with support contained in $[a, b]$.

$S_C\Gamma$ also has the structure of a discrete category with objects $\mathbb{R}$ considered as a discrete set, namely if $f : [a, b] \to [a, b]$ is in $S_C\Gamma$ then as a category the source is "a" and the target is "b."

In summary we have the following diagram:

$$\begin{array}{ccc} \mathbb{R} & \leftleftarrows & S_C\Gamma \\ \downdownarrows & & \downdownarrows \\ \mathbb{R} & \leftleftarrows & S_L\mathbb{R} \end{array}$$

The above diagram gives rise to a bisimplicial set $S_C\Gamma_{**}$ constructed in the same manner as in §3. The space $\|S_C\Gamma_{**}\|$ is the space BBG constructed by Mather in [12], (except we are using $\| \cdot \|$).

We will show that our proof of Lemma 2 can be modified to obtain a weak homotopy equivalence $\|S_C\Gamma_{**}\| \to B\Gamma$.

The hypothesis on P implies that $N\Gamma(q)$ is connected as a topological space for all q. Fix q and consider the simplicial set $p \to S^p N\Gamma(q)$. $\mathbb{R}$ is embedded as the subspace of $N\Gamma(q)$ consisting of points whose entries are all identity germs. Let $\dot{S}^p N\Gamma(q)$ denote the subset of $S^p N\Gamma(q)$ of all simplices $\sigma : \Delta^p \to N\Gamma(q)$ such that $\sigma(v) \in \mathbb{R} \subset N\Gamma(q)$ for all vertices v. Because $N\Gamma(q)$ is a connected space for all q the complex $p \to \dot{S}^p N\Gamma(q)$ has the same homology as $p \to S^p N\Gamma(q)$, (see [19, p. 392]).

From this point on the same constructions as in the proof of Lemma 2 yield an homology isomorphism $\|S_C\Gamma_{**}\| \to B\Gamma$:

$$\|S_C\Gamma_{**}\| \leftarrow \|O\dot{S}_{\mathscr{U}}\Gamma_{**}\| \to \|\dot{S}_{\mathscr{U}}\Gamma_{**}\| \to \|\dot{S}\Gamma_{**}\| \to \|S\Gamma_{**}\| \to B\Gamma.$$

Of course in the strongly connected case $\|S_C\Gamma_{**}\|$ and $B\Gamma$ are simply connected so one obtains a weak homotopy equivalence directly from the Whitehead Theorem.

We do not see how to deduce (ii) from Theorem 1 without in effect redoing Quillen's proof. Nevertheless it is possible to carry out all of Mather's applications to $H_2(B\Gamma)$ using our spectral sequence and the known calculations of $H_1(BG_C)$. Let $G = G_C$.

Theorem 2 (Mather). *Let P be strongly connected and assume that $H_1(BG) = 0$. Then $E^2_{p,1} = 0$ for all p.*

Proof. There is a homomorphism $f_p : G_p \to G_0 \times \cdots \times G_0$ ($p+1$ times) given by projection on the germs at $0, 1, \ldots, p$. The kernel of f_p is isomorphic to $G \times \cdots \times G$ (p-times). We are assuming that $H_1(BG) = 0$ so

$$H_1(B(G \times \cdots \times G)) = 0$$

also. As a consequence $(f_p)_* : H_1(BG_p) \to H_1(B(G_0 \times \cdots \times G_0))$ is an isomorphism. The complex $p \to H_1(BG_0^{p+1})$ is isomorphic to $p \to [H_1(BG_0)]^{p+1}$ which is an "infinite simplex" with vertices $H_1(BG_0)$. So all the homology groups of the complex are 0. $\square$

The hypotheses of Theorem 2 are known to be satisfied for the pseudogroups P_1^r, $0 \leq r \leq \infty$, $r \neq 2$, see [1, 11 and 13].

If G is acyclic a slight modification of this proof gives

Theorem 3 (Mather). *Let P be a strongly connected pseudogroup. Assume $\widetilde{H}_k(BG) = 0$ for all k. Then $\widetilde{H}_k(B\Gamma) = 0$ for all k and $B\Gamma$ is contractible.*

5. Diffeomorphisms of $\mathbb{R}$ and $B\Gamma$

It follows from results of Segal [18] that the homology of $B\Gamma$ is the same as the homology of the group of diffeomorphisms of $\mathbb{R}$. We will prove this by comparing the spectral sequence of Theorem 1 to the spectral sequence of an appropriate pair of groups [5]. A crucial step will be to use Segal's Proposition 3.2 which says that a certain family of diffeomorphisms of $\mathbb{R}$ is acyclic.

Let P be a strongly connected pseudogroup, Γ its associated topological groupoid, and K the discrete group of globally defined P-homeomorphisms of $\mathbb{R}$.

Define a simplicial group $p \to S_L K_p$ as follows. The set of objects of $S_L K_p$ is $S_L \mathbb{R}$. The set of morphisms from $[a, a']$ to $[b, b']$ is the set of $f \in K$ with $f(a) = b$, $f(a') = b'$.

As in §3 we obtain a simplicial set $S_L K_{**}$ by extending vertically by nerves: $S_L K_{p,q} = N_q S_L K$. There is a natural map of bisimplicial sets $Q : S_L K_{**} \to S_L \Gamma_{**}$ which is the obvious quotient map on each p, q-term. We already know that $\|S_L \Gamma_{**}\| = B\Gamma$. Now there is a homotopy equivalence $\|S_L K_{**}\| \to BK$ for both of those spaces are homotopy equivalent to $\|\{K/K_0\}_{**}\|$, see [5], where K_0 is the set of elements of K keeping $0 \in \mathbb{R}$ fixed.

It remains to show

Lemma 4. *The mapping $\|Q\| : \|S_L K_{**}\| \to \|S_L \Gamma_{**}\|$ induces isomorphisms on all integer homology groups.*

An immediate consequence is

Theorem 5 (Segal). *There is a map $BK \to B\Gamma$ inducing isomorphisms on all integer homology groups.*

Proof of Lemma 4. The spectral sequence associated to $S_L K_{**}$ is $E^1_{p,q} = H_q(BK_p) \Rightarrow H_{p+q}(BK)$ where K_p is the isotropy group of $[0, 1] \times \cdots \times [p-1, p]$. The spectral sequence associated to $S_L \Gamma_{**}$ is $E^1_{p,q} = H_q(BG_p) \Rightarrow H_{p+q}(B\Gamma)$.

There is a quotient homomorphism $f_p : K_p \to G_p$ and a natural map of E^1 terms induced by Q,

$$H_q(BK_p) \to H_q(BG_p).$$

Now Proposition 3.2 of [18] applies to all of the discrete groups $\ker(f_p)$ to show that they are acyclic. Hence $H_q(BK_p) \to H_q(BG_p)$ is an isomorphism for all p and q. By the comparison theorem for spectral sequences there is an isomorphism $H_{p+q}(BK) \to H_{p+q}(B\Gamma)$ which is induced by Q. This completes the proof of the lemma and the theorem. □

6. Remarks on $B\Gamma_1^\omega$

The classifying space $B\Gamma_1^\omega$, unlike the $B\Gamma_1^r$'s, $0 \leq r \leq \infty$, is a $K(\pi, 1)$. A fundamental unresolved question is: What is $H_2(B\Gamma_1^\omega)$? We prove in [9] that

$E^2_{p,1} = 0$ in our spectral sequence, hence $H_2(B\Gamma_1^\omega)$ is the quotient of $H_2(BG_0^\omega)$ by the image of $H_2(BG_1^\omega)$ under $d^1_{1,2}$. It is easy to see that $H_2(BG_0^\omega)$ has a summand $H_2(B\mathbb{R}^+)$ where $\mathbb{R}^+$ is the discrete multiplicative group of positive reals [7]. However, the image of $H_2(B\mathbb{R}^+)$ in $H_2(B\Gamma_1^\omega)$ is zero. In fact, the image of any cycle represented by a Γ_1^ω structure on the torus is zero [8]. There are, however, many nontrivial representations of surfaces of genus $g > 1$ in G_0^ω, which raises the possibility of additional homology.

We have been approaching the calculation of $H_2(BG_0^\omega)$ through the finite jet groups J_n, and have proved in [7] that $H_2(BJ_n) = H_2(B\mathbb{R}^+)$ for all n. The proof requires constructing an increasing number of boundaries as n gets larger in order to "kill" derivatives.

We briefly speculate on the implication of these results. Because of the above observation it seems likely that in the inverse limit of the J_n's and in G_0^ω it becomes impossible to bound all cycles. Moreover the relations coming from $H_2(BG_1^\omega)$ are few because of the constraint of analytic continuation. We expect therefore in contrast to the C^r results that $H_2(B\Gamma_1^\omega)$ is not zero.

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