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THE MIZOHATA STRUCTURE ON THE SPHERE

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ABSTRACT. We prove that a compact surface that admits a Mizohata structure is (homeomorphic to) a sphere and that there exists exactly one such structure $\mathscr{L}$ up to conjugation by diffeomorphisms. We also characterize the range and the kernel of the operator δ_0 induced by $\mathscr{L}$, i.e., obtained from the exterior derivative on functions by passing to the quotient modulo $\mathscr{L}^{\perp}$.

0. INTRODUCTION

The Mizohata vector field

$$M = \partial_t - it\partial_x \qquad (x, t) \in \mathbb{R}^2,$$

was considered in [3] as an example in local solvability. It is not locally solvable at any point of the x-axis, i.e., at the points where M and its complex conjugate $\overline{M}$ are linearly dependent. The vector field M has the following properties:

(0.1) The set Σ where M and $\overline{M}$ are linearly dependent is connected. Furthermore, M and $[M, \overline{M}]$ are linearly independent on Σ.

There exists a function $Z(x, t) = x + it^2/2$ such that

$$MZ = 0 \quad \text{and} \quad dZ \neq 0 \quad \text{everywhere.} \tag{0.2}$$

Furthermore, the restriction of Z to Σ is injective.

Let S be a compact surface (smooth, connected, two-dimensional manifold) and consider a complex one-dimensional subbundle $\mathscr{L}$ of $\mathbb{C}TS = \mathbb{C} \otimes TS$. We say that $\mathscr{L}$ is a Mizohata structure on S if

(0.3) The set Σ where $\mathscr{L} = \overline{\mathscr{L}}$ is connected. Furthermore, for any local section X of $\mathscr{L}$ defined on an open subset U of S, $X \wedge [X, \overline{X}] \neq 0$ on $\Sigma \cap U$.

There exists a smooth function $Z: S \to \mathbb{C}$ such that

$$\begin{aligned} XZ &= 0 \quad \text{for any local section of } \mathscr{L}, \\ dZ &\neq 0 \quad \text{everywhere on } S. \end{aligned} \tag{0.4}$$

Furthermore, the restriction of Z to Σ is injective.

Properties (0.3) and (0.4) imply that $\mathscr{L}$ is a nondegenerate, globally integrable structure.

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In this paper we prove

Theorem 1. *If S admits a Mizohata structure, then S is homeomorphic to the sphere. If $\mathscr{L}_1$ and $\mathscr{L}_2$ are Mizohata structures on S_1 and S_2 respectively, there exists a diffeomorphism $\phi: S_1 \to S_2$ such that $\phi_*: \mathbb{C}TS_1 \to \mathbb{C}TS_2$ takes $\mathscr{L}_1$ onto $\mathscr{L}_2$.*

Let E_k denote the space of complex k-forms on S, $k = 0, 1, 2$, and let $\mathscr{I}$ be the ideal generated by dZ in $E_0 \oplus E_1 \oplus E_2$. Since $d\mathscr{I} \subset \mathscr{I}$, the de Rham complex

$$0 \to E_0 \xrightarrow{d} E_1 \xrightarrow{d} E_2 \to 0$$

induces a complex

$$0 \to E_0^* \xrightarrow{\delta} E_1^* \to 0$$

where $E_k^* = E_k/E_k \cap \mathscr{I}$. Notice that $E_0^* = E_0$ and $E_2^* = 0$.

The differential operator δ is not locally solvable at the points of Σ, so there is no equivalent for δ of the Poincaré lemma. The next theorem studies the global solvability of δ. If $\omega \in E_1$ we denote by $\dot{\omega}$ the class of ω in E_1^*.

Theorem 2. *The following conditions are equivalent for $\dot{\omega} \in E_1^*$:*

(i) *There exists $f \in E_0$ such that $\delta f = \dot{\omega}$.*

(ii) *For all $v \in \operatorname{Ker}\delta$,*

$$\int_S dv \wedge \omega = 0.$$

Let Z be the function (0.4). We show in §1 that $\Omega := Z(S\backslash\Sigma)$ is a simply connected region of $\mathbb{C}$ with smooth boundary. We denote by $\mathscr{A}(\Omega)$ the holomorphic functions in Ω.

Theorem 3. *If $f \in E_0$, the following conditions are equivalent:*

(i) $\delta f = 0$.

(ii) *There exists $F \in \mathscr{A}(\Omega) \cap C^\infty(\overline{\Omega})$ such that $f = F \circ Z$.*

(iii) *For all $\dot{\omega}$ in the range of δ,*

$$\int_S f\, d\omega = 0.$$

The paper is organized as follows: in §1, §2 and §3 we prove Theorems 1, 2 and 3 respectively; in §4 we make some comments and in §A we prove some known facts about Fourier series in polar coordinates that are useful for the proof of Theorem 2.

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1. Proof of Theorem 1

If Σ is the subset of S where $\mathscr{L}$ becomes real, it follows that Σ is an imbedding of S^1 in S. In fact [5], for every $p \in \Sigma$ there exist local coordinates $(x, t): U \to \widetilde{U} \subset \mathbb{R}^2$ such that $x(p) = t(p) = 0$ and

$$\Sigma \cap U = \{p \in U: t(p) = 0\}, \qquad Z(x, t) = x + it^2/2 \quad \text{in } U. \tag{1.1}$$

In $S\backslash\Sigma$ Z has rank 2, so it is a local diffeomorphism. In particular, $\Omega := Z(S\backslash\Sigma)$ is an open subset of the plane. It follows from (1.1) and (0.4) that $\gamma := Z(\Sigma)$ is a smooth simple closed curve in $\mathbb{C}$. Since $Z(S)$ is closed we have

$$\partial\Omega \subset \overline{\Omega} \subset Z(S) = \Omega \cap \gamma,$$

whence

$$\partial\Omega \subset \gamma. \tag{1.3}$$

Let U be a connected component of $\mathbb{C}\backslash\gamma$ intersecting Ω. Thus $\Omega \cap U = U$. Indeed,

$$\overline{\Omega} \cap U = (\Omega \cup \partial\Omega) \cap U = \Omega \cap U$$

since $\partial\Omega \cap U$ is empty, by (1.3). As Ω is bounded, it follows that Ω cannot meet the unbounded component of $\mathbb{C}\backslash\gamma$ and must be equal to the bounded component. Hence, $\partial\Omega = Z(\Sigma) = \gamma$.

Let S^+ be a connected component of $S\backslash\Sigma$. We show that the restriction of Z of S^+ is a diffeomorphism onto Ω. Since Z has rank 2 in S^+, $Z(S^+)$ is open; on the other hand it is also closed in Ω, since

$$\overline{Z(S^+)} \cap \Omega \subset (Z(S^+) \cup \gamma) \cap \Omega = Z(S^+).$$

Thus, $Z(S^+) = \Omega$. If K is a compact subset of Ω, $Z^{-1}(K)$ is a compact subset of S that does not intersect Σ. Then the restriction of Z to S^+ is a proper local diffeomorphism of S^+ onto the simply connected set Ω, hence a diffeomorphism. In particular, S^+ is homeomorphic to a disk and its point-set boundary is Σ. Hence, Z maps homeomorphically $S^+ \cup \Sigma$ onto $\overline{\Omega}$. It follows that $S\backslash\Sigma$ has exactly two components, S^+ and S^- and S is obtained topologically by identification of two closed disks along their boundary, i.e., S is a sphere.

In order to compare two Mizohata structures it is convenient to normalize $\Omega = Z(S\backslash\Sigma)$. Let ψ be the Riemann mapping from Ω to the unit disk Δ. Since $\partial\Omega$ is regular, ψ extends to a diffeomorphism of $\overline{\Omega}$ onto $\overline{\Delta}$ [4]. Then $Z_1 = \psi \circ Z$ verifies

$$XZ_1 = 0 \quad \textit{for any local section of } \mathscr{L},$$

and

$$dZ_1 \neq 0 \quad \textit{everywhere}.$$

In other words, we may assume without loss of generality that $Z(S\backslash\Sigma) = \Delta$, $Z(\Sigma) = \partial\Delta$ and say that Z is normalized.

Consider two Mizohata structures $\mathscr{L}_1$ and $\mathscr{L}_2$ defined on the spheres S_1 and S_2 respectively. If $Z_1 \colon S_1 \to \Delta$ and $Z_2 \colon S_2 \to \Delta$ are the corresponding normalized Z functions, we may define a homeomorphism ϕ from S_1 to S_2 such that $Z_1 = Z_2 \circ \phi$ using the fact that $Z_j/S_j^+ \cup \Sigma_j$ is a homeomorphism, $j = 1, 2$ (and this is also true replacing $-$ for $+$). It is plain that ϕ is a diffeomorphism of $S_1\backslash\Sigma_1$ onto $S_2\backslash\Sigma_2$. If $p \in \Sigma_1$ and $q = \phi(p) \in \Sigma_2$, we may take coordinates (x, t) in a neighborhood of p satisfying (1.1) and (1.2) with Z_1 in the place of Z. We also consider coordinates $(\tilde{x}, \tilde{t})$ in a neighborhood of q with the same properties. Then $Z_1 = Z_2 \circ \phi$ implies that $x(r) = \tilde{x}(\phi(r))$ and $t(r) = \pm\tilde{t}(\phi(r))$ for r in a neighborhood of p. This shows that ϕ' is injective everywhere and ϕ is a diffeomorphism that takes $\mathscr{L}_1$ into $\mathscr{L}_2$.

2. Proof of Theorem 2

Let us see that (i) implies (ii). Assume that there exist $f, \lambda, \mu \in E_0$ such that

$$df = \omega + \lambda\, dZ, \qquad dv = \mu\, dZ.$$

Then,

$$\int_S dv \wedge \omega = \int_S dv \wedge df = \int_S d(v\, df) = 0$$

by Stokes' Theorem.

Before proving the other implication we exhibit a Mizohata structure. To that extent, we identify S with $\mathbb{R}^2 \cup \{\infty\}$, where a chart in a neighborhood of $\{\infty\}$ is given by

$$X(x, y) = \frac{x}{x^2 + y^2}, \qquad Y(x, y) = \frac{y}{x^2 + y^2},$$

for $(x, y) \neq \infty$ and $X(\infty) = Y(\infty) = 0$. We set

$$Z = \exp\left[\int_1^r g(s)\, ds + i\theta\right] \tag{2.1}$$

where (r, θ) are polar coordinates in $\mathbb{R}^2\backslash\{0\}$ and $g(s)$ is a function $\in C^\infty(0, \infty)$ verifying

$$\begin{aligned} &g(s) = 1/s, && 0 < s < 1/3, \\ &g(s) = 1 - s, && 2/3 < s < 4/3, \\ &g(s) = -1/s, && 2 < s < \infty, \\ &g(s) = 0 \quad \Leftrightarrow && s = 1. \end{aligned} \tag{2.2}$$

It is readily verified that Z extends smoothly to $\{0, \infty\}$ is we set $Z(0) = Z(\infty) = 0$ and that $dZ \neq 0$ everywhere.

If we define $\mathscr{L} = \{X \in \mathbb{C}TS\colon X(Z) = 0\}$, it is easy to check that $\mathscr{L}$ is a Mizohata structure and Σ is the boundary of the unit disk. According to Theorem 1, it will be enough to prove that (ii) implies (i) for this particular structure. Let $\omega \in E_1$ satisfy (ii). Since dZ and $d\overline{Z}$ generate $\mathbb{C}T^*S$ away from Σ and (ii) is a property depending only on the class $\dot{\omega}$, we may assume without loss of generality that

$$\omega = \tfrac{1}{2}\lambda d\overline{Z} \quad \text{if } r \leq 1/3 \text{ or } r \geq 2, \tag{2.3}$$

where $\lambda \in E_0$. For $0 < r < \infty$, dr and dZ generate the cotangent bundle and we may write $\omega = f\, dr + m\, dZ$. If $r \notin [1/3, 2]$, (2.3) implies that

$$f = \lambda \exp(-i\theta), \qquad \lambda \in E_0, \tag{2.4}$$

and this also holds for $r \in [1/3, 2]$ redefining λ conveniently. To exploit the hypotheses we take $v = Z^k$, $k = 1, 2, \ldots,$ so in particular $\delta v = 0$. Hence,

$$0 = \int_S dv \wedge \omega = \int_0^{2\pi} \int_0^\infty k Z^{k-1} f\, dZ \wedge dr,$$

or

$$\int_0^{2\pi} \int_0^\infty k \exp\left[k \int_1^r g(s)\, ds + ik\theta\right] f(r, \theta) i\, d\theta\, dr = 0. \tag{2.5}$$

If $u = u(r, \theta) \in E_0$,

$$du = L(u)\, dr + Z_\theta^{-1} u_\theta\, dZ, \qquad 0 < r < \infty,$$

where

$$L = \frac{\partial}{\partial r} + ig(r)\frac{\partial}{\partial \theta} \tag{2.6}$$

so in order to solve $\delta u = \dot{\omega}$ it is enough to find $u \in E_0$ such that

$$Lu = f \quad \text{for } 0 < r < \infty, \tag{2.7}$$

where L is given by (2.6). To solve (2.7) we expand f in Fourier series in θ,

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta},$$

$$f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(re^{it})\,dt, \qquad 0 < r < \infty.$$

If $u = \sum u_n e^{in\theta} \in E_0$ satisfies (2.7) we must have

$$u_n'(r) - ng(r)u_n(r) = f_n(r). \tag{2.8}$$

Set

$$u_n^+(r) = \int_{l^+(n)}^r \exp\left[n\int_\rho^r g(s)\,ds\right] f_n(\rho)\,d\rho, \qquad n = 0, \pm 1, \pm 2, \dots, \tag{2.9}$$

with

$$\begin{cases} l^+(n) = 0 & \text{if } n \leq 0, \\ l^+(n) = 1 & \text{if } n \geq 0. \end{cases} \tag{2.10}$$

Then u_n^+ satisfies (2.8) for every integer n. In the same way, we define

$$u_n^-(r) = \int_{l^-(n)}^r \exp\left[n\int_\rho^r g(s)\,ds\right] f_n(\rho)\,d\rho, \qquad n = 0, \pm 1, \pm, \dots, \tag{2.11}$$

with

$$\begin{cases} l^-(n) = \infty & \text{if } n < 0, \\ l^+(n) = 1 & \text{if } n \geq 0. \end{cases} \tag{2.12}$$

The choice of $l^+(n)$ (resp. $l^-(n)$) was made in order to have $n\int_\rho^r g(s)\,ds \leq 0$ in the formula that defines u_n^+ (resp. u_n^-) for $r \leq 1$ (resp. $r \geq 1$). Set $u^+(re^{i\theta}) = \sum u_n^+(r)e^{in\theta}$, $u^-(re^{i\theta}) = \sum u_n^-(r)e^{-in\theta}$, $S^+ = \{r < 1\}$, $S^- = \{r > 1\} \cup \{\infty\}$, $\Sigma = \{r = 1\}$.

Lemma 2.1. (i) $u^+ \in C^\infty(S^+ \cup \Sigma)$ *and* $Lu^+ = f$ *for* $0 < r \leq 1$.
(ii) $u^- \in C^\infty(S^- \cup \Sigma)$ *and* $Lu^- = f$ *for* $1 \leq r < \infty$.

We postpone the proof of the lemma. By direct computation,

$$u_n^+(1) - u_n^-(1) = \begin{cases} 0 & \text{if } n \geq 0, \\ \int_0^\infty \exp\left[n\int_\rho^1 g(s)\,ds\right] f_n(\rho)\,d\rho & \text{if } n < 0. \end{cases}$$

Thus, for $n < 0$ we have

$$u_n^+(1) - u_n^-(1) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \exp\left[n\int_\rho^1 g(s)\,ds - in\theta\right] f(\rho, \theta)\,d\rho\,d\theta = 0,$$

in virtue of (2.5) with $k = -n$. Thus, u^+ and u^- agree on Σ and the same happens to their derivatives since $Lu^\pm = f$. Therefore, u^+ and u^- define a function $u \in E_0$ that verifies (2.7).

We now prove (i) of Lemma 2.1 and leave the proof of (ii) to the reader. For the sake of completeness we include in the appendix the facts we need about Fourier series in polar coordinates. We want to show that

$$u_n^+(r) = r^{|n|}U_n^+(r^2), \qquad U_n^+ \in C^\infty[0, 1], \tag{2.13}$$

$$r^{-\gamma}|n|^\beta|\partial_r^\alpha u_n^+(r)| \le C_{\alpha\beta\gamma}, \qquad |n| \ge \alpha+\gamma,\ 0 < r \le 1. \tag{2.14}$$

Let $\lambda_n(r)$ be the coefficient of $e^{in\theta}$ in the expansion of $\lambda(re^{i\theta})$. It follows from (2.4) that

$$f_n = \lambda_{n+1}, \qquad n = 0, \pm 1, \pm 2, \dots. \tag{2.15}$$

On the other hand, since $\lambda \in C^\infty$, we have for $r \le 1$,

$$\lambda_n(r) = r^{|n|}\Lambda_n(r^2), \qquad \Lambda_n \in C^\infty[0, 1], \tag{2.16}$$

$$r^{-\gamma}|n|^\beta|\partial_r^\alpha \lambda_n(r)| \le C'_{\alpha\beta\gamma}, \qquad |n| \ge \alpha+\gamma,\ 0 < r \le 1. \tag{2.17}$$

For $n \ge 0$, we obtain

$$u_n^+(r) = \int_1^r \exp\left[n\int_\rho^r g(s)\,ds\right]\lambda_{n+1}(\rho)\,d\rho.$$

We may split the integration between 1 and r as $\int_1^{1/3} + \int_{1/3}^r$ and recall that $g(s) = 1/s$ if $0 < s < 1/3$. The contribution from $\int_1^{1/3}$ for $r < 1/3$ is given by

$$\int_1^{1/3} \exp\left[n\int_\rho^{1/3} g(s)\,ds\right](3r)^n\lambda_{n+1}(\rho)\,d\rho = C_n r^n. \tag{2.18}$$

For the other integral we have, for $r < 1/3$,

$$\begin{aligned}\int_{1/3}^r &\exp[n\log(r/\rho)]\rho^{n+1}\Lambda_{n+1}(\rho^2)\,d\rho \\ &= \frac{1}{2}r^n\int_{1/9}^{r^2}\Lambda_{n+1}(t)\,dt = r^n\Gamma_n(r^2),\end{aligned} \tag{2.19}$$

where $\Gamma_n \in C^\infty[0, 1]$, in virtue of (2.16). From (2.18) and (2.19) we get (2.13) for $n \ge 0$ with $U_n^+ = \Gamma_n + C_n$. We now consider the case $n < 0$. For $r < 1/3$ we have

$$\begin{aligned}u_n^+(r) &= \int_0^r \exp\left[n\int_\rho^r g(s)\,ds\right]\lambda_{n+1}(\rho)\,d\rho \\ &= \int_0^r (r/\rho)^n\rho^{-n-1}\Lambda_{n+1}(\rho^2)\,d\rho \\ &= \frac{1}{2}r^n\int_0^{r^2} t^{-(n+1)}\Lambda_{n+1}(t)\,dt = r^n\Gamma_n(r^2).\end{aligned}$$

It is plain from the definition of Γ_n that $\Gamma_n(t) = t^{-n}U_n^+(t)$ for some $U_n^+ \in C^\infty[0, 1]$, so $u_n^+(r) = r^{-n}U_n^+(r^2) = r^{|n|}U_n^+(r^2)$.

Finally, estimates (2.14) follow from the analogous estimates (2.17) and the fact that the exponential that appears in the definition of u_n^+ is bounded by 1. We leave details the reader.

3. Proof of Theorem 3

Let us see that (i) implies (ii). With the notation of §0, consider a function $Z: S \to \overline{\Delta}$, with nonvanishing differential orthogonal to $\mathscr{L}$. On S^+ we may use $Z_+ = Z/S^+$ as a coordinate map. Equation $\delta f = 0$ means that df is proportional to dZ or, equivalently, that $\partial f/\partial \overline{Z}_+ = 0$ on S^+ and $\partial f/\partial \overline{Z}_- = 0$ on S^-. Thus, the functions $F^+, F^-: \overline{\Delta} \to \mathbb{C}$ defined by

$$F^+ = f \circ Z_+^{-1}, \qquad F^- = f \circ Z_-^{-1},$$

are holomorphic in Δ, continuous in $\overline{\Delta}$ and coincide in $\partial\Delta$, since $Z/S^\pm \cup \Sigma$ maps homeomorphically $S^\pm \cup \Sigma$ onto $\overline{\Delta}$. Thus $F^+ = F^- := F$. Since $F/\partial\Delta$ is smooth, we conclude that $F \in \mathscr{A}(\Delta) \cap C^\infty(\overline{\Delta})$ and of course, $F \circ Z = f$. The Riemann mapping theorem shows that the assumption $\Omega = \Delta$ is not restrictive.

We now show that (ii) implies (iii). Let $f = F \circ Z$, $F \in \mathscr{A}(\Omega) \cap C^\infty(\overline{\Omega})$. Then $f \in E_0$ and $df = F'\,dZ$. If $\omega \in E_1$ and $\dot{\omega}$ is in the range of δ, we have that $\omega = du + \mu\, dZ$ for some u, $\mu \in E_0$. Thus,

$$\int_S f\, d\omega = \int_S f d(\mu\, dZ) = -\int_S F'\, dZ \wedge \mu\, dZ = 0.$$

To conclude the proof we must show that (iii) implies (i). If $\omega = \mu\, dZ$, it is trivial that $\dot{\omega} = 0$ for any $\mu \in E_0$, in particular, $\dot{\omega}$ is in the range of δ. Hence,

$$0 = \int_S f d(\mu\, dZ) = -\int_S \mu\, df \wedge dZ, \qquad \mu \in E_0.$$

This shows that $df \wedge dZ$ vanishes identically in S, i.e., $\delta f = 0$.

4. Comments

(1) The compactness of S is essential in Theorem 1. There exist many nonisomorphic, globally integrable, nondegenerate structures in $\mathbb{R}^2$ [2].

(2) If Z is defined by (2.1) and (2.2) and $\phi(s)$ is defined by

$$\phi(s) = 1/2, \qquad 0 < s < 1/3,$$
$$\phi(s) = s^2/2, \qquad 2 < s < \infty,$$

it follows that

$$L' = e^{i\theta}\phi(r)(\partial_t + ig(r)\partial_\theta), \qquad 0 < r < \infty,$$

extends smoothly and without zeros to $S = \mathbb{R}^2 \cup \{\infty\}$. This shows that $\mathscr{L}$ is a trivial bundle.

(3) Theorem 2 shows that the range of δ is closed and has infinite codimension.

(4) Writing (iii) of Theorem 3 as "$\int df \wedge \omega = 0$ *for all* $\dot{\omega}$ *in the range of* δ", we see that this condition is dual to (ii) of Theorem 2. Hence, the kernel of δ and the range of δ are the orthogonal complement of each other with respect to this pairing.

(5) Consider the operator $T\colon E_0 \to E_2$ defined by

$$E_0 \ni f \mapsto df \wedge dZ.$$

We wish to determine the range of T. It is clear that every $\nu \in E_2$ can be written as $\nu = \omega \wedge dZ$, with $\omega \in E_1$, and this representation depends only on the class $\dot{\omega}$. By Theorem 2, $\nu \in T(E_0)$ if and only if $\int dv \wedge \omega = 0$ for all v in the kernel of δ.

(6) Since $\mathscr{L}$ is, in particular, a locally integrable bundle the results in [1] show that every solution of the equation $\delta f = 0$ is, locally, the limit of a sequence of polynomials in Z. Thus, Theorem 3 could be regarded as a global form of that general principle, valid in this particular case.

A. Appendix

Let $\overline{\Delta}$ be the closed unit disk in the plane and consider a function $f \in C^\infty(\overline{\Delta})$. For $r \in [0, 1]$ and $n = 0, \pm 1, \pm 2, \ldots$, set

$$f_n(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\, d\theta. \tag{A.0}$$

Proposition A. *If $f \in C^\infty(\overline{\Delta})$, then*

(i) *there exists a sequence of functions $F_n \in C^\infty[0, 1]$ such that*

$$f_n(r) = r^{|n|} F_n(r^2), \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{A.1}$$

(ii) *There exists a sequence of positive constants $C_{\alpha\beta\gamma}$, $\alpha, \beta, \gamma \in N$, such that*

$$r^{-\gamma} |n|^\beta |\partial_r^\alpha f_n(r)| \le C_{\alpha\beta\gamma}, \qquad |n| \ge \alpha + \gamma,\ 0 < r \le 1. \tag{A.2}$$

(iii) *The partial sums of the series*

$$\sum_{-\infty}^{\infty} f_n(r) e^{in\theta} \tag{A.3}$$

are smooth and converge in $C^\infty(\overline{\Delta})$ to f. Conversely, if (f_n) is a sequence in $C^\infty[0, 1]$ satisfying (i) *and* (ii), *the series* (A.3) *converges in $C^\infty(\overline{\Delta})$.*

Writing $z = x + iy$, $\overline{z} = x - iy$, and using the Taylor expansion we have

$$f(z) = \sum_{p+q<\gamma} A_{pq} z^p \overline{z}^q + \sum_{p+q=\gamma} z^p \overline{z}^q B_{pq}(z), \qquad z \in \overline{\Delta},$$

with $A_{pq} \in \mathbb{C}$, $B_{pq} \in C^\infty(\overline{\Delta})$. For $z = re^{i\theta}$ this gives

$$f(re^{i\theta}) = \sum_{p+q<\gamma} A_{pq} r^{p+q} e^{i\theta(p-q)} + r^\gamma \sum_{p+q=\gamma} e^{i\theta(p-q)} B_{pq}(re^{i\theta}). \tag{A.4}$$

Plugging (A.4) into (A.0) with $|n| = \gamma$, we obtain

$$f_n(r) = \frac{r^n}{2\pi} \sum_{p+q=|n|} \int_0^{2\pi} e^{-in\theta} e^{i\theta(p-q)} B_{pq}(re^{i\theta})\, d\theta = r^{|n|} g_n(r).$$

It is clear from the definition that $g_n \in C^\infty[0, 1]$. If we let r vary on $[-1, 1]$ we see that $g_n(r) = g_n(-r)$, just by the substitution $\theta = \theta' + \pi$ in the integral that defines g_n. Hence, $g_n(r) = F_n(r^2)$, with $F \in C^\infty[0, 1]$ as required. Next we prove (A.2) by induction in α. Plugging (A.4) into (A.0) with $\gamma \leq |n|$ we get after multiplication by $r^{-\gamma} n^{\beta}$,

$$f_n(r) r^{-\gamma} n^{\beta} = \sum_{p+q=\gamma} \int_0^{2\pi} i^{\beta} \partial_\theta^{\beta} (e^{-in\theta}) e^{i\theta(p-q)} B_{pq}(re^{i\theta})\, d\theta/2\pi$$

which after $|n|$ integration by parts, gives (A.2) for $\alpha = 0$. Assume now that (A.2) has been proved for $\alpha - 1$, $(\alpha > 0)$, all nonnegative γ, β, and $f \in C^{\infty}[0, 1]$. Differentiating (A.0) we obtain

$$f_n' = \partial_r f_n = i[(\partial f/\partial z)_{n-1} - (\partial f/\partial \overline{z})_{n+1}]$$

which implies immediately

$$\partial_r^{\alpha} f_n = i[\partial_r^{\alpha-1} (\partial f/\partial z)_{n-1} - \partial_r^{\alpha-1} (\partial f/\partial \overline{z})_{n+1}].$$

Now (A.2) follows easily from the inductive hypothesis.

To prove (iii) we observe that in view of (A.1)

$$f_n(r) = \begin{cases} (x+iy)^n F_n(x^2+y^2), & \text{if } n \geq 0, \\ (x-iy)^{-n} F_n(x^2+y^2), & \text{if } n < 0, \end{cases}$$

so the partial sums in (A.3) are smooth. The estimates (A.2) imply right away that the series is uniformly and absolutely convergent. This remains true after applying the vector fields ∂_r and $r^{-1}\partial_\theta$ any number of times, so the series converges in $C^{\infty}(\overline{\Delta})$ to a limit that has to be f itself, since it is so when f is a polynomial in z and $\overline{z}$. The same reasoning proves the second assertion in (iii).

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