TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 334, Number 1, November 1992

COHOMOLOGICAL ASPECTS OF HYPERGRAPHS

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ABSTRACT. By a k-graph we will mean a collection of k-element subsets of some fixed set V. A k-graph can be regarded as a $(k-1)$-chain on 2^V, the simplicial complex of all subsets of V, over the coefficient group $\mathbb{Z}/2$, the additive group of integers modulo 2. The induced group structure on the $(k-1)$-chains leads to natural definitions of the coboundary δ of a chain, the cochain complex of $C = \{C^k, \delta\}$ and the usual cohomology groups $H^k(C; \mathbb{Z}/2)$. In particular, it is possible to construct what could be called "higher-order" coboundary operators $\delta^{(i)}$, where $\delta^{(i)}$ increases dimension by i (rather than just 1).

In this paper we will develop various properties of these $\delta^{(i)}$, and in particular, compute the corresponding cohomology groups for 2^V over $\mathbb{Z}/2$. It turns out that these groups depend in a rather subtle way on the arithmetic properties of i.

1. INTRODUCTION

Among the most fundamental objects occurring in combinatorics are the so-called *k-uniform hypergraphs*, or *k-graphs*, for short. A k-graph is simply a collection of (distinct) k-element subsets, called *edges*, of some fixed set V. Because of the great generality of this definition, virtually any problem in combinatorics can be phrased in terms of a corresponding question about an appropriate class of k-graphs. For example, much of the field of Ramsey theory (cf. [GRS90]) can be interpreted simply as the study of chromatic numbers of certain k-graphs (where the chromatic number of a k-graph is the minimum number of classes into which V can be partitioned so that no edge is contained entirely in one class). For a full discussion of k-graphs, the reader should consult Berge [B89].

From a somewhat different point of view, k-graphs can also be regarded as $(k-1)$-chains on 2^V, the simplicial complex of all subsets of V, over the coefficient group $\mathbb{Z}/2$, the additive group of integers modulo 2 (so that the orientation of simplices is irrelevant). From this perspective, the induced group structure on the $(k-1)$-chains leads to natural definitions of the coboundary δ of a chain, the cochain complex $C = \{C^k, \delta\}$ and the usual cohomology groups $H^k(C; \mathbb{Z}/2)$. (For an excellent discussion of these concepts, the reader is referred to Munkres [M84].)

Received by the editors September 27, 1990.
1991 *Mathematics Subject Classification*. Primary 05C65, 05E99, 55N20.

In particular, it is possible to construct a class of what could be called "higher-order" coboundary operators $\delta^{(i)}$, where $\delta^{(i)}$ increases dimension by i (rather than just 1). Thus, if G is a k-graph then $\delta^{(i)}G$ will be a $(k+i)$-graph. These higher order $\delta^{(i)}$ were in fact introduced by S. T. Hu [H49, H50, H52] in 1949, who showed that they satisfy all but one of the Eilenberg-Steenrod axioms for a cohomology theory.

It turns out that in recent work of the authors and R. M. Wilson [CGW89, CG90, CG91] investigating aspects of random-like behavior in k-graphs, these higher-order coboundary operators arose in a natural way, and played an important role in settling several fundamental conjectures there.

In this paper we will develop further properties of these $\delta^{(i)}$, and in particular, compute the corresponding cohomology groups for 2^V over $\mathbb{Z}/2$. As will be seen (Theorem 4), these groups depend in a rather subtle way on the arithmetic properties of i, and in particular, on the representation of i to the base 2. We point out that there is a considerable body of work dealing with cohomological aspects of 3-uniform hypergraphs (cf. [MS75, C77, C78, S76, ST81, ML83, Z81, We84, CW86]). In some sense, our results can be considered as the beginnings of a natural extension of this work to general hypergraphs.

2. Definitions and basic properties

Let V be a finite set of cardinality $|V| = n$, and let $\binom{V}{k}$ denote the family of k-element subsets of V. We denote by $C_k = C_k(V)$ the vector space over $\mathbb{Z}/2$ (the integers modulo 2) generated by the $X \in \binom{V}{k}$. The elements of C_k are called k-graphs (on V). Thus, each k-graph $G \in C_k$ can be written as

$$G = \sum_{X \in \binom{V}{k}} \chi_G(X)X \quad \text{where } \chi_G : \binom{V}{k} \to \mathbb{Z}/2 .$$

We will sometimes write this as $G = (V, \chi_G)$, or $G = G^{(k)}(n)$, if we wish to emphasize that G is a k-graph on a set V of n vertices. The elements of $E = E(G) := \chi_G^{-1}(1)$ are called the edges of G, and we will also occasionally write $G = (V, E)$.

For $k < 0$ or $k > n$, C_k consists of the single element 0, the identity element of $\mathbb{Z}/2$. With the convention that $\binom{V}{0}$ consists of the single element $\varnothing$, the generic element of C_0 is

$$G^{(0)} = \sum_{X \in \binom{V}{0}} \chi_{G^{(0)}}(X)X = \chi_{G^{(0)}}(\varnothing)\varnothing .$$

We define $G_0^{(0)}$ to be the 0-graph having $\chi_{G^{(0)}}(\varnothing) = 0$, so that $G_0^{(0)} = 0 \in C_0$. Similarly, we define $G_1^{(0)}$ to be the (other) 0-graph having $\chi_{G^{(0)}}(\varnothing) = 1$. Thus, $G_1^{(0)} = \varnothing \in C_0$. (This convention will be useful later.) The group addition in C_k satisfies

$$\chi_{G+G'} \equiv \chi_G + \chi_{G'} \pmod 2 \quad \text{for } G, G' \in C_k .$$

We will often suppress the dependence of quantities on k when the meaning is clear, e.g., 0 will denote the zero element in C_k for every k.

For $p \geq 1$, we define the *p-coboundary operator* $\delta^{(p)} : C_k \to C_{k+p}$ as follows:

For $G = (V, \chi_G) \in C_k$, $\delta^{(p)}F = (V, \chi_F) \in C_{k+p}$ where for $Y \in \binom{V}{k+p}$,

$$\chi_F(Y) := \sum_{X \in \binom{Y}{k}} \chi_G(X) . \tag{2.1}$$

(As remarked earlier, $\delta^{(p)}$ should actually be written $\delta_k^{(p)}$; we will omit the index k when context makes it clear.) It is easily checked that $\delta^{(p)}$ is a vector space homomorphism, so we have a natural cochain complex $(C, \delta^{(p)})$ on $C = \bigcup_k C_k$:

$$\dots \xrightarrow{\delta^{(p)}} C_k \xrightarrow{\delta^{(p)}} C_{k+p} \xrightarrow{\delta^{(p)}} \cdots \tag{2.2}$$

(cf. Munkres [M84]).

Actually, (2.2) represents p disjoint cochain complexes, depending on the residue class of k modulo p. One of our goals will be to compute the cohomology groups of $(C, \delta^{(p)})$ over $\mathbb{Z}/2$ (see §5).

Fact 2.1. $\delta^{(p)} \circ \delta^{(p)} = 0$ *where* 0 *denotes the map sending everything to the zero element in the corresponding* C_k.

Proof. For $G = \sum_{X \in \binom{V}{k}} \chi_G(X) X \in C_k$, we have

$$\begin{aligned} \delta^{(p)} \circ \delta^{(p)} G &= \sum_{Z \in \binom{V}{k+2p}} \left(\sum_{Y \in \binom{Z}{k+p}} \left(\sum_{X \in \binom{Y}{k}} \chi_G(X) \right) Z \right) \\ &= \sum_{X \subset Y \subset Z} \chi_G(X) Z = \sum_{X \subset Z} \binom{2p}{p} \chi_G(X) Z = 0 \end{aligned}$$

since $\binom{2p}{p} \equiv 0 \pmod 2$ for $p \geq 1$. □

More generally, we have the following. For $x \in \mathbb{Z}$, write $x = \sum_{i \geq 0} x(i) 2^i$, $x(i) \in \{0, 1\}$, in its usual binary expansion. Define for $x, y \in \mathbb{Z}$,

$$x \vee y := z \text{ where } z(i) = \max\{x(i), y(i)\}, \qquad i \geq 0 . \tag{2.3}$$

Also, define

$$\perp (x, y) := \begin{cases} 1 & \text{if } x(i)y(i) = 0 \text{ for all } i, \\ 0 & \text{otherwise}, \end{cases} \tag{2.4}$$

$$x * y := \begin{cases} x + y & \text{if } \perp (x, y) = 1, \\ * & \text{otherwise}, \end{cases} \tag{2.5}$$

$$\delta^{(*)} := 0 \quad \text{(the zero map)}, \tag{2.6}$$

$$B(x) := \{i | x(i) = 1\} . \tag{2.7}$$

Fact 2.2.

$$\begin{aligned} \begin{pmatrix} x + y \\ x \text{ is odd} \end{pmatrix} &\Leftrightarrow \perp (x, y) = 1 \\ &\Leftrightarrow x \vee y = x + y . \end{aligned}$$

This is a standard result in number theory (e.g., see [GKP89]). We remark that setting $p = 0$ in (2.1) shows that $\delta^{(0)}$ is the identity operator, i.e., $\delta^{(0)}H = H$, a fact we will occasionally use.

Fact 2.3.

$$\delta^{(p)} \circ \delta^{(q)} = \delta^{(p*q)} = \begin{cases} \delta^{(p+q)} & \textit{if } \perp(p, q) = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The proof is essentially the same as that of Fact 2.1, except that here we get $\binom{p+q}{p}$ instead of $\binom{2p}{p}$. Fact 2.2 then implies the desired conclusion. □

It also follows from Fact 2.3 that

$$\delta^{(p)} \circ \delta^{(q)} = \delta^{(q)} \circ \delta^{(p)} \ ; \tag{2.8}$$

(2.9)

$$\text{If } p = \sum_i 2^{p_i} \ , \ p_1 < p_2 < \cdots < p_r \text{ then } \delta^{(p)} = \delta^{(2^{p_1})} \circ \delta^{(2^{p_2})} \circ \cdots \circ \delta^{(2^{p_r})} \ .$$

Remark (2.9) already suggests the dependence of the properties of $\delta^{(p)}$ on the form of the binary expansion of p. Our first result (in the next section) will determine the kernel of $\delta^{(p)}$ when $|B(p)| = 1$, i.e., $p = 2^t$ for some $t \geq 0$.

3. The kernel of $\delta^{(a)} : a = 2^t$

The main result of this section is the following.

Theorem 1. *If $a = 2^t$, $t \geq 0$, and $|V| = n \geq (k+1)a$, $k \geq 0$, and $G = G^{(k)}(n) = (V, \chi_G) \in C_k$ then*

$$\delta^{(a)}G = 0 \Leftrightarrow G = \delta^{(a)}F \quad \textit{for some } F \in C_{k-a} \ . \tag{3.10}$$

Remark. (3.10) asserts that the kernel of $\delta_k^{(a)}$ is just the image of $\delta_{k-a}^{(a)}$, i.e.,

$$0 \to C_{k-a} \xrightarrow{\delta^{(a)}} C_k \xrightarrow{\delta^{(a)}} C_{k+a} \to 0$$

is a short exact sequence.

At the end of this section, we give examples showing why some restriction on n is necessary.

Proof. The proof will proceed by induction on k. We first consider the case $k = 0$.

Considering the two different 0-graphs $G_0^{(0)}$ and $G_1^{(0)}$, it is easy to see that only $G = G_0^{(0)}$ satisfies the hypothesis that $\delta^{(a)}G = 0$ (since $\delta^{(a)}G_1^{(0)}$ has edge set $\binom{V}{a}$). However, $G_0^{(0)} = \delta^{(a)}F^{(-a)}$, since any such graph $F^{(-a)}$ is 0 by definition. Therefore (3.10) holds for $k = 0$.

To quell a potentially uneasy feeling about starting the induction at such a trivial level, we next give a direct proof of (3.10) for $k = 1$. Let $G = (V, E)$ be a 1-graph with $n \geq 2a$ and assume $\delta^{(a)}G = 0$.

First, suppose $a = 1$. Thus, $\delta^{(1)}G = 0$ so that every pair $\{x, y\} \subset V$ contains an even number of elements of E. This implies that either $E = \varnothing$, in which case $G = \delta^{(1)}G_0^{(0)}$, or $E = V$, in which case $G = \delta^{(1)}G_1^{(0)}$, which shows that (3.10) holds in this case.

Now, suppose $a = 2^t > 1$. If $E = \varnothing$ then $G = 0 = \delta^{(a)}F^{(1-a)}$ since by definition $F^{(1-a)} = 0$ for $a > 1$, and (3.10) holds. If $E = V$ then for $|S| = a + 1$, $\chi_{\delta^{(a)}G}(S) = |S \cap E| = |S| \equiv 1 \pmod 2$, which contradicts the hypothesis that $\delta^{(a)}G = 0$. So assume $\varnothing \neq E \neq V$. In this case, however, it is impossible for $|S \cap E| \equiv 0 \pmod 2$ for all $S \subset V$ with $|S| = a + 1$ (which is implied by $\delta^{(a)}G = 0$). Thus, (3.10) holds for $k = 1$.

We now assume that (3.10) holds for all values less than some fixed $k \geq 2$, and G is k-graph on $n \geq a(k+1)$ vertices satisfying $\delta^{(a)}G = 0$.

Let $A \subset V$ be a fixed (arbitrary) subset of V with $|A| = a = 2^t$, and let

$$\overline{V} := V \setminus A, \quad G' = G + \delta^{(a)}G(A),$$

where $G(A)$ denotes the $(k-a)$-graph $(\overline{V}, \chi_{G(A)})$ defined by

$$\chi_{G(A)}(Y) = \chi_G(Y \cup A) \quad \text{for } Y \in \binom{\overline{V}}{k-a}.$$

Thus,

$$\delta^{(a)}G' = \delta^{(a)}G + \delta^{(a)}\delta^{(a)}G(A) = \delta^{(a)}G = 0.$$

If we prove that $G' = \delta^{(a)}F'$ then we have

$$\begin{aligned} G &= G' + \delta^{(a)}G(A) \quad (\text{over } \mathbb{Z}/2) \\ &= \delta^{(a)}F' + \delta^{(a)}G(A) = \delta^{(a)}(F' + G(A)), \end{aligned}$$

and (3.10) holds.

Note that no edge of G' contains A, since any such edge X of G has $X \setminus A$ as an edge of $G(A)$, and so is cancelled in $G' = G + \delta^{(a)}G(A)$.

So we may henceforth assume that this normalization has been made, and therefore that

$$G \text{ has no edge containing } A. \tag{3.11}$$

Observe that for all $B \subset A$, X is an edge of $G(B)$ if and only if $B \cup X$ is an edge of G.

Define $\overline{\delta}^{(a)} := \delta^{(a)}|_{\overline{V}}$. That is,

$$\overline{\delta}^{(a)}F = \sum_{Y \in \binom{\overline{V}}{k}} \sum_{X \in \binom{Y}{k-a}} \chi_F(X)Y. \tag{3.12}$$

Fact 3.1. *For* $\varnothing \neq B \subseteq A$,

$$\overline{\delta}^{(a)}G(B) = \sum_{C \subsetneq B} \overline{\delta}^{(a-b+c)}G(C) \tag{3.13}$$

where $b = |B|$, $c = |C|$.

Note that for $B = A$, (3.13) gives

$$\overline{\delta}^{(a)}G(A) = \overline{\delta}^{(a)}0 = 0 = \sum_{C \subsetneq A} \overline{\delta}^{(c)}G(C). \tag{3.14}$$

This implies that if $\overline{G} := G|_{\overline{V}}$ then by (3.11),

$$\overline{G} = \sum_{\varnothing \neq C \subsetneq A} \overline{\delta}^{(C)} G(C) . \tag{3.15}$$

Proof of Fact 3.1. Consider $W = B \cup \overline{W}$ where $W \subset \overline{V}$, $|\overline{W}| = k + a - b$. Observe that all edges of G in $B \cup \overline{W}$ are of the form $C \cup Z$, where $C \subseteq B$ and $Z \subseteq \overline{W}$. Thus, we have

$$\begin{aligned} 0 &= \chi_{\delta^{(a)}G}(W) \quad (\text{since } \delta^{(a)}G = 0) \\ &= \chi_{\delta^{(a)}G}(B \cup \overline{W}) \\ &= \sum_{C \subseteq B} \chi_{\delta^{(a-b+c)}G(C)}(\overline{W}) \\ &= \sum_{C \subseteq B} \chi_{\overline{\delta}^{(a-b+c)}G(C)}(\overline{W}) \quad (\text{since } \overline{W} \subseteq \overline{V}) . \end{aligned} \tag{3.16}$$

Therefore,

$$\begin{aligned} \chi_{\overline{\delta}^{(a)}G(B)}(\overline{W}) &= \sum_{C = B} \chi_{\overline{\delta}^{(a-b+c)}G(C)}(\overline{W}) \\ &= \sum_{C \subsetneq B} \chi_{\overline{\delta}^{(a-b+c)}G(C)}(\overline{W}) \quad \text{by (3.16)} . \end{aligned}$$

Since $\overline{W} \subseteq \overline{V}$ with $|\overline{W}| = k + a - b$ was arbitrary then we conclude

$$\overline{\delta}^{(a)} G(B) = \sum_{C \subsetneq B} \overline{\delta}^{(a-b+c)} G(C)$$

which proves Fact 3.1. □

Fact 3.2. *There exist graphs F_C for $\varnothing \neq C \subseteq A$ such that for all $B \subseteq A$, $B \neq \varnothing$, we have*

$$\overline{G}(B) = \sum_{c=1}^{b} \overline{\delta}^{(a-b+c)} F_c^* \tag{3.17}$$

where

$$F_c^* = \begin{cases} \displaystyle\sum_{\varnothing \neq C \subseteq B} F_C & \text{if } \perp(a-b, c) = 0, \\ \displaystyle\sum_{\varnothing \neq C \subsetneq B} F_C & \text{if } \perp(a-b, c) = 1 \end{cases}$$

and $c = |C|$.

Proof. Induction on $b := |B|$. Suppose $b = 1$. Applying Fact 3.1, we have

$$\overline{\delta}^{(a)} \overline{G}(B) = \overline{\delta}^{(a-1)} \overline{G} = 0 .$$

Since $\overline{G}(B)$ is a $(k-1)$-graph on $\overline{V}$ with $|\overline{V}| = n - a \geq (k+1)a - a = ka$ then we can apply Theorem 1 for $(k-1)$-graphs and conclude that $\overline{G}(B) =$

$\overline{\delta}^{(a)}F_B$ as required. This, in particular, *defines* F_B for $|B| = 1$. Next we assume that Fact 3.2 has been proved for all values less than some $b \geq 2$, and suppose $B \subseteq A$ has $|B| = b$.

From Fact 3.1 we have

$$\begin{aligned}\overline{\delta}^{(a)}\overline{G}(B) &= \sum_{C \subsetneq B} \overline{\delta}^{(a-b+c)}\overline{G}(C) \\ &= \overline{\delta}^{(a-b)}\overline{G} + \sum_{\varnothing \neq C \subsetneq B} \overline{\delta}^{(a-b+c)}\overline{G}(C) \\ &= \sum_{\substack{\varnothing \neq C \subsetneq A \\ c<b}} \perp(a-b, c)\overline{\delta}^{(a-b+c)}\overline{G}(C) + \sum_{\varnothing \neq C \subsetneq B} \overline{\delta}^{(a-b+c)}\overline{G}(C)\end{aligned}$$

since by (3.15), $\overline{G} = \sum_{\varnothing \neq C \subsetneq A} \overline{\delta}^{(c)}\overline{G}(C)$ and since $\perp(a-b, c) = 0$ if $c \geq b$. Next, for $c < b$, we obtain

$$\overline{\delta}^{(a-b+c)}\overline{G}(C) = \overline{\delta}^{(a-b+c)} \sum_{i=1}^{c} \overline{\delta}^{(a-c+i)} F_i^*$$

by the induction hypothesis (since $c < b$). Therefore

$$\overline{\delta}^{(a-b+c)}\overline{G}(C) = \overline{\delta}^{(a-b+c)}\overline{\delta}^{(a)} F_C$$

by the definition of F_i^* since for $i < c$ and $a - b + c < 2^t$, if

$$\perp(a-b+c, a-c+i) = 1$$

then we have $(a-b+c) + (a-c+i) = 2a - b + i < 2^t = a$, i.e., $a < b$ which is a contradiction. (This implies that for $i < c$, we have $\perp(a-b+c, a-c+i) = 0$.) Thus,

$$\begin{aligned}\overline{\delta}^{(a)}\overline{G}(B) &= \sum_{\substack{\varnothing \neq C \subsetneq A \\ c<b}} \perp(a-b, c)\overline{\delta}^{(a-b+c)}\overline{\delta}^{(a)} F_C \\ &\quad + \sum_{\varnothing \neq C \subsetneq B} \overline{\delta}^{(a-b+c)}\overline{\delta}^{(a)} F_C \\ &= \overline{\delta}^{(a)} \sum_{c=1}^{b-1} \overline{\delta}^{(a-b+c)} F_c^*\end{aligned}$$

by definition of F_c^* (where the two cases $\perp(a-b, c) = 0$ and $\perp(a-b, c) = 1$ are considered separately).

Therefore, we can apply Theorem 1 for $(k-b)$-graphs and conclude that

$$\overline{G}(B) = \sum_{c=1}^{b-1} \overline{\delta}^{(a-b+c)} F_c^* + \overline{\delta}^{(a)} F_B$$

for some F_B (this is the definition of F_B).

$$\overline{G}(B) = \sum_{c=1}^{b} \overline{\delta}^{(a-b+c)} F_c^*$$

as required, since $F_b^* = F_B$. This completes the induction step and Fact 3.2 is proved. □

Fact 3.3. *For* $0 \le c \le b < a = 2^t$, *and* $b \ge 2$

$$\sum_{i=1}^{c} \binom{b}{i} \equiv \binom{b-1}{c} \equiv \binom{a-b+c}{c} \equiv \perp (a-b, c) \pmod 2. \tag{3.18}$$

Proof.

$$\sum_{i=0}^{c} \binom{b}{i} \equiv \sum_{i=0}^{c} \left(\binom{b-1}{i} + \binom{b-1}{i-1} \right)$$

(since $b \ge 2$ where $\binom{x}{-1} := 0$)

$$\equiv \binom{b-1}{c} + \binom{b-1}{-1} \equiv \binom{b-1}{c} \pmod 2.$$

It remains to show (by Fact 2.2) that

$$\binom{b-1}{c} \equiv \binom{a-b+c}{c} \pmod 2. \tag{3.19}$$

However, this follows easily by induction on b. □

Definition. For $J \subseteq A$, $\varnothing \ne C \subsetneqq A$, define F_C^J to be the graph having as its edges all sets of the form $J \cup X$ where X is an edge of F_C, the graph defined in Fact 3.2.

Thus, for $J = \varnothing$, $F_C^{\varnothing} = F_C$. Also define $j := |J|$.

Fact 3.4.

$$G = \sum_{\substack{\varnothing \ne C \subsetneqq A \\ C \ne J \subsetneqq A \\ j \le c}} \delta^{(a+c-j)} F_C^J . \tag{3.20}$$

Proof. We first consider an arbitrary set $X \in \binom{\overline{V}}{k}$.

Then, since $\overline{G} = \sum_{\varnothing \ne C \subsetneqq A} \overline{\delta}^{(c)} \overline{G}(C)$,

$$\begin{aligned} \chi_G(X) = \chi_{\overline{G}}(X) &= \chi_{\sum_{\varnothing \ne C \subsetneqq A} \overline{\delta}^{(c)} \overline{G}(C)}(X) \\ &= \chi_{\sum_{\varnothing \ne C \subsetneqq A} \overline{\delta}^{(c)} \sum_{i=1}^{c} \overline{\delta}^{(a-c+i)} F_i^*}(X) \\ &= \chi_{\sum_{\varnothing \ne C \subsetneqq A} \overline{\delta}^{(a)} \overline{\delta}^{(c)} F_C}(X) \end{aligned} \tag{3.21}$$

because $\perp(c, a-c+i) = 0$ for $i < c$. On the other hand, for the right-hand side of (3.20),

$$X \text{ is an edge of } \sum_{\substack{\varnothing \neq C \subsetneq A \\ C \neq J \subsetneq A \\ j \leq c}} \delta^{(a+c-j)} F_C^J$$

if and only if

$$X \text{ is an edge of } \sum_{\varnothing \neq C \subsetneq A} \delta^{(a+c)} F_C$$

if and only if (by (3.21)) X is an edge of G.

Next, suppose $X = B \cup \overline{X}$ where $\varnothing \neq B \subseteq A$, $\overline{X} \subset \overline{V}$, $|X| = k$. Assume first that $B = A$. Then by (3.11) we have $\chi_G(X) = 0$. On the right-hand side of (3.20), for each edge Y of F_C in $\overline{X}$, the number of J for which $J \cup Y$ is in X is exactly

$$\binom{a}{0} + \binom{a}{1} + \cdots + \binom{a}{c} - 1$$

(the -1 coming from the requirement that $J \neq C$)

$$\equiv \binom{a-1}{c} - 1 \equiv 0 \pmod 2$$

by Fact 3.3, since all the digits in the binary expression of $a-1$ are 1.

Now, suppose $\varnothing \neq B \subsetneq A$. Then

$$\chi_G(B \cup \overline{X}) = \chi_{\overline{G}(B)}(\overline{X}) = \chi_{\sum_{c=1}^{b} \overline{\delta}^{(a-b+c)} F_c^*}(\overline{X}). \tag{3.22}$$

On the other hand, consider the right-hand side of (3.20) for $B \cup \overline{X}$, i.e.,

$$\chi_{\sum_{\substack{\varnothing \neq C \subsetneq A \\ C \neq J \subsetneq A \\ j \leq c}} \delta^{(a+c-j)} F_C^J}(B \cup \overline{X}) = \chi_{\sum_{\substack{\varnothing \neq C \subsetneq A \\ C \neq J \subsetneq B \\ j \leq c}} \delta^{(a+c-j)} F_C^J}(B \cup \overline{X}) \tag{3.23}$$

since any edge of F_C^J by definition contains J. For a fixed $C \subseteq B$, the number of J so that $j \leq c$ and $C \neq J \subseteq B$ is just

$$\lambda := \binom{b}{0} + \binom{b}{1} + \cdots + \binom{b}{c} - 1 \quad (\text{since } J \neq C),$$

while for $C \not\subseteq B$, the corresponding number is just $\lambda + 1$. By Fact 3.3.

$$\lambda + 1 \equiv \binom{b-1}{c} \equiv \binom{a-b+c}{c} \equiv \perp(a-b, c) \pmod 2.$$

Continuing (3.23) we get

$$
\begin{aligned}
&\chi_{\sum_{\substack{\varnothing\neq C\subsetneq A\\ C\neq J\subsetneq B\\ j\le c}}\delta^{(a+c-j)}F_C^J}(B\cup\overline{X})\\
&\qquad = \chi_{\sum_{\varnothing\neq C\subseteq B}\lambda\overline{\delta}^{(a-b+c)}F_C+\sum_{\varnothing\neq C\subsetneq B}(\lambda+1)\overline{\delta}^{(a-b+c)}F_C}(\overline{X})\\
&\qquad = \chi_{\sum_{\varnothing\neq C\subseteq B}\overline{\delta}^{(a-b+c)}F_C+\sum_{\varnothing\neq C\subseteq A}\perp(a-b\,,c)\overline{\delta}^{(a-b+c)}F_C}(\overline{X})\\
&\qquad = \chi_{\overline{\delta}^{(a-b+c)}\left(\sum_{\varnothing\neq C\subseteq B}F_C+\sum_{\varnothing\neq C\subseteq A}\perp(a-b\,,c)F_C\right)}(\overline{X})\\
&\qquad = \chi_{\sum_{c=1}^{b}\overline{\delta}^{(a-b+c)}F_c^*}(\overline{X}) \quad \text{by the definition of } F_c^*\\
&\qquad = \chi_G(B\cup\overline{X}) \quad \text{by (3.22).}
\end{aligned}
\tag{3.24}
$$

This completes the proof of Fact 3.4. □

To finish the proof of Theorem 1, observe by Fact 3.4 that in fact

$$G=\delta^{(a)}\left(\sum_{\substack{\varnothing\neq C\subsetneq A\\ C\neq J\subsetneq A\\ j\le c}}\delta^{(c-j)}F_c^J\right)$$

since $0\le c-j<a=2^t$ implies $\perp(a\,,c-j)=1$.

The other direction of (3.10), namely that

$$G=\delta^{(a)}F\Rightarrow\delta^{(a)}G=\delta^{(a)}\delta^{(a)}F=0$$

follows from Fact 2.1. This completes the proof of Theorem 1. □

There are a variety of examples known to show that some size restriction on n is necessary in order for the conclusion of Theorem 1 be valid. One such family of examples is the following. Define

$$V=\{x_1\,,x_2\}\cup\mathbb{Z}/2^{t+1}\quad\text{for } t\ge 1\,,$$

$G=G^{(2^t+1)}=(V\,,E)$ with the edge set

$$E=\{x_j\cup\{i+1\,,\dots\,,i+2^t\}\mid j=1\,,2 \text{ and } i\in\mathbb{Z}/2^{t+1}\}\,.$$

Thus, G has 2^{t+2} edges, each of which is a (2^t+1)-set in V. A simple calculation shows that $\delta^{(2^t)}G=0$. However, $G\neq\delta^{(2^t)}G^{(1)}$ for any 1-graph $G^{(1)}$ since no (2^t+1)-graph of the form $\delta^{(2^t)}G^{(1)}$ can have exactly 2^{t+2} edges.

4. The kernel of $\delta^{(a)}$: general a

In this section we complete our analysis of $\ker\delta(a)$. In order to do this, we require an auxiliary result, of interest in its own right.

Theorem 2. *Suppose* $a_1, a_2, \ldots, a_r \geq 0$ *and* G_i *is a* $(k - a_i)$*-graph on* V *with* $|V| = n \geq \frac{1}{4}(k+1)^2$. *Then*

$$\sum_{i=1}^{r} \delta^{(a_i)} G_i = 0 \Leftrightarrow \textit{there exist } K_{ij} = K_{ji} \textit{ such that} \tag{4.25}$$

$$\delta^{(a_i)} G_i = \sum_{\substack{j \\ j \neq i}} \delta^{(a_i \vee a_j)} K_{ij} \ . \tag{4.26}$$

Proof. The proof will be a multiple induction on r, $\sum_i a_i$ and k. The desired conclusion holds for $r = 1$ by Theorem 1. Also, Theorem 2 is immediate for $k = 0$, so we will always assume henceforth that $k \geq 1$. We will first require several facts which will be proved under our induction hypotheses.

Fact 4.1. *Suppose* $a \leq \max_i a_i$ *and* G *is a* k'*-graph with* $a + k' \leq k$. *Then*

$$\delta^{(a)} G = 0 \Leftrightarrow G = \sum_{t \in B(a)} \delta^{(2^t)} F_t \quad \textit{for some } F_t \ . \tag{4.27}$$

Proof. The result is immediate for $|B(a)| = 1$. Suppose $|B(a)| > 1$. Let $u := \min\{t | t \in B(a)\}$. By Theorem 1 (since $n \geq \frac{1}{4}(k+1)^2 \geq 2^u(k' + a - 2^u + 1)$),

$$\delta^{(a-2^u)} G = \delta^{(2^u)} F \quad \text{for some } F \ ,$$

i.e.,

$$\delta^{(a-2^u)} G + \delta^{(2^u)} F = 0 \ .$$

Now by induction (since $k' + a - 2^u < k$), (4.26) implies

$$\delta^{(a-2^u)} G = \delta^{(a)} F_u \ ,$$

i.e.,

$$\delta^{(a-2^u)}(G + \delta^{(2^u)} F_u) = 0 \ .$$

By induction within the proof of Fact 4.1, we have

$$G + \delta^{(2^u)} F_u = \sum_{t \in B(a-2^u)} \delta^{(2^t)} F_t \ .$$

Therefore,

$$G = \sum_{t \in B(a)} \delta^{(2^t)} F_t$$

as required. □

Fact 4.2. *Suppose all the edges of* G *and* F_i, $1 \leq i \leq r$, *are in* $\overline{V} := V \setminus \{v\}$. *If*

$$\overline{G} = G|_{\overline{V}} = \sum_{i=1}^{r} \overline{\delta}^{(b_i)} F_i$$

then

$$G = \sum_{i=1}^{r} \delta^{(b_i)} F_i + \sum_{i=1}^{r} \delta^{(b_i - 1)} F_i^+ \tag{4.28}$$

where F_i^+ *has edge set* $\{e \cup \{v\} | e \in E(F_i)\}$, *and* $\overline{\delta}^{(b)} := \delta^{(b)}|_{\overline{V}}$ *as given in* (3.12).

Proof. It is straightforward to verify (4.28) for the two possible cases, namely, edges which belong to $\overline{V}$, and edges of the form $e' \cup \{v\}$, $e' \subseteq \overline{V}$. □

Fact 4.3. *Assume* $a \le a_i$ *for some* i, *and* a *is even. Suppose* $\overline{\delta}^{(a)} F = \overline{\delta}^{(a \vee b)} G$, *and all edges of* F *and* G *are contained in* $\overline{V}$. *Then*

$$\delta^{(a+1)} F = \begin{cases} \delta^{(a \vee b+1)} G & \text{if } b \text{ is even}, \\ \delta^{(a \vee b)} G^+ & \text{if } b \text{ is odd}. \end{cases}$$

Proof. $\overline{\delta}^{(a)} F = \overline{\delta}^{(a \vee b)} G$ implies $\overline{F} = \overline{\delta}^{(a \vee b - a)} G + K'$ where $K' \in \ker \overline{\delta}^{(a)}$. Thus, by Fact 4.1, we have

$$\overline{F} = \overline{\delta}^{(a \vee b - a)} G + \sum_{t \in B(a)} \overline{\delta}^{(2^t)} K_t . \tag{4.29}$$

By Fact 4.2, we have

$$F = \delta^{(a \vee b - a)} G + \sum_{t \in B(a)} \delta^{(2^t)} K_t + \delta^{(a \vee b - a - 1)} G^+ + \sum_{t \in B(a)} \delta^{(2^t - 1)} K_t^+ .$$

Thus,

$$\begin{aligned} \delta^{(a+1)} F &= \delta^{(a+1)*(a \vee b - a)} G + \sum_{t \in B(a)} \delta^{(a+1)*2^t} K_t \\ &\quad + \delta^{(a+1)*(a \vee b - a - 1)} G^+ + \sum_{t \in B(a)} \delta^{(a+1)*(2^t - 1)} K_t^+ \\ &= \delta^{(a+1)*(a \vee b - a)} G + \delta^{(a+1)*(a \vee b - a - 1)} G^+ . \end{aligned} \tag{4.30}$$

We consider two cases:

Case 1: *b is even.* Therefore

$$\delta^{(a+1)} F = \delta^{(a+1)*(a \vee b - a)} G = \delta^{(a \vee b + 1)} G .$$

Case 2: *b is odd.* Therefore

$$\delta^{(a+1)} F = \delta^{(a+1)*(a \vee b - a - 1)} G^+ = \delta^{(a \vee b)} G^+$$

as required. This proves Fact 4.3. □

Our next step will be to "normalize" the statement of Theorem 2. We first claim that it is enough to prove Theorem 2 in the case that all the a_i are *distinct*. To see this, assume Theorem 2 holds in this case, and suppose $a_i = a_j$ for some $i < j$. Thus, by hypothesis

$$\delta^{(a_i)}(G_i + G_j) + \sum_{l \ne i, j} \delta^{(a_l)} G_l = 0 .$$

By induction (on the number of summands), we can find $W_{u,v}$ for $u, v \ne j$ so that

$$\delta^{(a_l)} G_l = \sum_{u \ne l} \delta^{(a_u \vee a_l)} W_{u,l}$$

and

$$\delta^{(a_i)}(G_i + G_j) = \sum_{u \neq i, j} \delta^{(a_i \vee a_u)} W_{i,u} .$$

Now define

$$K_{m,l} = \begin{cases} W_{m,l} & \text{if } m, l \neq j, \\ G_j & \text{if } \{m, l\} = \{i, j\}, \\ 0 & \text{if } \{m, l\} = \{u, j\}, \ u \neq i . \end{cases}$$

It is easily checked that (4.26) holds with these choices.

Next, we show that in fact, it suffices to prove Theorem 2 for the case that all the a_i are distinct powers of 2. For, suppose $a_1 = 2^t + b$ where $t \in B(a_1)$ and $b > 0$. Then, by hypothesis,

$$\delta^{(2^t)}(\delta^{(b)} G_1) + \sum_{i>1} \delta^{(a_i)} G_i = 0 .$$

By induction (on $\sum_i a_i$), there exist W_{ij} such that

$$\delta^{(2^t)}(\delta^{(b)} G_1) = \sum_{i>1} \delta^{(2^t \vee a_i)} W_{1i} \tag{4.31}$$

and for $i > 1$,

$$\delta^{(a_i)} G_i = \sum\nolimits_{j \neq i, 1} \delta^{(a_i \vee a_j)} W_{ij} + \delta^{(2^t \vee a_i)} W_{1i} . \tag{4.32}$$

By (4.31),

$$\delta^{(b)} G_1 = \sum_{i>1} \delta^{(2^t \vee a_i - 2^t)} W_{1i} + \delta^{(2^t)} K$$

for some K. We now apply the induction hypothesis of Theorem 2 (for a smaller value of k, namely $k - 2t$), and conclude there exist U_{ij}, $0 \leq i < j \leq r$, with

$$\delta^{(b)} G_1 = \sum_{i>1} \delta^{(2^t \vee a_i - 2^t) \vee b} U_{1i} + \delta^{(b \vee 2^t)} U_{01} \tag{4.33}$$

and for $i > 1$,

$$\delta^{(2^t \vee a_i - 2^t)} W_{1i} = \sum_{\substack{j \\ j \neq i, 1}} \delta^{(2^t \vee a_i \vee a_j - 2^t)} U_{ij} + \delta^{(2^t \vee a_i - 2^t) \vee b} U_{1i} + \delta^{(2^t \vee a_i)} U_{0i} .$$

Therefore,

$$\delta^{(a_1)} G_1 = \sum_{i>1} \delta^{(a_i \vee a_1)} U_{1i}$$

and for $i > 1$ (by (4.32)),

$$\begin{aligned} \delta^{(a_i)} G_i &= \sum_{j \neq i, 1} \delta^{(a_i \vee a_j)} W_{ij} + \delta^{(2^t \vee a_i)} W_{1i} \\ &= \sum_{j \neq i, 1} \delta^{(a_i \vee a_j)} W_{ij} + \sum_{j \neq i, 1} \delta^{(2^t \vee a_i \vee a_j)} U_{ij} + \delta^{(a_i \vee a_1)} U_{1i} . \end{aligned}$$

So, if we choose

$$K_{ij} = \begin{cases} W_{ij} + \delta^{(2^t)} U_{ij} & \text{if } i, j \neq 1, \\ U_{1i} & \text{if } i > 1, \ j = 1, \end{cases}$$

then (4.26) is easily verified.

Finally, suppose some a_i vanishes, say $a_1 = 0$. Then by choosing $K_{ij} = G_i$ if $i = 1, j > 1$, and $K_{ij} = 0$ otherwise, the implication (4.25) $\Rightarrow$ (4.26) is immediate.

Thus, we may assume in the proof of Theorem 2 that $a_i = 2^{b_i}$ with $0 \leq b_1 < b_2 < \cdots < b_r$. We now return to the main line of the proof.

Define for $1 \leq i \leq r$,

$$F_i' := G_i(v) \ , \quad F_i = G_i + \delta^{(1)} F_i'$$

where v is an arbitrary (fixed) vertex of V. It is easy to see that F_i and F_i' have edge sets entirely contained in $\overline{V} := V \setminus \{v\}$, and

$$G_i = F_i + \delta^{(1)} F_i' \ . \tag{4.34}$$

Hence,

$$\sum_i \delta^{(a_i)} G_i = 0$$

implies

$$\sum_i \delta^{(a_i)} F_i + \sum_i \delta^{(a_i+1)} F_i' = 0 \ . \tag{4.35}$$

We first apply (4.35) to edges of the form $\overline{X} \cup \{v\}$ for $\overline{X} \in \binom{\overline{V}}{k-1}$. Thus,

$$\sum_i \overline{\delta}^{(a_i-1)} F_i + \sum_i \overline{\delta}^{(a_i)} F_i' = 0 \ . \tag{4.36}$$

We consider two cases.

Case 1. $b_1 = 0$. Thus, $a_1 = 1$ and $\delta^{(a_1)} G_1 = \delta^{(1)} G_1 = \delta^{(1)} F_1$ (since we can assume $F_1' = 0$). Applying $\overline{\delta}^{(1)}$ to (4.36), we have

$$\overline{\delta}^{(1)} F_1 + \sum_{i>1} \overline{\delta}^{(a_i+1)} F_i' = 0 \ . \tag{4.37}$$

Together with (4.35) we find

$$\sum_{i>1} \overline{\delta}^{(a_i)} F_i = 0 \ . \tag{4.38}$$

We can now apply Theorem 2 since (4.38) involves $r - 1$ summands. This implies that there exist $W_{i,j}$, $i, j > 1$, so that for $i > 1$,

$$\overline{\delta}^{(a_i)} F_i = \sum_{\substack{j \\ j \neq 1, i}} \overline{\delta}^{(a_i \vee a_j)} W_{ij} \ .$$

Hence, by Theorem 1 (since $a_i = 2^{b_i}$),

$$\overline{F}_i := F_i|_{\overline{V}} = \sum_{\substack{j \\ j \neq 1, i}} \overline{\delta}^{(a_i \vee a_j - a_i)} W_{ij} + \overline{\delta}^{(a_i)} K_i \tag{4.39}$$

for some K_i. By Fact 4.2, we obtain

$$F_i = \sum_{\substack{j \\ j \neq 1, i}} \delta^{(a_i \vee a_j - a_i)} W_{ij} + \sum_{\substack{j \\ j \neq 1, i}} \delta^{(a_i \vee a_j - a_i - 1)} W_{ij}^+ + \delta^{(a_i)} K_i + \delta^{(a_i - 1)} K_i^+ .$$

Therefore, for $i > 1$,

$$\delta^{(a_i)} F_i = \sum_{\substack{j \\ j \neq 1, i}} \delta^{(a_i \vee a_j)} W_{ij} + \sum_{\substack{j \\ 1 < j < i}} \delta^{(a_i \vee (a_j - 1))} W_{ij}^+ + \delta^{(2a_i - 1)} K_i^+ , \tag{4.40}$$

and

$$\overline{\delta}^{(a_i - 1)} F_i = \sum_{\substack{j \\ j > i}} \overline{\delta}^{(a_j + a_i - 1)} W_{ij} + \overline{\delta}^{(2a_i - 1)} K_i . \tag{4.41}$$

Substituting into (4.36) we have

$$\overline{F}_1 + \sum_{\substack{i, j \\ 1 < j < i}} \overline{\delta}^{(a_i + a_j - 1)} W_{ij} + \sum_{i > 1} \overline{\delta}^{(2a_i - 1)} K_i + \sum_{i > 1} \overline{\delta}^{(a_i)} F_i' = 0 ,$$

i.e.,

$$\overline{F}_1 + \sum_{i > 1} \overline{\delta}^{(a_i)} \left(\sum_{\substack{j \\ 1 < j < i}} \overline{\delta}^{(a_j - 1)} W_{ij} + \overline{\delta}^{(a_i - 1)} K_i + \overline{F}_i' \right) = 0 . \tag{4.42}$$

Now, define for $i > 1$,

$$L_i = \sum_{\substack{j \\ 1 < j < i}} \overline{\delta}^{(a_j - 1)} W_{ij} + \overline{\delta}^{(a_i - 1)} K_i + \overline{F}_i' . \tag{4.43}$$

Then

$$\overline{F}_1 = \sum_{i > 1} \overline{\delta}^{(a_i)} L_i ,$$

and for $i > 1$,

$$\overline{F}_i' = \sum_{\substack{j \\ 1 < j < i}} \overline{\delta}^{(a_j - 1)} W_{ij} + \overline{\delta}^{(a_i - 1)} K_i + L_i .$$

Using Fact 4.2, we obtain

$$F_1 = \sum_{i > 1} \delta^{(a_i)} L_i + \sum_{i > 1} \delta^{(a_i - 1)} L_i^+ . \tag{4.44}$$

Also, for $i > 1$,

$$\overline{\delta}^{(a_i)} F_i' = \sum_{\substack{j \\ 1<j<i}} \overline{\delta}^{(a_j+a_i-1)} W_{ij} + \overline{\delta}^{(2a_i-1)} K_i + \overline{\delta}^{(a_i)} L_i .$$

By Fact 4.3,

$$\delta^{(a_i+1)} F_i' = \sum_{\substack{j \\ 1<j<i}} \delta^{(a_j+a_i-1)} W_{ij}^+ + \delta^{(2a_i-1)} K_i^+ + \delta^{(a_i+1)} L_i . \tag{4.45}$$

Now, choose

$$K_{ij} = \begin{cases} W_{ij} & \text{if } i, j > 1, \\ L_i & \text{if } i > 1, \ j = 1. \end{cases}$$

Thus for $i > 1$ we have (by (4.40) and (4.45)),

$$\begin{aligned} \delta^{(a_i)} G_i &= \delta^{(a_i)} F_i + \delta^{(a_i+1)} F_i' \\ &= \sum_{\substack{j \\ j \neq 1, i}} \delta^{(a_i \vee a_j)} W_{ij} + \sum_{\substack{j \\ 1<j<i}} \delta^{(a_i \vee (a_j-1))} W_{ij}^+ \\ &\quad + \delta^{(2a_i-1)} K_i^+ + \sum_{\substack{j \\ 1<j<i}} \delta^{(a_i+a_j-1)} W_{ij}^+ \\ &\quad + \delta^{(2a_i-1)} K_i^+ + \delta^{(a_i+1)} L_i \\ &= \sum_{\substack{j \\ j \neq i}} \delta^{(a_i \vee a_j)} K_{ij} . \end{aligned}$$

Finally, by (4.44) we get

$$\delta^{(a_1)} G = \delta^{(1)} F_1 = \sum_{i>1} \delta^{(a_i+1)} L_i = \sum_{i>1} \delta^{(a_i+a_1)} K_{1i}$$

and the proof for Case 1 is complete.

Case 2. $a_1 = 2^{b_1} > 1$. We apply $\overline{\delta}^{(1)}$ to (4.36). Thus,

$$\sum_i \overline{\delta}^{(a_i+1)} F_i' = 0 . \tag{4.46}$$

Hence, by (4.35) (restricted to $\overline{V}$), we have

$$\sum_i \overline{\delta}^{(a_i)} F_i = 0 . \tag{4.47}$$

Now, rearranging (4.36), we obtain

$$\sum_i \overline{\delta}^{(a_i)} F_i' = \sum_i \overline{\delta}^{(a_i-1)} F_i = \overline{\delta}^{(a_1-1)} \left(\sum_i \overline{\delta}^{(a_i-a_1)} F_i \right). \tag{4.48}$$

We next apply Theorem 2 to (4.48) (which has a smaller value of k). Thus, there exist W_{ij}, $0 \le i < j \le r$, such that

$$\overline{\delta}^{(a_1-1)}\left(\sum_i \overline{\delta}^{(a_i-a_1)} F_i\right) = \sum_i \overline{\delta}^{(a_i+a_1-1)} W_{i0} \ , \tag{4.49}$$

and

$$\overline{\delta}^{(a_i)} F_i' = \sum_{j \neq i} \overline{\delta}^{(a_i+a_j)} W_{ij} + \overline{\delta}^{(a_i+a_1-1)} W_{i0} \ . \tag{4.50}$$

Hence, by Fact 4.3,

$$\delta^{(a_i+1)} F_i' = \sum_{j \neq i} \delta^{(a_i+a_j+1)} W_{ij} + \delta^{(a_i+a_1-1)} W_{i0}^{+} \ . \tag{4.51}$$

Applying Fact 4.1 to (4.49), we have

$$\sum_i \overline{\delta}^{(a_i-a_1)} F_i = \sum_{i=1}^{r} \overline{\delta}^{(a_i)} W_{i0} + \sum_{t \in B(a_1-1)} \overline{\delta}^{(2^t)} K_t \tag{4.52}$$

for some K_t. Applying $\overline{\delta}^{(a_1)}$ to both sides of (4.52), we obtain

$$\overline{\delta}^{(a_1)} F_1 = \sum_{i>1} \overline{\delta}^{(a_i+a_1)} W_{i0} + \sum_{t<b_1} \overline{\delta}^{(2^t+a_1)} K_t \ . \tag{4.53}$$

Using (4.47), we have

$$\sum_{i>1} \overline{\delta}^{(a_i)} (\delta^{(a_1)} W_{i0} + F_i) + \sum_{t<b_1} \overline{\delta}^{(2^t)} (\delta^{(a_1)} K_t) = 0 \ . \tag{4.54}$$

Since $\sum_{t<b_1} 2^t < a_1$, we can apply the induction hypothesis of Theorem 2. Therefore, there exist X_{ij}, Y_{it}, $Y'_{t,t'}$, $0 \le t, t' < b_1$, $i, j > 1$, satisfying:

For $i > 1$,

$$\overline{\delta}^{(a_i)} (\overline{\delta}^{(a_1)} W_{i0} + F_i) = \sum_{\substack{j \\ j \neq 1, i}} \overline{\delta}^{(a_i+a_j)} X_{ij} + \sum_t \overline{\delta}^{(2^t+a_i)} Y_{it} \ , \tag{4.55}$$

$$\overline{\delta}^{(2^t+a_1)} K_t = \sum_{i>1} \overline{\delta}^{(a_i+2^t)} Y_{it} + \sum_{t' \neq t} \overline{\delta}^{(2^t+2^{t'})} Y'_{tt'} \ . \tag{4.56}$$

This implies by Theorem 1 that

$$\overline{\delta}^{(a_1)} K_t = \sum_{i>1} \overline{\delta}^{(a_i)} Y_{it} + \sum_{t' \neq t} \overline{\delta}^{(2^{t'})} Y'_{tt'} + \overline{\delta}^{(2^t)} J_t \tag{4.57}$$

for some J_t. Applying the induction hypothesis of Theorem 2 to (4.57) (since k is smaller), we have:

For $i > 1$,

$$\overline{\delta}^{(a_i)} Y_{it} = \sum_{j \neq i} \overline{\delta}^{(a_i + a_j)} Z_{ij}^{(t)} + \sum_{t'} \overline{\delta}^{(a_i + 2^{t'})} Z'_{tt'} \tag{4.58}$$

for $Z_{ij}^{(t)}$, $Z'_{tt'}$, $0 \leq t, t' < b_1$, $1 < j$. Therefore,

$$\overline{\delta}^{(a_i + 2^t)} Y_{it} = \sum_{j \neq 1, i} \overline{\delta}^{(a_i + a_j + 2^t)} Z_{ij}^{(t)} + \sum_{t' \neq t} \overline{\delta}^{(a_i + 2^t + 2^{t'})} Z'_{tt'} . \tag{4.59}$$

Substituting into (4.55) we obtain

$$\overline{\delta}^{(a_i)} (\delta^{(a_1)} W_{i0} + F_i) = \sum_{\substack{j \\ j \neq 1, i}} \overline{\delta}^{(a_i + a_j)} X_{ij} + \sum_{\substack{j, t \\ j \neq 1, i}} \overline{\delta}^{(a_i + a_j + 2^t)} Z_{ij}^{(t)} . \tag{4.60}$$

Therefore, by Fact 4.1, we have

$$\overline{F}_i = \overline{\delta}^{(a_1)} W_{i0} + \sum_{\substack{j \\ j \neq 1, i}} \overline{\delta}^{(a_j)} X_{ij} + \sum_{\substack{j, t \\ j \neq 1, i}} \overline{\delta}^{(a_j + 2^t)} Z_{ij}^{(t)} + \overline{\delta}^{(a_i)} T_i \tag{4.61}$$

for some T_i.

Hence, by Fact 4.2,

(4.62)

$$\begin{aligned} F_i = & \delta^{(a_1)} W_{i0} + \sum_{\substack{j \\ j \neq l, i}} \delta^{(a_j)} X_{ij} + \sum_{\substack{j, t \\ j \neq 1, i}} \delta^{(a_j + 2^t)} Z_{ij}^{(t)} + \delta^{(a_i)} T_i \\ & + \delta^{(a_1 - 1)} W_{i0}^+ + \sum_{\substack{j \\ j \neq 1, i}} \delta^{(a_j - 1)} X_{ij}^+ + \sum_{\substack{j, t \\ j \neq 1, i}} \delta^{(a_j + 2^t - 1)} Z_{ij}^{(t)^+} + \delta^{(a_i - 1)} T_i^+ . \end{aligned}$$

Thus, for $i > 1$,

$$\begin{aligned} \delta^{(a_i)} F_i = & \delta^{(a_i + a_1)} W_{i0} + \sum_{j \neq 1, i} \delta^{(a_i + a_j)} X_{ij} \\ & + \sum_{\substack{j \neq 1, i \\ t \in B(a_1 - 1)}} \delta^{(a_i + a_j + 2^t)} Z_{ij}^{(t)} + \delta^{(a_i + a_1 - 1)} W_{i0}^+ \\ & + \sum_{1 < j < i} \delta^{(a_i + a_j - 1)} X_{ij}^+ + \sum_{\substack{j \neq 1, i \\ t \in B(a_1 - 1)}} \delta^{(a_i + a_j + 2^t - 1)} Z_{ij}^{(t)^+} + \delta^{(2a_i - 1)} T_i^+ . \end{aligned} \tag{4.63}$$

Now, by (4.63) and (4.51), we have for $i > 1$,

$$\begin{aligned} \delta^{(a_i)} G_i = & \delta^{(a_i)} F_i + \delta^{(a_i - 1)} F'_i \\ = & \sum_{j \neq i, 1} \delta^{(a_i \vee a_j)} \left(X_{ij} + \sum_{t \in B(a_1 - 1)} \delta^{(2^t)} Z_{ij}^{(t)} + \sum_{t \in B(a_1 - 1)} \delta^{(2^t - 1)} Z_{ij}^{(t)^+} + \delta^{(1)} W_{ij} \right) \\ & + \delta^{(a_i \vee a_1)} \left(W_{i0} + \delta^{(1)} W_{i1} + \sum_{1 < j < i} \delta^{(a_j - a_1 - 1)} X_{ij}^+ + \delta^{(a_i - a_1 - 1)} T_i^+ \right) . \end{aligned}$$

Now, choose

$$H_{ij} = H_{ji} = \begin{cases} X_{ij} + \delta^{(1)} W_{ij} + \sum_{t \in B(a_1 - 1)} \delta^{(2^t)} Z_{ij}^{(t)} + \sum_{t \in B(a_1 - 1)} \delta^{(2^t - 1)} Z_{ij}^{(t)^+} & \\ & \text{for } i, j > 1, \\ W_{i0} + \delta^{(1)} W_{i1} + \sum_{1 < j < i} \delta^{(a_j - a_1 - 1)} X_{ij}^+ + \delta^{(a_i - a_1 - 1)} T_i^+ & \\ & \text{for } j = 1 \text{ and } i > 1. \end{cases}$$

Therefore,

$$\delta^{(a_i)} G_i = \sum_{j \neq i} \delta^{(a_i \vee a_j)} K_{ij} \quad \text{for } i > 1 .$$

Since

$$\delta^{(a_1)} G_1 = \sum_{i > 1} \delta^{(a_i)} G_i = \sum_{i > 1} \delta^{(a_i \vee a_1)} K_{i1}$$

then the required H_{ij} have been exhibited, i.e., (4.26) holds. This completes the induction step and the proof of Theorem 2 is complete. □

We can now use Fact 4.1 (which holds for all k) to characterize the kernel of $\delta^{(a)}$ for general a.

Theorem 3. *Suppose G is a k-graph on $n \geq \frac{1}{4}(k + a + 1)^2$ vertices. Then*

$$\delta^{(a)} G = 0 \Leftrightarrow G = \sum_{t \in B(a)} \delta^{(2^t)} K_t \quad \textit{for some choice of } K_t\textit{'s} .$$

Proof. "$\Rightarrow$" The lower bound (which is actually rather generous) comes from that of Theorem 2, since $\delta^{(a)} G$ is a $(k + a)$-graph. Fact 4.1 gives the desired implication.

"$\Leftarrow$" Immediate, using Fact 2.3. □

Perhaps one could characterize those G satisfying $\delta^{(a)} G = 0$ but with $G \neq \sum_{t \in B(a)} \delta^{2^t} K_t$.

5. The cohomology groups $H_k^{p,q}$

Given a portion of the (generalized) chain complex at C_k:

$$\cdots \to C_{k-p} \xrightarrow{\delta^{(p)}} C_k \xrightarrow{\delta^{(q)}} C_{k+q} \to \cdots ,$$

it is natural to ask about the cohomology group $H_k^{p,q} := \ker \delta^{(q)} / \operatorname{im} \delta^{(p)}$. Here, we assume that $B(p) \cap B(q) \neq \varnothing$, i.e., the binary expansions of p and q share a common one, since otherwise we can have $\delta^{(p)}(\delta^{(q)}(\cdot)) \neq 0$, i.e., $\operatorname{im} \delta^{(p)} \subsetneq \ker \delta^{(q)}$, so that $H_k^{p,q}$ is not well defined. It is easy to see under this assumption that $H_k^{p,q} \cong (\mathbb{Z}/2)^{d(p,q;k)}$ where $d(p, q; k)$ is the dimension of the quotient space $\ker \delta^{(q)} / \operatorname{im} \delta^{(p)}$ (with $\ker \delta^{(q)}$ and $\operatorname{im} \delta^{(p)}$ considered as vector spaces over $\mathbb{Z}/2$). Thus, we need to compute the dimensions of $\ker \delta^{(q)}$ and $\operatorname{im} \delta^{(p)}$. In order to do this, we need to introduce the following class of matrices $W = W_{r,s}$. For a fixed n-set V, the rows and columns of W are indexed by the sets $\binom{V}{r}$ and $\binom{V}{r}$, with $r \geq s$. For $X \in \binom{V}{r}$, $Y \in \binom{V}{s}$, the (X, Y) entry $W(X, Y)$

of W is defined by

$$W(X, Y) = \begin{cases} 1 & \text{if } X \supseteq Y, \\ 0 & \text{otherwise.} \end{cases}$$

These inclusion matrices occur commonly in algebraic combinatorics (e.g., see [K72, GJ73, GLL80, LR81, F90, Wi*]). What will be of interest to us is the mod 2 rank $w_{r,s}$ of W (i.e., the rank of the integer matrix W over $\mathbb{Z}/2$). This was first determined by Linial and Rothschild [LR81]. Subsequently, Wilson [Wi*] determined the (mod p) rank of W for every prime p, and expressed the rank $w_{r,s}$ in a form which will be especially convenient for our purposes (a very elegant proof also appears in Frankl [F90]).

Theorem 4 (Wilson [Wi*]). *For* $0 \le s \le r \le n - s$,

$$w_{r,s} = \sum_i \left(\binom{n}{i} - \binom{n}{i-1} \right) \tag{5.64}$$

summed over all i *such that* $\binom{r-i}{s-i}$ *is odd* (*where* $\binom{n}{-1} := 0$).

As an immediate consequence, we have

Fact 5.1. $\dim(\operatorname{im} \delta^{(p)}) = w_{k,k-p}$.

Our main effort will be in determining $\dim(\ker \delta^{(q)})$. To begin, write $B(q) = \{q_1 < q_2 < \cdots < q_r\}$, so that $q = \sum_{i=1}^r 2^{q_i}$. Form the matrix W^* by concatenating the r matrices $W_{k,k-2^{q_i}}$, $1 \le i \le r$, i.e.,

$$W^* = W_{k,k-2^{q_1}} W_{k,k-2^{q_2}} \cdots W_{k,k-2^{q_r}} .$$

It is not hard to see that by Theorem 3,

$$\dim(\ker \delta^{(q)}) = \operatorname{rank}_2 W^* \tag{5.65}$$

(where rank_2 denotes the mod 2 rank). Now by inclusion-exclusion we have

$$\begin{aligned} \operatorname{rank}_2 W^* = \sum_i w_{k,k-2^{q_i}} - \sum_{i<j} w_{k,k-2^{q_i}-2^{q_j}} + \sum_{i<j<l} w_{k,k-2^{q_i}-2^{q_j}-2^{q_l}} \\ - \cdots + (-1)^{r-1} w_{k,k-q} . \end{aligned} \tag{5.66}$$

Now, consider a typical term

$$w_{k,k-c} = \sum_i \left(\binom{n}{i} - \binom{n}{i-1} \right) \tag{5.67}$$

where $c = 2^{s_1} + \cdots + 2^{s_l}$, and the sum is taken over all i such that $\binom{k-i}{k-i-c}$ is odd.

Changing the summation index in (5.67) from i to $k - i$, and using Fact 2.2, we can rewrite (5.67) as

$$w_{k,k-c} = \sum_i \left(\binom{n}{k-i} - \binom{n}{k-i-1} \right) \tag{5.68}$$

summed over all i such that $\binom{i}{i-c} = \binom{i}{c}$ is odd, i.e., such that $\{s_1, \ldots, s_l\} = B(c) \subseteq B(i)$ (by Fact 2.2). Of course, for all our c's,

$$B(c) \subseteq B(q) .$$

Thus,

$$B(c) \subseteq B(i) \cap B(q) \,. \tag{5.69}$$

Letting $\{k-i\}$ denote $\binom{n}{k-i} - \binom{n}{k-i-1}$, we now count how often $\{k-i\}$ occurs in the various terms in (5.66), of which (5.67) is typical. Let $t := |B(i) \cap B(q)|$. If $t = 0$ then no c's contribute to the $\{k-i\}$ count, so suppose $t > 0$. In this case, there are exactly $\binom{t}{l}$ different c's with $B(c) \subseteq B(i) \cap B(q)$, $|B(c)| = l$, for which $\{k-i\}$ occurs in $w_{k,k-c}$. More exactly, it occurs with the sign $(-1)^{l-1}$, which comes from the corresponding l-tuple sum in (5.66). Therefore, the total contribution of $\{k-i\}$ in (5.66) is just

$$\sum_{l \geq 1} (-1)^{l-1} \binom{t}{l} = 1 \quad \text{if } t > 0 \,,$$

and 0 if $t = 0$.

In particular, we obtain

$$\operatorname{rank}_2 W^* = \sum_i \left(\binom{n}{k-i} - \binom{n}{k-i-1} \right) \tag{5.70}$$

summed over all i such that $B(q) \cap B(i) \neq \varnothing$.

By Fact 5.1, we need to express $w_{k,k-p}$ in a similar form. This is given by (5.68):

$$w_{k,k-p} = \sum_i \left(\binom{n}{k-i} - \binom{n}{k-i-1} \right) \tag{5.71}$$

summed over all i with $B(p) \subseteq B(i)$.

We can now put everything together for the main result of this section.

Theorem 5. *When* $B(p) \cap B(q) \neq \varnothing$, *and* $n \geq \frac{1}{4}(k+q+1)^2$ *then*

$$H_k^{p,q} = \ker \delta^{(q)} / \operatorname{im} \delta^{(p)} \cong (\mathbb{Z}/2)^{d(p,q;k)}$$

where

$$d(p, q; k) = \sum_i \left(\binom{n}{k-i} - \binom{n}{k-i-1} \right) \tag{5.72}$$

summed over all i *such that*

$$B(p) \not\subseteq B(i) \quad \textit{and} \quad B(q) \cap B(i) \neq \varnothing \,. \tag{5.73}$$

Proof. Since

$$d(p, q; k) = \dim(\ker \delta^{(q)}) - \dim(\operatorname{im} \delta^{(p)})$$

then by (5.70) and (5.71), we simply have to keep track of the coefficients of $\left(\binom{n}{k-i} - \binom{n}{k-i-1}\right)$ in the sums for $\dim(\ker \delta^{(q)})$ and $\dim(\operatorname{im} \delta^{(p)})$.

Since by hypotheses, $B(p) \cap B(q) \neq \varnothing$, then it is easy to see that the only indices i which contribute to the sum satisfy (5.73). □

Condition (5.73) can be expressed in words as saying that in the binary expansions of p, q and i, some 0 of i corresponds to a 1 of p, and some 1 of i corresponds to a 1 of q. Of course, if $p = q = 2^t$ then no such i exists,

so that the sum in (5.72) is empty, $d(p, q; k) = 0$, and $H_k^{p,q}$ is trivial (as we already know by Theorem 1).

6. Applications

In this section are describe several applications of the preceding ideas, which in fact provided some of our initial motivation for investigating cohomological aspects of hypergraphs.

To begin with, given a k-graph $G = (V, E)$ we define the multiplicative edge function $\mu = \mu_G : V^k \to \{1, -1\}$ by setting

$$\mu(x_1, \dots, x_k) = \begin{cases} -1 & \text{if } \{x_1, \dots, x_k\} \in E, \\ 1 & \text{otherwise.} \end{cases}$$

With $|V| = n$, we define the *deviation* of G, denoted by dev G, by

$$\text{dev } G := \frac{1}{n^{2k}} \sum_{\substack{v_i(0), v_i(1) \in V \\ 1 \le i \le k}} \prod_{\substack{\varepsilon_j \in \{0,1\} \\ 1 \le j \le k}} \mu(v_1(\varepsilon_1), \dots, v_k(\varepsilon_k)) . \tag{6.74}$$

It turns out that dev G is a fundamental invariant of a k-graph G, and gives in many ways a quantitative measure of how much G behaves like a "random" k-graph $G_{1/2}$ on V (e.g., one in which each k-set $X \in \binom{V}{k}$ is selected independently with probability $1/2$ to be an edge of $G_{1/2}$). In particular, $0 \le \text{dev } G \le 1$ always holds, and the closer dev G is to 0, the more like a random k-graph G is. Families of k-graphs $G^{(k)}(n)$ for which dev $G^{(k)}(n) \to 0$ as $n \to \infty$ are called *quasi-random.* (For a fuller discussion of these ideas, the reader can consult [CGW89, CG90, CG91].)

In [CG91], it was important to characterize those k-graphs $G^{(k)}$ with the largest possible deviation, i.e., satisfying

$$\text{dev } G^{(k)} = 1 . \tag{6.75}$$

The following result gives such a characterization.

Theorem 6.

$$\text{dev } G^{(k)} = 1$$

if and only if

$$G^{(k)} = \sum_{i=1}^{k} \delta^{(i)} K^{(k-i)}$$

for some choice of $(k-i)$*-graphs* $K^{(k-i)}$, $1 \le i \le k$.

For a proof, see [CG91].

One property a quasi-random family of k-graphs $G^{(k)}(n) = (V_n, E_n)$, $n \to \infty$, satisfies is the following. For any fixed k-graph $H^{(k)}(m) = (V, E)$, the number $\#\{H^{(k)}(m) < G^{(k)}(n)\}$ of maps $\lambda : V \to V_n$ such that $X \in E \Leftrightarrow \lambda(X) \in E_n$, $X \in \binom{V}{k}$, satisfies

$$\#\{H^{(k)}(m) < G^{(k)}(n)\} = (1 + o(1)) n^m / 2^{\binom{m}{k}} , \qquad n \to \infty .$$

In other words, all k-graphs on a fixed number m of vertices occur (asymptotically) equally often as induced subgraphs of $G^{(k)}(n)$ as $n \to \infty$. In fact, it

is shown in [CG91] that if this holds for all $H^{(k)}(2k)$ on $2k$ vertices, then it holds for all $H^{(k)}(m)$ for any fixed m. Furthermore, the value $2k$ is critical, in the sense that there exist non-quasi-random families $G^{(k)}(n)$ for which

$$\#\{H^{(k)}(s) < G^{(k)}(n)\} = (1 + o(1))n^s/2^{\binom{s}{k}}, \qquad n \to \infty, \tag{6.76}$$

for all $s \leq 2k - 1$.

One way of constructing such families for the case $s = 2k - 1$, when $k \neq 2^a$ for any a, is the following.

For $1 \leq t \leq k - 1$, choose a "random" t-graph $G^{(t)}_{1/2}$ on a set V_n of size n, i.e., each t-set $X \in \binom{V_n}{t}$ is designated as an edge of $G^{(t)}_{1/2}$ independently with probability $1/2$. Define

$$\hat{G}^{(k)}(n) = \sum_{t=1}^{k-1} \delta^{(t)} G^{(k-t)}_{1/2}(n). \tag{6.77}$$

Theorem 7 [CG91]. *For almost all choices of* $G^{(t)}_{1/2}(n)$, $\hat{G}^{(k)}(n)$ *satisfies* (6.76).

In the case that $k = 2^t$, a slight extension of this construction gives the required family (see [CG91] for details).

7. Concluding remarks

It would be natural to investigate the corresponding results for more general coefficient groups, e.g., $\mathbb{Z}/p$ or $\mathbb{Z}$, as opposed to $\mathbb{Z}/2$, which is the simplest choice (and the one for which we had natural applications). A good beginning in this direction has very recently been taken by Dale Darrow, to whom we also wish to thank for a careful reading of an earlier draft of this paper. It would appear that the continuation of these investigations in the directions of the work of Cameron [C77, C78] and others (who dealt with the case $k = 3$) looks quite promising.

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