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BRAUER-HILBERTIAN FIELDS

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ABSTRACT. Let F be a field of characteristic p ($p = 0$ allowed), and let $F(t)$ be the rational function field in one variable over F. We say F is *Brauer-Hilbertian* if the following holds. For every α in the Brauer group $\mathrm{Br}(F(t))$ of exponent prime to p, there are infinitely many specializations $t \to a \in F$ such that the specialization $\overline{\alpha} \in \mathrm{Br}(F)$ is defined and has exponent equal to that of α. We show every global field is Brauer-Hilbertian, and if K is Hilbertian and F is finite separable over $K(t)$, F is Brauer-Hilbertian.

INTRODUCTION

A field F is called *Hilbertian* if the following property holds: If $f(t, x) \in F[x, t]$ is an irreducible polynomial then there are infinitely many $a \in F$ such that the specialization $f(a, x)$ is irreducible as a polynomial in $F[x]$. To state this property in other language, let $P_a = (t - a) \subseteq F[t]$ be the prime ideal. Recall that $L \supseteq K$ is called a simple field extension if $L = K(b)$. Then F is Hilbertian if for all simple field extensions $L \supseteq F(t)$ there are infinitely many $a \in F$ such that P_a has a unique extension to L, $L/F(t)$ is unramified at P_a, and the residue field extension $\overline{L} \supseteq F$ has dimension $[\overline{L} : F]$ equal to the dimension $[L : F(t)]$. It is well known that any global field is Hilbertian and if K is any field and F is finite over the rational field $K(t)$ then F is Hilbertian (e.g., [FJ, p. 155]).

It is natural to consider a division algebra analog to the Hilbertian property. To discuss this, let us specify right now that in this paper all division algebras are finite dimensional over their centers. We write D/F to mean D has center F. The degree of D is the square root of its dimension over its center F and the exponent of D is the order of its class $[D]$ in the Brauer group $\mathrm{Br}(F)$. Suppose $R \subseteq F$ is a discrete valuation ring and F is its field of fractions. We say D is unramified at R if $[D]$ is in the image of $\mathrm{Br}(R)$ under the natural injection $\mathrm{Br}(R) \to \mathrm{Br}(F)$. Equivalently, D is unramified at R if $D = A \otimes_R F$ for A/R Azumaya. If D is unramified at R, A is as above, and $P \subseteq R$ is the maximal ideal, we write $\overline{D} = A/PA$, which is central simple over R/P. If $F = K(t)$, and $a \in K$ we set $K[t]_a$ to be $K[t]$ localized at the prime $(t - a)$. Of course, $K[t]_a$ is a discrete valuation ring. We write D_a to be $\overline{D}$ with respect

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to this discrete valuation ring if D is unramified at $(t-a)$.

Returning to our discussion, one's first guess at a division algebra Hilbertian property might be the following. If $D/F(t)$ is a division algebra then there are infinitely many $a \in F$ such that D is unramified at $(t-a)$ and D_a is another division algebra. The problem is that this fails for global fields. We are about to describe a division algebra $D/K(t)$, for a global field K, such that D has exponent a prime p but degree p^2. However, any specialization D_a must have exponent equal to degree if a division algebra and so D_a cannot be a division algebra. In order to construct such a D, let $\Delta(M/F, \sigma, b)$ be the cyclic algebra defined by $b \in F^*$, M/F cyclic Galois, and σ a generator of the Galois group $\mathrm{Gal}(M/F)$.

Proposition 0.1. *Let K be a global field, and D'/K a division algebra of prime degree p. Then there is a cyclic Galois extension L/K of degree p such that*

$$D = D'(t) \otimes_{K(t)} \Delta\big(L(t)/K(t), \sigma, t\big)$$

is a division algebra of degree p^2 and exponent p.

Remark. For $p = 2$ the above fact and proof were previously known to quadratic forms specialists. The fact can be deduced from, and the construction is implicit in, e.g., [E, p. 442–443].

Proof. D clearly has exponent p if D is a division algebra, so we must show that for some L, D is a division algebra. Choose an L such that L splits completely at a prime where D' ramifies. Such an L exists by class field theory (e.g., [AT, p. 103] or the techniques of [S, p. 279]). It follows that L does not split D' and so $D'' = D' \otimes_K L$ is a division algebra. It follows by, e.g., [P, p. 379] that D is a division algebra of exponent p. For the reader's convenience, we include the following proof. If σ is any generator of $\mathrm{Gal}(L/K)$, σ extends to an automorphism of D'' (also called σ) defined by letting σ act trivially on D'. Let R be the twisted polynomial ring $D''[x, \sigma]$. R must be a domain and have center $K[x^p]$. R has a division ring of fractions denoted by $D''(x, \sigma)$ which has center $K(x^p)$. Identifying t and x^p we have that D is isomorphic to $D''(x, \sigma)$ and so the proposition is proved. □

Hence the "division algebra Hilbertian" property defined above seems uninteresting. Instead, in this paper we will consider a Brauer group version of the Hilbertian property. For any $a \in F$, the natural injection $\mathrm{Br}(F[t]_a) \to \mathrm{Br}(F(t))$ allows us to view $\mathrm{Br}(F[t]_a)$ as a subgroup of $\mathrm{Br}(F(t))$. We define $\rho : \mathrm{Br}(F[t]_a) \to \mathrm{Br}(F)$ to be the map induced by $F[t]_a \to F$. Let $p = \mathrm{char}(F)$ be the characteristic of F. For the purposes of the following definition, if $p = 0$, any $\alpha \in \mathrm{Br}(F(t))$ has order "prime to p."

Definition. A field F is called *Brauer-Hilbertian* if for all $\alpha \in \mathrm{Br}(F(t))$ of order prime to p, there are infinitely many $a \in F$ such that $\alpha \in \mathrm{Br}(F[t]_a)$ and $\rho(\alpha)$ has order equal to the order of α.

Note that the restriction to p-prime elements is forced by our lack of understanding of the p-part of $\mathrm{Br}(F(t))$. The authors do not know if the results of this paper extend to this p-part. Also note that the property of being Brauer-Hilbertian neither implies nor is implied by the classical Hilbertian property. For example, the field $\mathbb{C}(t)$ is Hilbertian but not Brauer-Hilbertian. In the other direction, the field $\mathbb{C}$ itself is Brauer-Hilbertian because $\mathrm{Br}(\mathbb{C}(t)) = 0$ but certainly is not Hilbertian. Some readers may find the fact $\mathbb{C}$ is Brauer-Hilbertian a bit bizarre, arising as it does because the definition involves $\mathbb{C}(t)$ and not $\mathbb{C}(t_1, \ldots, t_n)$ for $n > 1$.

In this paper we show that any global field is Brauer-Hilbertian. In addition, we show that if K is any Hilbertian field and F is separable over $K(t)$ then F is Brauer-Hilbertian. This suggests that Brauer-Hilbertian is a sort of two-dimensional Hilbertian property. Finally, as a consequence we observe that the inequality of degree and exponent is the only obstruction to a division algebra Hilbertian property over global fields. That is, if F is Brauer-Hilbertian and $D/F(t)$ has exponent n equal to its degree and n is prime to the characteristic, there are infinitely many $a \in F$ such that D is unramified at $(t - a)$ and D_a is a division algebra (of exponent equal to degree). Of course, this applies to F a global field.

Since writing the first draft of this paper, the authors have found a number of older and contemporary results that intersect with the results presented here. In [Sc, p. 202] it is shown that if D is a quaternion division algebra over $K(t_1, \ldots, t_r)$, for K a number field, then D has a nontrivial specialization over K. In addition, such a specialization can always be found in any r arithmetic progressions in K.

In the Russian language paper [V], Voronovich shows global fields are Brauer-Hilbertian. Again, what is actually shown is a stronger result involving arithmetic progressions. The methods of [V] are related to the methods used here, as, for example, Theorem 1.7 is proved for the special case of rings of integers in global fields of finite characteristic. Language difficulties and a less general approach preclude using [V] as a reference to shorten any of the proofs here.

Finally, Serre informs us (private communication) that he has also shown $K(t)$ is Brauer-Hilbertian for K a number field. Furthermore (and this was both the point and the hard part) Serre has derived quantitative results about the points where an $\alpha \in \mathrm{Br}(K(t))$ has prescribed exponent.

1. Preliminaries

Let F be a field and $G(F)$ the Galois group of F in its separable closure, F_s. Define $\chi(F) = H^1(G(F), Q/Z)$, where Q/Z has the discrete topology and trivial $G(F)$ action. Thus $\chi(F) = \mathrm{Hom}(G(F), Q/Z)$. If $f \in \chi(F)$, then $f(G(F)) \subseteq Q/Z$ is finite and hence cyclic. If $L \subseteq F_s$ is the fixed field of the kernel of f, then L/F is cyclic Galois and we say f defines L/F. The character f is completely determined by L and the generator $\sigma \in \mathrm{Gal}(L/F)$ such that $f(\sigma) = (1/[L:F]) + Z$. The exact sequence $0 \to Z \to Q \to Q/Z \to 0$ and the cohomology long exact sequence shows that $H^1(G(F), Q/Z) \cong H^2(G(F), Z)$. The cup product (e.g., [B, p. 112]) defines a map:

$$H^2(G(F), Z) \times H^0(G(F), F_s^*) \to H^2(G(F), F_s^*).$$

Of course, $H^2(G(F), F_s^*)$ is isomorphic to the Brauer group $\mathrm{Br}(F)$ via the crossed product construction, and $H^0(G(F), F_s^*) \cong F^*$. If $f \in \chi(F)$ and $a \in F^*$, let $\delta(f) \in H^2(G(F), Z)$ be the image of f and let $\Delta(f, a) \in H^2(G(F), F_s^*)$ be the cup product of $\delta(f)$ and a. In algebra terms, $\Delta(f, a)$ is represented by the cyclic crossed product $\Delta(L/F, \sigma, a)$, where f defines L/F and $\sigma \in \mathrm{Gal}(L/F)$ satisfies $f(\sigma) = (1/[L:F]) + Z$.

Next, suppose K is a field with a discrete valuation $v : K^* \to Z$. Denote by R the valuation ring of v and by $\overline{K}$ the residue field. The natural map $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is an injection (e.g., [DI, p. 136]) and we will frequently view $\mathrm{Br}(R)$ as a subset of $\mathrm{Br}(K)$. If $\alpha \in \mathrm{Br}(K)$ is in $\mathrm{Br}(R)$ we say α is unramified at R. If we avoid the characteristic of $\overline{K}$, the quotient $\mathrm{Br}(K)/\mathrm{Br}(R)$ can be described. To this end, let p be the characteristic of $\overline{K}$. If $p \neq 0$, and A is a torsion abelian group, let A' denote the prime to p part of A. If $p = 0$, set $A' = A$. There is an exact sequence:

$$(1) \qquad 0 \to \mathrm{Br}(R)' \to \mathrm{Br}(K)' \xrightarrow{\chi} \chi(\overline{K})' \to 0.$$

Let us recall the definition of the character map χ. Denote by K_v the completion of K with respect to v. Let $L_v \supseteq K_v$ be the maximal unramified extension of K_v. The Galois group of L_v/K_v is canonically isomorphic to $G(\overline{K})$. The valuation v extends to L_v and defines a $G(\overline{K})$ map $L_v \to Z$. χ is the composition $\mathrm{Br}(K)' \to \mathrm{Br}(K_v)' \to H^2(G(\overline{K}), L_v^*)' \to H^2(G(\overline{K}), Z)' \cong H^1(G(\overline{K}), Q/Z)' = \chi(\overline{K})'$. In particular, if $\alpha \in \mathrm{Br}(K)$, then $\chi(\alpha)$ can be computed by first going to the completion K_v.

We concentrate for a moment on K_v. If $f \in \chi(\overline{K})$, then f can be considered a character of $\mathrm{Gal}(L_v/K_v)$ and so an "unramified" element of $\chi(K_v)$. Let $\pi \in K_v$ be a prime element. The map $Z \to L_v^*$ defined by sending $1 \to \pi$ yields a splitting of $v : L_v^* \to Z$. It follows that $\chi(\Delta(f, \pi)) = f$.

A large part of this section will be taken up with a formula relating the corestriction map and character maps. To this end, let us review the corestriction. For convenience, we will confine ourselves in this discussion to finite groups. So let $H \subseteq G$ be finite groups and M a G-module. Let $\mathrm{Res}_H^G(M)$ denote the H-module obtained by restriction. If N is an H-module, let $\mathrm{Ind}_H^G(N)$ be the induced G-module. As the H-module $\mathrm{Res}_H^G(M)$ has a G-module structure, there is an induced map $\psi : \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M)) \to M$. Shapiro's lemma says $H^q(H, \mathrm{Res}_H^G(M)) \cong H^q(G, \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M)))$ and ψ induces a map $H^q(G, \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))) \to H^q(G, M)$. The composition

$$H^q(H, \mathrm{Res}_H^G(M)) \to H^q(G, M)$$

is one possible definition of the corestriction map.

Suppose $F \subseteq K$ are fields with K/F finite separable. Choose $L \supseteq K$ such that L/F is Galois with group, say, G. Let $H \subseteq G$ be the subgroup corresponding to K. The relative Brauer groups $\mathrm{Br}(L/K)$ and $\mathrm{Br}(L/F)$ are isomorphic to $H^2(H, L^*)$ and $H^2(G, L^*)$ respectively. Define $\chi(L/K)$ and $\chi(L/F)$ to be $H^1(H, Q/Z)$ and $H^1(G, Q/Z)$ respectively. The cohomological corestriction maps induce maps $\mathrm{Cor}_{K/F} : \mathrm{Br}(L/K) \to \mathrm{Br}(L/F)$ and $\mathrm{Cor}_{K/F} : \chi(L/K) \to \chi(L/F)$. It is not hard to see that if $\alpha \in \mathrm{Br}(L/K)$ or $\chi(L/K)$, then $\mathrm{Cor}_{K/F}(\alpha)$ is independent of the choice of L. Thus there are induced maps $\mathrm{Cor}_{K/F} : \mathrm{Br}(K) \to \mathrm{Br}(F)$ and $\mathrm{Cor}_{K/F} : \chi(K) \to \chi(F)$.

Later on, we will discuss $\mathrm{Cor}_{K/F}$ for K/F inseparable. Now consider K to be a separable finite (commutative) F-algebra. That is, assume $K = K_1 \oplus \cdots \oplus K_s$ is a finite direct sum of finite separable field extensions of F. We view $F \subseteq K$ via the diagonal embedding. For brevity, we call such a K a finite separable extension of F. The important point is that if $F' \supseteq F$ is any field extension, $K \otimes_F F'$ is a finite separable extension of F'. Also, $\mathrm{Br}(K)$ is naturally isomorphic to $\mathrm{Br}(K_1)\oplus\cdots\oplus\mathrm{Br}(K_s)$. Define $\mathrm{Cor}_{K/F}:\mathrm{Br}(K)\to\mathrm{Br}(F)$ to be the sum of all the maps:

$$\mathrm{Cor}_{K_i/F}:\mathrm{Br}(K_i)\to\mathrm{Br}(F).$$

If K is a finite separable extension of F, then $K\subseteq L$, where L is a finite G-Galois extension of F and K is the fixed ring of a subgroup $H\subseteq G$. Moreover, if $\alpha\in\mathrm{Br}(K)$, there is such an L which splits α. By e.g., [S, p. 253], such an L has the form $\mathrm{Ind}_I^G(L'/F)$, where $I\subseteq G$ is a subgroup, L'/F is a finite Galois *field* extension with group I, and "Ind" indicates the induced Galois extension construction as in [S]. As an easy consequence of the field extension result, $\mathrm{Br}(L/K)$ and $\mathrm{Br}(L/F)$ are naturally isomorphic to $H^2(H, L^*)$ and $H^2(G, L^*)$ respectively. The next result shows that the cohomological corestriction agrees with the corestriction map defined above.

Proposition 1.1. *Let $F\subseteq K\subseteq L$ be as above. If $\alpha\in\mathrm{Br}(L/K)$, then $\mathrm{Cor}_{K/F}(\alpha)$ corresponds to the image of α under the cohomological corestriction $H^2(H, L^*)\to H^2(G, L^*)$.*

Proof. It is clear that $L^*=(\mathrm{Ind}_I^G(L'))^*$ is isomorphic to $\mathrm{Ind}_I^G(L'^*)$ as G-modules. Set M to be the G-module L'^*. To compute the cohomological corestriction we form $\mathrm{Ind}_H^G(\mathrm{Res}_H^G(\mathrm{Ind}_I^G(M)))$. Let E be a set of $H-I$ double coset representatives in G. That is, G is the disjoint union:

$$\bigcup_{g\in E} HgI.$$

For any $g\in E$, let gM be the (gIg^{-1})-module obtained by "twisting" $\mathrm{Res}_I^G(M)$ by g. By e.g., [B, p. 69],

$$\mathrm{Ind}_H^G(\mathrm{Res}_H^G(\mathrm{Ind}_I^G(M)))\cong\mathrm{Ind}_H^G\left(\bigoplus_{g\in E}\mathrm{Ind}_{H\cap gIg^{-1}}^H(\mathrm{Res}_{H\cap gIg^{-1}}^{gIg^{-1}}(gM))\right),\tag{2}$$

which is isomorphic to

$$\bigoplus_{g\in E}\mathrm{Ind}_{H\cap gIg^{-1}}^G(\mathrm{Res}_{H\cap gIg^{-1}}^{gIg^{-1}}(gM)).$$

Thus $H^2(H, L^*)\cong H^2(G, \mathrm{Res}_H^G(L^*))\to H^2(G, L^*)$ is the sum of maps:

$$\mathrm{Cor}_g: H^2\left(H\cap gIg^{-1},\ \mathrm{Res}_{H\cap gIg^{-1}}^{gIg^{-1}}(gM)\right)\to H^2(G, L^*).$$

Now gM is isomorphic to the (gIg^{-1})-module of units in the g conjugate, gL', of L' in L. It is easy to check that there is a bijection between E and the set of fields K_i. Moreover, K_i naturally embeds in gL' for the corresponding $g\in E$. In fact, K_i is the fixed field in $H\cap gIg^{-1}$ of gL'. Thus each Cor_g is just the corestriction map $\mathrm{Cor}:\mathrm{Br}(gL'/K_i)\to\mathrm{Br}(gL'/F)=\mathrm{Br}(L'/F)$. Thus after a lot of checking, the proposition is proved. □

An immediate consequence of 1.1 is

Corollary 1.2. *Let $K\supseteq F$ be a finite separable extension, and $F'\supseteq F$ a field. Then the following diagram commutes:*

$$\begin{array}{ccc}\mathrm{Br}(K) & \xrightarrow{\mathrm{Res}} & \mathrm{Br}(K\otimes_F F')\\ {\scriptstyle\mathrm{Cor}}\downarrow & & \downarrow{\scriptstyle\mathrm{Cor}}\\ \mathrm{Br}(F) & \xrightarrow{\mathrm{Res}} & \mathrm{Br}(F')\end{array}$$

Proof. Let $L \supseteq K \supseteq F$ be such that L/F is Galois with group G. Let $H \subseteq G$ correspond to K. From the discussion of [B, pp. 80–84] it is clear the corestriction is natural. Thus we have the following commutative diagram:

$$\begin{array}{ccc} H^2(H, L^*) & \longrightarrow & H^2(H, (L \otimes_F F')^*) \\ \text{Cor} \downarrow & & \downarrow \text{Cor} \\ H^2(G, L^*) & \longrightarrow & H^2(G, (L \otimes_F F')^*) \end{array}$$

and the result follows. □

Now let us return to the case when $K \supseteq F$ is a field, but K is not separable. Let $K' \subseteq K$ be the maximal separable extension of F in K. If $q = [K : K']$, then we can define the field monomorphism $\theta : K \to K'$ by $\theta(a) = a^q$. The map θ induces a map $\mathrm{Br}(K) \to \mathrm{Br}(K')$ which we call $\mathrm{Cor}_{K/K'}$. We define $\mathrm{Cor}_{K/F}$ to be the composition $\mathrm{Cor}_{K'/F}\mathrm{Cor}_{K/K'}$. If $K_s \supseteq K'$ is the separable closure of K', then $K_s \otimes_{K'} K$ is the separable closure of K and so we may identify the absolute Galois group G_K of K with that of K'. We define $\mathrm{Cor}_{K/K'} : \chi(K) \to \chi(K')$ by setting $\mathrm{Cor}_{K/K'}(f) = f^q$. Finally, $\mathrm{Cor}_{K/F} : \chi(K) \to \chi(F)$ is the composition $\mathrm{Cor}_{K'/F}\mathrm{Cor}_{K/K'}$. It is not hard to see that on both Brauer groups and characters, if $L \supseteq K \supseteq F$ are fields, $\mathrm{Cor}_{L/F} = \mathrm{Cor}_{K/F}\mathrm{Cor}_{L/K}$. In other words, the corestriction respects towers of fields. Secondly, if $n = [L : K]$ and $\mathrm{Res}_{L/K}$ refers to the restriction map $\chi(K) \to \chi(L)$ or $\mathrm{Br}(K) \to \mathrm{Br}(L)$, then $\mathrm{Cor}_{L/K}\mathrm{Res}_{L/K} = n$. This result is standard in the cohomological or separable case and extends easily. Finally, the definition of corestriction immediately above shows that if K/F is purely inseparable, and $F' \supseteq F$ is a field extension linearly disjoint from K/F, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Br}(K) & \longrightarrow & \mathrm{Br}(K \otimes_F F') \\ \text{Cor} \downarrow & & \downarrow \text{Cor} \\ \mathrm{Br}(F) & \longrightarrow & \mathrm{Br}(F') \end{array} \tag{3}$$

For the rest of this section, we will consider the interaction of discrete valuations and the corestriction. Let R, P be a discrete valuation ring with field of fractions K. Let $v : K^* \to Z$ denote the associated valuation. In the next result, we are particularly interested in the case that R contains a field F and the composition $F \to R \to R/P$ is an isomorphism. If $F' \supseteq F$ is a finite field extension, $R' = F' \otimes_F R$ is a discrete valuation ring with maximal ideal $P' = F' \otimes_F P$, residue field $R'/P' \cong F'$, and field of fractions $K' = F' \otimes_F K$. Use the symbol ρ to denote both the induced maps $\mathrm{Br}(R) \to \mathrm{Br}(F)$ and $\mathrm{Br}(R') \to \mathrm{Br}(F')$.

Proposition 1.3. *Suppose $L \supseteq F'$ is a finite field extension and $\alpha \in \mathrm{Br}(R')$ is split by $T = L \otimes_{F'} R'$. Assume α has order prime to p if p is the nonzero characteristic of F. Then $\mathrm{Cor}_{K'/K}(\alpha)$ is in $\mathrm{Br}(R)$ and*

$$\rho(\mathrm{Cor}_{K'/K}(\alpha)) = \mathrm{Cor}_{F'/F}(\rho(\alpha)).$$

Proof. Using our definition of corestriction we can quickly reduce to the cases F'/F is separable or purely inseparable. By our assumption on the order of α, we may assume L/F' is separable.

Case 1. *F'/F is separable.* We may assume L/F is G-Galois and let $H \subseteq G$ correspond to F'. Then α is in the image of $H^2(H, T^*)$ and the naturality of the cohomological corestriction shows that the diagram

$$\begin{array}{ccc} H^2(H, T^*) & \longrightarrow & H^2(H, L^*) \\ \text{Cor} \downarrow & & \downarrow \text{Cor} \\ H^2(G, T^*) & \longrightarrow & H^2(G, L^*) \end{array}$$

commutes and this case is done.

Case 2. *F'/F is purely inseparable of degree q.* Let π denote both the ring homomorphisms $R' \to R$ and $F' \to F$ defined by $\pi(x) = x^q$. Then the following diagram commutes:

$$\begin{array}{ccc} R' & \longrightarrow & F' \\ \pi \downarrow & & \downarrow \pi \\ R & \longrightarrow & F \end{array}$$

Applying the Brauer group functor yields this case. □

A main goal of this section is to prove a result linking the corestriction and the character map. To that end, let K once again be a field with a discrete valuation $v : K^* \to Z$. Let $L \supseteq K$ be a finite not necessarily separable field extension. Denote by $L' \subseteq L$ the maximal separable subfield of L/K. Let Γ be the finite set of extensions of v to L'. Any $w \in \Gamma$ has a unique extension to L and so we identify Γ with the set of extensions of v to L. Recall that L_w and L'_w are the completions of L and L' with respect to $w \in \Gamma$. Set $\delta(w) = [L : L']/[L_w : L'_w]$ and call $\delta(w)$ the defect of w. The residue field $\overline{K}$ is naturally a subfield of $\overline{L}_w$. Recall that if A is an abelian group and p is the characteristic of $\overline{K}$, then A' is the prime to p part of A if $p \neq 0$ and $A' = A$ otherwise. Denote by $\chi : \text{Br}(K)' \to \chi(\overline{K})'$ and $\chi_w : \text{Br}(L)' \to \chi(\overline{L}_w)'$ the respective character maps.

Theorem 1.4. *If $\alpha \in \text{Br}(L)'$, then*

$$\chi\left(\text{Cor}_{L/K}(\alpha)\right) = \prod_{w \in \Gamma} \text{Cor}_{\overline{L}_w/\overline{K}_v}\left(\chi_w(\alpha)\right)^{\delta(w)}. \tag{4}$$

Proof. Recall that K_v is the completion of K with respect to v. We will, in this argument, repeatedly use the fact that $\chi(\alpha) = \chi'(\alpha')$, where χ' is the character map for K_v and α' is the image of α in $\text{Br}(K_v)$. If $M'' \supseteq M' \supseteq M$ is a tower of fields, we know that on characters or Brauer groups $\text{Cor}_{M''/M} = \text{Cor}_{M'/M}\text{Cor}_{M''/M'}$. It follows that if $L \supseteq L' \supseteq K$, it suffices to prove this theorem for L/L' and L'/K.

We will perform this proof by using the above arguments to reduce to a series of cases. First of all, if $K \subseteq L' \subseteq L$ is such that L'/K is the maximal separable subextension of L/K, we can restrict our attention to L/L' and L'/K. That is, we may assume that L/K is either purely inseparable or separable. The same argument shows that if L/K is purely inseparable, we may assume $L = K(a^{1/p})$ with p the characteristic of K.

Case 1. *L/K is purely inseparable of degree p and L, K_v are not linearly disjoint over K.* It follows that L is isomorphic to a subfield of K_v and we may assume that $K \subseteq L \subseteq K_v$. We also use "$v$" to denote the unique extension of v to L (and K_v). Let χ_L denote the character map of L, as χ and χ' have been defined for K and K_v. Note that $\overline{L} = \overline{K}$. This implies that the corresponding corestriction on characters is the identity map. By, e.g., [J] or [D, p. 110], any $\alpha \in \mathrm{Br}(L)$ is the image under restriction of $\beta \in \mathrm{Br}(K)$. Since $K \subseteq L \subseteq K_v$, $\chi_L(\alpha) = \chi'(\alpha') = \chi(\beta)$, where α' is the common image of α, β in $\mathrm{Br}(K_v)$. Thus $\chi(\mathrm{Cor}_{L/K}(\alpha)) = \chi(\beta^p) = \chi(\alpha)^p$ which verifies (4) in this case because p is the defect.

In all remaining cases, $L \otimes_K K_v$ is a direct sum of fields. By Corollary 1.2, or (3), we may assume $K = K_v$ is complete. If L/K is separable, then there is a maximal unramified subfield $L' \subseteq L$. Considering L'/K first, we may assume L/K is unramified.

Case 2. *L/K is unramified and K is complete.* Let $\alpha \in \mathrm{Br}(L)'$. Since α has an unramified splitting field, we may choose $M \supseteq L \supseteq K$ such that M/K is unramified Galois and M splits α. If $G \supseteq H$ are the Galois groups of M/F and M/L then we have the following commutative diagram:

$$\begin{array}{ccccc} H^2(H, M^*) & \longrightarrow & H^2(H, Z) & \cong & H^1(H, Q/Z) \\ \mathrm{Cor}\downarrow & & \downarrow \mathrm{Cor} & & \downarrow \mathrm{Cor} \\ H^2(G, M^*) & \longrightarrow & H^2(G, Z) & \cong & H^1(G, Q/Z) \end{array}$$

The left square commutes because corestriction is natural and the right square commutes because corestriction commutes with the boundary map (left to the reader in [B, p. 81]). This proves Case 2.

Case 3. *K is complete and L/K has no unramified subfield.* Then $\overline{L}/\overline{K}$ is purely inseparable. Let $R \subseteq S$ be the valuation rings of $K \subseteq L$. Since $\mathrm{Br}(R) \cong \mathrm{Br}(\overline{K})$ and $\mathrm{Br}(S) \cong \mathrm{Br}(\overline{L})$ (e.g., [Se, p. 186]), the natural map $\mathrm{Br}(R) \to \mathrm{Br}(S)$ is a surjection. Using (1), we view $\mathrm{Br}(R)$ and $\mathrm{Br}(S)$ as subgroups of $\mathrm{Br}(K)$ and $\mathrm{Br}(L)$. If $\alpha \in \mathrm{Br}(S)$, then α is the image of $\beta \in \mathrm{Br}(R)$ so $\mathrm{Cor}_{L/K}(\alpha) = \beta^n$, where $n = [L : K]$. It follows that $\mathrm{Cor}_{L/K}(\mathrm{Br}(S)) \subseteq \mathrm{Br}(R)$.

If $\alpha \in \mathrm{Br}(L)$, let $f' = \chi_L(\alpha) \in \chi(\overline{L})$. Let $\pi \in L$ be a prime element and form $\Delta(f', \pi)$. Recall that we are identifying characters of $\overline{L}$ (or $\overline{K}$) with unramified characters of L (or K). Since $\chi_L(\Delta(f', \pi)) = f'$, $\alpha = \alpha'\Delta(f', \pi)$, where $\alpha' \in \mathrm{Br}(S)$. Thus $\chi(\mathrm{Cor}_{L/K}(\alpha)) = \chi(\mathrm{Cor}_{L/K}(\Delta(f', \pi)))$. As $\overline{L}/\overline{K}$ is purely inseparable, f' is the restriction of $f \in \chi(F)$. If L/K is separable, the fact that $\Delta(f', \pi)$ is a cup product and [B, p. 112] imply that $\mathrm{Cor}_{L/K}(\Delta(f', \pi)) = \Delta(f, N_{L/K}(\pi))$, where $N_{L/K} : L \to K$ is the norm. $v(N_{L/K}(\pi)) = [L : K]/e(L/K) = [\overline{L} : \overline{K}] =$ (say) r. Thus $\chi(\mathrm{Cor}_{L/K}(f', \pi)) = \chi(\Delta(f, N_{L/K}(\pi))) = f^r$, which verifies (4) in this case.

Continuing with Case 3, assume L/K is purely inseparable of degree p. If $\Delta(f', \pi)$ has order m, which by assumption is prime to p, choose r such that rp is congruent to 1 mod m. Then $\Delta(f', \pi)$ is the image of $\Delta(f^r, \pi^p) \in \mathrm{Br}(K)$. Thus $\mathrm{Cor}_{L/K}(\Delta(f', \pi)) = \Delta(f^r, \pi^p)^p = \Delta(f, \pi^p)$. We have $\chi(\Delta(f, \pi^p)) = f^s$, where $s = 1$ if $e(L/K) = p$ and $s = p$ if $e(L/K) = 1$. But in these cases $[\overline{L} : \overline{K}] = 1$ and p respectively so (4) is again verified. All together, 1.4 is proved. □

Before we get to the main results of this paper, we need one final observation. Let $K \subseteq L$ be a finite purely inseparable field extension with $a \in L$ such that $L = K(a)$. Let p be the characteristic of K. Assume $R \subseteq K$ is a Dedekind domain with $K = q(R)$. Let S be the integral closure of R in L and assume S is a finitely generated R-module. If $P \subseteq R$ is a prime ideal let

$P' = \{s \in S \mid s^q \in P \text{ for some } q = p^r\}$. Then P' is the unique prime of S lying over $P \subseteq R$. Note that the localization $S_{P'}$ is also $S_P = S \otimes_R R_P$. For convenience we will often use the same symbol for $P \subseteq R$ and its associated $P' \subseteq S$.

Lemma 1.5. *For all but finitely many primes P of R, $S_P = R_P[a]$.*

Proof. Replacing R by some $R(1/r)$ for $0 \neq r \in R$, we may assume $S \supseteq R[a]$. $S/R[a]$ is a finitely generated torsion R-module, and so $(S/R[a]) \otimes_R R_P = (0)$ for all but finitely many primes P of R. For these P, $S_P = R_P[a]$. □

In the proof of 2.4 of the next section we will be given a prime $P \subseteq R$ such that the corresponding residue fields satisfy $\overline{K}_P = \overline{L}_P$. We will want a $b \in K$ such that $a - b$ is a prime element of S_P. This will be possible because of the next result.

Theorem 1.6. *Suppose $R \subseteq S$ are Dedekind domains such that the fraction fields $K \subseteq L$ satisfy L/K purely inseparable and $L = K(a)$. Suppose $P \subseteq R$ is a prime such that $S_P = R_P[a]$ and $\overline{K}_P = \overline{L}_P$. Then there is a $b \in R$ such that $a - b$ is a prime element of S_P.*

Proof. Let $v_P : L \to Z$ be the normalized valuation associated with the unique extension of P. If $v_P(a) = 0$ then $v_P(a - r) > 0$ for some $r \in R_P$. Since $R_P[a] = R_P[a - r]$, we can replace a by $a - r$ and assume $v_P(a) > 0$. If $e(L/K) = 1$ let $\pi \in R_P$ be prime and note that $v_P(a + \pi) = 1$ or $v_P(a) = 1$, so we are done in this case. Thus we may assume $e(L/K) > 1$. Let $\theta \in S_P$ satisfy $v_P(\theta) = 1$. Write $\theta = r_0 + r_1 a + \cdots + r_n a^n$. If $v_P(r_0) = 0$, θ would be a unit. Thus $v_P(r_0) > 0$, implying $v_P(r_0) > v_P(\theta)$. We can replace θ by $\theta - r_0$ and conclude $\theta = a(g(a))$. It follows that $v_P(a) = 1$, proving the theorem. □

An immediate consequence of 1.5 and 1.6 is our goal:

Theorem 1.7. *Let $R \subseteq S$ be Dedekind domains whose corresponding field of fractions extension $K \subseteq L$ is purely inseparable. Assume $L = K(a)$ and that S is finitely generated as an R-module. Suppose $\mathscr{P}$ is an infinite set of primes of R such that for $P \in \mathscr{P}$, $\overline{K}_P = \overline{L}_P$. Then for all but finitely many $P \in \mathscr{P}$, there is a $b \in R$ with $a - b$ a prime element of S_P.*

2. Specializing

This paper is concerned with specializing elements of $\mathrm{Br}(F(t))$, where $F(t)$ is the rational function field in the one-variable t. We begin, therefore, with a description of the Brauer group of $F(t)$. As in §1, we will frequently need to avoid elements of order a power of p if F has nonzero characteristic p. So, as in §1, for any torsion abelian group A, let A' be the prime to p part of A if $p \neq 0$ and let $A' = A$ otherwise. To describe the Brauer group $\mathrm{Br}(F(t))$ we recall that there is a split exact sequence (e.g., [FS, p. 51]):

$$0 \to \mathrm{Br}(F)' \to \mathrm{Br}(F(t))' \xrightarrow{\chi} \bigoplus_{P \subseteq F[t]} \chi(F[t]/P)' \to 0,$$

which has the following description. $\mathrm{Br}(F)' \to \mathrm{Br}(F(t))'$ is induced by $F \subseteq F(t)$. The direct sum is over all primes $P \subseteq F[t]$. To each such P, there is a discrete valuation ring $F[t]_P$ and an associated character map $\chi_P : \mathrm{Br}(F(t))' \to \chi(F[t]/P)$. χ is the sum of the χ_P's.

The goal of the next result is to define a set of generators of $\mathrm{Br}(F(t))'$ which only ramify at one prime.

Proposition 2.1. *Suppose $P \subseteq F[t]$ is a prime, $F' = F[t]/P$, and $f \in \chi(F')$. Denote by $a \in F'$ the image of t so $F' = F(a)$. Use the inclusion $F' \subseteq F'(t)$ to regard $f \in \chi(F'(t))$ also. Let f define L'/F' (or $L'(t)/F'(t)$). If $\alpha = \mathrm{Cor}_{F'(t)/F(t)}(\Delta(f, t-a))$ then $\chi_P(\alpha) = f$ and $\chi_Q(\alpha) = 0$ for all $Q \neq P$.*

Proof. If $P' = (t-a) \subseteq F'[t]$, then P' is a prime extending P. Also, $\chi_{P'}(\Delta(f, t-a)) = f$. If $Q' \subseteq F'[t]$ is another prime, then $t-a$ is a unit in $F'[t]_{Q'}$ so $\chi_{Q'}(\Delta(f, t-a)) = 0$. If $F'' \subseteq F'$ is the maximal separable subfield of F'/F, let $q = [F' : F'']$. Then $F'' = F(a^q)$. The prime $(t-a) \subseteq F'[t]$ extends $(t^q - a^q) \subseteq F''[t]$ and so $F'(t)/F''(t)$ is totally ramified at $(t-a)$. In particular, $F'(t)/F''(t)$ has defect 1. Thus 2.1 follows from 1.4. □

If $\alpha = \mathrm{Cor}_{F'(t)/F(t)}(\Delta(f, t-a))$ is as in 2.1, we call α a *basic* element of $\mathrm{Br}(F(t))'$. We call the prime P used to define α the prime of α. It is clear that $\mathrm{Br}(F)'$ and the basic elements generate $\mathrm{Br}(F(t))'$. Our next result concerns specializing basic elements. To this end, suppose $b \in F$ and set $P_b = (t-b) \subseteq F[t]$. Let $F[t]_b$ be the localization of $F[t]$ at P_b and $\rho_b : \mathrm{Br}(F[t]_b) \to \mathrm{Br}(F)$ the induced map. Regard $\mathrm{Br}(F[t]_b)$ as a subgroup of $\mathrm{Br}(F(t))$. If $\alpha \in \mathrm{Br}(F(t))$, we say $\rho_b(\alpha)$ is defined if and only if $\alpha \in \mathrm{Br}(F[t]_b)$. If $\alpha = \mathrm{Cor}_{F'(t)/F(t)}(\Delta(f, t-a))$ is a basic element, $\rho_b(\alpha)$ is defined if and only if P_b is not the prime of α. That is, $\rho_b(\alpha)$ is defined if and only if $b \neq a$.

Lemma 2.2. *If $\alpha = \mathrm{Cor}_{F'(t)/F(t)}(\Delta(f, t-a))$ is a basic element and $b \neq a$, then $\rho_b(\alpha) = \mathrm{Cor}_{F'/F}(\Delta(f, b-a))$.*

Proof. P_b extends uniquely to the prime $P'_b = (t-b) \subseteq F'[t]$. Defining ρ_b on $\mathrm{Br}(F'[t]_b)$ in the obvious way, it is clear that $\rho_b(\Delta(f, t-a)) = \Delta(f, b-a)$. Now 2.2 follows from 1.3. □

In order to carry our analysis further, we must assume more about F. Suppose $R \subseteq F$ is a Dedekind domain with field of fractions F. The next result is a technical one which gives a setup in which we can control the ramification of specializations with respect to primes of R. To set up the situation, let $\alpha = \mathrm{Cor}_{F'(t)/F(t)}(\Delta(f, t-a))$ be a basic element. By definition, $F' = F(a)$. Assume f defines L/F' and let $L' \supseteq L$ be such that L'/F is normal. Let $a = a_1, \dots, a_s$ be the conjugates of a in L'. Denote by $R' \subseteq F'$, $T \subseteq L$, and $T' \subseteq L'$ the integral closures of R. Finally, let $L'' \subseteq L'$ be the maximal separable extension over F.

Lemma 2.3. *Let $\alpha = \mathrm{Cor}_{F'(t)/F(t)}(\Delta(f, t-a))$, L, etc. be as above. Suppose $Q' \subseteq T'$ is a prime and set $P = Q' \cap R$, $P' = Q' \cap R'$, and $Q = Q' \cap T$. Let $F_P \subseteq F'_{P'} \subseteq L_Q \subseteq L'_{Q'}$ be the respective completions, and let $\overline{F} \subseteq \overline{F}' \subseteq \overline{L}$ be the respective residue fields. Suppose $c \in L'$ is a Q' unit and $b \in F$. Let $v' : F' \to Z$ be the valuation associated with P'. Consider the following conditions:*

(I) *L''/F is unramified with respect to all extensions of P to L''.*
(II) *$a_1, \dots, a_s$ are all Q' integral and distinct modulo Q'.*
(III) *$v'(b-c) = 1$.*
(IV) *$\overline{F} = \overline{F}'$ and $[\overline{L} : \overline{F}'] = [L : F]$.*

If (I), (II) *and* (III) *hold and* c *is distinct from all* $a_1, \ldots, a_s$ *modulo* Q', *then* $\chi_P(\rho_b(\alpha)) = 0$. *If* (I), (II), (III), *and* (IV) *hold and* $c = a_1 = a$, *then* $\chi_P(\rho_b(\alpha))$ *has order equal to that of* f.

Proof. Assume (I), (II), and (III) hold. For each $a_i \in L'$, there is an embedding $\sigma_i : F' \to L'$ such that $\sigma_i(a) = a_i$. Q' and σ_i induce a prime P_i on F'. All extensions of P have the form P_i. Thus if c is distinct from $a_i \bmod Q'$ for all i, then $b - a$ is a unit with respect to all extensions of P. As L/F' is unramified, $\Delta(f, b-a)$ is unramified with respect to all extensions of P. This case then follows from 1.4.

Assume (IV) also holds and $c = a_1$. Then $b - a$ is a unit with respect to all primes but $P_1 = P'$. Moreover $\chi_{P'}(\Delta(f, b-a))$ defines $\overline{L}/\overline{F}'$. By assumption, $\overline{F} = \overline{F}'$. If δ is the defect in the purely inseparable part of F'/F, then δ is prime to the order of f. Thus by 1.4, $\chi_{P'}(\mathrm{Cor}_{F'/F}(\Delta(f, b-a)))$ defines $\overline{L}/\overline{F}$. By assumption, $[\overline{L} : \overline{F}] = [L : F']$ which is the order of f. The result follows from 2.2. □

Let $L \supseteq F' \supseteq F$ be fields, and Q a prime of L. Q induces an extension of residue fields $\overline{L} \supseteq \overline{F}' \supseteq \overline{F}$. We say $L \supseteq F' \supseteq F$ is *good* at Q if $\overline{F} = \overline{F}'$ and $[\overline{L} : \overline{F}'] = [L : F']$. This is just (IV) of 2.3. Recall that F was defined to be Brauer-Hilbertian if for all $\alpha \in \mathrm{Br}(F(t))'$, there are infinitely many $b \in F$ such that $\rho_b(\alpha)$ is defined and has order equal to α. One can show that a field is Brauer-Hilbertian by using the following result.

Theorem 2.4. *Suppose that* F *is the field of fractions of a Dedekind domain with the following property*:

> *Suppose* $L \supseteq F' \supseteq F$ *is a tower of finite field extensions,* L/F' *is cyclic Galois,* $F' = F(a)$, *and* $T \subseteq L$ *is the integral closure of* R *in* L. *Then* T *is a finite* R*-module and there are infinitely many primes* $Q \subseteq T$ *such that* $L \supseteq F' \supseteq F$ *is good with respect to* Q.

Then F *is Brauer-Hilbertian.*

Remark. If R is a Nagata ring (e.g., [M, p .231]) then T above is necessarily a finitely generated R-module.

Proof. Write $\alpha = \beta\alpha_1, \ldots, \alpha_s$, where $\beta \in \mathrm{Br}(F)'$ and $\alpha_1, \ldots, \alpha_s$ are basic elements with distinct primes. Write

$$\alpha_i = \mathrm{Cor}_{F_i'(t)/F(t)}(\Delta(f_i, t - a_i)),$$

where $F_i' = F(a_i)$. Let $g_i(t) \in F[t]$ be the minimal monic polynomial of a_i, so $g_i(t)$ generates the prime of α_i. As the primes of the α_i are distinct, $g_i(t) \neq g_j(t)$ if $i \neq j$. Suppose f_i defines L_i/F_i' and let L be the normal closure of all the L_i over F. Set $L' \subseteq L$ to be the maximal separable extension of F in L. Denote by $\{a_{ij} \mid 1 \leq j \leq s(i)\}$ the set of all distinct conjugates of $a_i = a_{i1}$ in L. Note that the a_{ij} are all distinct because the $g_i(t)$ are. Let S be a finite set of primes of R which contain:

(a) All primes in R which ramify in L'.

(b) All primes in R which lie under primes in L which appear with positive or negative exponents in any a_{ij}.

(c) All primes in R which are the restrictions of primes in L which appear in $a_{ij} - a_{i'j'}$ for any distinct (i, j), (i', j').

(d) All primes where β ramifies.

In particular, if $P \notin S$ and Q is a prime in L lying over P, all a_{ij} are Q units and distinct modulo Q.

Let $F_i'' \subseteq F_i'$ be the maximal separable extensions of F in F_i'. Set $R_i' = T \cap F_i'$ and $R_i'' = T \cap F_i''$. By 1.7 and our assumptions there are primes Q_i of L such that:

(1) $P_i = Q_i \cap R$ is not in S.
(2) $L \supseteq F_i' \supseteq F$ is good with respect to Q_i.
(3) $P_i \neq P_j$ if $i \neq j$.
(4) There is a $Q_i \cap R_i''$ integer $a_i'' \in F_i''$ such that $a_i - a_i''$ is a prime element in F_i' with respect to $Q_i \cap R_i'$.

Using the above, we construct infinitely many $b \in R$ such that $a_i - b$ is a prime element in F_i' with respect to $Q_i \cap R_i'$ but $(a_{jk} - b)$ is a Q_i unit if $(j, k) \neq (i, 1)$. To find such b, let $v_i : L \to Z$ be the valuation associated with Q_i. Choose a_i'' as in (4) above. Since F, F_i'' have equal Q_i completions, there are $b_i \in F$ with $v_i(a_i'' - b_i) > v_i(a_i - a_i'')$. Finally, use weak approximation to find $b \in F$ such that $v_i(b - b_i) > v_i(a_i'' - b_i)$. There are clearly infinitely many such b. Now $v_i(a_i - b) = v_i(a_i - a_i'')$ so $a_i - b$ is a prime element with respect to $Q_i \cap R_i'$. By (b) and (c) above, $a_{jk} - b$ is a Q_i unit if $(j, k) \neq (i, 1)$. Thus the $b \in F$ which we require are constructed.

Given $b \in F$ as above, the following hold. First of all, $\rho_b(\alpha_i)$ is defined for all i. Let χ_i be the character map associated with $Q_i \cap R$. By 2.3, $\chi_i(\rho_b(\alpha_i))$ has order the order of f_i. By 2.3 again, $\chi_i(\rho_b(\alpha_j)) = 0$ if $j \neq i$. Thus $\chi_i(\rho_b(\alpha)) = \chi_i(\rho_b(\alpha_i))$. Let q' be the highest power of a prime p' dividing the order, n, of α. We must show that q' divides the order of $\rho_b(\alpha)$. If q' divides the order of some f_i, we are done. If not, then $n' = n/p'$ satisfies $f_i^{n'} = 1$ for all i. By construction, the order of α_i is the order of f_i which is the order of $\rho_b(\alpha_i)$. Thus $\beta^{n'} \neq 1$ and so $\rho_b(\alpha)^{n'} \neq 1$. □

Of course, we want to use 2.4 to show certain fields are Brauer-Hilbertian. First of all, we consider global fields.

Theorem 2.5. *Let F be a global field. Then F is Brauer-Hilbertian.*

Proof. Let $R \subseteq F$ be a (or the) ring of integers. Suppose $L \supseteq F' \supseteq F$ is a tower of fields with L/F' cyclic. Let $L'' \supseteq F'' \supseteq F$ be such that $L'' \subseteq L$ and $F'' \subseteq F'$ are maximal separable extensions of F. Of course, L''/F'' is cyclic and $L = L'' \otimes_{F''} F'$. Furthermore, F' and F'' have equal residue fields with respect to any prime because these residue fields are perfect. Thus it suffices to assume $F' = F''$, or that F'/F is separable. Let $L' \supseteq L$ be the Galois closure of L/F with group G. Choose $\sigma \in G$ such that the image of σ generates the Galois group of L/F'. By Tchebotarev density (e.g., [T, p. 163]), there are infinitely many primes P of R which have extensions Q to L such that L'/F is unramified and the decomposition group $H \subseteq G$ is generated by σ with σ corresponding to the "Frobenius" element in the Galois group of the completions. Then $L \supseteq F' \supseteq F$ is good for such Q. R is a Nagata ring by [M, p. 240] so 2.4 applies and 2.5 is proved. □

Our next example gives another class of Brauer-Hilbertian fields. As mentioned in the introduction, these examples suggest that Brauer-Hilbertian is a kind of 2-dimensional version of the Hilbertian property.

Theorem 2.6. *Let K be a Hilbertian field and F a finite separable extension of the rational function field $K(x_1, \ldots, x_r)$, $r > 0$. Then F is Brauer-Hilbertian.*

Proof. Since $K(x_1, \ldots, x_{r-1})$ is Hilbertian, we may assume F is finite over $K(x)$. If R is the integral closure of $K[x]$ in F, R is Nagata by [M, p. 240]. Let $L \supseteq F' \supseteq F$ be a tower of fields, and $T \subseteq L$ the integral closure of $K[x]$. If $L \supseteq F' \supseteq K(x)$ is good with respect to $Q \subseteq T$, then the same is true of $L \supseteq F' \supseteq F$. If F'/F is simple, then $F'/K(x)$ is simple because $F/K(x)$ is separable. Thus we may assume $F = K(x)$ and $R = K[x]$.

Assume $F' = F(a)$. F' is the field of fractions of $S = K[x, y]/(g(x, y))$ where a is the image of y. The degree of g in y is precisely the degree of a over F. Consider the K map $\phi : K[y] \to S$ defined by $\phi(y) = a$.

Case 1. *ϕ is injective.* Then ϕ induces what we may take to be a tower of fields $K(y) \subseteq F' \subseteq L$. Clearly $F' = K(y)(x)$. As L/F' is separable, $L/K(y)$ is also a simple extension.

Since K is Hilbertian, there are infinitely many $b \in K$ such that specializing y at b yields a tower $K \subseteq \overline{F}' \subseteq \overline{L}$ with $[\overline{F}' : K] = [F' : K(y)]$ and $[\overline{L} : \overline{F}'] = [L : F']$. As there are infinitely many such b, we can assume b is not in any given finite set. Thus we may assume x is integral over $K[y]_b$. If $n = [F' : K(y)]$, we may also assume the image of $1, x, \ldots, x^{n-1}$ in $\overline{F}'$ is linearly independent over K. Thus $\overline{F}' = K(\bar{x})$, where $\bar{x}$ is the image of x. If Q is the unique prime of L over $(y - b)$ then Q lies over a prime P of $K[x]$. We have $K[x]/P = K(\bar{x}) = \overline{F}'$. Thus $L \supseteq F' \supseteq F$ is good with respect to Q and this case is done.

Case 2. *ϕ is not injective.* That is, a is algebraic over K. Write $K' = K(a)$, so $F' = K' \otimes_K F = K'(x)$. Since K' is also Hilbertian, there are infinitely many $b' \in K'$ such that specializing at b' yields $K' \subseteq \overline{L}$ with $[\overline{L} : K'] = [L : F']$. If $Q \subseteq L$ is the unique extension of the prime $(x - b')$ of $K'[x]$, then $F \subseteq F' \subseteq L$ is good with respect to Q. Thus this case and the whole theorem are proved. □

Remark. In finding good primes in 2.6, we never used the fact that L/F' is assumed cyclic. Plugging this fact into the argument of 2.4 we can show: If F is as in 2.6, and $\alpha_1, \ldots, \alpha_s \in \mathrm{Br}(F(t))$, then there are infinitely many $b \in F$ such that $\rho_b(\alpha_i)$ is defined and has order equal to that of α_i for all i.

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