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ANOMALIES ASSOCIATED TO THE POLAR DECOMPOSITION OF $GL(n, \mathbf{C})$

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ABSTRACT. Let D be a selfadjoint elliptic differential operator on a hermitian bundle over a compact manifold. For positive D, the variation of the functional determinant of D under positive definite hermitian gauge transformations is calculated. This corresponds to computing a gauge anomaly in the nonunitary directions of the polar decomposition of the frame bundle $GL(E)$. The variation of the eta invariant for general D is also calculated. If D is not selfadjoint, the integrand in the heat equation proof of the Atiyah-Singer Index Theorem is interpreted as an anomaly for D^*D. In particular, the gauge anomaly for semiclassical Yang-Mills theory is computed.

1. INTRODUCTION

Let $D: \Gamma(E_1) \to \Gamma(E_2)$ be an elliptic differential operator acting on sections of hermitian bundles E_i over a compact Riemannian manifold M. If the E_i are built canonically from a third bundle E, sections of the frame bundle (or complex gauge group) $GL(E)$ of E act on E via conjugation. In various geometric situations, this action naturally arises within one factor of the polar decomposition of $GL(E) = P(E)U(E)$ into positive definite hermitian transformations and unitary transformations. For example, in gauge theory the E_i are bundles of forms with values in $\mathrm{Hom}(E, E)$, and sections of $U(E)$ act on the covariant derivatives d_A associated to a connection A on E. For the complex of differential forms with the Laplacian introduced by Witten [13], $E = T^*M$ and the E_i are the bundles of odd and even forms. In this case and for conformally covariant operators such as the Dirac operator, the action is (roughly) conjugation by positive functions, which form a subbundle $C_+^\infty(M) \subset P(E)$.

In this paper we will compute anomalies (the variation of the functional determinant) and variations of eta invariants for elliptic operators under the action of the complex gauge group. This extends work in [9], in which we showed that the eta invariant $\eta(0)$ and the functional determinant $\exp(-\zeta'(0))$ of conformally covariant elliptic operators are conformal invariants. In particular, in §§2 and 3, this invariance is extended to deformations of operators $D \mapsto \psi_1 D \psi_2$, where the ψ_i are commuting endomorphisms in $P(E)$ (Proposition 2.2, 3.1). Here $E = E_i$ and D is selfadjoint. We also correct an error in the discussion of the eta invariant in [9].

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In §2 we also consider the case $E \neq E_i$ and compute the anomaly for the Laplacian D^*D. By using integral kernels for $\ker D$ and $\ker D^*$, the Atiyah-Singer Index Theorem for D can be easily rewritten as the integral of a certain density involving both local and nonlocal terms. (If the complex is acyclic, only local terms appear.) This density is the anomaly in $P(E)$ directions (Theorem 2.10). We consider this result to be an interpretation of the integrand in the "local" heat equation approach to the Index Theorem. For example, the anomaly for the Dirac Laplacian in $C_+^\infty(M)$ directions is given by the $\widehat{A}$-polynomial as a differential form, provided there are no harmonic spinors.

These calculations are applied to compute the gauge anomaly for the functional determinants which appear in semiclassical Yang-Mills theory. The anomaly for the Faddeev-Papov ghost determinant vanishes for unitary gauge transformations (trivially) or if the base manifold is odd dimensional. Under a slightly altered action of the gauge group, the anomaly for the semiclassical determinant ratio is a refinement of the characteristic form whose integral is the index of the basic elliptic complex in Yang-Mills theory. We do not know how to compute the anomaly in general, but the characteristic form, which gives the anomaly in $C_+^\infty(M)$ directions, is essentially calculated in [1].

We would like to thank Peter Gilkey for several helpful discussions and the referee for pointing our that work in [11] leads to simpler proofs of some of our results.

2. Anomalies for elliptic operators and elliptic complexes

Let E be a hermitian vector bundle over an orientable compact Riemannian manifold M of dimension m and let $D\colon \Gamma(E) \to \Gamma(E)$ be a strictly positive selfadjoint elliptic differential operator of order r acting on smooth sections of E. For example, if $B\colon \Gamma(E) \to \Gamma(F)$ is a positive elliptic differential operator, we can take $D = B^*B$ on $\Gamma(E)$. $L^2(E)$ has an orthonormal basis of D-eigensections $\{\phi_i\}$ of E with positive eigenvalues $\lambda_i \to \infty$. The zeta functions for D are defined by $\zeta(s) = \sum_n \lambda_n^{-s}$ and $\zeta(s, x) = \sum_n \lambda_n^{-s} |\phi_n(x)|_x^2$, where $|\phi_n|_x$ denotes the norm in the fiber E_x for $x \in M$. Note that $\zeta(s) = \int_M \zeta(s, x)\, dx$. These functions converge for $\mathrm{Re}(s) \gg 0$ and have a meromorphic continuation to all of $\mathbf{C}$ with zero a regular value [12]. Whereas $\zeta(0, x)$ is given by an expression in the jets of the total symbol of D and the metrics on E and M at x, $\zeta'(0)$ is not the integral of such a local expression. Recall that $\exp(-\zeta'(0))$ is interpreted as the regularized determinant of D, at least if D has trivial kernel, since it equals the determinant for finite-dimensional linear transformations.

In [9] we considered conformally covariant operators; that is, operators associated to metrics on M which transform via $D \mapsto e^{w_1 f} D e^{w_2 f}$ for some w_1, $w_2 \in \mathbf{R}$ under a conformal transformation of the metric $g \mapsto e^f g$ for $f\colon M \to \mathbf{R}$. In odd dimensions $\zeta'(0)$ was shown to be invariant under such conformal transformations. Moreover, if ρ is an acyclic n-dimensional unitary representation of $\pi_1(M)$ with associated flat bundle V_ρ and if $\zeta_\rho(s)$ is the zeta function for the extension of D to the bundle $E \otimes V_\rho$, then in even dimensions $\zeta'(0) - \frac{1}{n}\zeta_\rho'(0)$ is a conformal invariant.

These results can be interpreted and extended using the polar decomposition of the frame bundle of E. The $\mathrm{GL}(n, \mathbf{C})$-bundle $\mathrm{GL}(E)$ of invertible bundle maps of E splits into the fiberwise product of bundles $P(E)$, $U(E)$ corresponding to the polar decomposition $\mathrm{GL}(n, \mathbf{C}) = P \cdot U$ of $\mathrm{GL}(n, \mathbf{C})$ into positive definite hermitian transformations and unitary transformations. A section of $U(E)$ is a gauge transformation, while multiplication by e^{wf} as above gives a section of $P(E)$. In this paper a section of $\mathrm{GL}(E)$ will be called a gauge transformation. Let X be a connected smooth (or smooth enough) manifold of possibly infinite dimension and let ψ_1, $\psi_2 \colon X \to P(E)$ be maps as smooth as possible with $[\psi_1(x), \psi_2(x)] = 0$ in each fiber of E. Define a family of operators D_p by $D_p = \psi_1(p) D \psi_2(p)$. It is assumed that there exists $p \in X$ such that $\psi_1(p) = \psi_2(p) = \mathrm{Id}$. For example, with $X = C^\infty(M)$, $\psi_i(p)$ could be the functions $e^{w_i p}$ above, but in general D_p need not be associated to a metric on M.

Although D_p is not selfadjoint, it has a zeta function $\zeta(s) = \zeta_p(s)$ with all the nice properties of D's.

Lemma 2.1. *The spectrum of D_p acting on $L^2(E)$ is contained in $\mathbf{R}^+$ and consists of discrete points of finite multiplicity. The spectrum diverges to infinity fast enough for $\zeta(s)$ to be defined for* $\mathrm{Re}(s) \gg 0$. *The zeta function has a meromorphic continuation to* $\mathbf{C}$ *with zero as a regular value.*

Proof. Let λ be in the spectrum of D_p. Equivalently, there exist $\phi_n \in L^2(E)$ with $\|\phi_n\| = 1$ and $\varepsilon_n \to 0$ with $\|D_p\phi_n - \lambda\phi_n\| < \varepsilon_n$. ($\|\cdot\|$ is the L^2 norm on $\Gamma(E)$.) Since ψ_1 and ψ_2 commute, $D_p^* = \psi_1^{-1}\psi_2 D_p \psi_2^{-1}\psi_1$. By considering the approximate eigenfunctions $\psi_1^{-1}\psi_2\phi_n$, we see that λ is in the spectrum of D_p^*. Define error terms $f_n = D_p\phi_n - \lambda\phi_n$ and $g_n = D_p^*\psi_1^{-1}\psi_2\phi_n - \lambda\psi_1^{-1}\psi_2\phi_n$. With respect to the global inner product $\langle\ ,\ \rangle$ on $\Gamma(E)$,

$$\begin{aligned}\lambda\langle\phi_n, \psi_1^{-1}\psi_2\phi_n\rangle + \langle f_n, \psi_1^{-1}\psi_2\phi_n\rangle &= \langle D_p\phi_n, \psi_1^{-1}\psi_2\phi_n\rangle \\ = \langle\phi_n, D_p^*\psi_1^{-1}\psi_2\phi_n\rangle &= \bar{\lambda}\langle\phi_n, \psi_1^{-1}\psi_2\phi_n\rangle + \langle\phi_n, g_n\rangle.\end{aligned}$$

Applying Cauchy-Schwarz to the error terms gives

$$\limsup_{n\to\infty} \lambda\langle\phi_n, \psi_1^{-1}\psi_2\phi_n\rangle = \limsup_{n\to\infty} \bar{\lambda}\langle\phi_n, \psi_1^{-1}\psi_2\phi_n\rangle.$$

$|\langle\phi_n, \psi_1^{-1}\psi_2\phi_n\rangle|$ is bounded by Cauchy-Schwarz. Moreover, since ψ_1 and ψ_2 are simultaneously diagonalizable, $\psi_1^{-1}\psi_2$ is positive definite in each fiber, so there exists a function $c(x) > 0$ on M such that $|\langle\phi_n, \psi_1^{-1}\psi_2\phi_n\rangle| \geq \int_M c(x)|\phi_n(x)|_x^2\,dx > 0$. Thus $\lambda \in \mathbf{R}$. (This paragraph is valid for complete manifolds provided $\psi_i \in L^2(\mathrm{Hom}(E, E))$.)

We now show that λ is a discrete eigenvalue. Since the ψ_i are invertible, D_p is elliptic. By the definition of ϕ_n, they and $D_p\phi_n$ are bounded in the L^2 norm, so by Gårding's inequality the ϕ_n are bounded in the Sobolev space H_r. Thus there exists a section ϕ with $\phi_n \to \phi$ in $H_{r-\delta}$ for any $\delta > 0$. This implies that $D_p\phi_n$ converges to $D_p\phi$ in $H_{-\delta}$ and to $\lambda\phi$ in L^2. Therefore $D_p\phi = \lambda\phi$ in $H_{-\delta}$ and hence by elliptic regularity in C^∞.

In fact, the spectrum of D_p is strictly positive: the map $\omega \mapsto \psi_2^{-1}\omega$ takes the (zero) kernel of D isomorphically to the kernel of D_p, so there is no spectral flow of eigenvalues from positive to nonnegative.

The argument of [6, p. 46] shows that the eigenvalues grow fast enough so that the heat operator e^{-tD_p} is well defined with a smooth kernel $e_p(t, x, y)$

decaying exponentially in t. In particular, the multiplicity of each eigenvalue is finite. The Mellin transform

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(e^{-tD_p})\, dt$$

now defines the zeta function for $\operatorname{Re}(s) \gg 0$. The construction of the asymptotic expansion for $e_p(t, x, s)$ as $t \to 0$ in [6, §1.7] is valid for D_p, since the essential point is to have the spectrum of the operator contained in an interval $[C, \infty]$. The meromorphic continuation of $\zeta(s)$ to $\mathbf{C}$ follows from the existence of the asymptotic expansion.

Remark. For any α, $\beta \in \Gamma(P(E))$, $\alpha D \beta$ has a well-defined eta function (see §3). This gives a map from $\Gamma(P(E)) \times \Gamma(P(E))$ to meromorphic functions on $\mathbf{C}$, and on the connected component of $(\mathrm{Id}, \mathrm{Id})$ the eta function is a zeta function by Lemma 2.1.

We now give two proofs of the invariance of the functional determinant. The first proof, while longer, gives added information about $\zeta(0)$ (Lemma 2.3). The second, more direct proof follows from a key lemma in [11].

To begin the first proof, since the spectrum of D_p is unchanged under conjugation, we may replace D_p by $Q_p = \psi(p) D$, where $\psi(p) = \psi_2^{-1}(p)\psi_1(p)$, to compute spectral invariants.

Proposition 2.2. *Let D_p be defined as above. If the dimension of M is odd, then $\zeta'(0)$ is independent of p. If p is an n-dimensional acyclic representation of $\pi_1(M)$, then $\zeta'(0) - \frac{1}{n}\zeta_\rho'(0)$ is independent of p.*

The first step is to show that the integrated local expression $\zeta(0)$ is invariant for D_p. In contrast, it is shown in §3 that the η-invariant need not be constant on connected components of $\Gamma(P(E)) \times \Gamma(P(E))$.

Lemma 2.3. *Let D_p be as above. Then $\zeta(0)$ is independent of p.*

Proof. For $v \in T_pX$, $\delta\zeta_p(0)(v)$ denotes the variation of $\zeta(0) = \zeta_p(0)$ in the direction v; i.e., $\delta\zeta_p(0)(v) = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} \zeta_{\alpha(\varepsilon)}(0)$, where α is a curve in X with $\alpha(0) = p$, $\alpha'(0) = v$. By the Mellin transform, the variation of $\zeta(0)$ at Q_p is given by

$$\begin{aligned}
\delta\zeta(0) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \delta \operatorname{Tr}(e^{-tQ_p})\, dt \\
&= -\frac{1}{\Gamma(s)} \int_0^\infty t^s \operatorname{Tr}(\delta\psi D e^{-tQ_p})\, dt \\
&= -\frac{1}{\Gamma(s)} \int_0^\infty t^s \operatorname{Tr}(\delta\psi \cdot \psi^{-1} Q_p e^{-tQ_p})\, dt \qquad (2.4)\\
&= \frac{1}{\Gamma(s)} \int_0^\infty t^s \partial_t \operatorname{Tr}(\delta\psi \cdot \psi^{-1} e^{-tQ_p})\, dt \\
&= -s \cdot \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(\delta\psi \cdot \psi^{-1} e^{-tQ_p})\, dt
\end{aligned}$$

with all integrals evaluated at $s = 0$. The commutativity of the operators within the trace is standard. There are no boundary terms in the integration by parts since $\operatorname{Tr}(\delta\psi\psi^{-1}e^{-tQ_p})$ has exponential decay at infinity and is $O(t^{-m/r})$ as $t \to 0$ (see (2.5)). The interchanging of δ and $\int_0^\infty$ also follows from (2.5).

It suffices to show that the last integral has only a simple pole at $s = 0$. As usual, the integral from one to infinity is analytic at $s = 0$, so only the interval from zero to one can contribute a pole. The heat kernel for Q_p has the asymptotic expansion

$$e_p(t, x, x) \sim \sum_{k=0}^{\infty} a_k(x) t^{(k-m)/r} \tag{2.5}$$

with $a_k(x) \in \mathrm{Hom}(E_x, E_x)$. Thus for any $C > 0$ there exists an integer $N \gg 0$ and a function $h(s)$ holomorphic in the half-plane $\mathrm{Re}(s) > -C$ such that

$$\int_0^1 t^{s-1} \operatorname{Tr}(\delta\psi \cdot \psi^{-1} e^{-tQ_p})\, dt = \sum_{k=0}^{N} \frac{\int_M \operatorname{tr}_x(\delta\psi \cdot \psi^{-1} a_k)\, dx}{s + (k-m)/r} + h(s),$$

which has only a simple pole with $k = m$ at $s = 0$.

Proof I *of* 2.2. Since $\Gamma(s)\zeta(s) = [\frac{1}{s} + c + O(s)][\zeta(0) + s\zeta'(0) + O(s^2)]$ near $s = 0$, by the lemma $\delta\zeta'(0) = \delta\Gamma(s)\zeta(s)|_{s=0}$. Repeating the calculation in Lemma 2.3 with the gamma function omitted gives

$$\delta\zeta'(0) = -\int_M \operatorname{tr}_x(\delta\psi \cdot \psi^{-1} a_m)\, dx = -\int_M \operatorname{tr}_x(\delta\psi \cdot \psi^{-1} \zeta(0, x))\, dx$$

since $a_m(x) = \zeta(0, x)$. If m is odd, $a_m(x) \equiv 0$, and for all m, $a_{m,\rho}(x) = n \cdot a_m(x)$, where the $a_{k,\rho}$ are the coefficients in the heat kernel asymptotics for the extension of D_p to $E \otimes V_\rho$.

Proof II *of* 2.2. We recall the following result of Schwarz [11, Lemma 8].

Lemma 2.6. *Let $A_q(p)$ be positive selfadjoint elliptic operators (depending on a parameter $p \in \mathbf{R}$) acting on sections of hermitian bundles over a compact manifold M for $0 \le q \le m$, and let $P_q^A(p)$ denote projection in L^2 onto the kernel of $A_q(p)$. Pick $\lambda_q \in \mathbf{R}$. Assume there exist operators $R_r(p)$ and $T_r(p)$ such that*

$$\frac{d}{dp} \sum_{q=0}^{m} \lambda_q \cdot \operatorname{Tr}(e^{-tA_q(p)} - P_q^A(p)) = t\frac{d}{dt} \sum_{r=0}^{n} \operatorname{Tr}(R_r(p)(e^{-tT_r(p)} - P_r^T(p))).$$

Let $\zeta_q^p(s)$ be the zeta function for $A_q(p)$. If there is an asymptotic expansion of the form

$$\operatorname{tr}_x(R_r(p)(e^{-tT_r(p)} - P_r^T(p))) = \sum_l b_l(R_r(p), T_r(p), x) t^{-l} + O(t^\varepsilon)$$

for some finite set of nonnegative integers $\{l\}$ and some $\varepsilon > 0$, and if

$$\left| \frac{d}{dp} \operatorname{Tr}(R_r(p)(e^{-tT_r(p)} - P_r^T(p))) \right| = O(t^{-N})$$

as $t \to \infty$ for any $N \in \mathbf{Z}^+$, then

$$\frac{d}{dp} \sum_{q=0}^{m} \lambda_q \cdot (\zeta_q^p)'(0) = -\sum_{r=0}^{n} \operatorname{tr}_x(b_0(R_r(p), T_r(p), x)).$$

The lemma follows easily from the explicit construction of the meromorphic continuation of the zeta function for $A = A_q$:

$$\zeta(s) = \Gamma(s)^{-1}\left[\sum_{l=-m/2}^{K} \frac{\int_M \mathrm{tr}_x(b_l)}{s-l} t^l + \int_1^\infty t^{s-1}\,\mathrm{Tr}(e^{-tA} - P^A)\,dt \right.$$
$$\left. + \int_0^1 t^{s-1}\left(\mathrm{Tr}(e^{-tA} - P^A) - \sum_{l=-m/2}^{K}\int_M \mathrm{tr}_x(b_l)\right) dt\right].$$

We apply this lemma by setting $m = 1$, $\lambda_1 = 1$, and $A_q(p) = Q_p$. Then as in (2.4) and (2.5) the hypotheses of the lemma are satisfied with $n = 1$, $R_1 = \delta\psi\psi^{-1}$, and $T_1(p) = Q_p$. Now the last sentence of the first proof yields the proposition.

We now turn to general elliptic complexes and an interpretation of a local version of the Atiyah-Singer Index Theorem as measuring anomalies for certain determinant ratios (more precisely, for certain differences of $\zeta'(0)$'s). This work is motivated by Witten's modification of the Laplacian on forms by a Morse function f [13]:

$$\Delta = dd^* + d^*d \mapsto e^{-f}\, de^{2f}\, d^* e^{-f} + e^f d^* e^{-2f}\, de^f.$$

Given an elliptic complex of hermitian bundles over a compact m-dimensional Riemannian manifold M,

$$\Gamma(\Lambda_0) \xrightarrow{d_0} \Gamma(\Lambda_1) \xrightarrow{d_1} \cdots \xrightarrow{d_{p-1}} \Gamma(\Lambda_p) \tag{2.7}$$

with $p > 1$, there is the associated two-step complex

$$\Gamma(\Lambda_+) \xrightarrow{D} \Gamma(\Lambda_-)$$

formed by setting $\Lambda_+ = \bigoplus \Lambda_{2k}$, $\Lambda_- = \bigoplus \Lambda_{2k+1}$, and $D = \bigoplus(d_{2k} + d^*_{2k-1})$. By the heat equation approach to the Index Theorem,

$$\text{index } D = \dim\ker D - \dim\ker D^* = \int_M \mathrm{tr}_x(a_m^+ - a_m^-), \tag{2.8}$$

where a_m^+ and a_m^- are the coefficients of t^0 in the asymptotic expansion for the heat kernels of D^*D and DD^*, respectively. Thus the top dimensional form $\mathrm{tr}_x(a_m^+ - a_m^-)$ is in the cohomology class of the index polynomial for D. For elliptic operators naturally associated to the metric on M (e.g. Dirac operator, signature operator, $\overline{\partial}$-operator, Gauss-Bonnet operator) this form equals the index polynomial constructed via Chern-Weil theory from the curvature of the metrics.

(2.8) can be rewritten in a more suggestive form by introducing integral kernels $\beta^\pm(x)$ for the orthogonal projections P_+ and P_- onto $\ker D$ and $\ker D^*$. Choose an orthonormal basis $\{\omega_i^\pm\}$ for $\ker D$ and $\ker D^*$, respectively, and set

$$\beta^\pm(x, y) = \sum \omega_i^\pm(x) \otimes \overline{\omega}_i^\pm(y).$$

For $\beta^\pm(x) = \beta^\pm(x, x)$, $\int_M \mathrm{tr}_x\,\beta^+(x) = \dim\ker D$ and similarly for $\beta^-(x)$. $\mathrm{tr}_x\,\beta^\pm(x)$ are called local Betti numbers in [8, §5], although they cannot be computed by information at x alone, whereas $\mathrm{tr}_x\, a_m^{\pm}(x)$ can. In any case, the Index Theorem can be stated as the integral of "local" expressions:

$$\int_M \mathrm{tr}_x[\beta^+(x) - \beta^-(x) - a_m^+(x) + a_m^-(x)] = 0. \tag{2.9}$$

Let $\Gamma(\mathrm{GL}(\Lambda_+)) \cap \Gamma(\mathrm{GL}(\Lambda_-))$ be denoted by $\Gamma(\mathrm{GL}(\Lambda_\pm))$. An element $g \in \Gamma(\mathrm{GL}(\Lambda_\pm))$ acts on D by $g\colon D \mapsto g^{-1}Dg$. The space $\Gamma(\mathrm{GL}(\Lambda_\pm))$ always includes the space of positive functions $C_+^\infty(M)$ but in geometric situations is often larger. For example, for the complex of differential forms ($D = d+d^*$) the action of $C_+^\infty(M)$ extends to an action of $\Gamma(\mathrm{GL}(T^*M))$ acting on differential forms. (This is not Witten's action: here

$$\Delta \mapsto e^f de^{-2f} de^f + e^f de^{-2f} d^* e^f + e^f d^* e^{-2f} de^f + e^f d^* e^{-2f} d^* e^f$$

if $g = e^f$.) Similarly, in the gauge theory example below, $\Lambda_\pm$ are the bundles of even and odd forms with values in $\mathrm{Hom}(E, E)$ and g ranges over $\mathrm{GL}(E)$. Replacing $g^{-1}Dg$ by $g^{w_1}Dg^{w_2}$ for some $w_i \in \mathbf{R}$ and $g \in C_+^\infty(M)$ handles the case of conformally covariant differential operators such as the Dirac operator.

We want to interpret the integrand in (2.9) as an anomaly, i.e., as a measure of the variation of a certain "determinant ratio" associated to the elliptic complex. To be precise, let $\zeta(s)$ be the zeta functions for $\Delta = D^*D$ and define the anomaly for the complex (2.7) to be the gradient vector field of $\zeta'(0)$ on $\Gamma(\mathrm{GL}(\Lambda_\pm))$ with respect to the inner product $\langle \phi, \psi \rangle = \int_M \mathrm{tr}_x(\phi\psi^*)\,dx$ on $\Gamma(\mathrm{GL}(\Lambda_\pm))$. The adjoint ψ^* and tr_x are computed with respect to the hermitian structure on Λ_-. This vector field, which is an element of $\Gamma(\mathrm{Hom}(\Lambda_\pm)) = \Gamma(\mathrm{Hom}(\Lambda_+, \Lambda_+)) \cap \Gamma(\Lambda_-, \Lambda_-))$, vanishes in unitary directions, so it suffices to determine the vector field in positive definite hermitian (p.d.h.) directions. The anomaly $A(x)$ is characterized by the equation

$$d\zeta'(0)(B) = \int_M \mathrm{tr}_x(AB^*)\,dx$$

for all $B \in \Gamma(\mathrm{Hom}(\Lambda_\pm))$. In particular, on $C_+^\infty(M) = \exp(C^\infty(M))$ the anomaly is characterized by

$$d\zeta'(0)(f) = \int_M f \cdot \mathrm{tr}_x A\,dx$$

for all $f \in C^\infty(M)$.

To calculate the anomaly, let $g(p)$ be a curve in $\Gamma(\mathrm{GL}(\Lambda_\pm))$ with $g(0) = \mathrm{Id}$. Set $D_p = g(p)^{-1}Dg(p)$ with associated zeta function $\zeta_p(s)$ for $D_p^*D_p$. Setting $m = 1$, $\lambda_1 = 1$, and $A = A_1(p) = D_p^*D_p$ in Lemma 2.6, we get

$$\frac{d}{dp}\mathrm{Tr}(e^{-tA}) = -t\,\mathrm{Tr}(\delta A e^{-tA}).$$

Since

$$\delta D^*D = \delta(g^*)D^*D + D^*\delta(g^{-1})^*D + D^*\delta g^{-1}D + D^*D\delta g,$$

we find

$$\begin{aligned}\frac{d}{dp}\mathrm{Tr}(e^{-tA}) &= t\,\mathrm{Tr}[(\delta g^* + \delta g)(D^*De^{-tD^*D} - DD^*e^{-tDD^*})]\\ &= t\frac{d}{dt}\mathrm{Tr}[\delta(g^*g)(e^{-tD^*D} - P_+ - e^{-tDD^*} + P_-)].\end{aligned}$$

Here we have used the identities

$$\mathrm{Tr}(D^*Be^{-tD^*D}) = \mathrm{Tr}(Be^{-tD^*D}D^*) = \mathrm{Tr}(BD^*e^{-tDD^*})$$

for any differential operator B, and $\delta g + \delta g^{-1} = 0$, $\delta g^* + \delta(g^*)^{-1} = 0$, which follows from differentiating $gg^{-1} = \mathrm{Id}$, $g^*(g^*)^{-1} = \mathrm{Id}$. The hypothesis of Lemma 2.6 is now satisfied with $n = 2$, $R_1 = R_2 = \delta(g^*g)$, $T_1 = D^*D$, and $T_2 = DD^*$. Thus Lemma 2.6 implies

$$\delta\zeta'(0) = -\int_M \mathrm{tr}_x[\delta(g^*g)(a_m^+ - \beta^+ - a_m^- + \beta^-)]\,dx,$$

where a_k^+ and a_k^- denote coefficients in the asymptotics of the heat kernels of D^*D and DD^*, respectively. If g is a family of p.d.h. transformations passing through the identity, $\delta(g^*g) = 2\delta g$. Since the family g is selfadjoint, so is δg. Thus the gradient of $\zeta'(0)$ is $-2(a_m^+ - a_m^- - \beta^+ - \beta^-)$ in p.d.h. directions.

Theorem 2.10. *The anomaly for the elliptic complex* (2.7) *is zero in unitary directions and is* $-2(\beta^+(x) - \beta^-(x) - a_m^+(x) + a_m^-(x))$ *is positive definite hermitian directions. In particular, if the elliptic complex is acyclic and the dimension of the base manifold is odd, the anomaly vanishes.*

Examples. (i) Let $\partial\!\!\!/ : S_+ \to S_-$ be the Dirac operator from plus to minus spinors on an even-dimensional spin manifold. The elliptic anomaly for $\zeta'(0)$ for the Dirac Laplacian $\partial\!\!\!/^*\partial\!\!\!/$ in p.d.h. directions is $-2(\beta_+(x) - \beta_-(x) - \hat{a}(x))$, where $\mathrm{tr}_x\,\hat{a}(x)$ is the $\widehat{A}$-polynomial at x. In particular, if the base metric has positive scalar curvature, the anomaly on $C_+^\infty(M)$ is twice the $\widehat{A}$-polynomial as a differential form (cf. [8, §5] for a different interpretation of the $\widehat{A}$-polynomial as an anomaly).

(ii) The elliptic anomaly on $C_+^\infty(M)$ for the Laplacian on even forms on M is $-2(\beta_+(x) - \beta_-(x) - \mathrm{tr}_x\,E(x))$, where $\mathrm{tr}_x\,E(x)$ is the Euler form on M. If ρ is an acyclic unitary representation of $\pi_1(M)$, the anomaly on $C_+^\infty(M)$ for forms with values in the associated flat bundle is $-2(\dim\rho)\,\mathrm{tr}_x\,E(x)$. Thus the Euler form measures the variation of $\zeta'(0)$ on even forms under our modification of Witten's deformation of the Laplacian.

Remark. Theorem 2.10 is similar in spirit to [11, Theorem 1]. However, we can obtain a bit more information by reproving 2.10 along the lines of the first proof of Proposition 2.2. In particular, the zeta functions for D^*D and DD^* satisfy $\delta\zeta(0) = 0$, i.e., the "dimension anomaly" vanishes in all directions.

We now apply these techniques to semiclassical Yang-Mills theory. Let $\Lambda^p(E)$ denote the space of p-forms with values in the bundle $\mathrm{Hom}(E, E)$. For a connection A on E with curvature $F = F_A \in \Lambda^2(E)$, the classical Yang-Mills functional is $\int_M |F|_x^2\,dx$. The norm $|\cdot|_x$ on $\Lambda_x^2(E)$ is induced by the metrics on M and E; if $\dim M = 4$, this norm is just $|F \wedge *F|_x^2$. A induces covariant derivatives $d_A^p\colon \Lambda^p(E) \to \Lambda^{p+1}(E)$ with d_A^0 (also denoted A) the natural extension of A to a connection on $\mathrm{Hom}(E, E)$. Let $\mathscr{A}$ denote the space of connection on E, $\mathscr{G} = \mathrm{Aut}(E)$ the group of (unitary) gauge transformations, and $\mathscr{M} \subset \mathscr{A}/\mathscr{G}$ the moduli space of self-dual connections.

If $\mathbf{\Phi}\colon \mathscr{A} \to \mathbf{R}$ is a gauge invariant function, the partition function for the quantized theory

$$Z(\mathbf{\Phi}) = \int_{\mathscr{A}} \mathbf{\Phi} \exp\left(-\int |F|^2/\hbar\right) dA$$

formally reduces to

$$\int_{\mathscr{A}/\mathscr{G}} \Phi \exp\left(-\int |F|^2/\hbar\right) \sqrt{\det\Delta}\, dv$$

up to a normalizing constant, where $\Delta = A^*A = (d_A^0)^* d_A^0$ and dA and dv are formal measures on $\mathscr{A}$ and $\mathscr{A}/\mathscr{G}$. Δ is called the Faddeev-Papov ghost determinant. Assume $\dim M = 4$. Define $\Lambda^2_-(E) \subset \Lambda^2(E)$ to be the space of anti-self-dual forms (i.e., $*\omega = -\omega$) and set P_- to be the projection from $\Lambda^2(E)$ to $\Lambda^2_-(E)$. In the semiclassical approximation one lets $\hbar \to 0$ and formally applies stationary phase to the last integral to obtain

$$Z(\Phi) \sim \exp\left(-\frac{k}{\hbar}\right) \int_{\mathscr{M}} \Phi \sqrt{\frac{\det\Delta}{\det\Delta_-}}\, dm$$

for some constant k. Here $\Delta_- = (P_- d_A^1)(P_- d_A^1)^*$ and dm denotes the natural metric on the space of self-dual connections. Since the moduli space is often finite dimensional, the last integral at least makes sense. For details, see [7; 10, §7 and Appendix II; 11, §5].

In the discussion above, we have used that at a point $[\mathscr{A}]$ of $\mathscr{A}/\mathscr{G}$ or of $\mathscr{M}$ the functional determinants may be computed at any connection A within the equivalence class $[\mathscr{A}]$; in other words, there is no gauge anomaly. For under the gauge transformation $g \in \mathscr{G}$, A transforms to $g^{-1}Ag$ and Δ transforms to $g^{-1}\Delta g$, so $\det\Delta = \det g^{-1}\Delta g$.

In gauge theory one usually mods out only by the unphysical (i.e., length preserving) unitary gauge transformations. If the definition of a gauge transformation is relaxed to include all sections of $\mathrm{GL}(E)$, anomalies for both determinants may be present. The Laplacian Δ and Δ_- are associated to the basic elliptic complex

$$\Lambda^0(E) \xrightarrow{A} \Lambda^1(E) \xrightarrow{P_- d_A^1} \Lambda^2_-(E)$$

of [1]. Since one sets the determinant to be zero if the operator has nonzero kernel, we may assume $\ker A = \ker(P_- d_A^1)^* = 0$. According to Theorem 2.10, there is no gauge anomaly for the ghost determinant if the dimension of M is odd. In dimension four, the anomaly for the action $A \mapsto g^{-1}Ag$ for the logarithm of the ratio $\sqrt{\det\Delta/\det\Delta_-}$ is not calculable by our methods. However, if we define the action of the extended gauge group to be

$$A \mapsto g^{-1}Ag, \qquad d_A^1 \mapsto g d_A^1 g^{-1} \tag{2.11}$$

on covariant derivatives, then this action is consistent with the usual action of the unitary gauge transformations as far as determinant calculations are concerned. Since the roles of g and g^{-1} have been switched in the action on d_A^1, the variation of $\ln\det\Delta_-$ changes sign. Now the anomaly for the determinant ratio agrees with the anomaly for $B = A + (P_- d_A^1)^* \colon \Lambda^0(E) \oplus \Lambda^2(E) \to \Lambda^1(E)$ in Theorem 2.10, since $B^*B = \Delta \oplus \Delta_-$. By (2.10), under the action (2.11)

$$\delta \ln\sqrt{\det\Delta/\det\Delta_-} = \int_M \mathrm{tr}_x[\delta(g^*g)(\mathscr{T}(x) - \beta^-(x))]\, dx,$$

with $\mathscr{T}(x)$ the difference of the zeroth order terms in the asymptotics of the heat kernels of e^{-tB^*B} and e^{-tBB^*} and β^- the integral kernel for $\ker B^*$.

Up to bundle isomorphism, B is a Dirac operator coupled to the self-dual connection A [1]. Therefore, if we assume for simplicity that $\ker B^* = 0$, then $\mathrm{tr}_x \mathscr{I}(x)$, the anomaly for the determinant ratio on $C_+^\infty(M)$, is the index polynomial for the coupled Dirac operator computed as a differential form via Chern-Weil theory. This polynomial as a characteristic class is determined by the Index Theorem in [1] as follows: let $p_1(E)$ equal the first Pontrjagin class of E and let χ and τ denote the Euler characteristic and the signature of M, respectively. Then

$$\int_M \mathrm{tr}_x \mathscr{I}(x) = p_1(E) - \frac{1}{2}(\dim E)^2(\chi - \tau).$$

$\mathrm{tr}_x \mathscr{I}(x)$ as a differential form is independent of the bundle isomorphism above, since the isomorphism cancels in the trace.

Corollary 2.13. *Assume that the kernels A, A^*, $P_- d_A^1$, and $(P_- d_A^1)^*$ all vanish. Under the action* (2.11)*, the gauge anomaly on $C_+^\infty(M)$ for the logarithm of the determinant ratio $\sqrt{\det\Delta/\det\Delta_-}$ in the semiclassical approximation to Yang-Mills theory is given by $\mathrm{tr}_x \mathscr{I}(x) = p_1(E)(x) - \frac{1}{2}(\dim E)^2(\chi - \tau)(x)$ computed as a differential form in the curvature of M and the curvature of the self-dual connection.*

It would be interesting to identify $\mathscr{I}(x)$ itself as an endomorphism of E_x.

As an example, let P be a $U(n)$ principal bundle over M and let E be the vector bundle associated to the adjoint representation of $U(n)$. $\ker A = 0$ in this case, and $\ker(P_- d_A^1)^* = 0$ if M's metric has positive scalar curvature [1]. $\int_M \mathscr{I}(x)$ equals the dimension of the moduli space of self-dual connections if it is nonempty. If $\ker B^* = 0$, the moduli space consists of at most isolated points. Nevertheless, the index polynomial will be nontrivial in general, so the determinant ratio has a nontrivial anomaly at each of these points.

3. $\eta(0)$

We now drop the assumption that D be positive and define eta functions $\eta(s) = \sum_n \mathrm{sgn}(\lambda_n)|\lambda_n|^{-s}$ and $\eta(s, x) = \sum_n \mathrm{sgn}(\lambda_n)|\lambda_n|^{-s}|\phi_n(x)|_x^2$ with the sum taken over the nonzero spectrum. Let D_p be as in §2. We also drop the assumption that $D_p = D$ for some p. The discussion in [6, p. 46] shows that $e^{-tD_p^2}$ is well defined with a smooth kernel. Since

$$\eta(s) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2}\, \mathrm{Tr}(D_p e^{-tD_p^2})\, dt,$$

the eta function for D_p converges for $\mathrm{Re}(s) \gg 0$ and has a meromorphic continuation to $\mathbf{C}$ [2].

In this section we will show the invariance of the eta invariant $\eta(0)$ of D_p for $m+r$ odd and produce a counterexample to the invariance for $m+r$ even.

For $m+r$ odd, $\eta(0)$ reduced mod $\mathbf{Z}$ is a homotopy invariant for arbitrary families of differential operators. This follows from [2, Proposition 2.12] and a parity count for the appropriate asymptotics. For the family of operators D_p, the lack of spectral flow implies that $\eta(0)$ itself is invariant. However, we need an explicit calculation of the invariance to construct the counterexample. Note that the methods of [11] used in §2 do not apply to the eta invariant.

Proposition 3.1. *Let D_p be as above. If $m+r$ is odd, then $\eta(0)$ is independent of p.*

Proof. Since

$$\eta(s) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}(Q_p e^{-tQ_p^2})\, dt,$$

we have

$$\begin{aligned}
\delta\eta(0) &= \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} [\operatorname{Tr}(\delta\psi \cdot \psi^{-1} Q_p e^{-tQ_p^2}) \\
&\qquad - 2t \operatorname{Tr}(Q_p^2 \delta\psi \cdot \psi^{-1} Q_p e^{-tQ_p^2})]\, dt \\
&= \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}(\delta\psi \cdot \psi^{-1} Q_p e^{-tQ_p^2})\, dt \\
&\quad + \frac{2}{\Gamma((s+1)/2)} \int_0^\infty t^{(s+1)/2} \partial_t \operatorname{Tr}(\delta\psi \cdot \psi^{-1} Q_p e^{-tQ_p^2})\, dt \\
&= \frac{-s}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}(\delta\psi \cdot \psi^{-1} Q_p e^{-tQ_p^2})\, dt \\
&= \frac{-s}{\Gamma((s+1)/2)} \int_0^1 t^{(s-1)/2} \operatorname{Tr}(\delta\psi \cdot \psi^{-1} Q_p e^{-tQ_p^2})\, dt
\end{aligned}$$

at $s = 0$. The local trace $\operatorname{tr}_x(Q_p e^{-tQ_p^2})$ has an asymptotic expansion

$$\sum_{k\geq 0} b_k(x) t^{(k-2r-m)/2}$$

[6, p. 59; 9], so

$$\delta\eta(0) \sim \sum_k \frac{-s}{\Gamma((s+1)/2)} \int_0^1 \int_M t^{(s-1)/2} \operatorname{tr}_x(\delta\psi \cdot \psi^{-1} b_k(x)) t^{(k-2r-m)/2r}$$

at $s = 0$. Thus

$$\delta\eta(0) = \pi^{-1/2} \int_M \operatorname{tr}_x(\delta\psi \cdot \psi^{-1} B(x))\, dx, \tag{3.2}$$

where $B(x) = b_{r+m}(x)$ is the coefficient of $k^{-1/2}$ in the asymptotics. Since D_p is a *differential* operator with real spectrum, a standard parity count shows that $B(x) \equiv 0$ if $r+m$ is odd.

The coefficient $B(x)$ is familiar from another context. Separating the integral in

$$\eta_p(s, x) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \operatorname{tr}_x(Q_p e^{-tQ_p^2})\, dt$$

into $\int_0^1 + \int_1^\infty$ and substituting the asymptotic expression for the local trace into the first integral, one obtains for fixed $N \in \mathbb{Z}^+$,

$$\eta_p(x) = \frac{1}{\Gamma((s+1)/2)} \sum_{k=0}^N b_k(x)(s+1/2) + (k-2r-m)/2r + \frac{1}{\Gamma((s+1)/2)} h(s),$$

where $h(s)$ is a holomorphic function of s. In particular, $B(x)$ is the residue of the local eta function at $s = 0$.

It is a deep result of Atiyah-Patodi-Singer [2] and Gilkey [4] that $\eta(0)$ is a regular value of the eta function, i.e., the residue $\int_M B(x)$ of the global eta function is zero. In contrast, Gilkey has given examples of differential operators Q with $B(x_0) \neq 0$ for most $x_0 \in M$ [3]. If $\psi(p)$ is multiplication by a positive function pf for $p \in \mathbf{R}^+$, then f may be chosen so that $\mathrm{tr}_x(\delta\psi\psi^{-1}\cdot)$ is arbitrary close to a delta function at x_0. Thus the right-hand side of (3.2) will be nonzero for this family of operators.

In [9] it was claimed that the eta invariant $\eta(0)$ is an invariant for conformally covariant operators. While the argument was correct if $m+r$ is odd, the discussion for $m+r$ even must fail because of the example just given. In fact, the proof of [9, Theorem 3.9] is incorrect, since the operator $A \# B$ considered there fails to be pseudo-differential (see [6, §4.3]).

Professor Gilkey has pointed out that the local residue $B(x)$ does vanish identically if the family Q_p is naturally associated to the metric, i.e., in the case P is the set of metrics on M. Thus the results in [9] for specific geometric operators such as the Dirac operator and the conformal Laplacian are correct with this additional fact.

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