TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 334, Number 2, December 1992

H^p- AND L^p-VARIANTS OF MULTIPARAMETER CALDERÓN-ZYGMUND THEORY

ANTHONY CARBERY AND ANDREAS SEEGER

ABSTRACT. We consider Calderón-Zygmund operators on product domains. Under certain weak conditions on the kernel a singular integral operator can be proved to be bounded on $H^p(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R})$, $0 < p \leq 1$, if its behaviour on L^2 and on certain scalar-valued and vector-valued rectangle atoms is known. Another result concerns an extension of the authors' results on L^p-variants of Calderón-Zygmund theory [1, 23] to the product-domain-setting. As an application, one obtains estimates for Fourier multipliers and pseudo-differential operators.

1. INTRODUCTION

The first purpose of this paper is to extend to the multiparameter setting the results obtained by the authors in [1, 2, and 23] concerning the classical (one-parameter) Calderón-Zygmund theory of L^p spaces. One of the results of [1 and 23] may be stated as follows. Let Φ be a smooth bump function on $\mathbb{R}$ supported in [1, 4] such that $\sum_{m\in\mathbb{Z}} \Phi(|t|/2^m) \equiv 1$ on $\mathbb{R}\backslash\{0\}$.

Theorem A. *Suppose m is a (bounded) function on $\mathbb{R}^n$ such that if $\tilde{m}_i(\xi) = m(2^i\xi)\Phi(|\xi|)$, we have*

$$\sup_{i\in\mathbb{Z}}(\|\tilde{m}_i\|_{p-p} + \|\tilde{m}_i\|_{\Lambda_\varepsilon}) < \infty$$

for some $1 \leq p < 2$ and some $\varepsilon > 0$. Then m is a Fourier multiplier of $L^r(\mathbb{R}^n)$ for $p < r < p'$.

(Here, $\|\cdot\|_{p-p}$ is the L^p multiplier norm and Λ_ε is the space of Lipschitz continuous functions of order ε.) Theorem A strengthens the classical Hörmander multiplier theorem and gives a good "almost orthogonality" criterion for Fourier multipliers in the sense that L^p boundedness of each "dyadic piece" $\tilde{m}_i$ implies L^r boundedness of the original multiplier m provided we have a tiny amount of smoothness. (This smoothness cannot be entirely dispensed with because of a classical counterexample of Littman, McCarthy and Rivière [20].) Thus one of our aims is to prove the analogue of Theorem A when the one-parameter family of dilations $\xi \mapsto 2^i\xi$ is replaced by the n-parameter family $(\xi_1, \dots, \xi_n) \mapsto (2^{i_1}\xi_1, \dots, 2^{i_n}\xi_n)$. To give a taste of what is to come, we state (for simplicity only) a result in $\mathbb{R}^2$.

Received by the editors December 1, 1989 and, in revised form, September 17, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42B20; Secondary 35S05, 42B15, 42B25.
Key words and phrases. Calderón-Zygmund theory, product spaces.

Theorem B. *Suppose* m *is a function on* $\mathbb{R}^2$ *such that if*

$$\tilde{m}_i(\xi) = m(2^i\xi_1, \xi_2)\Phi(|\xi_1|), \quad \tilde{m}_j(\xi) = m(\xi_1, 2^j\xi_2)\Phi(|\xi_2|),$$

and

$$\tilde{m}_{ij}(\xi) = m(2^i\xi_1, 2^j\xi_2)\Phi(|\xi_1|)\Phi(|\xi_2|),$$

we have

$$\sup_{i\in\mathbb{Z}} \|\tilde{m}_i\|_{p-p} < \infty, \qquad \sup_{j\in\mathbb{Z}} \|\tilde{m}_j\|_{p-p} < \infty,$$

and

$$\sup_{i,j\in\mathbb{Z}} \|\tilde{m}_{ij}\|_{\Lambda'_\varepsilon} < \infty.$$

Then m *is a Fourier multiplier of* $L^r(\mathbb{R}^2)$ *for* $p < r < p'$.

(Here again, $\|\cdot\|_{p-p}$ is the $L^p(\mathbb{R}^2)$ multiplier norm and Λ'_ε is the 2-parameter Lipschitz space with differences taken in both variables.)

The second purpose of the paper is to obtain a strengthened H^p-theory ($0 < p \le 1$) of product domain Calderón-Zygmund singular integrals Much work on this topic has been done recently by Journé [15, 16, 17], R. Fefferman [6, 7], Soria [24], and Pipher [22], with Journé's work being especially important. The moral of R. Fefferman's point of view [7] is that despite the fact that H^p cannot be characterized by "rectangle atoms," nevertheless a linear operator which "behaves well" with respect to rectangle atoms will be bounded from H^p to L^p, *at least in the* 2-*parameter setting.* Journé [17] has shown that in three parameters, this philosophy breaks down, but, however, remains valid (see also H. Lin [19]) for convolution operators. Perhaps the principal achievement of this paper is to show that with a different interpretation of rectangle atoms—indeed as vector-valued rectangle atoms—the Fefferman philosophy *remains valid with any number of parameters.* We stress at this point that the analysis of this paper still relies heavily on Journé's geometric ideas contained in [15].

Let us make things a little more precise. The H^p-space on the product domain $\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R} = \mathbb{R}^n$ consists of those tempered distributions f whose multiparameter area integral $S(f)$ is in L^p. Here,

$$Sf(x) = \left(\iint_{\Gamma(x)} |\psi_t * f(y)|^2\, dy \frac{dt}{t_1\cdots t_n}\right)^{1/2},$$

where $\Gamma(x)$ is the cone $\{(y, t) \in \mathbb{R}^n \times \mathbb{R}^n_+ \mid |y_i - x_i| < t_i,\ i = 1, \ldots, n\}$, and

$$\psi_t(x) = \prod_{i=1}^n \frac{1}{t_i}\psi\left(\frac{x_i}{t_i}\right)$$

where ψ is a C^∞ function of compact support in $[-\frac{1}{4}, \frac{1}{4}]$ satisfying

$$\int_{\mathbb{R}} \psi(t)p(t)\, dt = 0$$

for all polynomials p of degree $\le M$ for sufficiently large M. (If $0 < p \le 1$, we require $M > n/p$.) Of course a similar definition could have been made for the product space $\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_n}$, but for simplicity only we assume each $d_i = 1$. Also, this definition of $S(f)$ and H^p makes sense for f taking values in a Hilbert space H, and all statements we shall make remain valid in this setting.

We now introduce vector-valued rectangle atoms.

Definition. A (p, R) rectangle atom on $\mathbb{R}^m$ with values in a Hilbert space H is a function $a : \mathbb{R}^m \to H$ which is supported on some rectangle R in $\mathbb{R}^m$ (here and always a *rectangle* means a *rectangle with sides parallel to the coordinate axes*) which satisfies

(i) $\int_R |a(x)|_H^2 \, dx \leq |R|^{1-2/p}$;

(ii) $\int_{\mathbb{R}} a(x_1, \dots, x_j, \dots, x_m) p(x_j) \, dx_j = 0$ for almost all $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m)$, $j = 1, \dots, m$, and for all polynomials p of degree $\leq M$, where M is sufficiently large (we require $M > n/p$).

It is easy to see that every (p, R) rectangle atom is in $H^p(\mathbb{R}^m, H)$; see [4]. We shall refer to R as the rectangle associated to a. We use the letter m instead of n in the above definition because we shall be considering rectangle atoms living on a factor subspace $\mathbb{R}^m$ of $\mathbb{R}^n$ with values in $L^2(\mathbb{R}^{n-m})$.

To be more systematic, given a subset $\alpha \subseteq \{1, 2, \dots, n\}$, we denote by $\mathbb{R}^\alpha$ the subspace of $\mathbb{R}^n$ spanned by the unit coordinate vectors e_s, $s \in \alpha$. Similarly we define $\mathbb{R}_+^\alpha$, $\mathbb{N}^\alpha$, $\mathbb{Z}^\alpha$. We shall denote by α' the complement of α in $\{1, \dots, n\}$ so that $\mathbb{R}^n = \mathbb{R}^\alpha \oplus \mathbb{R}^{\alpha'}$ for all α. $\Pi_\alpha : \mathbb{R}^n \to \mathbb{R}^\alpha$ is the natural projection; when α is a singleton, say $\alpha = \{s\}$, we shall often write Π_s instead of $\Pi_{\{s\}}$. If R is a dyadic rectangle in $\mathbb{R}^\alpha$, we denote by $2^{L_s(R)}$ the sidelength of the projection $\Pi_s R$ and by $L_\alpha(R)$ the vector in $\mathbb{Z}^\alpha$ with components $L_s(R)$, $s \in \alpha$.

Now consider a singular integral operator with kernel $K(x, y)$ so that

$$Tf(x) = \int K(x, y) f(y) \, dy.$$

For $\alpha \subseteq \{1, \dots, n\}$ and $m \in \mathbb{Z}^\alpha$, let T_m^α be the operator with kernel

$$K(x, y) \prod_{s \in \alpha} \Phi\left(\frac{|x_s - y_s|}{2^{m_s+2}}\right) = K(x, y) \Phi_m^\alpha(\bar{x} - \bar{y}) \quad (\text{with } \bar{x}, \bar{y} \in \mathbb{R}^\alpha).$$

When $\alpha = \{1, \dots, n\}$, T_m^α will be denoted by T_m, and $L_\alpha(R)$ by $L(R)$.

We are now ready to state our first result.

Theorem 1. *Let T be a singular integral operator which is bounded on $L^2(\mathbb{R}^n)$. Let $0 < p \leq 1$ and $\varepsilon > 0$. Suppose that for all $\alpha \subseteq \{1, \dots, n\}$, $1 \leq |\alpha| \leq n-1$, for all (p, R) $L^2(\mathbb{R}^{\alpha'})$-valued rectangle atoms a associated to some rectangle R in $\mathbb{R}^\alpha$, for all $l_\alpha \in \mathbb{N}^\alpha$ we have*

$$\|T^\alpha_{L_\alpha(R)+l_\alpha} a\|_{L^p(L^2)} \leq c \prod_{s \in \alpha} 2^{-\varepsilon l_s}.$$

Suppose furthermore that for all $\mathbb{C}$-valued (p, S) rectangle atoms b associated to some rectangle S in $\mathbb{R}^n$, for all $l \in \mathbb{N}^n$ and for all $r = 1, \dots, n$, we have

$$\left\| \sum_{l_r > 0} T_{L(S)+l_r} b \right\|_p \leq c \prod_{s \neq r} 2^{-\varepsilon l_s}.$$

Then T is bounded from H^p to L^p.

Some remarks concerning Theorem 1 are in order.

Remark 1. Results for Calderón-Zygmund operators on H^p- and BMO-spaces in the multiparameter $(n \geq 3)$ case have been obtained by Journé [15] and

Pipher [22]. They considered classes of Calderón-Zygmund operators which are defined by an iteration based on an induction over the dimension. Those iterated conditions are not appropriate in the Hilbert-space valued setting, and do not contain, for example, the standard Hörmander-Marcinkiewicz multiplier theorem (Corollary 5.2). In contrast, Theorem 1 immediately extends to the vector-valued setting, with rectangle atoms in $H^p(\mathbb{R}^n, H_0)$ and $H^p(\mathbb{R}^m, L^2(H_0))$ and operator-valued kernels with values in $\mathscr{L}(H_0, H_1)$.

We should like to remark that, in the special case of BMO-estimates, a result like Theorem 1 could already have been obtained by the technique that Journé used to prove L^p-boundedness ($p \geq 2$) for the Littlewood-Paley-function associated to arbitrary intervals.

Remark 2. Theorem 1, as stated, is weaker in the case $n = 2$ than Fefferman's result [7] because we require the notion of vector-valued atoms. However there is a sharper version (see Theorem 3.1 below) which is very close to Fefferman's theorem. In fact, it requires the notion of $L^2(\mathbb{R}^{\alpha'})$-valued atoms in $\mathbb{R}^\alpha$ only for those α with $|\alpha| \leq n-2$. This illustrates the difference between the geometrically simple case $n = 2$ and the higher-parameter case.

Complementary to the H^p theory is an L^p theory, $1 < p < 2$, which eventually yields results such as Theorem B above. With this in mind, we introduce a multiparameter Littlewood-Paley decomposition. Let ϕ be a smooth function such that $\phi^2 = \Phi$. For $\alpha \subseteq \{1, \dots, n\}$ and $j \in \mathbb{Z}^\alpha$ we define

$$(Q_j^\alpha f)\hat{}(\xi) = \prod_{s \in \alpha} \phi(2^{j_s}|\xi|)\hat{f}(\xi)$$

and

$$(P_j^\alpha f)\hat{}(\xi) = \prod_{s\in\alpha}\left[1 - \sum_{k_s \leq j_s} \phi^2(2^{k_s}|\xi|)\right]\hat{f}(\xi).$$

Then we have

Theorem 2. *Suppose that T is an L^2 bounded singular integral operator. Suppose that for all β, $\gamma \subseteq \{1, \dots, n\}$ with $\beta \cap \gamma = \phi$, $\beta \cup \gamma \neq \phi$, for all $l \in \mathbb{N}^{\beta\cup\gamma}$, all $m \in \mathbb{N}^\beta$, and that for some $\varepsilon > 0$ and some large N we have*

$$\sum_{k\in\mathbb{Z}^{\beta\cup\gamma}} \sup_{j\in\mathbb{Z}^{\beta\cup\gamma}} \|Q_{j+k}^{\beta\cup\gamma} T_{j+l}^{\beta\cup\gamma} Q_{\pi_\beta j+m}^{\beta} \otimes P_{\pi_\gamma j}^{\gamma}\|_{r-r} \leq c \prod_{s\in\beta\cup\gamma} 2^{-\varepsilon l_s} \prod_{s\in\beta} 2^{N m_s},$$

for $r = p$ and for $r = 2$ (with the obvious interpretation if β or $\gamma = \varnothing$). Then T is bounded on L^p.

Section 2 of this paper contains the preliminaries from the Chang-Fefferman Hardy space theory which we need and also the relevant geometric lemma (a variant of Journé's). Sharper versions of Theorems 1 and 2 are proved in §§3 and 4 respectively. Section 5 contains the applications to Fourier multipliers and pseudodifferential operators. Finally, in §6, an example is presented to show that one cannot replace "strip-hypotheses" in multiparameter multiplier theorems by purely local hypotheses. That is, we cannot replace the first two hypotheses of Theorem B by $\sup_{i,j\in\mathbb{Z}} \|\tilde{m}_{ij}\|_{p-p} < \infty$ and still have a true theorem. In fact, if we are to insist on local hypotheses, the standard Hörmander-Marcinkiewicz theorem cannot be essentially improved upon.

2. Preliminaries from Hardy-space theory and geometry

In this section we recall some definitions and facts about the Chang-Fefferman atomic decomposition of H^p functions and prove a variant of Journé's covering lemma.

We first need some notation. Given a dyadic rectangle R in $\mathbb{R}^n$ and $\alpha \subseteq \{1, \dots, n\}$, we associate a rectangle R_+^α in $\mathbb{R}^n \times \mathbb{R}_+^\alpha$ by defining $R_+ = \{(x, \bar{t}) | x \in R, \ \bar{t} \sim 2^{L_\alpha(R)}\}$, where $\bar{t} \sim 2^{L_\alpha(R)}$ means $2^{L_s(R)} \le t_s \le 2^{L_s(R)+1}$ for all $s \in \alpha$, $(\bar{t} = (t_s)_{s\in\alpha})$. We also set $\chi_{R,\bar{t}}^\alpha(x) = \chi_{R_+^\alpha}(x, \bar{t})$ and drop the α when $\alpha = \{1, \dots, n\}$.

The following kind of estimate occurs in [3] and is proved by a duality argument and Plancherel's theorem. Let Ψ be a C^∞ even function of compact support in $[-\frac{1}{4}, \frac{1}{4}]$ such that $\int |\widehat{\Psi}(t)|^2 \, dt/t = 1$, and let $\hat{\psi}_t^\alpha(\xi) = \prod_{s\in\alpha} \widehat{\Psi}(t_s \xi_s)$.

Lemma 2.1 [3]. *Let $\mathscr{R}$ be a collection of distinct dyadic rectangles in $\mathbb{R}^n$. Suppose that for each $R \in \mathscr{R}$, for each $t \in \mathbb{R}_+^n$, we have a function $e_{R,t} : \mathbb{R}^n \to \mathbb{C}$. Then*

$$\left\| \sum_{R\in\mathscr{R}} \int_{\mathbb{R}_+^n} \psi_t * (e_{R,t}\chi_{R,t}) \frac{dt}{t_1 \cdots t_n} \right\|_2^2 \le c \sum_{R\in\mathscr{R}} \int_{\mathbb{R}_+^n} \|e_{R,t}\|_2^2 \frac{dt}{t_1 \cdots t_n}.$$

Moreover, if $\alpha \subseteq \{1, \dots, n\}$ and $\mathscr{R}$ is a collection of distinct dyadic rectangles in $\mathbb{R}^n$ with common $L_{\alpha'}$, and for each $R \in \mathscr{R}$, $\bar{t} \in \mathbb{R}_+^\alpha$, we have a function $e_{R,\bar{t}} : \mathbb{R}^n \to \mathbb{C}$, then

$$\left\| \sum_{R\in\mathscr{R}} \int_{\mathbb{R}_+^\alpha} \psi_{\bar{t}}^\alpha *_\alpha (e_{R,\bar{t}}\chi_{R,\bar{t}}^\alpha) \frac{d\bar{t}}{\prod_{s\in\alpha} t_s} \right\|_2^2 \le c \prod_{R\in\mathscr{R}} \int_{\mathbb{R}_+^\alpha} \|e_{R,\bar{t}}\|_2^2 \frac{d\bar{t}}{\prod_{s\in\alpha} t_s}$$

(*where $*_\alpha$ denotes convolution only in the α-variables*). □

For $\mu \in \mathbb{Z}$, let $\Omega_\mu = \{x | Sf(x) > 2^\mu\}$ and let $\mathscr{R}_\mu$ be the collection of all dyadic rectangles R in $\mathbb{R}^n$ satisfying $|R \cap \Omega_\mu| > \frac{1}{2}|R|$, but $|R \cap \Omega_{\mu+1}| \le \frac{1}{2}|R|$. Then we have

Lemma 2.2 [3].

$$\sum_{R\in\mathscr{R}_\mu} \int \|[(\psi_t * f)\chi_{R,t}]\|_2^2 \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n} \le c 2^{2\mu} |\Omega_\mu|. \quad \square$$

These lemmas were used to derive an atomic decomposition for H^1 functions. We give a definition of an atom associated to an open set similar to the one which may be found in [3].

Definition. Let $\Omega \subseteq \mathbb{R}^n$ be an open set of finite measure. A function b_Ω is called a (p, Ω) atom (associated to Ω) if b_Ω can be decomposed as $b_\Omega = \sum_R \int \psi_t * e_{R,t} \, dt / \prod_{s=1}^n t_s$ where $(x, t) \mapsto e_R(x, t)$ is supported in R_+ and R runs over a collection of dyadic rectangles supported in Ω, and where

$$\left(\sum_R \int \|e_{R,t}\|_2^2 \frac{dt}{t_1 \cdots t_n} \right)^{1/2} \le |\Omega|^{1/2 - 1/p}.$$

By Lemma 2.1, a (p, Ω) atom b_Ω satisfies $\|b_\Omega\|_2 \le c|\Omega|^{1/2-1/p}$; hence any $(p-R)$ atom for R a rectangle is also a (p, R) *rectangle* atom.

Now any reasonable function has an atomic decomposition via the Calderón reproducing formula. That is, $f = \sum_{\mu\in\mathbb{Z}} a_{\Omega_\mu}$ where

$$a_{\Omega_\mu} = \sum_{R\in\mathscr{R}_\mu} \int \psi_t * [(\psi_t * f)\chi_{R,t}] \frac{dt}{t_1\cdots t_n}$$

(for suitably chosen ψ) and by Lemma 2.2 the functions $[c|\Omega_\mu|^{1/p}2^\mu]^{-1} a_{\Omega_\mu}$ are $(p, \widetilde{\Omega}_\mu)$ atoms associated to the open set $\widetilde{\Omega}_\mu = \{x | M\chi_{\Omega_\mu}(x) > \frac{1}{2}\}$, M denoting the strong maximal operator. (Note that by the strong maximal theorem, $|\widetilde{\Omega}_\mu| \le c|\Omega_\mu|$.)

If an operator T satisfies $\|Tb_\Omega\|_p \le c$ for all atoms b_Ω, then, for $0 < p \le 1$, it follows that $\|Tf\|_p^p \le \sum \|Ta_\mu\|_p^p \le c\sum |\Omega_\mu| 2^{\mu p} \le c\|Sf\|_p^p = c\|f\|_p^p$, and so T is bounded from H^p into L^p.

For future use, we set, in accordance with the above discussion,

$$e_{R,t} = (\psi_t * f)\chi_{R,t}, \qquad e_R = \int \psi_t * e_{R,t} \frac{dt}{t_1\cdots t_n}$$

so that the atom b_Ω can be decomposed as the sum of elementary particles $\sum e_R$.

We now turn to geometry and a variant of Journé's lemma.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, $\mathscr{R}$ a collection of dyadic rectangles supported in Ω. We define $\widetilde{\Omega} = \{x | M\chi_\Omega(x) > 10^{-n}\}$ and inductively, $\widetilde{\Omega}^{(j)} = (\widetilde{\Omega}^{(j-1)})^\sim$, where $\widetilde{\Omega}^{(0)} = \Omega$. For $R = I_1 \times I_2 \times \cdots \times I_n \in \mathscr{R}$, let $\widehat{I}_1 = \widehat{I}_1(R, \Omega)$ be the largest dyadic interval containing I_1 such that

$$|(\widehat{I}_1 \times I_2 \times \cdots \times I_n) \cap \Omega| > \tfrac{1}{2}|\widehat{I}_1 \times I_2 \times \cdots \times I_n|.$$

Notice that $\widehat{I}_1 \times I_2 \times \cdots \times I_n \subseteq \widetilde{\Omega}$. Inductively let $\widehat{I}_s = \widehat{I}_s(R, \Omega)$ be the largest dyadic interval containing I_s such that

$$|(\widehat{I}_1 \times \widehat{I}_2 \times \cdots \times \widehat{I}_s \times I_{s+1} \times \cdots \times I_n) \cap \widetilde{\Omega}^{(s-1)}| > \tfrac{1}{2}|\widehat{I}_1 \times \widehat{I}_2 \times \cdots \times \widehat{I}_s \times I_{s+1} \times \cdots \times I_n|,$$

and notice that $\widehat{I}_1 \times \cdots \times \widehat{I}_s \times I_{s+1} \times \cdots \times I_n \subseteq \widetilde{\Omega}^{(s)}$ (which allows $\widehat{I}_{s+1}$ to be defined). Further, let $\kappa_s(R, \Omega) = L_s(\widehat{I}_s) - L_s(I_s)$.

Given the subset $\{1, \ldots, k\}$ of $\{1, \ldots, n\}$, we write $\mathbb{R}^{\{1,\ldots,k\}}$ as $\mathbb{R}^{[1\cdots k]}$, $\mathbb{R}^{\{1,\ldots,k\}'}$ as $\mathbb{R}^{[k+1\cdots n]}$, and similarly $\mathbb{Z}^{[1\cdots k]}$, $\mathbb{N}^{[1\cdots k]}$. If now $m \in \mathbb{N}^{[1\cdots k]}$ and Q is dyadic in $\mathbb{R}^{[1\cdots k]}$, we let $\mathscr{D}_{Q,m}^{[1\cdots k]}$ be the union of all dyadic rectangles S in $\mathbb{R}^{[k+1\cdots n]}$ such that $Q \times S \in \mathscr{R}$ and such that $\kappa_i(Q \times S, \Omega) = m_i$, $i = 1, \ldots, k$. Further, for $l \in \mathbb{N}^{[1\cdots k]}$, let $\mathscr{B}_{Q,l}^{[1\cdots k]} = \bigcup_{m<l} \mathscr{D}_{Q,m}^{[1\cdots k]}$ (where $m < l$ means $m_i < l_i$, $i = 1, \ldots, k$). (Both $\mathscr{D}_{Q,m}^{[1\cdots k]}$ and $\mathscr{B}_{Q,l}^{[1\cdots k]}$ depend of course on Ω and $\mathscr{R}$.)

Lemma 2.3. *For* $k = 1, 2, \ldots, n-1$ *and* $l \in \mathbb{N}^{[1\cdots k]}$,

$$\sum |Q||\mathscr{B}_{Q,l}^{[1\cdots k]}| \le c \prod_{s=1}^{k} l_s^{k-s+1} |\Omega|,$$

where the sum is extended over the dyadic Q *in* $\mathbb{R}^{[1\cdots k]}$.

Proof. First let $k = 1$, $l \in \mathbb{N}$. For a dyadic interval I of $\mathbb{R}^1$, let $E_I(\Omega)$ be the union of all dyadic S in $\mathbb{R}^{[2\cdots n]}$ such that $I \times S \subseteq \Omega$. Let $I(l)$ be the unique dyadic interval containing I such that $L_1(I(l)) = L_I(I) + l$. We first claim that

$$\mathscr{B}^{[1]}_{I,l} \subseteq [E_I(\Omega) \backslash E_{I(l)}(\Omega)]^{\sim}. \tag{2.1}$$

To see this, let $y \in \mathscr{B}^{[1]}_{I,l}$ and let S be a dyadic rectangle in $\mathbb{R}^{[2\cdots n]}$, $y \in S$, with $I \times S \in \mathscr{R}$ and $\kappa_i(I \times S) < l$. Then, since $I(l) \times E_{I(l)}(\Omega) \subseteq \Omega$,

$$|(I(l) \times S) \cap (I(l) \times E_{I(l)}(\Omega))| \le |(I(l) \times S) \cap \Omega| \le \tfrac{1}{2}|I(l) \times S|,$$

(the second inequality holding since $I(l)$ is too big to be $\widehat{I}_1(I \times S, \Omega)$). Hence, $|S \cap E_{I(l)}(\Omega)| \le \frac{1}{2}|S|$ and consequently (since $S \cup E_{I(l)} \subseteq E_I$), $|(E_I \backslash E_{I(l)}) \cap S| \ge \frac{1}{2}|S|$, which shows that $M(\chi_{E_I \backslash E_{I(l)}})(y) \ge \frac{1}{2} > \frac{1}{10^n}$, establishing (2.1). ($M$ here and $\sim$ of (2.1) refer to the strong maximal function in $\mathbb{R}^{[2\cdots n]}$.) We observe that the sets $I \times (E_I \backslash E_{I(l)})$ are disjoint if I runs over all dyadic intervals with sidelength $L_1(I) = ml + i$, for i fixed, for variable $m \in \mathbb{Z}$ (since if $I_1 \subsetneq I_2$ and I_1 and I_2 have lengths of the form 2^{ml}, $m \in \mathbb{Z}$, then $I_1(l) \subseteq I_2$, and so $(E_{I_1} \backslash E_{I_1(l)}) \cap (E_{I_2} \backslash E_{I_2(l)}) \subseteq E_{I_2} \backslash E_{I_1(l)} = \varnothing$). By the strong maximal theorem we have

$$\begin{aligned}\sum |I||\mathscr{B}^{[1]}_{I,l}| &\le c \sum_I |I||E_I \backslash E_{I(l)}| \\ &\le c \sum_{i=0}^{l-1} \sum_{m \in \mathbb{Z}} \sum_{\substack{I \\ L(I) = ml+i}} |I \times (E_I \backslash E_{I(l)})| \\ &\le cl|\Omega|,\end{aligned}$$

thus finishing the proof in the case $k = 1$.

Now assume that $2 \le k \le n-1$ and that the lemma has been proved for $1, 2, \ldots, k-1$. Take $l = (\bar{l}, l_k) = (l_1, \ldots, l_k) \in \mathbb{N}^k$. We write dyadic intervals in $\mathbb{R}^{[1\cdots k]}$ as $\overline{Q} \times I_k$ with $\overline{Q} \subseteq \mathbb{R}^{[1\cdots k-1]}$, $I_k \subseteq \mathbb{R}^{\{k\}}$.

We fix $\bar{m} \in \mathbb{N}^{[1\cdots k]}$ with $m_s < l_s$, $s = 1, \ldots, k-1$, and consider a dyadic rectangle $R = \overline{Q} \times S = I_1 \times \cdots \times I_n$ where $R \in \mathscr{R}$ and $\kappa_s(R, \Omega) = m_s$, $s = 1, \ldots, k-1$, and we write $\hat{\overline{Q}} = \widehat{I}_1(\overline{Q} \times S, \Omega) \times \cdots \times \widehat{I}_{k-1}(\overline{Q} \times S, \Omega)$. In other words, $\widehat{I}_s = I_s(m_s)$ and so $\hat{\overline{Q}}$ is independent of the choice of S used. By the definition of $\widehat{I}_s$, we have

$$\hat{\overline{Q}} \times \mathscr{D}^{[1\cdots k-1]}_{\overline{Q}, \bar{m}} \subseteq \widetilde{\Omega}^{(k-1)}.$$

For each $\overline{m}$, let $I'_{k,\bar{m}} = I'_{k,\bar{m}}(R)$ be the largest dyadic interval in $\mathbb{R}^{\{k\}}$ containing I_k such that

$$|(I'_{k,\bar{m}} \times I_{k+1} \times \cdots \times I_n) \cap \mathscr{D}^{[1\cdots k-1]}_{\overline{Q}, \bar{m}}| > \tfrac{1}{2}|I'_{k,\bar{m}} \times I_{k+1} \times \cdots \times I_n|.$$

We have

$$\begin{aligned}|(\hat{\hat{Q}} \times I'_{k,\bar{m}} \times I_{k+1} \times \cdots \times I_n) \cap \widetilde{Q}^{(k-1)}| &\\ &\geq |\hat{\hat{Q}} \times (I'_{k,\bar{m}} \times \cdots \times I_n \cap \mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}}| \\ &> \tfrac{1}{2}|\hat{\hat{Q}}||I'_{k,\bar{m}} \times \cdots \times I_n| \\ &= \tfrac{1}{2}|\hat{\hat{Q}} \times I'_{k,\bar{m}} \times \cdots \times I_n|,\end{aligned}$$

so that by the maximality condition in the definition of $\widehat{I}_k$ we can conclude that $I'_{k,\bar{m}} \subseteq \widehat{I}_k$.

In analogy with (2.1) we now claim that for all $Q = \overline{Q} \times I_k$ dyadic in $\mathbb{R}^{[1\cdots k]}$ and all $m \in \mathbb{N}^{[1\cdots k]}$, we have

$$\mathscr{D}^{[1\cdots k]}_{Q,m} \subseteq [E^{[k]}_{I_k}(\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}}) \backslash E^{[k]}_{I_k(m_k+1)}(\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}})]^{\sim}, \tag{2.2}$$

where, for J an interval in $\mathbb{R}^{\{k\}}$ and $\Sigma \subseteq \mathbb{R}^{[k\cdots n]}$, $E^{[k]}_J(\Sigma)$ is the union of all dyadic S in $\mathbb{R}^{[k+1\cdots n]}$ such that $J \times S \subseteq \Sigma$.

To establish (2.2), we should like to show that if $T = I_{k+1} \times \cdots \times I_n \subseteq \mathscr{D}^{[1\cdots k]}_{Q,m}$, then $T \subseteq E^{[k]}_{I_k}(\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}})$ and that

$$|T \cap E^{[k]}_{I_k(m_k+1)}(\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}})| \leq \tfrac{1}{2}|T|.$$

The first of these two assertions is clear from the definitions, and the above arguments show that $I_k \subseteq I'_{k,\bar{m}}(Q \times T) \subseteq \widehat{I}_k = I_k(m_k) \subsetneq I_k(m_k+1)$, and hence that

$$|(I_k(m_k+1) \times I_{k+1} \times \cdots \times I_n) \cap \mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}}| \leq \tfrac{1}{2}|I_k(m_k+1) \times I_{k+1} \times \cdots \times I_n|,$$

or $|(I_k(m_k+1) \times T) \cap \mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}}| \leq \tfrac{1}{2}|(I_k(m_k+1) \times T|$. Now

$$I_k(m_k+1) \times E^{[k]}_{I_k(m_k+1)}(\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}}) \subseteq \mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}},$$

and so

$$\begin{aligned}|(I_k(m_k+1) \times T) \cap (I_k(m_k+1) \times E^{[k]}_{I_k(m_k+1)}(\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}}))| &\\ &\leq \tfrac{1}{2}|I_k(m_k+1) \times T|.\end{aligned}$$

Dividing both sides of this inequality by $|I_k(m_k+1)|$ establishes (2.2).

An immediate consequence of (2.2) is

$$\mathscr{D}^{[1\cdots k]}_{Q,m} \subseteq [E^{[k]}_{I_k}(\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}}) \backslash E^{[k]}_{I_k(l_k)}(\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}})]^{\sim}$$

whenever $m_k < l_k$. Arguing as in the first step,

$$\begin{aligned}\sum_Q |Q||\mathscr{B}^{[1\cdots k]}_{Q,l}| &\leq \sum_{\overline{Q}} \sum_{I_k} |\overline{Q}||I_k| \sum_{\bar{m}<\bar{l}} \left| \bigcup_{m_k=1}^{l_k-1} \mathscr{D}^{[1\cdots k]}_{\overline{Q}\times I_k,(\bar{m},m_k)} \right| \\ &\leq c \sum_{\bar{m}<\bar{l}} \sum_{\overline{Q}} |\overline{Q}| \sum_{I_k} |I_k| \left| E^{[k]}_{I_k}(\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}}) \backslash E^{[k]}_{I_k(l_k)}(\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}}) \right| \\ &\leq c \sum_{\bar{m}<\bar{l}} \sum_{\overline{Q}} |\overline{Q}| l_k |\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}}|,\end{aligned}$$

and, since $\mathscr{D}^{[1\cdots k-1]}_{\overline{Q},\bar{m}} \subseteq \mathscr{B}^{[1\cdots k-1]}_{\overline{Q},\tilde{\bar{m}}}$ $(\tilde{m}_s = m_s + 1)$ we may apply the inductive hypothesis to obtain

$$\begin{aligned}\sum_{Q} |Q||\mathscr{B}^{[1\cdots k]}_{Q,l}| &\le c l_k \sum_{\bar{m}<\bar{l}} \sum_{\overline{Q}} |\overline{Q}||\mathscr{B}^{[1\cdots k-1]}_{\overline{Q},\tilde{\bar{m}}}| \\ &\le c l_k \sum_{\bar{m}<\bar{l}} \prod_{r=1}^{k-1} (m_r+1)^{k-1-r+1}|\Omega| \\ &\le c l_k \sum_{\bar{m}<\bar{l}} \prod_{r=1}^{k-1} l_r^{k-r}|\Omega| \\ &= c l_k \prod_{r=1}^{k-1} l_r^{k-r} \#\{\bar{m} < \bar{l}\}|\Omega| \\ &= c l_k \prod_{r=1}^{k-1} l_r^{k-r} l_1 \cdots l_{k-1}|\Omega| \\ &= c \prod_{r=1}^{k} l_r^{k-r+1}|\Omega|,\end{aligned}$$

which is what we had to prove. □

Now let $\mathscr{R}$ be some collection of dyadic rectangles supported in Ω such that any two rectangles with the same projections on $\mathbb{R}^{[1\cdots n-1]}$ are disjoint. Define $\widehat{I}_s(R, \Omega)$ for $R \in \mathscr{R}$, $1 \le s \le n-1$ as before. Let $\mathscr{M}(l, \mathscr{R})$ be the collection of all those rectangles in $\mathscr{R}$ satisfying $\kappa_s(R, \Omega) < l_s$, $s = 1, \ldots, n-1$. Then we have the following variant of Journé's lemma [16]:

Lemma 2.4.

$$\sum_{R\in\mathscr{M}(l,\mathscr{R})} |R| \le c \prod_{i=1}^{n-1} l_i^{n-i}|\Omega|.$$

Proof. Two different dyadic rectangles in $\mathscr{M}(l, \mathscr{R})$ with the same projection Q in $\mathbb{R}^{[1\cdots n-1]}$ are disjoint and will be contained in $Q\times\mathscr{B}^{1\cdots n-1]}_{Q,l}$. The assertion now follows from the case $k = n-1$ of Lemma 2.3. □

In the above two lemmas, we have made geometrical constructions (dilations of rectangles in some directions) depending on an ordering of these directions. Of course, we could have chosen any permutation of the standard ordering used here, and still have obtained an analogous result; we shall need this observation in the next section.

3. H^p-ESTIMATES

We first formulate a version of Theorem 1, with, among other things, improved decay assumptions. For $\alpha \subsetneqq \{1, \ldots, n\}$ with $\alpha = \{\alpha_1 < \cdots < \alpha_\nu\}$ we set $d(l, \alpha) = l_{\alpha_1}^{\nu} l_{\alpha_2}^{\nu-1} \cdots l_{\alpha_\nu}$. Then we have

Theorem 3.1. *Let T be a singular integral operator which is bounded on L^2 with norm at most A. Suppose that for all α, $1 \le |\alpha| \le n-2$, all $l \in \mathbb{N}^\alpha$, all*

$L^2(\mathbb{R}^{\alpha'})$-valued (p, R) rectangle atoms a we have

$$\|T^\alpha_{L_\alpha(R)+l}a\|_{L^p(L^2)} \le A^\alpha_l.$$

Suppose furthermore that for all α, $|\alpha| = n-1$, and all $\mathbb{C}$-valued (p, S) rectangle atoms b we have

$$\|T^\alpha_{L_\alpha(S)+l}b\|_p \le B^\alpha_l$$

and that for all $1 \le r \le n$, for all $\tilde{l} \in \mathbb{N}^{\{r\}'}$,

$$\left\|\sum_{l_r>0} T_{L(S)+l}b\right\|_p \le C^{\{r\}'}_{\tilde{l}}.$$

If

$$\sup_{|\alpha|\le n-2}\sum_{l\in\mathbb{N}^\alpha}[A^\alpha_l d(l, \alpha)^{1/p-1/2}]^p \le A^p,$$
$$\sup_{|\alpha|=n-1}\sum_{l\in\mathbb{N}^\alpha}[B^\alpha_l d(l, \alpha)^{1/p-1/2}]^p \le A^p,$$

and

$$\sup_{1\le r\le n}\sum_{\tilde{l}\in\mathbb{N}^{\{r\}'}}[C^{\{r\}'}_{\tilde{l}} d(l, \{r\}')^{1/p-1/2}]^p \le A^p,$$

then T is bounded from H^p to L^p with norm at most CA.

We first show how Theorem 1 can be obtained from Theorem 3.1. First of all, $d(l, \alpha) \le c_\varepsilon \prod_{s\in\alpha} 2^{l_s\varepsilon}$ for every $\varepsilon > 0$. Hence we only have to verify that under the assumptions of Theorem 1 we have

$$\|T^\alpha_{L_\alpha(S)+l}b\|_p \le c_\varepsilon \prod_{s\in\alpha} 2^{-l_s\varepsilon} \quad \text{for } |\alpha| = n-1.$$

Suppose without loss of generality that $\alpha = \{2, \dots, n\}$ and let b be a $\mathbb{C}$-valued rectangle atom associated to $S = I \times J$ with $I \subseteq \mathbb{R}^{\{1\}}$ and $J \subseteq \mathbb{R}^{\{1\}'}$. Let $\bar{I}$ be the interval with the same centre as I but with twenty times the sidelength. Then by Hölder's inequality,

$$\begin{aligned}\|T^\alpha_{L_\alpha(S)+l}b\|_p^p &\le c\int_{\bar{I}}\left[\int_{\mathbb{R}^\alpha}|T^\alpha_{L_\alpha(S)+l}b|^2\right]^{p/2} dx|I|^{1-p/2} \\ &\quad + \int_{\bar{I}^c\times\mathbb{R}^\alpha}|T^\alpha_{L_\alpha(S)+l}b|^p\,dx \\ &= A + B.\end{aligned}$$

Now $|I|^{-1/2+1/p}\|b\|_2^{-1}b$ may be considered as an $L^2(\mathbb{R}^\alpha)$-valued rectangle atom associated to I and so $A \le c\prod_{s=2}^n 2^{-l_s\varepsilon}$ by the hypothesis of Theorem 1. Moreover,

$$B \le \left\|\sum_{l_1>0} T_{L(S)+l}b\right\|_p^p \le c\prod_{s=2}^n 2^{-l_s\varepsilon},$$

again by the hypothesis of Theorem 1.

In order to prove Theorem 3.1 we shall use an induction argument which reduces estimates for $(p, I \times A)$ atoms (with A living in an m-dimensional space with $m \geq 2$) to estimates for L^2-valued (p, I) rectangle atoms and for $\mathbb{C}$-valued rectangle atoms, and, if $m \geq 3$, to estimates for $(p, I \times J \times B)$ atoms for B living in a space of dimension $m - \dim J$.

Proposition 3.2. (a) *Let b_Ω be a (p, Ω) atom in $\mathbb{R}^n$. Suppose that T is bounded on $L^2(\mathbb{R}^n)$ with norm at most A. Suppose furthermore that for all $\alpha \subseteq \{1, \dots, n\}$, $1 \leq |\alpha| \leq n-2$, for all $l \in \mathbb{N}^\alpha$, for all dyadic rectangles Q in $\mathbb{R}^\alpha$, for all open sets A in $\mathbb{R}^{\alpha'}$ and all $(p, Q \times A)$ atoms, we have*

$$\|T^\alpha_{L(Q)+l} a\|_p \leq \Gamma^\alpha_l, \qquad l \in \mathbb{N}^\alpha. \tag{3.1}$$

Moreover, suppose that for all α, $|\alpha| = n-1$, for all $l \in \mathbb{N}^\alpha$, for all dyadic S in $\mathbb{R}^n$ and all $\mathbb{C}$-valued (p, S) rectangle atoms, we have

$$\|T^\alpha_{L_\alpha(S)+l} a\|_p \leq D^\alpha_l \tag{3.2}$$

and also, writing $l = (\tilde{l}, l_n)$, $\tilde{l} \in \mathbb{N}^{\{n\}'}$,

$$\left\| \sum_{l_n \geq 0} T_{L(S)+l} a \right\|_p \leq E_{\tilde{l}}. \tag{3.3}$$

Then

$$\|T b_\Omega\|_p^p \leq c \left[A^p + \sup_{1 \leq |\alpha| \leq n-2} \sum_{l \in \mathbb{N}^\alpha} (\Gamma^\alpha_l)^p d(l, \alpha)^{1-p/2} + \sup_{|\alpha| = n-1} \sum_{l \in \mathbb{N}^\alpha} (D^\alpha_l)^p d(l, \alpha)^{1-p/2} + \sum_{\tilde{l} \in \mathbb{N}^{\{n\}'}} E^p_{\tilde{l}} d(\tilde{l}, \{n\}')^{1/-p/2} \right].$$

(b) *Suppose $b_{I \times A}$ is a $(p, I \times A)$ atom where $I \subseteq \mathbb{R}^\beta$, $A \subseteq \mathbb{R}^\gamma$, $\beta \cap \gamma = \varnothing$, $\beta \cup \gamma = \{1, \dots, n\}$, $\gamma = \{\gamma_1 < \cdots < \gamma_\nu\}$ with $\nu \geq 2$. Suppose that for all $L^2(\mathbb{R}^\gamma)$-valued (p, I) rectangle atoms, we have*

$$\|Ta\|_{L^p(\mathbb{R}^\beta, L^2(\mathbb{R}^\gamma))} \leq A, \tag{3.4}$$

and that for all $\alpha \subseteq \gamma$, $|\alpha| \leq |\gamma| - 2$, for all $l \in \mathbb{N}^\alpha$, all Q dyadic in $\mathbb{R}^\alpha$, for all open sets B in $\mathbb{R}^{\gamma - \alpha}$ and for all $(p, I \times Q \times B)$ atoms a we have

$$\|T^\alpha_{L_\alpha(Q)+l} a\|_p \leq \Gamma^\alpha_l. \tag{3.5}$$

Suppose furthermore that for all $\alpha \subseteq \gamma$, $|\alpha| = |\gamma| - 1$, for all $l \in \mathbb{N}^\alpha$, all dyadic S in $\mathbb{R}^\gamma$ and all $(p, I \times S)$ rectangle atoms, we have

$$\|T^\alpha_{L_\alpha(S)+l} a\|_p \leq D^\alpha_l, \tag{3.6}$$

and if $l = (\tilde{l}, l_{\gamma_\nu})$, $\tilde{l} \in \mathbb{N}^{\gamma - \{\gamma_\nu\}}$,

$$\left\| \sum_{l_{\gamma_\nu} > 0} T^\gamma_{L_\gamma(s)+l} a \right\|_p \leq E^{\gamma - \{\gamma_\nu\}}_{\tilde{l}}. \tag{3.7}$$

Then

$$\|Tb_{I\times A}\|_p^p \le c\left[A^p + \sup_{|\alpha|\le|\gamma|-2}\sum_{l\in\mathbb{N}^\alpha}(\Gamma_l^\alpha)^p d(l\,,\,\alpha)^{1-p/2} + \sup_{|\alpha|=|\gamma|-1}\sum_{l\in\mathbb{N}^\alpha}(D_l^\alpha)^p d(l\,,\,\alpha)^{1-p/2} + \sum_{\tilde{l}\in\mathbb{N}^{\gamma-\{\gamma_\nu\}}}(E_{\tilde{l}}^{\gamma-\{\gamma_\nu\}})^p d(\tilde{l}\,,\,\gamma-\{\gamma_\nu\})^{1-p/2}\right].$$

Proof of Theorem 3.1. By the discussion of §2, it suffices to show $\|Tb_\Omega\|_p \le C$ for all $(p\,,\,\Omega)$ atoms. Proposition 3.2(a) reduces this to the L^2-boundedness of T, appropriate estimates of the form (3.2), and (3.3)—all of which are hypotheses of Theorem 3.1—and estimates of the form (3.1). To check (3.1) we apply Proposition 3.2(b) to the operator $T^\alpha_{L_\alpha(s)+l}$, reducing matters to (3.4), (3.6), and (3.7)—all of which follow directly from the hypotheses of Theorem 3.1—and (3.5). Hence Proposition 3.2 gives an iterative scheme for proving Theorem 3.1. We leave the verification of the constants to the reader. □

In order to prove Proposition 3.2, we carry out a further geometric construction which is essentially the same as in [22].

Suppose $\gamma\subseteq\{1\,,\,\dots\,,\,n\}$, $\gamma=\{\gamma_1<\cdots<\gamma_m\}$. Let $<$ be any total ordering on the subset of γ such that $\varnothing<\cdots<\bar{\gamma}<\gamma$, where if $m\ge 2$, $\bar{\gamma}$ contains γ_m. (This last requirement is merely a technical convenience.) For $\alpha\subseteq\gamma$, let $\bar{\alpha}$ denote its predecessor with respect to the above ordering and define $N(\varnothing)=0$, $N(\alpha)=N(\bar{\alpha})+|\alpha|$ (so that $N(\gamma)=\sum_{k=0}^m k\binom{m}{k}=m2^{m-1}$).

Let $\Omega\subseteq\mathbb{R}^\gamma$ be an open set and let $\mathscr{R}$ be a family of dyadic rectangles supported in Ω. For each $R\in\mathscr{R}$ and each $\alpha\subseteq\gamma$ we will define several "enlargements" $u_r^\alpha(R\,,\,\Omega)$ $(r=0\,,\,\dots\,,\,|\alpha|)$ and $w^\alpha(R\,,\,\Omega)$, such that for every α we will be able to apply Lemma 2.3 to the family of rectangles $\{u_0^\alpha(R\,,\,\Omega)\}_{R\in\mathscr{R}}$.

We set $w^\varnothing(R)=R$. Suppose we have defined $w^{\bar{\alpha}}(R)\subseteq\widetilde{\Omega}^{(N(\bar{\alpha}))}$. Given $\alpha=\{\alpha_1<\cdots<\alpha_\nu\}$, we set $u_0^\alpha(R\,,\,\Omega)=w^{\bar{\alpha}}(R)$ and if $u_0^\alpha(R\,,\,\Omega)=I_{\gamma_1}\times\cdots\times I_{\gamma_m}$, we proceed to define $\widehat{I}_{\alpha_1}\,,\,\widehat{I}_{\alpha_2}\,,\,\dots\,,\,\widehat{I}_{\alpha_\nu}$ as in §2. That is, $\widehat{I}_{\alpha_1}$ is the largest dyadic interval containing I_{α_1} such that

$$|I_{\gamma_1}\times\cdots\times\widehat{I}_{\alpha_1}\times\cdots\times I_{\gamma_m}\cap\widetilde{\Omega}^{(N(\bar{\alpha}))}|>\tfrac{1}{2}|I_{\gamma_1}\times\cdots\times\widehat{I}_{\alpha_1}\times\cdots\times I_{\gamma_m}|$$

and we define $u_1^\alpha(R\,,\,\Omega)=I_{\gamma_1}\times\cdots\times\widehat{I}_{\alpha_1}\times\cdots\times I_{\gamma_m}:=J_{\gamma_1}\times\cdots\times J_{\gamma_m}$, which is clearly supported in $\widetilde{\Omega}^{(N(\bar{\alpha})+1)}$. If $\nu>1$ we let $\widehat{J}_{\alpha_2}$ be the largest dyadic interval containing J_{α_2} such that

$$|J_{\gamma_1}\times\cdots\times\widehat{J}_{\alpha_2}\times\cdots\times J_{\gamma_m}\cap\widetilde{\Omega}^{(N(\bar{\alpha})+1)}|>\tfrac{1}{2}|J_{\gamma_1}\times\cdots\times\widehat{J}_{\alpha_2}\times\cdots\times J_{\gamma_m}|$$

and set $u_2^\alpha(R\,,\,\Omega)=J_{\gamma_1}\times\cdots\times J_{\gamma_m}\subseteq\widetilde{\Omega}^{(N(\bar{\alpha})+2)}$. Similarly, we proceed to define $u_r^\alpha(R\,,\,\Omega)$ supported in $\widetilde{\Omega}^{(N(\bar{\alpha})+r)}$ for $r=3\,,\,\dots\,,\,\nu$. We set

$$w^\alpha(R\,,\,\Omega)=\begin{cases}u_\nu^\alpha(R\,,\,\Omega) & \text{if } \alpha\ne\gamma\,,\\ u_{|\gamma|-1}^\alpha(R\,,\,\Omega) & \text{if } \alpha=\gamma.\end{cases}$$

We also set $\kappa_s^\alpha(R\,,\,\Omega)=L_s w^\alpha(R\,,\,\Omega)-L_s u_0^\alpha(R\,,\,\Omega)$, and $\nu_s^\alpha(R\,,\,\Omega)=L_s w^\gamma(R)-L_s u_0^\alpha(R\,,\,\Omega)$ for $s\in\gamma$, so that we clearly have

$$\kappa_s^\alpha(R\,,\,\Omega)\le\nu_s^\alpha(R\,,\,\Omega). \tag{3.8}$$

Notice also that $\kappa_s^\gamma(R, \Omega) = \nu_s^\gamma(R, \Omega)$ and that $\kappa_{\gamma_m}^\gamma(R, \Omega) = 0$.

Lemma 3.3. (a) *For Q dyadic in $\mathbb{R}^\alpha$, $l \in \mathbb{N}^\alpha$ $(\alpha \subseteq \gamma)$, Ω open in $\mathbb{R}^\gamma$, let $\mathscr{A}_{Q,l}^{\alpha,\gamma}(\Omega)$ be the union of all projections $\Pi_{\gamma-\alpha} u_0^\alpha(R, \Omega)$ where $\nu_s^\alpha(R, \Omega) < l_s$, $s \in \alpha$, and $\Pi_\alpha u_0^\alpha(R, \Omega) = Q$. Then*

$$\sum_Q |Q||A_{Q,l}^{\alpha,\gamma}(\Omega)| \leq cd(l, \alpha)|\Omega|.$$

(b) *For $\tilde{l} \in \mathbb{N}^{\gamma-\{\gamma_m\}}$, let $\mathscr{M}_{\tilde{l}}^\gamma(\Omega)$ be the family of all dyadic S in $\mathbb{R}^\gamma$ such that $S = u_0^\gamma(R, \Omega)$ for some $R \in \mathscr{R}$ and such that $\nu_s^\gamma(R, \Omega) = \kappa_s^\gamma(R, \Omega) < l_s$ for $s \neq \gamma_m$. Then*

$$\sum_{S \in \mathscr{M}_{\tilde{l}}^\gamma(\Omega)} |S| \leq cd(\tilde{l}, \gamma - \{\gamma_m\})|\Omega|.$$

Proof. Part (a) follows immediately from Lemma 2.3 applied to the rectangles $\{u_0^\alpha(R)\}_{R\in\mathscr{R}}$ which are contained in the set $\widetilde{\Omega}^{(N(\bar{\alpha}))}$. To see part (b), observe that since $\bar{\gamma}$ contains γ_m, the maximality condition in the definition of $u_{m-1}^{\bar{\gamma}}(R, \Omega)$ ensures that two different S in $\mathscr{M}_{\tilde{l}}^\gamma(\Omega)$ with the same projections in $\mathbb{R}^{\gamma-\{\gamma_m\}}$ are disjoint. Now apply Lemma 2.4. □

Proof of Proposition 3.2. We prove only part (a); the proof of part (b) is exactly similar and will be omitted. Let b_Ω be a (p, Ω) atom; we consider Tb_Ω separately in $\widetilde{\Omega}^{(n2^{n-1})}$ and in $\mathbb{R}^n \backslash \widetilde{\Omega}^{(n2^{n-1})}$. As in [6], the estimate in $\widetilde{\Omega}^{(n2^{n-1})}$ is an easy consequence of the L^2-boundedness of T and the strong maximal theorem:

$$\begin{aligned}\|Tb_\Omega\|_{L^p(\widetilde{\Omega}^{(n2^{n-1})})}^p &\leq |\widetilde{\Omega}^{(n2^{n-1})}|^{1-p/2}\|Tb_\Omega\|_2^p \\ &\leq CA^p|\Omega|^{1-p/2}\|b_\Omega\|_2^p \leq CA^p.\end{aligned}$$

To estimate Tb_Ω in $\mathbb{R}^n \backslash \widetilde{\Omega}^{(n2^{n-1})}$, we use the formula

$$\sum_{m\in\mathbb{Z}^n} a_m = \sum_{1\leq|\alpha|\leq n} (-1)^{|\alpha|-1} \sum_{\substack{m\in\mathbb{Z}^m,\ m_s>0 \\ \text{for } s\in\alpha}} a_m + \sum_{\substack{m\in\mathbb{Z}^m,\ m_s\leq 0 \\ 1\leq s\leq n}} a_m$$

(which may be proved by applying the formula

$$\prod_j (1 - P_j) = 1 - \sum_j P_j + \sum_{j\neq k} P_j P_k - \sum_{j\neq k\neq l} P_j P_k P_l \cdots$$

with $P_j = \chi_{\{m_j>0\}}$) to write, as in [15], for $x \notin \widetilde{\Omega}^{(n2^{n-1})}$,

$$\begin{aligned}Tb_\Omega(x) &= \sum_{1\leq|\alpha|\leq n} (-1)^{|\alpha|-1} \sum_R \sum_{m_s > L_s w^\gamma(R,\Omega)} T_m^\alpha e_R(x) \\ &= \sum_{1\leq|\alpha|\leq n} (-1)^{|\alpha|-1} I_\alpha\end{aligned}$$

with $\gamma = \{1, \dots, n\}$ and

$$b_\Omega = \sum_R e_R = \sum_R \int \psi_t * e_{R,t} \frac{dt}{t_1 \cdots t_n},$$

since the missing term

$$\sum_{1\le|\alpha|\le n}\sum_{R}\sum_{m_s\le L_s w^\gamma(R,\Omega)} T_m^\alpha e_R$$

is supported in $\widetilde{\Omega}^{(n2^{n-1})}$. For $1\le|\alpha|\le n-2$ we substitute $l_s=m_s-L_s u_0^\alpha(R,\Omega)$ and write

$$\begin{aligned} I_\alpha &= \sum_{R}\sum_{l\in\mathbb{N}^\alpha}\sum_{l_s>L_s w^\gamma(R)-L_s u_0^\alpha(R)} T^\alpha_{l+L_\alpha u_0^\alpha(R)} e_R \\ &= \sum_{l\in\mathbb{N}^\alpha}\sum_{R:\,\nu_s^\alpha(R,\Omega)<l_s} T^\alpha_{l+L_\alpha u_0^\alpha(R)} e_R. \end{aligned}$$

Let $\mathscr{U}^\alpha_{Q,l}$ be the family of all rectangles R such that $\nu_s^\alpha(R,\Omega)<l_s$, $s\in\alpha$, and $\Pi_\alpha u_0^\alpha(R)=Q$. Then, if we set $a^\alpha_{Q,l}=\sum_{R\in\mathscr{U}^\alpha_{Q,l}} e_R$, we have

$$I_\alpha=\sum_{l\in\mathbb{N}^\alpha}\sum_{Q\text{ dyadic in }\mathbb{R}^\alpha} T^\alpha_{l+L(Q)} a^\alpha_{Q,l}.$$

Now

$$|Q\times\mathscr{A}^{\alpha,\gamma}_{Q,l}|^{-1/p+1/2}\left(\sum_{R\in\mathscr{U}^\alpha_{Q,l}}\int\|e_{R,t}\|_2^2\frac{dt}{t_1\cdots t_n}\right)^{-1/2} a^\alpha_{Q,l}$$

is a constant multiple (for a fixed absolute constant) of a $(p, Q\times\mathscr{A}^{\alpha,\gamma}_{Q,l})$ atom. Consequently, we have

$$\|T^\alpha_{l+L_\alpha(Q)} a^\alpha_{Q,l}\|_p^p \le C(\Gamma_l^\alpha)^p|Q\times\mathscr{A}^{\alpha,\gamma}_{Q,l}|^{1-p/2}\left(\sum_{R\in\mathscr{U}^\alpha_{Q,l}}\int\|e_{R,t}\|_2^2\frac{dt}{t_1\cdots t_n}\right)^{p/2},$$

and by applying Hölder's inequality and Lemma 3.3(a), we get

$$\begin{aligned} &\sum_Q \|T^\alpha_{l+L_\alpha(Q)} a^\alpha_{Q,l}\|_p^p \\ &\le c(\Gamma_l^\alpha)^p\left(\sum_Q |Q||\mathscr{A}^{\alpha,\gamma}_{Q,l}|\right)^{1-p/2}\left(\sum_Q\sum_{R\in\mathscr{U}^\alpha_{Q,l}}\int\|e_{R,t}\|_2^2\frac{dt}{t_1\cdots t_n}\right)^{p/2} \\ &\le c(\Gamma_l^\alpha)^p d(l,\alpha)^{1-p/2}, \end{aligned}$$

since b_Ω is a (p,Ω) atom. The bound for I_α follows by summation on $l\in\mathbb{N}^\alpha$.

For $|\alpha|=n-1$ we use a slightly different argument. Let $\mathscr{M}(\mathscr{A}^{\alpha,\gamma}_{Q,l})$ be the family of all *one-dimensional* dyadic intervals which are maximal in $\mathscr{A}^{\alpha,\gamma}_{Q,l}$. Each $R\in\mathscr{U}^\alpha_{Q,l}$ is contained in a unique $Q\times I$ with $I\in\mathscr{M}(\mathscr{A}^{\alpha,\gamma}_{Q,l})$. For $J\in\mathscr{M}(\mathscr{A}^{\alpha,\gamma}_{Q,l})$ let

$$a^\alpha_{Q,J,l}=\sum_{R\in\mathscr{U}^\alpha_{Q,l},\,\pi_{\alpha'}R\subseteq J} e_R;$$

clearly $a^\alpha_{Q,J,l}|Q\times J|^{-1/p+1/2}\|a_{Q,J,l}\|_2^{-1}$ is a $(p, Q\times J)$ atom. Hence,

$$\|T^\alpha_{l+L(Q)} a^\alpha_{Q,J,l}\|_p^p\le(D_l^\alpha)^p\|Q\times J|^{1-p/2}\|a^\alpha_{Q,J,l}\|_2^p.$$

Therefore,

$$\|I_\alpha\|_p^p \le \sum_{l\in\mathbb{N}^\alpha}(D_l^\alpha)^p\sum_Q|Q|^{1-p/2}\left(\sum_J|J|^{1-p/2}\|a_{Q,J,l}^\alpha\|_2^p\right)$$

$$\le \sum_{l\in\mathbb{N}^\alpha}(D_l^\alpha)^p\sum_Q|Q|^{1-p/2}|\mathscr{A}_{Q,l}^{\alpha,\gamma}|^{1-p/2}\left(\sum_{J\in\mathscr{M}(\mathscr{A}_{Q,l}^{\alpha,\gamma})}\|a_{Q,J,l}^\alpha\|_2^2\right)^{p/2}$$

(since different $J\in\mathscr{M}(\mathscr{A}_{Q,l}^{\alpha,\gamma})$ are disjoint)

$$\le \sum_{l\in\mathbb{N}^\alpha}(D_l^\alpha)^p|\Omega|^{1-p/2}\left(\sum_Q\sum_{J\in\mathscr{M}(\mathscr{A}_{Q,l}^{\alpha,\gamma})}\|a_{Q,J,l}^\alpha\|_2^2\right)^{p/2}$$

$$\le c\sum_{l\in\mathbb{N}^\alpha}(D_l^\alpha)^p d(l,\alpha)^{1-p/2},$$

by Lemma 3.3, Lemma 2.1, and the definition of the atom b_Ω.

Finally, we must estimate I_γ for $\gamma=\{1,\dots,n\}$. For $S\in\mathscr{M}_{\tilde{l}}^\gamma(\Omega)$, let $a_S=\sum_{u_0^\gamma(R,\Omega)=S}e_R$. Then

$$I_\gamma=\sum_R\sum_{m_s>L_sw^\gamma(R)}T_me_R$$

$$=\sum_S\sum_{\substack{R\\u_0^\gamma(R)=S}}\sum_{\substack{m_s>L_s(S)+\nu_s^\gamma(R,\Omega)\\s=1,\dots,n-1}}\sum_{m_n>L_nu_0^\gamma(R)}T_me_R$$

(since $L_nw^\gamma(R)=L_nu_0^\gamma(R)$)

$$=\sum_{\tilde{l}\in\{n\}'}\sum_{S\in\mathscr{M}_{\tilde{l}}^\gamma(\Omega)}\sum_{l_n\ge0}T_{L(S)+l}a_S.$$

Since $|S|^{1/2-1/p}\|a_S\|_2^{-1}a_S$ is a (p,S) atom, we get

$$\left\|\sum_{l_n\ge0}T_{L(S)+l}a_S\right\|_p^p\le cE_{\tilde{l}}|S|^{1-p/2}\|a_S\|_2^p,$$

and, as before, by Hölder's inequality, Lemma 3.3, and Lemma 2.1,

$$\sum_{S\in\mathscr{M}_{\tilde{l}}^\gamma(\Omega)}\left\|\sum_{l_n\ge0}T_{L(S)+l}a_S\right\|_p^p\le cE_{\tilde{l}}^p\left(\sum|S|\right)^{1-p/2}\left(\sum\|a_S\|_2^2\right)^{p/2}$$

$$\le E_{\tilde{l}}^p d(\tilde{l},\{n\}')^{1-p/2}.$$

Thus

$$\|I_\gamma\|_p^p\le\sum_{\tilde{l}\in\{n\}'}E_{\tilde{l}}^p d(\tilde{l},\{n\}')^{1-p/2},$$

concluding the proof of Proposition 3.2 and Theorem 3.1. □

4. L^p-ESTIMATES

Recall the definition of Q_j^α and P_j^α from §1, and that of $d(l, \alpha)$ from §3. We now state an improved version of Theorem 2, which is new even in the classical one-parameter setting.

Theorem 4.1. *Suppose T is a singular integral operator which is bounded on L^2 with norm at most A, and let $1 < p < 2$. Suppose that for all $\beta, \gamma \subseteq \{1, \dots, n\}$ with $1 \le |\beta \cup \gamma| \le n-1$, $\beta \cap \gamma = \varnothing$ (allowing the possibility that β or $\gamma = \varnothing$), for all $m \in \mathbb{N}^\beta$, $l \in \mathbb{N}^{\beta\cup\gamma}$, and some large N,*

$$\sum_{k\in\mathbb{Z}^{\beta\cup\gamma}} \sup_{j\in\mathbb{Z}^{\beta\cup\gamma}} \|Q_{j+k}^{\beta\cup\gamma} T_{j+l}^{\beta\cup\gamma} Q_{\pi_\beta j+m}^{\beta} \otimes P_{\pi_\gamma j}^{\gamma}\|_{r-r} \le A_l^{\beta,\gamma}(r) \prod_{s\in\beta} 2^{m_s N}$$

for $r = p$ and $r = 2$, and suppose that for all β, γ with $\beta \cup \gamma = \{1, \dots, n\}$, $\beta \cap \gamma = \varnothing$, $m \in \mathbb{N}^\beta$, $\tilde{l} \in \mathbb{N}^{\{n\}'}$ and some large N,

$$\sum_{k\in\mathbb{Z}^{\{1,\dots,n\}}} \sup_{j\in\mathbb{Z}^{\{1,\dots,n\}}} \left\| \sum_{l_n>0} Q_{j+k} T_{j+l} Q_{\pi_\beta j+m}^{\beta} \otimes P_{\pi_\gamma j}^{\gamma} \right\|_{r-r} \le B_{\tilde{l}}^{\beta,\gamma}(r) \prod_{s\in\beta} 2^{m_s N}$$

for $r = p$ and $r = 2$. Let

$$C(r) = \sup_{1\le|\beta\cup\gamma|\le n-1} \sum_{l\in\mathbb{N}^{\beta\cup\gamma}} A_l^{\beta,\gamma}(r) d(l, \beta\cup\gamma)^{1/r-1/2} + \sup_{\beta\cup\gamma=\{1,\dots,n\}} \sum_{\tilde{l}\in\mathbb{N}^{\{n\}'}} B_{\tilde{l}}^{\beta,\gamma}(r) d(l, \{n\}')^{1/r-1/2}.$$

If $A + C(2) + C(p) < \infty$, then T is bounded on L^p.

Remark. In the one-parameter case, the theorem reduces to the following statement. Suppose T is an L^2-bounded operator such that

$$\sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \left\| \sum_{l\ge 0} Q_{j+k} T_{j+l} P_j \right\|_{r-r} \le C$$

and

$$\sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \left\| \sum_{l\ge 0} Q_{j+k} T_{j+l} Q_{j+m} \right\|_{r-r} \le C 2^{mN}$$

for $m \in \mathbb{N}$, some large N, for $p \le r \le 2$. Then T is bounded on L^p.

Proof. Let $f \in L^p$; then by the Calderón reproducing formula, we may write

$$f = \sum_{\mu\in\mathbb{Z}} \sum_{R\in\mathscr{R}_\mu} \int \psi_t * [(\psi_t * f)\chi_{R,t}] \frac{dt}{t_1 \cdots t_n} \tag{4.1}$$

where ψ, R_μ, and Ω_μ are defined as in §2, and where we additionally assume $\int \Psi(t) t^i \, dt = 0$, $i = 0, \dots, M$, with $M > N$. For $\mu \in \mathbb{Z}$, $t \in (\mathbb{R}_+)^n$, and R running over the collection of all dyadic rectangles in $\mathbb{R}^n$, we consider vector-valued functions $F = \{e_{R,t}^\mu\}$ as members of one of the function spaces Y_r with norm

$$\|F\|_{Y_r} = \left(\sum_\mu \left[|\Omega_\mu|^{1/r-1/2} \left(\sum_R \int \|e_{R,t}^\mu\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t_1\cdots t_n} \right)^{1/2} \right]^r \right)^{1/r}$$

where $|\Omega_\mu|$ are fixed weights coming from a *fixed* $f \in L^p$. We shall apply estimates for general F to the particular choice $F(f) = \{(\psi_t * f)\chi^\mu_{R,t}\}$ where $\chi^\mu_{R,t} = \chi_{R,t}$ (see §2) if $R \in \mathscr{R}_\mu$ and is zero otherwise. Then we may rewrite (4.1) as

$$f = \sum_{\mu\in\mathbb{Z}} \sum_{\text{all } R} \int \psi_t * [F(f)]^\mu_{R,t} \frac{dt}{t_1\cdots t_n}, \tag{4.2}$$

and, by Lemma 2.2 we have

$$\|F(f)\|_{Y_p} \le C\|S(f)\|_p \le C'\|f\|_p. \tag{4.3}$$

For general F we consider the operator

$$\widetilde{T}(F) = \sum_\mu \sum_R T\left[\int \psi_t * (e^\mu_{R,t}\chi^\mu_{R,t}) \frac{dt}{t_1\cdots t_n}\right],$$

and set

$$e^\mu_R(F) = \int \psi_t * (e^\mu_{R,t}\chi^\mu_{R,t}) \frac{dt}{t_1\cdots t_n}.$$

Hence, by (4.2), $Tf = \widetilde{T}(F(f))$. As in §3 we may decompose $\widetilde{T}$ by the formula

$$\widetilde{T}(F) = V(F) + \sum_{1\le|\alpha|\le n} (-1)^{|\alpha|-1} \sum_\mu \sum_R \sum_{\substack{m_s > L_s\omega(R,\widetilde{\Omega}_\mu)\\ s\in\alpha}} T^\alpha_m(e^\mu_R(F)) \tag{4.4}$$

where $\omega(R, \widetilde{\Omega}_\mu)$ is the $\omega^{\{1,\dots,n\}}(R, \widetilde{\Omega}_\mu)$ of §3 and where

$$V(F) = \sum_\mu \sum_R \sum_{\substack{m_s \le L_s\omega(R,\widetilde{\Omega}_\mu)\\ s=1,\dots,n}} T_m(e^\mu_R(F)).$$

For $\tilde{\mu} \in \mathbb{Z}$, let

$$F_{\tilde\mu} = \begin{cases} e^\mu_{R,t}, & \mu = \tilde\mu, \\ 0, & \mu \ne \tilde\mu, \end{cases}$$

so that $V(F) = \sum_{\tilde\mu\in\mathbb{Z}} V(F_{\tilde\mu})$.

For $j \in \mathbb{Z}^\alpha$, $l \in \mathbb{N}^\alpha$ $(|\alpha| \le n)$, let $\mathscr{U}^\alpha_{j,l}$ be the family of all dyadic rectangles R in $\mathbb{R}^n$ with the property that if $\mathscr{R}_\mu$ contains R, then $\nu^\alpha_s(R, \widetilde{\Omega}_\mu) < l_s$, $s \in \alpha$, and $L_\alpha(u^\alpha_0(R, \widetilde{\Omega}_\mu)) = j$ (with u^α_0 as in §3). If $\beta \subseteq \alpha$, $\beta \ne \varnothing$, $i \in \mathbb{N}^\beta$, let $\mathscr{U}^{\alpha,\beta}_{j,i,l}$ be the family of all dyadic $R \in \mathscr{U}^\alpha_{j,l}$ such that $L_\beta(R) = \Pi_\beta j - i$. Similarly for $j \in \mathbb{Z}^n$, $\tilde{l} \in \mathbb{N}^{\{n\}'}$, let $\mathscr{U}_{j,\tilde{l}}$ be the family of all dyadic rectangles R in $\mathbb{R}^n$ with the property that if $\mathscr{R}_\mu$ contains R, then $\nu_s(R, \widetilde{\Omega}_\mu) < l_s$, $1 \le s \le n-1$, and $L(u_0(R, \widetilde{\Omega}_\mu)) = j$ (with $\nu_s = \nu_s^{\{1,\dots,n\}}$ and $u_0 = u_0^{\{1,\dots,n\}}$). If $\beta \subseteq \{1, \dots, n\}$ and $i \in \mathbb{N}^\beta$, let $\mathscr{U}^\beta_{j,i,\tilde{l}}$ be the subfamily of rectangles R in $\mathscr{U}_{j,\tilde{l}}$ such that $L_\beta(R) = \Pi_\beta j - i$.

Setting $e_R(F) = \sum_\mu e_R^\mu(F)$ (only one term in the sum is nonzero) we consider the operators

$$\sigma_{j,l}^\alpha(F) = \sum_{R\in\mathscr{U}_{j,l}^\alpha} e_R(F), \qquad \sigma_{j,i,l}^{\alpha,\beta}(F) = \sum_{R\in\mathscr{U}_{j,i,l}^{\alpha,\beta}} e_R(F),$$
$$\tau_{j,\tilde{l}}(F) = \sum_{R\in\mathscr{U}_{j,\tilde{l}}} e_R(F), \qquad \tau_{j,i,\tilde{l}}^\beta(F) = \sum_{R\in\mathscr{U}_{j,i,\tilde{l}}^\beta} e_R(F).$$

With this notation, we may, as in §3, rewrite (4.4) as

$$\begin{aligned}\widetilde{T}(F) - V(F) = &\sum_{1\le|\alpha|\le n-1} (-1)^{|\alpha|-1} \sum_{l\in\mathbb{N}^\alpha}\sum_{j\in\mathbb{Z}^n} T_{j+l}^\alpha \sigma_{j+l}^\alpha(F) \\ &+ (-1)^{n-1} \sum_{\tilde{l}\in\mathbb{N}^{\{n\}'}} \sum_{j\in\mathbb{Z}^n}\sum_{l_n>0} T_{j+l}\tau_{j,\tilde{l}}(F).\end{aligned} \tag{4.5}$$

We shall need some estimates for the operators σ and τ.

Lemma 4.2. *Let* $1 \le r \le 2$.

(i) $(\sum_{j\in\mathbb{Z}^\alpha} \|\sigma_{j,l}^\alpha(F)\|_r^r)^{1/r} \le c d(l,\alpha)^{1/r-1/2}\|F\|_{Y_r}$.

(ii) *If* $\varnothing \ne \beta \subseteq \alpha$, $i, m \in \mathbb{N}^\beta$,

$$\left(\sum_{j\in\mathbb{Z}^\alpha} \|Q_{\pi_\beta j+m}^\beta \sigma_{j,i,l}^{\alpha,\beta}(F)\|_r^r\right)^{1/r} \le C\prod_{s\in\beta} 2^{-(m_s+i_s)M} d(l,\alpha)^{1/r-1/2}\|F\|_{Y_r}.$$

(iii) $(\sum_{j\in\mathbb{Z}^n} \|\tau_{j,\tilde{l}}(F)\|_r^r)^{1/r} \le c d(\tilde{l}, \{n\}')^{1/r-1/2}\|F\|_{Y_r}$.

(iv) *If* $\beta \ne \varnothing$, $i, m \in \mathbb{N}^\beta$,

$$\left(\sum_{j\in\mathbb{Z}^n} \|Q_{\pi_\beta j+m}^\beta \tau_{j,i,\tilde{l}}^\beta(F)\|_r^r\right)^{1/r} \le C\prod_{s\in\beta} 2^{-(m_s+i_s)M} d(l,\{n\}')^{1/r-1/2}\|F\|_{Y_r}.$$

Proof. We prove only (ii) in the case that $\alpha - \beta \ne \varnothing$; all the other cases may be handled in the same way. Let $W_t^\beta = \psi_t^\beta *_\beta f$; then clearly for $t_s \sim 2^{j_s-i_s}$, $s\in\beta$, the L^p operator norm of $Q_{\pi_\beta j+m}^\beta W_t^\beta$ is dominated by $\prod_{s\in\beta} 2^{-M|m_s+i_s|}$. For $\bar{t} = \pi_\beta t$, let

$$a_{j,i,l}^{\alpha,\beta,\bar{t}} = \sum_{R\in\mathscr{U}_{j,i,l}^{\alpha,\beta}} \int_{t'\in\mathbb{R}_+^{\{\beta\}'}} \psi_{t'}^{\beta'} *_{\beta'} [e_{R,t}^\mu \chi_{R,t}^\mu] \frac{dt'}{\prod_{s\notin\beta} t_s}.$$

Hence, considering the case $r=2$ of (ii),

$$\left(\sum_j \|Q_{\pi_\beta j+m}^\beta \sigma_{j,i,l}^{\alpha,\beta}(F)\|_2^2\right)^{1/2} \le \sum_j \int_{\substack{\bar{t} \\ t_s\sim 2^{j_s-i_s} \\ s\in\beta}} \|Q_{\pi_\beta j+m}^\beta W_t^\beta\|_2^2 \|a_{j,i,l}^{\alpha,\beta,\bar{t}}\|_2^2 \frac{d\bar{t}}{\prod_{s\in\beta} t_s},$$

(by the Cauchy-Schwarz inequality), and, by using Lemma 2.1 with respect to the β' variables, we can dominate this by $\prod_{s\in\beta} 2^{-M(m_s+i_s)}\|F\|_{Y_2}$.

Now we consider the case $r = 1$ of (ii); once this is established the general case follows by interpolation. For a dyadic J in $\mathbb{R}^\alpha$ with $L_\alpha(J) = j$ and $\bar{t} \in (\mathbb{R}_+)^\beta$, we set

$$a^{\alpha,\beta,\bar{t}}_{J,i,l,\mu} = \sum_{\substack{R\in\mathscr{U}^{\alpha,\beta}_{j,i,l}\cap\mathscr{R}_\mu \\ \Pi_\alpha u^\alpha_0(R,\widetilde{\Omega}_\mu)=J}} \int \psi_{t'} *_{\beta'} [e^\mu_{R,t}\chi^\mu_{R,t}] \frac{dt'}{\prod_{s\notin\beta} t_s}.$$

Notice that $a^{\alpha,\beta,\bar{t}}_{J,i,l,\mu}$ is supported in $J\times\widetilde{\mathscr{A}^\alpha_{J,l}}(\widetilde{\Omega}_\mu)$, where $\mathscr{A}^\alpha_{J,l}$ is the $\mathscr{A}^{\alpha,\{1,\dots,n\}}_{J,l}$ of Lemma 3.3. Hence,

$$\sum_{j\in\mathbb{Z}^\alpha} \|Q^\beta_{\pi_\beta j+m}\sigma^{\alpha,\beta}_{j,i,l}(F)\|_1$$

$$\le C\sum_{\mu\in\mathbb{Z}}\sum_{\substack{J \text{ dyadic}\\ \text{in } \mathbb{R}^\alpha}} \int_{\bar{t}_s\sim 2^{i_s}} \|Q^\beta_{\pi_\beta L_\alpha(J)+m} W^\beta_{\bar{t}}(\|\cdots\|_1)a^{\alpha,\beta,\bar{t}}_{J,i,l,\mu}\|_1 \frac{d\bar{t}}{\prod_{s\in\beta} t_s}$$

$$\le C\sum_\mu\sum_J\prod_{s\in\beta} 2^{-(m_s+i_s)M}|J\times\widetilde{\mathscr{A}^\alpha_{J,l}}(\widetilde{\Omega}_\mu)|^{1/2}\left(\int \|a^{\alpha,\beta,\bar{t}}_{J,i,l,\mu}\|_2^2\frac{d\bar{t}}{\prod_{s\in\beta} t_s}\right)^{1/2}$$

$$\le C\prod_{s\in\beta} 2^{-(m_s+i_s)M}\sum_\mu\left(\sum_j |J\times\mathscr{A}^\alpha_{J,l}(\widetilde{\Omega}_\mu)|\right)^{1/2}\left(\sum_j\int \|a^{\alpha,\beta,\bar{t}}_{J,i,l,\mu}\|_2^2\frac{d\bar{t}}{\prod_{s\in\beta} t_s}\right)^{1/2}$$

(by the strong maximal theorem and two applications of Cauchy-Schwarz)

$$\le C\prod_{s\in\beta} 2^{-(m_s+i_s)M}\sum_\mu |\Omega_\mu|^{1/2} d(l,\alpha)^{1/2}\left(\sum_{R\in\mathscr{R}_\mu}\int \|e^\mu_{R,t}\|_2^2\frac{dt}{t_1\cdots t_n}\right)^{1/2}$$

(by Lemmas 2.1 and 3.3)

$$= C\prod_{s\in\beta} 2^{-(m_s+i_s)M} d(l,\alpha)^{1/2}\|F\|_{Y_1}. \quad \square$$

Next we shall prove an estimate for $\widetilde{T}(F) - V(F)$ which will follow from the case $r = p$ of the following lemma.

Lemma 4.3. *Let* $1 < p \le 2$ *and* $r \in \{2, p\}$.

(i) *If* $1 \le |\alpha| \le n-1$ *and* $l \in \mathbb{N}^\alpha$,

$$\left\|\sum_{j\in\mathbb{Z}^\alpha} T^\alpha_{j+l}\sigma^\alpha_{j,l}(F)\right\|_r \le C \sup_{\substack{\beta\cup\gamma=\alpha\\ \beta\cap\gamma=\varnothing}} A^{\beta,\gamma}_l(r) d(l,\alpha)^{1/r-1/2}\|F\|_{Y_r}.$$

(ii) *If* $\tilde{l} \in \mathbb{N}^{\{n\}'}$,

$$\left\|\sum_{j\in\mathbb{Z}^n}\sum_{l_n>0} T_{j+l}\tau_{j,\tilde{l}}(F)\right\|_r \le C \sup_{\beta\cup\gamma=\{1,\dots,n\}} B^{\beta,\gamma}_{\tilde{l}}(r) d(l,\{n\}')^{1/r-1/2}\|F\|_{Y_r}.$$

Proof. As in [1 and 23], we combine Littlewood-Paley theory and Calderón-Zygmund theory. Thus

$$\left\|\sum_{j\in\mathbb{Z}^\alpha} T^\alpha_{j+l}\sigma^\alpha_{j,l}(F)\right\| \le \sum_k \left\|\sum_{j\in\mathbb{Z}^\alpha} Q^\alpha_{j+k}Q^\alpha_{j+k}T^\alpha_{j+l}\sigma^\alpha_{j,l}(F)\right\|_r$$

$$\le C\sum_k \left(\sum_{j\in\mathbb{Z}^\alpha} \|Q^\alpha_{j+k}T^\alpha_{j+l}\sigma^\alpha_{j+l}(F)\|_r^r\right)^{1/r}$$

by Littlewood-Paley theory and the embedding $l^r \subseteq l^2$. Using the identity

$$I = \sum_{\beta\subseteq\alpha}\sum_{m\in\mathbb{N}^\beta} Q^{\beta 2}_{\pi_\beta j+m}\otimes P^{\beta'}_{\pi_{\beta'} j} \quad \text{on } \mathbb{R}^n$$

(valid for all j), we see this expression can in turn be dominated by

$$C\sum_{\substack{\beta\cup\gamma=\alpha\\ \beta\cap\gamma=\varnothing}}\sum_{m\in\mathbb{N}^\beta}\sum_k\left(\sum_{j\in\mathbb{Z}^\alpha}\|Q^\alpha_{j+k}T^\alpha_{j+l}Q^\beta_{\pi_\beta j+m}\otimes P^\gamma_{\pi_\gamma j}\|^r_{r-r}\|Q^\beta_{\pi_\beta j+m}\sigma^\alpha_{j,l}(F)\|^r_r\right)^{1/r}$$

$$\le C\sum_{\beta\cup\gamma}\sum_{m\in\mathbb{N}^\beta} A^{\beta,\gamma}_l(r)\prod_{s\in\beta}2^{m_s N}\left(\sum_j\|Q^\beta_{\pi_\beta j+m}\sigma^\alpha_{j,l}(F)\|^r_r\right)^{1/r}$$

(by the hypothesis of Theorem 4.1)

$$\le C\sum_{\beta\cup\gamma}\sum_{m\in\mathbb{N}^\beta}\sum_{i\in\mathbb{N}^\beta} A^{\beta,\gamma}_l(r)\prod_{s\in\beta}2^{m_s N}\left(\sum_j\|Q^\beta_{\pi_\beta j+m}\sigma^{\alpha,\beta}_{j,i,l}(F)\|^r_r\right)^{1/r}$$

$$\le C\sum_{\beta\cup\gamma}\sum_{m,i\in\mathbb{N}^\beta} A^{\beta,\gamma}_l(r)\prod_{s\in\beta}2^{m_s(N-M)}2^{-i_s M}d(l,\alpha)^{1/r-1/2}\|F\|_{Y_r}$$

by Lemma 4.2. Since we have chosen M to be greater than N, we may sum up and obtain part (i). Part (ii) is proved in a similar way. □

We have shown that $\|\widetilde{T}(F)-V(F)\|_p \le CC(p)\|F\|_{Y_p}$, and we now consider $V(F)$ and show

$$\|V(F)\|_r \le C[A+C(2)]\|F\|_{Y_r} \tag{4.6}$$

for $1\le r\le 2$. Once we have done this, we are finished, since then

$$\begin{aligned}\|Tf\|_p = \|\widetilde{T}(F(f))\|_r &\le \|\widetilde{T}(F(f))-V(F(f))\|_p + \|V(F(f))\|_p\\ &\le C(A+C(2)+C(p))\|F(f)\|_{Y_p}\\ &\le C(A+C(2)+C(p))\|f\|_p \quad \text{by (4.3).}\end{aligned}$$

We establish (4.6) also by interpolation. For $r=2$ we have

$$\|V(F)\|_2 \le \|\widetilde{T}(F)\|_2 + \|V(T)-\widetilde{T}(F)\|_2$$

$$\le A\left\|\sum_\mu\sum_R\int \psi_t * e^\mu_{R,t}\chi^\mu_{R,t}\frac{dt}{t_1\cdots t_n}\right\|_2 + CC(2)\|F\|_{Y_2} \tag{4.7}$$

(by assumption and Lemma 4.3 in the case $r=2$)

$$\le C[A+C(2)]\|F\|_{Y_2} \quad \text{(by Plancherel's theorem).}$$

Now let $r = 1$ and write $V(F) = \sum_{\mu\in\mathbb{Z}} V(F_\mu)$, observing that $V(F_\mu)$ is supported in $\widetilde{\Omega}_\mu^{(n2^{n-1}+1)}$. Hence,

$$\begin{aligned}\|V(F)\|_1 &\le C\sum_\mu |\Omega_\mu|^{1/2}\|V(F_\mu)\|_2 \\ &\le C\sum_\mu |\Omega_\mu|^{1/2}[A + C(2)]\|F_\mu\|_{Y_2} \quad \text{(by (4.7))} \\ &\le C[A + C(2)]\|F\|_{Y_1},\end{aligned}$$

which establishes (4.6), and hence Theorem 4.1. □

5. Fourier multipliers and pseudodifferential operators

In this section we give some applications of Theorems 1.1 and 1.2 and prove H^p- and L^p-results for convolution and pseudodifferential operators. The computations needed to check the hypotheses of Theorems 1.1 and 1.2 are easy modifications of those carried out in [1 and 2] in the one-parameter case; so we will be very concise and omit most of the details.

We need some notation: M_p, $1 \le p \le \infty$, denotes the standard space of Fourier multipliers m in L^p. The norm is given by the norm of the operator T where $(Tf)\hat{}(\xi) = m(\xi)\hat{f}(\xi)$. Similarly we define for $\alpha \subset \{1, \dots, n\}$ the space M_{12}^α consisting of those m such that the multiplier transformation T is bounded on the mixed norm space $L^1(\mathbb{R}^\alpha, L^2(\mathbb{R}^{\alpha'}))$. (Note that by [13], $M_1 \subseteq M_{12}^\alpha$.) The dyadic decomposition Φ_l (or Φ_l^α) is used to express some Lipschitz conditions with respect to multiplier norms.

Proposition 5.1. *Suppose that* m *is a bounded function,* $\|m\|_\infty \le A$, *and that for some* $\delta > 0$,

$$\sup_{t\in(\mathbb{R}_+)^n} \|[\phi m(t\cdot)] * \widehat{\Phi}_l\|_{M_1} \le A\prod_{s=1}^n 2^{-l_s\delta} \tag{5.1}$$

and that for each $\alpha \subset \{1, \dots, n\}$

$$\sup_{t^\alpha\in(\mathbb{R}_+)^\alpha} \|[\phi^\alpha m(t^\alpha\cdot)] *_\alpha \widehat{\Phi}_l^\alpha\|_{M_{12}^\alpha} \le A\prod_{s\in\alpha} 2^{-l_s\delta}. \tag{5.2}$$

Then we have the inequality

$$\|\mathscr{F}^{-1}[m\hat{f}]\|_{H^p} \le CA\|f\|_{H^p} \quad \text{for } \frac{1}{1+\delta} < p \le 1.$$

Proof. It suffices to show that $T : H^p \to L^p$ since H^p may be characterized as a space of distributions whose iterated Hilbert transforms are in L^p (see [11, 15]).

In order to keep the notation simple, we assume $n = 2$ and the general case is proved in the same way. Then we have to check the four hypotheses of the theorem. We only examine the hypothesis involving mixed norms in $L^p(L^2)$; the other inequalities are obtained in the same manner.

Let a be an L^2-valued rectangle atom. We may consider a as a function supported in $I \times \mathbb{R}$, where $I = \{x_1,\ a \le x_1 \le b\}$. By translation and dilation invariance, we may assume that $I = [-1, 1]$. Then we have to show the inequality

$$\|T_l^1 a\|_{L^p(L^2)} \le c2^{-\varepsilon l_1} \tag{5.3}$$

for some $\varepsilon > 0$. In this proof, T_l^1 denotes the operator with kernel in $m^\vee(x-y)\Phi_l^1((x_1-y_1)/2^l)$ (we shall write l, k, i instead of l_1, k_1, i_1). Since $T_l^1 a$ is supported in the strip $\{(x_1, x_2), |x_1| \le 2^{l+1}\}$, we have by Hölder's inequality,

$$\|T_l^1 a\|_{L^p(L^2)} \le c2^{l(1-p)}\|T_l^1 a\|^p_{L^1(L^2)}.$$

Using the dyadic decomposition (Q_k^1) on the multiplier side with respect to the ξ_1-variable, we can write

$$T_l^1 a = \sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}[TQ_k^1 Q_k^1]_l^1 Q_i^1 Q_i^1 a$$

and if $\tilde{\phi} = \phi\phi$ and

$$m_{ikl}(\xi_1, \xi_2) = \phi(2^i\xi_1)\int 2^l\widehat{\Phi}(2^l(\xi_1 - y_1))m(y_1, \xi_2)\tilde{\phi}(2^k y_1)\,dy_1,$$

we get

$$\|T_l^1 a\|_{L^p(L^2)} \le 2^{l(1-p)}\left[\sum_i\sum_{k\in\mathbb{Z}}\|m_{ikl}\|_{M_{12}}\|Q_i^1 a\|_{L^1(L^2)}\right]^p.$$

If $i > 0$, the expression $Q_i^1 a$ is small because the rectangles have cancellation in the x_1-direction: using Taylor's formula, we obtain

$$\|Q_i^1 a\|_{L^1(L^2)} \le c\min(1, 2^{-iM(p)}) \tag{5.4}$$

where $M(p) > 1 - \frac{1}{p}$. By dilation invariance, we have to examine

$$\begin{aligned}\tilde{m}_{ikl}(\xi_1, \xi_2) &= m_{ikl}(2^{-i}\xi_1, \xi_2)\\ &= \phi(\xi_1)\int 2^{l-k}\widehat{\Phi}(2^{l-i}(\xi_1 - 2^{i-k}y_1))\tilde{\phi}(y_1)m(2^{-k}y_1, \xi_2)\,dy_1.\end{aligned} \tag{5.5}$$

The essential terms occur if i is close to k; we use the estimate

$$\begin{aligned}\|\tilde{m}_{ikl}\|_{M_{12}} &\le \|\phi(2^{i-k}\cdot)\|_{M_{12}}\|\tilde{\phi}m(2^{-k}\cdot) * \widehat{\Phi}_{l-k}\|_{M_{12}}\\ &\le c\min\{1, 2^{(k-l)\delta}\}\end{aligned} \tag{5.6}$$

(see also the Remark below). From (5.5) it is easy to check that for every j, $N > 0$, we have the estimates

$$\begin{aligned}|\partial_{\xi_1}^j[\tilde{m}_{ikl}] &\le c_{j,N}2^{l-k}\min(1, 2^{-(l-i)N}) \quad \text{if } k \ge i+5,\\ |\partial_{\xi_1}^j[\tilde{m}_{ikl}]| &\le c_{j,N}\min(1, 2^{-(l-k)N}) \quad \text{if } k \le i-5.\end{aligned}$$

Recalling that $\tilde{m}_{ikl}$ has compact support in $\{\frac{1}{4} \le |\xi_1| \le 4\}$, we obtain

$$\begin{aligned}\|m_{ikl}\|_{M_{12}} &\le c\sup_{\xi_2}\left[\int_{\xi_1}|m_{ikl}(\xi_1, \xi_2)|^2 + |\partial_{\xi_1}m_{ikl}(\xi_1, \xi_2)|^2\,d\xi_1\right]^{1/2}\\ &\le \begin{cases} c_N 2^{l-k}\min(1, 2^{-(l-i)N}) & \text{if } k \ge i+5,\\ c_N\min(1, 2^{-(l-k)N}) & \text{if } k \le i-5.\end{cases}\end{aligned} \tag{5.7}$$

Putting the estimates (5.4), (5.6), and (5.7) together, we get

$$\begin{aligned}\|T_l^1 a\|_1 \le c2^{l(1-p)} &\left(\sum_i \sum_{|k-i|\le 5} 2^{(k-l)\delta} \min(1, 2^{-iM(p)})\right)^p \\ &+ c2^{l(1-p)} \left(\sum_i \sum_{k\ge i+5} 2^{l-k} \min(1, 2^{-i(l-i)N}) \min(1, 2^{-iM(p)})\right)^p \\ &+ c2^{l(1-p)} \left(\sum_i \sum_{k\le i-5} \min(1, 2^{-(l-i)N}) \min(1, 2^{-iM(p)})\right)^p\end{aligned}$$

and since $\delta > 1 - \frac{1}{p}$, $M(p) > 1 - \frac{1}{p}$, we get the bound (5.3).

Similarly one can do the estimates required for the rectangle atoms. For these estimates we use (5.1) and the restriction theorem of de Leeuw ([18]; see also Jodeit [14]). □

Remark. The condition (5.1) is equivalent to the condition

$$\sup_{t\in(\mathbb{R}_+)^n} \|\Delta_{h_1}^{[1]} \cdots \Delta_{h_n}^{[n]}[\phi m(t\cdot)]\|_{M_1} \le c \prod_{s=1}^n |h_s|^\varepsilon,$$

all $h \in \mathbb{R}^n$, some $\varepsilon > 0$, where $\Delta_{h_s}^{[s]}$ denotes the difference operator $\Delta_{h_s}^{[s]} f(x) = f(x + h_s e_s) - f(x)$ in the x_s-direction. A similar remark applies to (5.2). This observation can be used to show that the hypotheses of Proposition 5.1 are really independent of the choice of ϕ (see [2, 23] for similar arguments).

Proposition 5.1 implies as a corollary an H^p-version of the Hörmander multiplier theorem in product spaces. The same result was proved by R. Fefferman and K. C. Lin [9] in the two-parameter case, using R. Fefferman's [7] result on rectangle atoms. For $p = 1$ the corollary already follows from the BMO-estimates for convolution operators proved by H. Lin [19].

Let us define the multiparameter Sobolev-space $\mathscr{L}_\gamma^p$ by

$$\|g\|_{\mathscr{L}_\gamma^p} = \left\| \mathscr{F}^{-1} \left[\prod_{s=1}^n (1 + |\xi_s|^2)^{\gamma/2} \hat{g} \right] \right\|_p .$$

Then we have

Corollary 5.2 (Hörmander-Marcinkiewicz type multiplier theorem). *Suppose that for* $\gamma > \frac{1}{p} - \frac{1}{2}$, $0 < p \le 1$,

$$\sup_{t\in(\mathbb{R}_+)^n} \|\phi m(t_1\cdot, \dots, t_n\cdot)\|_{\mathscr{L}_\gamma^2} \le A.$$

Let $(Tf)^\frown(\xi) = m(\xi)\hat{f}(\xi)$. *Then* $\|Tf\|_{H^p} \le cA\|f\|_{H^p}$.

Proof. Suppose again $n = 2$ for simplicity. For fixed t_1, let $\widehat{K}_{t_1} = \phi m(t_1\cdot)$.

We apply the Cauchy-Schwarz inequality and Plancherel's theorem to get

$$\|K_{t_1}\Phi^1_{l_1} * g\|_{L^1(L^2)}$$

$$\leq \int_{y_1}\int_{x_1}\left(\int_{x_2}\left|\int_{y_2} K_{t_1}\Phi^1_{l_1}(x-y)g(y_1, y_2)\,dy_2\right|^2 dx_2\right)^{1/2} dx_1\,dy_1$$

$$\leq c2^{l_1/2}\int_{y_1}\left(\iint_x\left|\int_{y_2} K_{t_1}\Phi^1_{l_1}(x-y)g(y)\,dy_2\right|^2 dx\right)^{1/2} dy_1$$

$$= c2^{l_1/2}\int\left(\iint\left|\widehat{K}_{t_1} * \widehat{\Phi}^1_{l_1}(\xi_1, \xi_2)\int g(y)e^{iy_2\cdot\xi_2}\,dy_2\right|^2 d\xi\right)^{1/2} dy_1$$

$$\leq c2^{l_1/2}\sup_{\eta_2}\left(\int_{\xi_1}|\widehat{K}_{t_1} * \widehat{\Phi}^1_{l_1}(\xi_1, \eta_2)|^2\,d\xi_1\right)^{1/2}$$

$$\cdot\left(\int_{y_1}\left[\int_{\xi_2}\left|\int g(y)e^{iy_2\cdot\xi_2}\,dy_2\right|^2 d\xi_2\right]^{1/2} dy_1\right)$$

$$\leq c2^{l_1/2}\sup_{t_2}\sup_{\eta_2}\left(\int_{\xi_1}|\phi m(t_1\cdot, t_2\cdot) * \widehat{\Phi}^1_{l_1}(\xi_1, \eta_1)|^2\,d\xi_1\right)^{1/2}\|g\|_{L^1(L^2)}.$$

Similarly one may prove

$$\|K\Phi_l\|_1 \leq c2^{l_1/2}2^{l_2/2}\left(\int |m * \widehat{\Phi}_l|^2\,d\xi\right)^{1/2}.$$

From these estimates, the corollary follows by Proposition 5.1. □

We now give a generalization of Proposition 5.1 for pseudodifferential operators

$$Tf = \sigma(x, D)f = \int \sigma(x, \xi)\hat{f}(\xi)e^{2\pi i\xi\cdot x}\,d\xi.$$

Let us first introduce some notation. In order to keep this simple, we restrict ourselves to the two-parameter case. Let X_p be the space of symbols σ such that $\sigma(\cdot, D)$ is bounded in L^p. The norm in this space is the operator-norm of $\sigma(\cdot, D)$. Let X_{p2} be the space of symbols σ such that $\sigma(\cdot, D)$ is bounded on $L^p(L^2)$, where the L^p-norm is taken with respect to the x_1-variable. Further, set $\|\sigma\|_{X_{2p}} = \|\tau_\sigma\|_{X_{2p}}$ where $\tau_\sigma(x_1, x_2, \xi_1, \xi_2) = \sigma(x_2, x_1, \xi_2, \xi_1)$. (Notice that X_{2p} is a space of operators in $L^p(L^2)$, not in $L^2(L^p)$!) Further, we introduce the localized and dilated symbols

$$\tilde{\sigma}^1_{k_1}(x, \xi) = \phi^{[1]}(\xi_1)\sigma(2^{k_1}x_1, x_2, 2^{-k_1}\xi_1, \xi_2),$$
$$\tilde{\sigma}^2_{k_2}(x, \xi) = \phi^{[2]}(\xi_2)\sigma(x_1, 2^{k_2}x_2, \xi_1, 2^{-k_2}\xi_2),$$
$$\tilde{\sigma}_k(x, \xi) = \phi(\xi_1)\phi(\xi_2)\sigma(2^{k_1}x_1, 2^{k_2}x_2, 2^{-k_1}\xi_1, 2^{-k_2}\xi_2).$$

As in Proposition 5.1, Lipschitz conditions with respect to the x- and ξ-variables are expressed by certain decay conditions of the norms $\|\sigma_k *_\xi \widehat{\Phi}_l\|$, $\|\widehat{\Phi}_l *_x \tilde{\sigma}_k\|$, etc. as $l_1, l_2 \to \infty$. (For a detailed discussion, see [2, §3].)

Proposition 5.3. (a) *Suppose that* $\varepsilon > 0$ *and* $T = \sigma(\cdot, D)$ *satisfies the following conditions*:

$$\|\sigma\|_{X_2} \le A, \tag{5.8}$$

$$\sup_{k_1} \|\tilde{\sigma}^1_{k_1} *_{\xi_1} \widehat{\Phi}^1_{l_1}\|_{X_{12}} \le A2^{-\varepsilon l_1}, \tag{5.9}$$

$$\sup_{k_2} \|\tilde{\sigma}^2_{k_2} *_{\xi_2} \widehat{\Phi}^2_{l_2}\|_{X_{12}} \le A2^{-\varepsilon l_2}, \tag{5.10}$$

$$\sup_{k_1, k_2} \|\tilde{\sigma}_k *_\xi \widehat{\Phi}_l\|_{X_1} \le A2^{-\varepsilon l_1}2^{-\varepsilon l_2}, \tag{5.11}$$

$$\sup_{k_1} \sup_{x_2, \xi_2} \|\tilde{\sigma}_{k_1}(\cdot, x_2, \cdot\, \xi_2)\|_{X_2(\mathbb{R})} \le A, \tag{5.12}$$

$$\sup_{k_1} \sup_{x_2, \xi_2} \|\tilde{\sigma}_{k_1} *_{\xi_1} \widehat{\Phi}^1_{l_1}(\cdot, x_2, \cdot, \xi_2)\|_{X_1(\mathbb{R})} \le A2^{-\varepsilon l_1}, \tag{5.13}$$

$$\sup_{k_2} \sup_{x_1, \xi_1} \|\tilde{\sigma}_{k_2}(x_1, \cdot, \xi_1, \cdot)\|_{X_2(\mathbb{R})} \le A, \tag{5.14}$$

$$\sup_{k_2} \sup_{x_1, \xi_1} \|\tilde{\sigma}_{k_2} *_{\xi_2} \widehat{\Phi}^2_{l_2}(x_1, \cdot, \xi_1, \cdot)\|_{X_1(\mathbb{R})} \le A2^{-\varepsilon l_2}, \tag{5.15}$$

$$\sup_{x, \xi} |\sigma(x, \xi)| \le A. \tag{5.16}$$

Then T *is bounded from* H^p *to* L^p, $1/(1+\varepsilon) < p \le 1$, *the operator norm being bounded by* $c_p A$.

(b) *Suppose that* (5.8)-(5.16) *hold with* X_1, X_{12}, X_{21} *replaced by* X_∞, $X_{\infty 2}$, $X_{2\infty}$ *and suppose that furthermore*

$$\sup_{k} \sup_{x_2, \xi_2} \|\widehat{\Phi}^2_{l_2} *_{x_2} \tilde{\sigma}_k * \widehat{\Phi}^1_{l_1}(\cdot, x_2, \cdot, \xi_2)\|_{X_\infty(\mathbb{R})} \le A2^{-\varepsilon l_1}2^{-\varepsilon l_2}, \tag{5.17}$$

$$\sup_{k} \sup_{x_1, \xi_1} \|\widehat{\Phi}^1_{l_1} *_{x_1} \tilde{\sigma}_k * \widehat{\Phi}^2_{l_2}(x_1, \cdot, \xi_1, \cdot)\|_{X_\infty(\mathbb{R})} \le A2^{-\varepsilon l_1}2^{-\varepsilon l_2}, \tag{5.18}$$

$$\sup_{k} \sup_{x, \xi} |\widehat{\Phi}_l *_x \tilde{\sigma}_k(x, \xi)| \le A2^{-\varepsilon l_1}2^{-\varepsilon l_2}, \tag{5.19}$$

$$\sup_{k_1} \sup_{x_1, \xi_1} \|\widehat{\Phi}^1_{l_1} *_{x_1} \tilde{\sigma}^1_{k_1}(x_1, \cdot, \xi_1, \cdot)\|_{X_2(\mathbb{R})} \le A2^{-\varepsilon l_1}, \tag{5.20}$$

$$\sup_{k_1} \sup_{x, \xi} |\widehat{\Phi}^1_{l_1} *_{x_1} \tilde{\sigma}^1_{k_1}(x, \xi)| \le A2^{-\varepsilon l_1}, \tag{5.21}$$

$$\sup_{k_2} \sup_{x_2, \xi_2} \|\widehat{\Phi}^2_{l_2} *_{x_2} \tilde{\sigma}^2_{k_2}(\cdot, x_2, \cdot, \xi_2)\|_{X_2(\mathbb{R})} \le A2^{-\varepsilon l_2}, \tag{5.22}$$

$$\sup_{k_2} \sup_{x, \xi} |\widehat{\Phi}^2_{l_2} *_{x_2} \tilde{\sigma}^2_{k_2}(x, \xi)| \le A2^{-\varepsilon l_2}. \tag{5.23}$$

Then T^* *is bounded from* H^p *to* L^p, $1/(1+\varepsilon) < p \le 1$, *the operator norm only depending on* A. □

We will not give the proof of Proposition 5.3 here. The result follows by straightforward modifications of computations in [1, §5] and [2, §4]. The hypotheses are more complicated than in the multiplier result because there is no version of the de Leeuw restriction theorem on multipliers available in the context of the pseudodifferential operators. Furthermore it is usually nontrivial to verify the L^2-boundedness (5.8). In order to do this one may use a product version of the T1-theorem of David and Journé, proved by Journé [15], or certain product versions of the Calderón-Vaillancourt theorems, proved in [2]. For the H^p-boundedness we state a corollary generalizing the product $S^0_{1\delta}$-theorem. For simplicity we shall use the abbreviation $\mathscr{D}^\varepsilon_x$ to indicate a (fractional) derivative in the x-variable. Since we do not prove endpoint results, the reader may replace this by a Lipschitz condition as in Proposition 5.2. The corollary extends certain previously known L^p-results [25, 21, 2] to the Hardy-space setting.

Corollary 5.4. *Suppose that* $0 < p \le 1$, $\gamma > \frac{1}{p}$ *and for some* $\varepsilon > 0$,

$$\sup_k \sup_x \|\tilde{\sigma}_k(x, \cdot)\|_{\mathscr{L}^1_\gamma} < \infty, \tag{5.24}$$

$$\left(\sum_{k_1} \sup_{k_2} \sup_x \|\mathscr{D}^\varepsilon_{x_1} \tilde{\sigma}_k(x, \cdot)\|^2_{\mathscr{L}^1_\gamma}\right)^{1/2} < \infty, \tag{5.25}$$

$$\left(\sum_{k_2} \sup_{k_1} \sup_x \|\mathscr{D}^\varepsilon_{x_2} \tilde{\sigma}_k(x, \cdot)\|^2_{\mathscr{L}^1_\gamma}\right)^{1/2} < \infty, \tag{5.26}$$

$$\left(\sum_{k} \sup_x \|\mathscr{D}^\varepsilon_{x_1} \mathscr{D}^\varepsilon_{x_2} \tilde{\sigma}_k(x, \cdot)\|^2_{\mathscr{L}^1_\gamma}\right)^{1/2} < \infty. \tag{5.27}$$

Then $T = \sigma(\cdot, D)$ *is bounded from* H^p *to* L^p. *If* $\mathscr{L}^1_\gamma$ *is replaced by* $\mathscr{L}^2_{\gamma-1/2}$ *and if* $\varepsilon > \frac{1}{p} - 1$ *then* T^* *is bounded from* H^p *to* L^p. □

The proof of Corollary 5.4 consists of verifying the conditions in Proposition 5.3. This can be done by slightly modifying arguments in Coifman and Meyer [5, p. 14] and [2, §§3.2 and 3.3]. We omit the details.

The $\mathscr{L}^1_\gamma$ result for T does not require Theorem 1 and has already been deduced from Journé's and Pipher's theorems [15 and 22]. However the $\mathscr{L}^2_\beta$ result for T^* does require Theorem 1. Corollary 5.4 contains product space versions of the $S^0_{1\delta}$ theorems. In fact, suppose σ has support in $\{|\xi_1| \ge 4, \ |\xi_2| \ge 4\}$. If

$$\left|\left(\frac{\partial}{\partial \xi}\right)^\alpha \left(\frac{\partial}{\partial x}\right)^\beta \sigma(x, \xi)\right| \le C_{\alpha,\beta} |\xi_1|^{\delta\beta_1 - \alpha_1} |\xi_2|^{\delta\beta_2 - \alpha_2}$$

for some $0 \le \delta \le 1$, we say then that σ is in the product $S^0_{1\delta}$ class. It is easy to see that such a σ satisfies

$$\sup_x \|\mathscr{D}^\varepsilon_{x_1} \mathscr{D}^\varepsilon_{x_2} \tilde{\sigma}_k(x, \cdot)\|_{\mathscr{L}^1_\gamma} \le C 2^{-k_1(1-\delta)\varepsilon} 2^{-k_2(1-\delta)\varepsilon},$$

etc. and hence, in the case $0 \le \delta < 1$, satisfies the hypotheses of Corollary 5.4 for all $\gamma > 0$.

We remark that every Lipschitz type condition in Propositions 5.1 and 5.3 may be replaced by a Dini-type condition (see [25] for certain results of this nature for L^p, $1 < p < \infty$).

We now turn to L^p-results, $1 < p < 2$. We restrict ourselves to Fourier multipliers, although similar results may be clearly obtained for pseudodifferential operators in analogy with those in [2].

Proposition 5.5. *Let $1 < p < 2$ and suppose that for all α, $1 \le |\alpha| \le n$, and some $\delta > 0$, we have*

$$\sup_{t^\alpha \in (\mathbb{R}_+)^\alpha} \|\phi^\alpha m(t^\alpha \cdot) *_\alpha \widehat{\Phi}_l^\alpha\|_{M_p} \le A \prod_{s\in\alpha} 2^{-l_s\delta}. \tag{5.28}$$

If $(Tf)\hat{}(\xi) = m(\xi)\hat{f}(\xi)$, then $\|Tf\|_p \le CA\|f\|_p$. □

The proof consists of verifying the hypotheses of Theorem 2 or Theorem 4.1. This is done exactly as in [1, §§3 and 5], to which we refer the interested reader for details. Finally, we observe that Theorem B of the Introduction is an immediate consequence of Proposition 5.5, since the hypotheses of Theorem B for a given p imply, by interpolation, those of Proposition 5.5 for each r with $p < r < 2$.

6. A COUNTEREXAMPLE

We now wish to show that the "semilocal" assumptions in Propositions 5.1 and 5.2 cannot be replaced by purely "local" ones unless much more smoothness is assumed.

Proposition 6.1. *For each $1 < p < 2$ there is a bounded function m such that*

(i) $\sup_{t_1, t_2 > 0} \|\phi m(t_1 \cdot, t_2 \cdot)\|_{M_1} < \infty$,
(ii) $\sup_{t_1, t_2 > 0} \|\phi m(t_1 \cdot, t_2 \cdot)\|_{\Lambda'_{1/p-1/2}} < \infty$,
(iii) $m \notin M_p$.

Proof. We define

$$m(\xi_1, \xi_2) = \sum_{j=1}^{\infty} \hat{\theta}(\xi_1)\phi(2^{-j}\xi_2)e^{ij\xi_1} j^{1/2-1/p}$$

where θ is supported in $\{|x_1| \le \frac{1}{10}\}$ and ϕ is supported in $\{\frac{9}{10} \le \xi_2 \le \frac{11}{10}\}$. It is easy to see that m satisfies (i) and (ii). By a result of Herz and Rivière [13], it suffices to show that T associated to m is not a bounded operator on the mixed norm space $L^p(L^2)$. Choose a positive sequence a_j with $(\sum |a_j|^2)^{1/2} \le 1$ but $\sum a_j^p j^{p/2-1} = \infty$, and define $\hat{f}(\xi_1, \xi_2) = \hat{\theta}(\xi_1)\sum_j \phi(2^{-j}\xi_2)a_j 2^{-j/2}$. Then $f \in L^p(L^2)$ with norm $\le c(\sum |a_j|^2)^{1/2} \le C$. Further, by Plancherel's theorem with respect to x_2 and the fact that

$$\{\theta * \theta(\cdot - j)\}_{j\in\mathbb{Z}}$$

have disjoint supports, we obtain

$$\begin{aligned}\|Tf\|_{L^p(L^2)}^p &= \int_{x_1}\left[\int_{\xi_2}\left|\sum_j \theta * \theta(x_1 - j)\phi(2^{-j}\xi_2)2^{-j/2}a_j j^{1/2-1/p}\right|^2 d\xi_2\right]^{p/2} dx_1 \\ &\ge \sum_j \int_{x_1} |\theta * \theta(x_1 - j)|^p \, dx_1 a_j^p j^{p/2-1} = \infty. \quad \square\end{aligned}$$

We conclude with two remarks. Firstly, condition (ii) of Proposition 6.1 is sharp in the sense that $\sup_{t_1, t_2 > 0} \|\phi m(t_1 \cdot, t_2 \cdot)\|_{\mathscr{L}_\gamma^q} < \infty$ for $\frac{1}{q} = |\frac{1}{p} - \frac{1}{2}|$, $\gamma > |\frac{1}{p} - \frac{1}{2}|$ implies $m \in M_p$, and of course we have $\|\cdot\|_{\mathscr{L}_\gamma^q} \le c\|\cdot\|_{\Lambda'_\gamma}$, for functions with compact support. Secondly, Proposition 5.1 and the proof of Proposition 6.1 suggest the question of whether one can relax the M_p hypotheses on dyadic strips in Proposition 5.5 to M_p conditions on dyadic rectangles and M_{p2} conditions on the strips. This seems to be an open problem.

REFERENCES

1. A. Carbery, *Variants of the Calderón-Zygmund theory for L^p-spaces*, Rev. Mat. Iberoamericana **2** (1986), 381–396.
2. A. Carbery and A. Seeger, *Conditionally convergent series of linear operators on L^p-spaces and L^p-estimates for pseudodifferential operators*, Proc. London Math. Soc. **57** (1988), 481–510.
3. S.-Y. A. Chang and R. Fefferman, *A continuous version of duality of H^1 and BMO on the bidisc*, Ann. of Math. **112** (1980), 179–201.
4. ——, *The Calderón-Zygmund decomposition on product domains*, Amer. J. Math. **104** (1982), 445–468.
5. R. R. Coifman and Y. Meyer, *Au-delà des opérateurs pseudodifférentiels*, Astérisque **57** (1978).
6. R. Fefferman, *Calderón-Zygmund theory for product domains: H^p-spaces*, Proc. Nat. Acad. Sci. **83** (1986), 840–843.
7. ——, *Harmonic analysis on product spaces*, Ann. of Math. **126** (1987), 109–130.
8. ——, *A note on a lemma of Zo*, Proc. Amer. Math. Soc. **96** (1986), 241–246.
9. R. Fefferman and K. C. Lin, *A sharp Marcinkiewicz multiplier theorem*, Ann. Institut Fourier (to appear).
10. R. Fefferman and E. M. Stein, *Singular integrals on product spaces*, Adv. in Math. **45** (1982), 117–143.
11. R. Gundy and E. M. Stein, *H^p-theory for the polydisc*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), 1026–1029.
12. L. Hörmander, *Estimates for translation invariant operators in L^p-spaces*, Acta Math. **104** (1960), 93–139.
13. C. Herz and N. M. Rivière, *Estimates for translation invariant operators on spaces with mixed norms*, Studia Math. **44** (1972), 511–515.
14. M. Jodeit, *A note on Fourier multipliers*, Proc. Amer. Math. Soc. **27** (1971), 423–424.
15. J. L. Journé, *Calderón-Zygmund operators on product spaces*, Rev. Mat. Iberoamericana **1** (1985), 55–91.
16. ——, *A covering lemma for product spaces*, Proc. Amer. Math. Soc. **96** (1986), 593–598.
17. ——, *Two problems of Calderon-Zygmund theory on product spaces*, Ann. Inst. Fourier **38** (1988), 111–132.
18. K. de Leeuw, *On L^p-multipliers*, Ann. of Math. **81** (1965), 364–379.
19. H. Lin, *Weighted inequalities on product domains*, Thesis, Univ. of Chicago, 1987.
20. W. Littman, C. McCarthy, and N. M. Rivière, *L^p-multiplier theorems*, Studia Math. **30** (1968), 193–217.
21. J. Marschall, *Weighted parabolic Triebel spaces of product type, Fourier multipliers and pseudodifferential operators*, Preprint.
22. J. Pipher, *Journé's covering lemma and its extension to higher dimensions*, Duke Math. J. **53** (1986), 683–690.
23. A. Seeger, *Some inequalities for singular convolution operators in L^p-spaces*, Trans. Amer. Math. Soc. **308** (1988), 259–272.

24. F. Soria, *A note on a LIttlewood-Paley inequality for arbitrary intervals in* $\mathbb{R}^2$, J. London Math. Soc. **36** (1987), 137–142.

25. M. Yamazaki, *The* L^p*-boundedness of pseudodifferential operators with estimates of parabolic type and product type*, J. Math. Soc. Japan **38** (1986), 199–225.

Department of Mathematics, University of Sussex, Falmer, Brighton BN1 9QH, United Kingdom

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706