TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 334, Number 2, December 1992

ON THE STRUCTURE OF TWISTED GROUP C^*-ALGEBRAS

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ABSTRACT. We first give general structural results for the twisted group algebras $C^*(G, \sigma)$ of a locally compact group G with large abelian subgroups. In particular, we use a theorem of Williams to realise $C^*(G, \sigma)$ as the sections of a C^*-bundle whose fibres are twisted group algebras of smaller groups and then give criteria for the simplicity of these algebras. Next we use a device of Rosenberg to show that, when Γ is a discrete subgroup of a solvable Lie group G, the K-groups $K_*(C^*(\Gamma, \sigma))$ are isomorphic to certain twisted K-groups $K^*(G/\Gamma, \delta(\sigma))$ of the homogeneous space G/Γ, and we discuss how the twisting class $\delta(\sigma) \in H^3(G/\Gamma, \mathbb{Z})$ depends on the cocycle σ. For many particular groups, such as $\mathbb{Z}^n$ or the integer Heisenberg group, $\delta(\sigma)$ always vanishes, so that $K_*(C^*(\Gamma, \sigma))$ is independent of σ, but a detailed analysis of examples of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$ shows this is not in general the case.

In recent years the twisted group C^*-algebras associated to a locally compact group G and a multiplier σ on G have attracted a great deal of attention. These algebras were introduced in [2] in connection with a Mackey-style analysis of the representation theory of group extensions, and more recently have appeared, for example, as models of the noncommutative differential geometry of Connes [9] and in mathematical physics [5]. Here we shall prove some new results concerning the structure of twisted group algebras and their K-theory. In particular, we shall discuss the twisted group algebras of a discrete cocompact subgroup Γ of a solvable Lie group G, and compute their K-theory in terms of the twisted K-theory of the homogeneous space G/Γ.

To analyse the structure of $C^*(G, \sigma)$, we first choose a normal subgroup N and decompose $C^*(G, \sigma)$ as $C^*(G, C^*(N, \operatorname{Res} \sigma), \mathscr{T})$, a twisted covariance algebra in the sense of Green [17]. When N is central, G is amenable and $\sigma = \operatorname{Inf} \omega$ is inflated from a multiplier ω of G/N, a theorem of Williams [57] then allows us to realize $C^*(G, \sigma)$ as the algebra of sections of a C^*-bundle over $\widehat{N}$ whose fibres are twisted group algebras of the smaller group G/N. Even when $\sigma = 1$ this yields interesting results: by applying our theorem to an appropriate universal extension of G, we can often assemble the twisted group algebras of G as the fibres of a C^*-bundle. For $G = \mathbb{Z}^2$, this result is well known (e.g. [14, 1]), and for discrete amenable G it has recently been established by Rieffel [50].

When there is no suitable central subgroup to which we can apply this analy-

Received by the editors October 26, 1989 and, in revised form, September 19, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22D25, 46C80.
This research was supported by the Australian Research Grants Scheme.
The first author was partially supported by NSF grant IMS-8610730(1).

sis, we can alternatively pick an abelian normal subgroup N and use theorems of Green [17] and Gootman-Rosenberg [16] to give criteria for simplicity of $C^*(G, \sigma)$. This works particularly well for nilpotent groups, which always have large abelian subgroups. Our theorems apply to a variety of Heisenberg groups, and to semidirect products of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$, generalising and complementing other work of the first author [36, 37].

Much of the interest in twisted group algebras has concerned their K-theory; in particular, partly because

$$K_*(C^*(\mathbb{Z}^n, \sigma)) \cong K_*(C^*(\mathbb{Z}^n)) \cong K^*(\mathbb{T}^n)$$

for any multiplier σ of $\mathbb{Z}^n$, the algebras $C^*(\mathbb{Z}^n, \sigma)$ have been called "noncommutative tori" and have been extensively studied (e.g., [14, 11, 6, 49]). In a similar spirit, it was shown in [36] that the twisted group algebras $C^*(\Gamma, \sigma)$ of the integer Heisenberg group Γ have the same K-groups as $C^*(\Gamma)$, and these can be computed using a device of Rosenberg [52]. His idea is to embed a discrete group Γ in a solvable Lie group, and then use Green's imprimitivity theorem [18] and the Thom isomorphism of Connes [10] to obtain

$$K_*(C^*(\Gamma)) \cong K_*(C_0(G/\Gamma) \rtimes G) \cong K^{*+\dim G}(G/\Gamma).$$

We shall show by a similar trick that $K_*(C^*(\Gamma, \sigma))$ is isomorphic to the K-theory of an induced C^*-algebra $B(\sigma)$ with spectrum G/Γ. If

$$B(\sigma) \cong C_0(G/\Gamma, \mathscr{K}),$$

we have $K_*(B(\sigma)) = K^*(G/\Gamma)$, but in general $B(\sigma)$ is a continuous-trace algebra with possibly nonzero Dixmier-Douady class $\delta \in \check{H}^3(G/\Gamma, \mathbb{Z})$, and $K_*(B(\sigma))$ is the twisted K-theory $K^*(G/\Gamma, \delta)$ studied in [12, 51, 53]. The map $\sigma \to \delta(B(\sigma))$ can be identified with a homomorphism $\delta\colon H^2(\Gamma, \mathbb{T}) \to \check{H}^3(G/\Gamma, \mathbb{Z})$ studied in [56, §1 and 44, §4], and we shall describe the range and kernel of δ by showing that it fits into an exact sequence

$$H^2(\Gamma, \mathbb{R}) \longrightarrow H^2(\Gamma, \mathbb{T}) \xrightarrow{\delta} \check{H}^3(G/\Gamma, \mathbb{Z}) \longrightarrow \check{H}^3(G/\Gamma, \mathbb{R}).$$

It follows immediately that $\delta = 0$ for many groups Γ, such as $\mathbb{Z}^n$ or the integer Heisenberg group, although we shall give an example of a group $\Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}$ and a multiplier $\sigma \in Z^2(\Gamma, \mathbb{T})$ such that $\delta(\sigma) \neq 0$ and $K_*(C^*(\Gamma, \sigma)) \ncong K^*(G/\Gamma) = K_*(C^*(\Gamma))$.

The structure theorems for $C^*(G, \sigma)$ are the contents of our first section. These depend on deep results of Green [17], and in an appendix at the end we have spelled out in detail the connection between our twisted group C^*-algebras and his twisted covariance algebras. We also depend crucially on a description of the group $H^2(G, \mathbb{T})$ of multipliers in terms of the groups $H^2(N, \mathbb{T})$ and $H^2(G/N, \mathbb{T})$. For discrete G, the required facts are all easy consequences of the Lyndon-Hochschild-Serre spectral sequence, but the corresponding results for locally compact G do not appear to be so well known, and we therefore present these in a second appendix. In §2, we discuss the computation of $K_*(C^*(\Gamma, \sigma))$ as the twisted K-theory of an orbit space, as described above.

In our final section we discuss semidirect products of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$. We explicitly write down all the multipliers, using the approach described in Appendix 2, and give substantially new criteria for the simplicity of $C^*(\Gamma, \sigma)$.

We then characterise the multipliers of Γ which are the exponentials of cocylces in $Z^2(\Gamma, \mathbb{R})$, and hence find a specific group $\Gamma = Z^3 \rtimes \mathbb{Z}$ and a multiplier $\sigma \in Z^2(\Gamma, \mathbb{T})$ for which $\delta(\sigma) \neq 0$. Finally, we establish, using results of Rieffel [48] and the Pimsner-Voiculescu exact sequence, that for this example $K_*(C^*(\Gamma, \sigma)) \not\cong K_*(C^*(\Gamma))$.

Notation. If A is a C^*-algebra, $M(A)$ will denote its multiplier algebra and $U(A)$ the group of unitary elements of A; for $u \in U(A)$ or $UM(A)$, Ad u is the automorphism $a \to uau^*$ of A. All actions of groups on A will be pointwise norm continuous. We shall denote by 1 the identity element of A or $M(A)$, and by id the identity mapping between algebras.

Every locally compact group G we consider will be second countable, and $H^n(G, M)$ will denote the Moore cohomology groups of G with coefficients in a Polish G-module [31]; we write $Z^n(G, M)$ for the group of Borel cocycles, and we shall suppose they satisfy the appropriate cocycle identity everywhere rather than almost everywhere. We shall also refer to a cocycle $\sigma \in Z^2(G, \mathbb{T})$ as a multiplier of G. Since our groups are second countable, for any closed subgroup H there is a Borel section $c\colon G/H \to H$ for the orbit map; we shall often write $c(s)$ for $c(sH)$, and all our sections will satisfy $c(H) = e$.

1. General structure theorems

The starting point of our analysis of $C^*(G, \sigma)$ is a decomposition of $C^*(G, \sigma)$ relative to a normal subgroup N of G, and before stating it we need to introduce some notation. Given $\sigma \in Z^2(G, \mathbb{T})$, Res σ will denote its restriction to $N \times N$, and $\tilde{\sigma}$ will denote the antisymmetrised form of σ, defined by

$$\tilde{\sigma}(r, s) = \sigma(r, s)\sigma(s, s^{-1}rs)^{-1} \quad \text{for } r, s \in G.$$

Elementary but messy calculations using the cocycle identity show that

$$\tilde{\sigma}(r, s)\tilde{\sigma}(s^{-1}rs, t) = \tilde{\sigma}(r, st), \tag{1.1}$$

$$\tilde{\sigma}(rs, t) = \sigma(r, s)^{-1}\sigma(t^{-1}rt, t^{-1}st)\tilde{\sigma}(r, t)\tilde{\sigma}(s, t). \tag{1.2}$$

(Note that (1.2) is slightly different from the corresponding equation (iii) in [20, Proposition 1.2].) For the definition of G_σ, see Appendix 1.

Proposition 1.1. *Suppose G is a second countable locally compact group and N a closed normal subgroup. Let $\sigma \in Z^2(G, \mathbb{T})$. Then $C^*(G, \sigma)$ is isomorphic to the twisted covariance algebra $C^*(G_\sigma, C^*(N, \text{Res } \sigma), \mathscr{T})$ where G_σ acts on $C^*(N, \text{Res } \sigma)$ by*

$$\alpha_{(z,r)}(f)(n) = \Delta_{G,N}(r)\tilde{\sigma}(r^{-1}nr, r^{-1})f(r^{-1}nr) \quad \textit{for } f \in L^1(N),$$

and $\mathscr{T}\colon N_\sigma \to UM(C^*(N, \text{Res } \sigma))$ is given by

$$(\mathscr{T}(z, n)f)(m) = z\sigma(n, n^{-1}m)f(n^{-1}m) \quad \textit{for } (z, n) \in N_\sigma,\ f \in L^1(N).$$

Proof. We identity $C^*(G, \sigma)$ with $C^*(G_\sigma, \mathbb{C}, \mathscr{T}_\sigma)$, as in Appendix 1. Then [17, Proposition 1] implies that

$$C^*(G_\sigma, \mathbb{C}, \mathscr{T}_\sigma) \cong C^*(G_\sigma, C^*(N_\sigma, \mathbb{C}, \mathscr{T}_\sigma), \mathscr{T}),$$

where G_σ acts on $C_c(N_\sigma, \mathbb{C}, \mathscr{T}_\sigma)$ by

$$\begin{aligned}\alpha_{(z,r)}(f)(w, n) &= \Delta_{G_\sigma, N_\sigma}((z, r))f((z, r)^{-1}(w, n)(z, r)) \\ &= \Delta_{G,N}(r)f(w\sigma(r^{-1}, r)^{-1}\sigma(r^{-1}, n)\sigma(r^{-1}n, r), r^{-1}nr) \\ &= \Delta_{G,N}(r)\sigma(r^{-1}, r)\sigma(r^{-1}, n)^{-1}\sigma(r^{-1}n, r)^{-1}f(w, r^{-1}nr) \\ &= \Delta_{G,N}(r)\tilde{\sigma}(r^{-1}nr, r^{-1})f(w, r^{-1}nr),\end{aligned}$$

and the twist $\mathscr{T}$ on N_σ is given by

$$\begin{aligned}(\mathscr{T}(z, n)f)(w, m) &= f((z, n)^{-1}(w, m)) \\ &= f(z^{-1}\sigma(n, n^{-1})^{-1}w\sigma(n^{-1}, m), n^{-1}m) \\ &= z\sigma(n, n^{-1})\sigma(n^{-1}, m)^{-1}f(w, n^{-1}m) \\ &= z\sigma(n, n^{-1}m)_f(w, n^{-1}m).\end{aligned}$$

The result follows by restricting these formulas to $\{1\} \times N$ and using the results in Appendix 1.

Our initial strategy for analysing $C^*(G, \sigma)$ is to apply this decomposition with N a large abelian subgroup for which $\operatorname{Res} \sigma$ is trivial, so that

$$C^*(N, \operatorname{Res} \sigma) \cong C^*(N) \cong C_0(\widehat{N}).$$

If further G is amenable and the action of G_σ on $\widehat{N}$ is trivial, then we use a theorem of Williams [57] to deduce that $C^*(G_\sigma, C_0(\widehat{N}), \mathscr{T})$ is given by a C^*-bundle over $\widehat{N}$. Note that this result is interesting even when the multiplier σ is trivial: for we can then take $N = Z(G)$, and we obtain a decomposition of $C^*(G)$ as the algebra of sections of a C^*-bundle over $\widehat{Z}$.

Theorem 1.2. *Let G be an amenable second countable locally compact group and $\sigma \in Z^2(G, \mathbb{T})$. Suppose that N is a closed central subgroup of G such that $\tilde{\sigma}(n, s) = 1$ for all $n \in N$, $s \in G$. Let $c\colon G/N \to G$ be a Borel section, and for $\gamma \in \widehat{N}$ define $d_2(\gamma) \in Z^2(G/N, \mathbb{T})$ by*

$$d_2(\gamma)(sN, tN) = \gamma(c(s)c(t)c(st)^{-1}).$$

Then σ is equivalent to $\operatorname{Inf} \omega$ for some $\omega \in Z^2(G/N, \mathbb{T})$, and $C^*(G, \sigma)$ is isomorphic to the section algebra $\Gamma_0(E)$ of a C^*-bundle E over $\widehat{N}$ with fibre $E_\gamma \cong C^*(G/N, \omega d_2(\gamma))$.

Proof. Proposition A2 implies that σ is equivalent to a cocycle of the form $\operatorname{Inf} \omega$. Proposition 1.1 says that $C^*(G, \sigma)$ is isomorphic to $C^*(G_\sigma, C^*(N), \mathscr{T})$ where G_σ acts trivially on $C^*(N)$ because N is central and $\tilde{\sigma}(n, \cdot) = 1$ for $n \in N$. If I_γ denotes the kernel of the representation

$$f \to \hat{f}(\gamma) = \int f(n)\gamma(n)\, dn$$

of $C^*(N)$, then $\gamma \to I_\gamma$ is a homeomorphism of $\widehat{N}$ onto $\operatorname{Prim} C^*(N)$, and Theorem 2.1 of [57] therefore implies that $C^*(G_\sigma, C^*(N), \mathscr{T})$ is given by a C^*-bundle E over $\widehat{N}$ with fibre E_γ isomorphic to $C^*(G_\sigma, C^*(N)/I_\gamma, \mathscr{T}_\gamma)$, where $\mathscr{T}_\gamma$ is the composition of $\mathscr{T}$ with the canonical map of $M(C^*(N))$ into $M(C^*(N)/I_\gamma)$. Of course, $f \to \hat{f}(\gamma)$ induces an isomorphism of $C^*(N)/I_\gamma$

onto $\mathbb{C}$, and composing $\mathscr{T}_\gamma$ with this gives a twisting map $\mathscr{T}_\gamma'\colon N_\sigma \to \mathbb{T} = U(\mathbb{C})$; computations show that

$$\mathscr{T}_\gamma'(z, n) \cdot \hat{f}(\gamma) = (\mathscr{T}_\gamma(z, n)f)^\wedge(\gamma) = z\gamma(n)\hat{f}(\gamma).$$

If we now define $d\colon G_\sigma/N_\sigma = G/N \to G_\sigma$ by $d(sN) = (1, c(sN))$, then by Proposition A1 we have $C^*(G_\sigma, \mathbb{C}, \mathscr{T}_\sigma') \cong C^*(G/N, \omega_\gamma)$, where

$$\begin{aligned}\omega_\gamma(sN, tN) &= \mathscr{T}_\gamma'(d(s)d(t)d(st)^{-1})\\ &= \mathscr{T}_\gamma'(\sigma(c(s), c(t)), c(s)c(t)c(st)^{-1})\\ &= \sigma(c(s), c(t))\gamma(c(s)c(t)c(st)^{-1})\\ &= \omega(sN, tN)d_2(\gamma)(sN, tN).\end{aligned}$$

This establishes the result.

Corollary 1.3. *Let G be a discrete amenable group. Then there is a compact Hausdorff topology on $H^2(G, \mathbb{T})$ and a C^*-bundle E over $H^2(G, \mathbb{T})$ with fibres $E_\omega \cong C^*(G, \omega)$ for $\omega \in H^2(G, \mathbb{T})$.*

Proof. (The first part of our discussion follows that in [30, §III.2].) The group $Z^2(G, \mathbb{T})$ of cocycles is compact in the topology of pointwise convergence (i.e., the product topology), and $B^2(G, \mathbb{T})$ is a closed subspace, so the quotient $H^2(G, \mathbb{T})$ is also a compact abelian group. Since $C^1(G, \mathbb{T}) = \mathbb{T}^G$, $C^1(G, \mathbb{T})^\wedge$ is a free abelian group, and so is its subgroup $B^2(G, \mathbb{T})^\wedge$; thus the dual sequence of

$$0 \longrightarrow B^2(G, \mathbb{T}) \longrightarrow Z^2(G, \mathbb{T}) \longrightarrow H^2(G, \mathbb{T}) \longrightarrow 0$$

splits, and hence the sequence itself splits. This means there is a continuous section $b\colon H^2(G, \mathbb{T}) \to Z^2(G, \mathbb{T})$, and then

$$\sigma(s, t)(\beta) = b(\beta)(s, t) \tag{1.3}$$

defines a 2-cocycle with values in $H^2(G, \mathbb{T})^\wedge$. We let H be the extension of G by $H^2(G, \mathbb{T})^\wedge$ defined by the cocycle σ—that is, H is $H^2(G, \mathbb{T})^\wedge \times G$ with product defined by

$$(\gamma, s)(\chi, t) = (\gamma\chi\sigma(s, t), st).$$

We can easily compute the homomorphism $d_2\colon H^2(G, \mathbb{T}) = H^2(G, \mathbb{T})^{\wedge\wedge} \to H^2(G, \mathbb{T})$ corresponding to the extension H : define a section $c\colon G \to H$ by $c(s) = (1, s)$, and then for $\omega \in H^2(G, \mathbb{T})$ we have

$$d_2(\omega)(s, t) = \hat{\hat{\omega}}(c(s)c(t)c(st)^{-1}) = \sigma(s, t)(\omega) = b(\omega)(s, t),$$

so d_2 is the identity. Now by applying Theorem 1.2 to the group H and the central subgroup $H^2(G, \mathbb{T})^\wedge$, we obtain a realisation of $C^*(H)$ as the algebra of sections of a C^*-bundle E over $H^2(G, \mathbb{T})$ with fibres $E_\omega \cong C^*(G, d_2(\omega)) = C^*(G, \omega)$.

Remark. This result is essentially due to Rieffel [50, 2.8], who used instead of $H^2(G, \mathbb{T})$ the space $Z^2(G, \mathbb{T})$ of all normalised cocylces. We shall show elsewhere [39] that many of the other results in [50] can also be deduced from Williams' theorem [57].

The group H constructed in the proof of the corollary has the property that Inf: $H^2(G, \mathbb{T}) \to H^2(H, \mathbb{T})$ is the zero map; this follows immediately

from the restriction-inflation sequence because d_2 is an isomorphism. Moore called a group with this property a *splitting extension* of G, and investigated the question of when locally compact groups have such extensions. The restriction-inflation sequence shows that, for any splitting extension

$$1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1, \tag{1.4}$$

d_2 induces an isomorphism of $\widehat{N}/\ker d_2$ onto $H^2(G, \mathbb{T})$ and hence induces a locally compact, but not necessarily Hausdorff, topology on $H^2(G, \mathbb{T})$. Moore proved that this topology was independent of the splitting extension chosen [30, Theorem 2.2], and later he showed that it also agreed with the topology on $H^2(G, \mathbb{T})$ inherited from the natural topology on cochains [32, Theorem 6]. If we have a splitting extension (1.4) such that $d_2\colon \widehat{N} \to H^2(G, \mathbb{T})$ is injective, then $H^2(G, \mathbb{T})$ is Hausdorff and we can identify N with $H^2(G, \mathbb{T})^\frown$. Now we can apply Theorem 1.2 as in the corollary to deduce that $C^*(H)$ is given by a C^*-bundle over $H^2(G, \mathbb{T})$ with fibres $\{C^*(G, \omega) : \omega \in H^2(G, \mathbb{T})\}$.

Our corollary, therefore, will hold for any locally compact group G for which there is a splitting extension (1.4) with d_2 injective. Such an extension was called a *representation group* for G by Moore, and earlier by Schur, because the projective representation theory of G is essentially the same as the ordinary representation theory of H [30, Theorem 2.6]. Moore showed that an almost-connected group has a representation group if and only if $H^2(G, \mathbb{T})$ is Hausdorff [30, Proposition 2.7]; even if G is a nilpotent Lie group, this need not happen [30, p. 85], but it is automatic if G is abelian [32, Theorem 7]. Hence the corollary holds also for compact or connected locally compact abelian groups. Our method of proof will not work for every abelian group, since they need not have splitting extensions at all if G is not almost connected [30, p. 85], and we do not know if the corollary holds in this generality.

Examples 1.4. (1) $G = \mathbb{Z}^2$. For $\theta \in \mathbb{R}$, we can define $\omega_\theta \in Z^2(G, \mathbb{T})$ by

$$\omega_\theta((n_1, p_1), (n_2, p_2)) = \exp(2\pi i \theta p_1 n_2),$$

and $\theta \to \omega_\theta$ induces a continuous homomorphism of $\mathbb{T} \cong H^2(G, \mathbb{T})$ into $Z^2(G, \mathbb{T})$. If we identify $H^2(G, \mathbb{T})^\frown$ with $\widehat{\mathbb{T}} = \mathbb{Z}$, the cocycle (1.3) becomes

$$\sigma((n_1, p_1), (n_2, p_2)) = p_1 n_2,$$

and the corresponding extension of $\mathbb{Z}^2$ by $\mathbb{Z}$ is $\mathbb{Z}^3$ with the product

$$(m_1, n_1, p_1)(m_2, n_2, p_2) = (m_1 + m_2 + p_1 n_2, n_1 + n_2, p_1 + p_2), \tag{1.5}$$

in other words, the usual integer Heisenberg group H. (In fact, it follows from [30, Theorem 3.2] that this is the only representation group for $\mathbb{Z}^2$.) In this case, we obtain from 1.2 the well-known decomposition of $C^*(H)$ as the algebra of sections of a C^*-bundle over $\mathbb{T}$ with fibres the rotation algebras $A_\theta = C^*(\mathbb{Z}^2, \omega_\theta)$ (e.g., [1]).

(2) $G = \mathbb{R}^2$. Similarly, the real Heisenberg group H is a representation group for $G = \mathbb{R}^2$. To see this, view H as $\mathbb{R}^3$ with multiplication as in (1.5), and let N be the centre $\{(x, 0, 0) : x \in \mathbb{R}\}$. Then computing d_2 using the section $c(y, z) = (0, y, z)$ gives

$$d_2(\theta)((y_1, z_1), (y_2, z_2)) = \exp(i\theta z_1 y_2) \quad \text{for } \theta \in \mathbb{R} = \widehat{\mathbb{R}} = \widehat{\mathbb{N}},$$

and d_2 is the usual isomorphism of $\mathbb{R}$ into $H^2(\mathbb{R}^2, \mathbb{T})$. Theorem 1.2 gives the usual decomposition of $C^*(H)$ in terms of a C^*-bundle over $\mathbb{R}$ with fibre $C_0(\mathbb{R}^2)$ over 0 and $\mathscr{K}(L^2(\mathbb{R}))$ elsewhere.

(3) Let G be the integer Heisenberg group itself. It follows from [36, 1.1] that the homomorphism $\sigma\colon \mathbb{T}^2 \to Z^2(G, \mathbb{T})$ defined by

$$\begin{aligned}\sigma(\lambda, \mu)((m_1, n_1, p_1), (m_2, n_2, p_2)) \\ = \lambda^{m_2 p_1 + p_1(p_1-1)n_2/2} \mu^{n_1(m_2+p_1 n_2)+p_1 n_2(n_2-1)/2}\end{aligned} \tag{1.6}$$

induces an isomorphism of $\mathbb{T}^2$ onto $H^2(G, \mathbb{T})$. (Alternatively, we could deduce this using the methods of (4) below.) This map naturally lifts to a cocycle with values in $\mathbb{Z}^2 = H^2(G, \mathbb{T})^\frown$, and the representation group of Γ is $\mathbb{Z}^5$ with a rather wierd-looking multiplication.

For a fixed $\sigma(\lambda, \mu)$, we can apply Theorem 1.2 to the subgroup

$$\begin{aligned} N &= \{z \in Z(G) : \tilde{\sigma}(z, s) = 1 \text{ for } s \in G\} \\ &= \{(m, 0, 0) : \lambda^{pm}\mu^{nm} = 1 \text{ for all } p, n \in \mathbb{Z}\}\end{aligned}$$

which is nontrivial when

$$\lambda = \exp 2\pi i\alpha, \quad \mu = \exp 2\pi i\beta$$

for some $\alpha, \beta \in \mathbb{Q}$. Then $N = q\mathbb{Z}$ for some integer q, and we obtain a description of $C^*(G, \sigma(\lambda, \mu))$ as the section algebra of a C^*-bundle over $\mathbb{T}$. A detailed study of these algebras has been given in [36]: $C^*(G, \sigma(\lambda, \mu))$ is isomorphic to $C^*(G, \sigma(\exp(2\pi i/q), 1))$ [36, 1.5], which is Morita equivalent to $C^*(G)$ itself [36, p. 56].

(4) For the real Heisenberg group H, we can compute $H^2(H, \mathbb{T})$ by applying the spectral sequence for Moore cohomology to the centre $Z \cong \mathbb{R}$. We know that $H^2(Z, \mathbb{T}) = H^3(H/Z, \mathbb{T}) = 0$, so $\ker \mathrm{Res} = H^2(H, \mathbb{T})$ and $d\colon H^2(H, \mathbb{T}) \to H^1(H/Z, \widehat{Z})$ is surjective. The restriction-inflation sequence

$$\begin{aligned} 0 \longrightarrow H^1(H/Z, \mathbb{T}) \xrightarrow{\mathrm{Inf}} H^1 H, \mathbb{T}) \xrightarrow{\mathrm{Res}} H^1(Z, \mathbb{T}) \xrightarrow{d_2} H^2(H/Z, \mathbb{T}) \\ \xrightarrow{\mathrm{Inf}} H^2(H, \mathbb{T}) \longrightarrow H^1(H/Z, \widehat{Z}) \longrightarrow 0\end{aligned}$$

becomes

$$0 \longrightarrow (H/Z)^\frown \xrightarrow{\mathrm{id}} (H/Z)^\frown \xrightarrow{\mathrm{Res}} \widehat{Z} = \mathbb{R} \xrightarrow{d_2} \mathbb{R} \xrightarrow{\mathrm{Inf}} H^2(H, \mathbb{T}) \longrightarrow \mathbb{R}^2 \longrightarrow 0.$$

The first Inf is onto, so Res $= 0$ and the first d_2 is injective; since every monomorphisms of $\mathbb{R}$ into itself is an isomorphism, this forces the second Inf to be 0 and $d\colon H^2(H, \mathbb{T}) \to \mathbb{R}^2$ is an isomorphism. If for $(\alpha, \beta) \in \mathbb{R}^2$ we set

$$\begin{aligned}\sigma(\alpha, \beta)((x_1, y_1, z_1), (x_2, y_2, z_2)) \\ = \exp[2\pi i\alpha(x_2 z_1 + z_1(z_1 - 1)y_2/2) \\ + 2\pi i\beta(y_1(x_2 + z_1 y_2) + z_1 y_2(y_2 - 1)/2)],\end{aligned}$$

then σ is a homomorphism of $\mathbb{R}^2$ into $H^2(H, \mathbb{T})$, which inverts d since

$$\tilde{\sigma}(\alpha, \beta)((x, 0, 0), (x_2, y_2, z_2)) = \exp(-2\pi i(z_2\alpha + y_2\beta)x),$$

and we have found a specific parametrisation of $H^2(H, \mathbb{T})$. As in (3), we can use this to write down a multiplication on $\mathbb{R}^5$ which makes $\mathbb{R}^5$ a representation group for H; this group is called $G_{5,4}$ in Nielsen's list of low-dimensional

nilpotent Lie groups [58], and the representation group in (3) above is the integer subgroup of $G_{5,4}$. (This reference should also help identify representation groups in other cases: for example, we believe that $G_{6,15}$ [58, p. 78] is a representation group for $\mathbb{R}^3$.)

The multiplier $\sigma(\alpha, \beta)$ is nontrivial if at least one of α, β is nonzero, and then there is no nontrivial central subgroup to which we can apply Theorem 1.2. However, using the decomposition of Proposition 1.1 with respect to an appropriate normal subgroup K, we can still show that $C^*(H, \sigma(\alpha, \beta))$ is given by a C^*-bundle over $\mathbb{R}$, and indeed we have $C^*(H, \sigma(\alpha, \beta)) \cong C_0(\mathbb{R}, \mathscr{K}(L^2(\mathbb{R}))$.

To see this, we take

$$K = \ker\{\phi(\sigma)\colon H \to \widehat{Z}\} = \{(x, \lambda\alpha, -\lambda\beta) : (x, \lambda) \in \mathbb{R}^2\},$$

and observe that $(x, y, z) \to z\alpha + y\beta$ induces an isomorphism of H/K onto $\mathbb{R}$: the action of $\mathbb{R} \cong H/K$ on K by conjugation becomes

$$r \cdot (x, \lambda\alpha, -\lambda\beta) = (x + \lambda r, \lambda\alpha, -\lambda\beta).$$

There is an isomorphism $\psi\colon \mathbb{R}^2 \to K$ defined by

$$\psi(x, \lambda) = (x - \lambda(\lambda - 1)\beta\alpha/2, \alpha\lambda, -\lambda\beta),$$

and this carries the action of $\mathbb{R}$ on K into the action on $\mathbb{R}^2$ given by $r \cdot (x, \lambda) = (x + r\lambda, \lambda)$. If $\gamma_{\delta,\varepsilon}$ denotes the character of $\mathbb{R}^2$ which sends (x, λ) to $\exp 2\pi i(\delta x + \varepsilon\lambda)$, and $r = z\alpha + y\beta \in \mathbb{R}$, then a calculation shows

$$\begin{aligned}\phi(\sigma)(r)(x, \lambda\alpha, -\lambda\beta) &= \tilde{\sigma}((x, \lambda\alpha, -\lambda\beta), (0, y, z)) \\ &= \exp 2\pi i[r(-\lambda^2\alpha\beta/2 + \alpha\lambda/2 + \beta\lambda/2 - x) + \lambda r^2/2].\end{aligned}$$

If $\alpha \neq 0$, the homomorphism $c\colon \mathbb{R} \to H$ defined by $c(r) = (0, 0, r/\alpha)$ is a splitting for $H \to H/K = \mathbb{R}$, so $H \cong K \rtimes H/K$ (if $\alpha = 0$, take $c(r) = (0, r/\beta, 0)$). Since $\sigma = 1$ on the range of c, the map $r \to (1, c(r))$ is a homomorphism into H_σ, and composing it with the action α of Proposition 1.1 gives an action β of H/K on $C^*(K, \operatorname{Res}\sigma)$ such that $C^*(H, \sigma) \cong C^*(K, \operatorname{Res}\sigma) \rtimes H/K$. As we also have $\sigma = 1$ on K, $C^*(K, \operatorname{Res}\sigma) \cong C_0(\widehat{K})$, and computations show that β induces the action of H/K on $\widehat{K}$ given by

$$s \cdot \gamma(k) = \tilde{\sigma}(k, c(s))\gamma(s^{-1}ks) \quad \text{for } s \in H/K.$$

Now we identify K with $\mathbb{R}^2$, H/K with $\mathbb{R}$, and compute

$$\begin{aligned}r \cdot \gamma_{\delta,\varepsilon}(x, \lambda) &= \phi(\sigma)(r)(\psi(x, \lambda))\gamma_{\delta,\varepsilon}((-r) \cdot (x, \lambda)) \\ &= \phi(\sigma)(r)(x - \lambda(\lambda - 1)\beta\alpha/2, \lambda\alpha, -\lambda\beta)\gamma_{\delta,\varepsilon}(x - r\lambda, \lambda) \\ &= \exp 2\pi i[r(\alpha\lambda/2 + \beta\lambda/2 - x - \lambda\beta\alpha/2) + \delta(x - r\lambda) + \varepsilon\lambda] \\ &= \gamma_{\delta - r, \varepsilon - r\delta + (\alpha+\beta-\alpha\beta)r/2 + r^2/2}(x, \lambda).\end{aligned}$$

Thus $C^*(H, \sigma) \cong C_0(\mathbb{R}^2) \rtimes \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ via

$$r \cdot (\delta, \varepsilon) = (\delta - r, \varepsilon - r\delta + (\alpha + \beta - \alpha\beta)r/2 + r^2/2).$$

Once can easily verify that this action is topologically conjugate to the action $r \cdot (\delta, \varepsilon) = (\delta - r, \varepsilon)$ via the homeomorphism

$$h(\delta, \varepsilon) = (\delta + (\alpha + \beta - \alpha\beta)/2, \varepsilon + \delta^2/2),$$

and it follows that

$$C^*(H, \sigma(\alpha, \beta)) \cong C_0(\mathbb{R}^2) \rtimes \mathbb{R} \cong (C_0(\mathbb{R}) \rtimes \mathbb{R}) \otimes C_0(\mathbb{R})$$
$$\cong \mathscr{K}(L^2(\mathbb{R})) \otimes C_0(\mathbb{R}).$$

(5) Although Theorem 1.2 gives no useful information for any twisted group algebra of the real Heisenberg group H except $C^*(H)$ itself, it does often apply to higher-dimensional Heisenberg groups. For suppose $V = \mathbb{R}^n$ for some $n \geq 2$, and $G = \mathbb{R} \times V \times V$ with product

$$(r_1, x_1, y_1)(r_2, x_2, y_2) = (r_1 + r_2 + y_1^t x_2, x_1 + x_2, y_1 + y_2).$$

Then with $Z = \{(r, 0, 0)\} \cong \mathbb{R}$, we have $H^2(G/Z, \mathbb{T}) \cong H^2(V^2, \mathbb{T}) \cong \mathbb{R}^{n(2n-1)}$ (for A upper triangular in $M_{2n}(\mathbb{R})$, we can take $\omega_A(u, v) = \exp(u^t A v)$). Thus in the restriction-inflation sequence, $d_2 \colon H^1(Z, \mathbb{T}) = \mathbb{R} \to H^2(G/Z, \mathbb{T})$ has a large cokernel, and G has many nontrivial multipliers of the form $\operatorname{Inf} \omega_A$ to which we can apply Theorem 1.2.

For any G we can always apply Theorem 1.2 to the central subgroup

$$N = \{z \in Z(G) : \tilde{\sigma}(z, s) = 1 \text{ for all } s \in G\},$$

unless, of course, N is trivial. Even when the theorem does apply, however, the fibre algebras are twisted group algebras for which the corresponding central subgroup can be trivial, and it is therefore of some interest to consider this case. Our other main structural result is a criterion for simplicity of $C^*(G, \sigma)$ which is designed to apply in such situations. In particular, it will give necessary and sufficient conditions for simplicity of $C^*(G, \sigma)$ when G is a 2-step nilpotent group (Corollary 1.6).

Theorem 1.5. *Suppose G is an amenable second countable locally compact group, N is a closed normal abelian subgroup of G, and*

$$S = \{s \in N : \tilde{\sigma}(s, n) = 1 \textit{ for all } n \in N\}$$

is the symmetriser subgroup of $\sigma|_{N \times N}$. We may without loss of generality suppose $\sigma|_{S \times S} = 1$. Then there is a homeomorphism of Prim $C^*(N, \operatorname{Res} \sigma)$ *onto $\widehat{S}$ which carries the action of G/N described in Proposition* 1.1 *into that given by*

$$(r \cdot \gamma)(n) = \tilde{\sigma}(n, r)\gamma(r^{-1}nr) \quad \textit{for } \gamma \in \widehat{S},\ n \in S,\ r \in G.$$

If this action is free and minimal, then $C^*(G, \sigma)$ is simple. Conversely, if $C^*(G, \sigma)$ is simple, then this action is minimal but not necessarily free.

Proof. Since S is abelian and $\tilde{\sigma} = 1$ on $S \times S$, the multiplier $\sigma|_{S \times S}$ is trivial and we can multiply by a coboundary to ensure $\sigma|_{S \times S} = 1$ [26, Lemma 7.2]. By [17, Proposition 34], induction induces a homeomorphism Ind of $\widehat{N}_\sigma / Z^\perp$ onto Prim $C^*(N_\sigma, \mathbb{C}, \mathscr{T}_\sigma) =$ Prim $C^*(N, \operatorname{Res} \sigma)$, where Z is the centre of N_σ. But it is easy to check that $Z = S_\sigma = \mathbb{T} \times S$, and we therefore have a homeomorphism of $\widehat{S} = \widehat{N}_\sigma / Z^\perp$ onto Prim $C^*(N_\sigma, \mathbb{C}, \mathscr{T}_\sigma)$. The map Ind is defined on ideals by $\operatorname{Ind}(\ker L) = \ker(\operatorname{Ind} L)$ [17, Proposition 9], and the Green-Rieffel inducing process is equivalent to Mackey's. Thus it follows from

[4, Lemma 1.2] that we have a commutative diagram of homeomorphisms

$$\begin{array}{ccc} \operatorname{Prim} C^*(Z, \mathbb{C}, \mathscr{T}_\sigma) & \xrightarrow{\text{Ind}} & \operatorname{Prim} C^*(N_\sigma, \mathbb{C}, \mathscr{T}_\sigma) \\ \downarrow & & \downarrow \\ \operatorname{Prim} C^*(S) = \widehat{S} & \xrightarrow{\sigma-\text{Ind}} & \operatorname{Prim} C^*(N, \operatorname{Res} \sigma); \end{array}$$

here σ-Ind sends $\gamma \in \widehat{S}$ to the representation in

$$H_\sigma = \left\{ \xi \colon N \to \mathbb{C} \;\middle|\; \begin{array}{l} \xi \text{ is Borel, } \int_{N/S} |\xi(n)|^2 \, d(nS) < \infty \text{ and} \\ \xi(sn) = \overline{\sigma(s, n)} \xi(n) \gamma(s) \text{ for } s \in S, \; n \in N \end{array} \right\}$$

given by

$$(\sigma\text{-Ind}(\gamma)(f)\xi)(m) = \int_N f(m)\sigma(m, n)\xi(mn)\, dn \quad \text{for } f \in L^1(N).$$

To see that S is normal in G, so that $r \cdot \gamma$ makes sense, observe that $S_\sigma = Z$ is central in the normal subgroup N_σ, and hence normal in G_σ.

We next verify that the actions of G/N match up under the homeomorphism σ-Ind: specifically, we shall construct an intertwining operator W for $(\sigma\text{-Ind } \gamma) \circ \alpha_r^{-1}$ and $\sigma\text{-Ind}(r \cdot \gamma)$. Indeed, if for $\xi \in H_\gamma$ we define $W\xi \colon N \to \mathbb{C}$ by

$$W\xi(m) = \tilde{\sigma}(m, r)\xi(r^{-1}mr) \quad \text{for } m \in N,$$

then we can easily verify that $W\xi \in H_{r \cdot \gamma}$ and that W is then a scalar multiple of a unitary operator from H_γ to $H_{r \cdot \gamma}$. We now compute

$$\begin{aligned} W[(\sigma\text{-Ind } \gamma)(\alpha_r^{-1}(f))](\xi)(m) &= \tilde{\sigma}(m, r)(\sigma\text{-Ind } \gamma(\alpha_r^{-1}(f))(\xi)](r^{-1}mr) \\ &= \tilde{\sigma}(m, r) \int_N \Delta_{G,N}(r^{-1}) \tilde{\sigma}(rnr^{-1}, r) f(rnr^{-1}) \sigma(r^{-1}mr, n) \xi(r^{-1}mrn)\, dn \\ &= \tilde{\sigma}(m, r) \int_N \tilde{\sigma}(k, r) f(k) \sigma(r^{-1}mr, r^{-1}kr) \xi(r^{-1}mkr)\, dk \\ &= \int_N \tilde{\sigma}(mk, r) \sigma(m, k) f(k) \xi(r^{-1}mkr)\, dk \quad \text{(using equation (1.2))} \\ &= [\sigma\text{-Ind}(r \cdot \gamma)(f) W\xi](m). \end{aligned}$$

Thus $\sigma\text{-Ind}(r \cdot \gamma)$ is equivalent to $r \cdot (\sigma\text{-Ind } \gamma)$, and we have proved that the homeomorphism σ-Ind is G/N-equivariant.

Since G is amenable, so is $G_\sigma/N_\sigma \cong G/N$, and thus by [16, Theorem 4.2] the twisted covariant system $(C^*(N, \operatorname{Res} \sigma), G_\sigma, \mathscr{T})$ is Effros-Hahn regular. By hypothesis, it is essentially free in the sense that $G_\sigma/N_\sigma = G/N$ acts freely on $\operatorname{Prim} C^*(N, \operatorname{Res} \sigma) = \widehat{S}$. Thus by [17, Theorem 24 and the remark following], there is a homeomorphism of the quasiorbit space $\mathscr{Q}(\widehat{S}/G)$ onto $\operatorname{Prim} C^*(G_\sigma, C^*(N, \operatorname{Res} \sigma), \mathscr{T})$. But since we are assuming that G/N acts minimally on $\widehat{S}$, the quasiorbit space will be a point and $C^*(G, \sigma) \cong C^*(G_\sigma, C^*(N, \operatorname{Res} \sigma), \mathscr{T})$ is simple.

To see the converse, we observe that a closed G-invariant subset of $\widehat{S}$ will give rise to a G_σ-invariant ideal I in $C^*(N, \operatorname{Res} \sigma)$, and then $C^*(G_\sigma, I, \mathscr{T})$

will be a nontrivial ideal in $C^*(G_\sigma, C^*(N, \operatorname{Res}\sigma), \mathscr{T})$. We justify the last statement by example: if $G = \mathbb{Z}^2$, $N = \{e\}$, and

$$\sigma((m, n), (p, q)) = \exp 2\pi i\alpha np$$

for some irrational number α, then $\sigma|_{N\times N} = 1$, G/N does not act freely on $\widehat{N} = \{0\}$, but $C^*(G, \sigma)$ is certainly simple.

Corollary 1.6. *Let G be a two-step nilpotent locally compact group, $\sigma \in Z^2(G, \mathbb{T})$, and $N = [G, G]$. Then $\phi(\sigma)(t) = \tilde{\sigma}(\cdot, t)$ belongs to $\widehat{N}$ for all $t \in G$; suppose that $\phi(\sigma)\colon G \to \widehat{N}$ has dense range. Then $K = \{s \in G : \tilde{\sigma}(n, s) = 1$ for all $s \in N\}$ is a normal abelian subgroup of G containing N, and we may therefore suppose without changing the class of σ that $\sigma|_{K\times K}$ is constant on N-cosets, in which case $\tilde{\sigma}(\cdot, t) \in \widehat{K}$ for all $t \in G$. Let H be a maximal isotropic subgroup of K relative to σ (the existence of H is guaranteed by* [20, *Theorem* 1.6]). *Then H is normal in G, and $C^*(G, \sigma)$ is simple if and only if G acts minimally on $\widehat{H}$ for the action*

$$r \cdot \gamma(h) = \tilde{\sigma}(h, r)\gamma(r^{-1}hr), \qquad r \in G,\ h \in H,\ \gamma \in \widehat{H}.$$

Remark. Since N is central, if $\phi(\sigma)$ does not have range we can apply Theorem 1.2 to [range $\phi(\sigma)]^\perp$ to decompose $C^*(G, \sigma)$ as sections of a C^*-bundle, in which case $C^*(G, \sigma)$ is certainly not simple.

Proof. We have $[N, G] = \{e\}$ because G is step 2, and N is therefore central. Thus for any $n \in N$, $\tilde{\sigma}(\cdot, n)$ is a homomorphism of G into the abelian group $\mathbb{T}$, and which is therefore 1 on $[G, G] = N$. Hence $N \subset K$. Because G/N is abelian, K/N is normal in G/N, and it follows that K is normal in G. Since $\tilde{\sigma}(\cdot, t)$ is identically 1 from K to $\widehat{N}$ we may by Proposition A2 suppose that $\sigma|_{K\times K}$ has the form $\operatorname{Ind}\omega$ for some $\omega \in Z^2(K/N, \mathbb{T})$, so that $\sigma|_{K\times K}$ is constant on N-cosets in either variable. Thus for k, $l \in K$ and $t \in G$ we have

$$\sigma(t^{-1}kt, t^{-1}lt) = \sigma(t^{-1}kt(t^{-1}k^{-1}tk), t^{-1}lt(t^{-1}l^{-1}tl)) = \sigma(k, l).$$

By equation (1.2), it follows that $\tilde{\sigma}(\cdot, t)$ is a homomorphism on K for fixed $t \in G$, and hence is equal to 1 on $[K, K] \subset N$. But $[K, K]^\perp$ is a closed subgroup of $\widehat{N}$, and range $\phi(\sigma) \subset [K, K]^\perp$ can only be dense if $[K, K]^\perp = \widehat{N}$. By duality, this implies $[K, K] = \{e\}$, and K is abelian.

Now let H be a maximal isotropic subgroup of K, i.e., $H = \{t \in K : \tilde{\sigma}(t, s) = 1$ for all $s \in H\}$. Again, we may adjust σ by a coboundary to ensure $\sigma|_{H\times H} = 1$, and, provided we chose the coboundary to be constant on N-cosets, $\tilde{\sigma}$ will be left unchanged. Because $N \subset K$, we have $N \subset H$, and H is normal in G. If G acts minimally on $\widehat{H}$, then all the stabilisers are equal, and they contain H since $\tilde{\sigma}|_{H\times H} = 1$. Furthermore, if we define $\phi_H(\sigma)\colon G \to \widehat{H}$ by $\phi_H(\sigma)(t) = \tilde{\sigma}(\cdot, t)$, then $\ker\phi_H(\sigma) \subset K$ since

$$\tilde{\sigma}(\cdot, t)|_H = 1 \Rightarrow \tilde{\sigma}(\cdot, t)|_N = 1 \Rightarrow t \in K;$$

thus $\ker\phi_H(\sigma) = H$ by maximality of H, and the stabiliser of $1 \in \widehat{H}$ is also equal to H. Hence G/H acts freely on $\widehat{H}$, Theorem 1.5 applies, and $C^*(G, \sigma)$ is simple. The converse also follows from that theorem.

We shall later use Theorem 1.5 to discuss the simplicity of the twisted group algebras of semidirect products $\mathbb{Z}^n \rtimes \mathbb{Z}$. However, we can also easily deduce from it that most twisted group algebras of the integer Heisenberg group are simple; other proofs of this are given in [36, Theorem 1.6 and p. 56].

Corollary 1.7. *Suppose Γ is the integer Heisenberg group, α, $\beta \in [0, 1]$, and $\sigma = \sigma(\exp 2\pi i\alpha, \exp 2\pi i\beta)$ is the multiplier of Γ described in equation* (1.6). *If either of α or β is irrational, then $C^*(\Gamma, \sigma)$ is simple.*

Proof. We discuss two special cases separately. If α and β are not rationally related, we apply Theorem 1.5 with $N = Z = [\Gamma, \Gamma] \cong \mathbb{Z}$. The action of $\mathbb{Z}^2 = \Gamma/N$ on $\widehat{Z}$ is given by

$$(n, p) \cdot \gamma = \tilde{\sigma}(\cdot, (0, n, p))\gamma;$$

viewing $\widehat{Z}$ as $\mathbb{T}$ and computing, this action becomes

$$(n, p) \cdot e^{2\pi i\theta} = e^{2\pi i(-p\alpha - n\beta + \theta)}.$$

The irrationality of α and β implies that this action is minimal, and the independence of α and β that the action is free, so Theorem 1.5 applies and $C^*(\Gamma, \sigma)$ is simple.

Now suppose at least one of α, β is irrational and there exist $a, b, c \in Q$ such that $a\alpha + b\beta = c = 0$. By [36, Proposition 1.5] we may without loss of generality assume α is irrational and $\beta = p/q$ (this is not necessary for the following argument, but makes the calculation easier). Let $N = \{(m, n, 0) : m, n \in \mathbb{Z}\}$; computation shows that the symmetriser S of $\sigma|_{N\times N}$ is $\{(m, nq, 0) : m, n \in \mathbb{Z}\}$. For $(z, w) \in \mathbb{T}^2$, let γ denote the character of S which sends $(m, nq, 0)$ to $z^m w^n$. Then

$$\begin{aligned}(0, 0, 1) \cdot \gamma(m, nq, 0) &= \tilde{\sigma}((m, nq, 0), (0, 0, 1))\gamma(m - qn, qn, 0)\\ &= (e^{-2\pi i\alpha} z)^m (e^{2\pi i(\alpha q - q(q-1)/2)} z^{-q} w)^n,\end{aligned}$$

and the action of $\mathbb{Z} = \Gamma/N$ on $\widehat{S} = \mathbb{T}^2$ is therefore generated by the transformation $(z, w) \to (e^{2\pi i\alpha} z, \mu z^{-q} w)$ for some $\mu \in \mathbb{T}$. Therefore $\mathbb{Z}$ acts freely and minimally on $\mathbb{T}^2$ by [12, Theorem 2.1], and Theorem 1.5 implies that $C^*(\Gamma, \sigma)$ is simple, as claimed.

2. The K-theory of twisted group algebras

We shall describe the K-theory of the twisted group C^*-algebras $C^*(\Gamma, \sigma)$ of a discrete subgroup Γ of a solvable simply-connected Lie group. Many discrete groups arise this way: a well-known theorem of Mal′cev [29] says that any finitely generated torsion-free discrete nilpotent group Γ can be embedded as a discrete cocompact subgroup of a simply-connected nilpotent Lie group. Our description of $K_*(C^*(\Gamma, \sigma))$ involves stabilising to reduce to the case of an ordinary crossed product $\mathscr{K} \rtimes \Gamma$. This trick works for any twisted crossed product in the sense of [38], so we shall start by considering the crossed product $A \rtimes_{\alpha, u} \Gamma$ by a twisted action (α, u) of Γ on a C^*-algebra A, and later specialise to the case of $C^*(\Gamma, \sigma) = \mathbb{C} \rtimes_{\mathrm{id}, \sigma} \Gamma$.

To state our results, we require some definitions. Suppose H is a closed subgroup of a locally compact group G and $\beta : H \to \operatorname{Aut} B$ is an action of

H on a C^*-algebra B. The induced C^*-algebra $\operatorname{Ind} B = \operatorname{Ind}_H^G(B, \beta)$ is the algebra of all continuous functions $f: G \to B$ such that

$$f(gh) = \beta_h^{-1}(f(g)) \quad \text{for } g \in G, \ h \in H,$$

and $gH \to \|f(g)\|$ vanishes at infinity on G/H. The algebra $\operatorname{Ind} B$ carries a natural action λ of G by left translation, and a theorem of Green (see, for example, [42, §1.5]) asserts that $B \rtimes_\beta H$ is Morita equivalent to $(\operatorname{Ind} B) \rtimes_\lambda G$.

Proposition 2.1. *Let Γ be a discrete subgroup of a solvable simply-connected Lie group G, and (α, u) be a twisted action of Γ on a C^*-algebra A. Then there is an action β of Γ on $A \otimes \mathscr{K}$ such that $(A \rtimes_{\alpha,u} \Gamma) \otimes \mathscr{K}$ is isomorphic to $(A \otimes \mathscr{K}) \rtimes_\beta \Gamma$, and for any such β we have*

$$K_*(A \rtimes_{\alpha,u} \Gamma) \cong K_{*+\dim G}(\operatorname{Ind}_\Gamma^G(A \otimes \mathscr{K}, \beta)).$$

Proof. The existence of the action β is proved in [38, §3], and the theorem of Green cited above implies that $(A \otimes \mathscr{K}) \rtimes_\beta \Gamma$ is Morita equivalent to the crossed product $\operatorname{Ind}(A \otimes \mathscr{K}) \rtimes_\lambda G$. Now we can apply Connes' Thom isomorphism theorem as in [10, Corollary 7] to compute the K =theory:

$$\begin{aligned} K_*(A \rtimes_{\alpha,u} \Gamma) &\cong K_*((A \otimes \mathscr{K}) \rtimes_\beta \Gamma) \\ &\cong K_*(\operatorname{Ind}(A \otimes \mathscr{K}) \rtimes_\beta G) \\ &\cong K_{*+\dim G}(\operatorname{Ind}(A \rtimes \mathscr{K})). \end{aligned}$$

Corollary 2.2. *If the action of Γ on $A \otimes \mathscr{K}$ extends to an action of G, then*

$$K_*(A \rtimes_{\alpha,u} \Gamma) \cong K_{*+\dim G}(C_0(G/\Gamma) \otimes A \otimes \mathscr{K}).$$

Proof. If δ is the extension of β to G, then $\Phi(f)(g) = \delta_g^{-1}(f(g))$ defines an isomorphism Φ of $C_0(G/\Gamma, A \otimes \mathscr{K})$ onto $\operatorname{Ind}(A \otimes \mathscr{K})$.

Remark. This corollary can be proved without involving induced C^*-algebras: when β extends to an action of G, Green's imprimitivity theorem [17, §2] immediately implies that $(A \otimes \mathscr{K}) \rtimes \Gamma$ and $(C_0(G/\Gamma) \otimes (A \otimes \mathscr{K})) \rtimes G$ are Morita equivalent. The point of our proposition is to give information when β does not extend. The commutative case shows that $K_*(\operatorname{Ind} B)$ can differ from $K_*(C_0(G/\Gamma) \otimes B)$, and we shall see in Example 2.13 that they can differ even when Γ acts trivially on $\operatorname{Prim} B$.

We now apply our observations to twisted group algebras—the case where $A = \mathbb{C}$ and $u = \sigma$ is an ordinary two-cocycle. Then we can take $\beta: \Gamma \to \operatorname{Aut} \mathscr{K}$ to be $\operatorname{Ad} V$, where V is any nondegenerate $\overline{\sigma}$-representation of Γ (see [55, §1]). The induced C^*-algebra $\operatorname{Ind}_\Gamma^G(\mathscr{K}, \operatorname{Ad} V)$ is a continuous-trace algebra with spectrum G/Γ, whose Dixmier-Douady class can be described as the image of $\overline{\sigma}$ under a homomorphism

$$\delta: H^2(\Gamma, \mathbb{T}) \to \check{H}^3(G/\Gamma, \mathbb{Z}).$$

(See [56 or 44, Lemma 4.3]; the class $\delta(\overline{\sigma})$ was there denoted $\delta(\overline{\sigma}, G)$.) The K-theory of $\operatorname{Ind} \mathscr{K}$ is therefore that of the stable continuous-trace C^*-algebra with spectrum G/Γ and Dixmier-Douady class $\delta(\overline{\sigma})$—in other words, the twisted K-theory $K^*(G/\Gamma, \delta(\overline{\sigma}))$ of Rosenberg [51]. These twisted K-groups are in general quite different from the ordinary ones (again, see Example 2.13 below), and in §3 we shall give an example of a group Γ and a multiplier σ for which $K_*(C^*(\Gamma, \sigma)) \cong K^*(G/\Gamma, \delta(\overline{\sigma}))$ is not the same as $K_*(C^*(\Gamma))$. To sum up:

Theorem 2.3. *Let Γ be a discrete subgroup of a solvable simply-connected Lie group G, and let σ be a multiplier on Γ. Then*

$$K_*(C^*(\Gamma, \sigma)) \cong K^{*+\dim G}(G/\Gamma, \delta(\overline{\sigma})).$$

It is well known that for many groups Γ and multipliers σ, we in fact have $K_*(C^*(\Gamma, \sigma)) \cong K_*((C^*(\Gamma))$, and this will follow from our theorem if we can show $\delta(\overline{\sigma})$ vanishes in these cases.

Proposition 2.4. *Suppose Γ is a closed subgroup of a locally compact group G. Then $\delta\colon H^2(\Gamma, \mathbb{T}) \to \check{H}^3(G/\Gamma, \mathbb{Z})$ vanishes on the image of* Res: $H^2(G, \mathbb{T}) \to H^2(\Gamma, \mathbb{T})$.

Proof. We return to the notation of the discussion preceding Theorem 2.3. If $\sigma = \operatorname{Res} \omega$, we can choose V to be the restriction of an $\overline{\omega}$-representation W of G, and then $\beta = \operatorname{Ad} V$ extends to the action $\operatorname{Ad} W$ of G on $\mathscr{K}$. As in the proof of Corollary 2.2, we then have $\operatorname{Ind}_\Gamma^G(\mathscr{K}, \operatorname{Ad} V) \cong C_0(G/\Gamma, \mathscr{K})$, and hence $\delta(\overline{\sigma}) = 0$.

This result applies in particular when $\Gamma = \mathbb{Z}^n$, $G = \mathbb{R}^n$ and σ is any cocycle on Γ: for we may suppose σ has the form $\sigma(v, w) = \exp(2\pi i v^t M w)$ for some skew-symmetric matrix M with entries $[0, \frac{1}{2})$ (see [3]), and the same formula defines a multiplier on $\mathbb{R}^n$. In this case, we recover from our theorem the well-known isomorphism of $K_*(C^*(\mathbb{Z}^n, \sigma))$ with $K^*(\mathbb{T}^n)$. The proposition also applies with Γ and G the integer and real Heisenberg groups—this follows easily from the descriptions of their multipliers given in Examples 1.4(3), (4). It is therefore natural to ask whether $\ker \delta = \operatorname{im} \operatorname{Res}$ in general. It turns out that this is not always the case (see Remark 3.8(3)), but it is for many groups (Corollary 2.10). In general, the kernel of δ is described by the following exact sequence.

Theorem 2.5. *Let Γ be a discrete cocompact subgroup of a solvable simply-connected Lie group G, and let $\delta = H^2(\Gamma, \mathbb{T}) \to \check{H}^3(G/\Gamma, \mathbb{Z})$ be the homomorphism $\delta\colon \sigma \to \delta(\sigma, G)$ of* [44, §4]. *Then there is an exact sequence*

$$H^2(\Gamma, \mathbb{R}) \xrightarrow{\pi_*} H^2(\Gamma, \mathbb{T}) \xrightarrow{\delta} \check{H}^3(G/\Gamma, \mathbb{Z}) \xrightarrow{i_*} \check{H}^3(G/\Gamma, \mathbb{R}),$$

where $i\colon \mathbb{Z} \to \mathbb{R}$ is the inclusion and $\pi\colon \mathbb{R} \to \mathbb{T}$ the quotient map.

The idea of our proof of this is as follows. Because G is contractible G/Γ is a classifying space for Γ, and hence there are isomorphisms $\lambda^*\colon H^*(\Gamma, A) \to \check{H}^*(G/\Gamma, A)$ for any discrete abelian group A. These isomorphisms are natural in A, and hence associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{R} \xrightarrow{\mathbb{T}} \mathbb{T} \longrightarrow 0$$

of discrete abelian groups there is a commutative diagram

$$\begin{tikzcd}
H^2(\Gamma, \mathbb{R}) \arrow[r, "\pi_*"] \arrow[d, "\lambda^*"'] & H^2(\Gamma, \mathbb{T}) \arrow[r] \arrow[d, "\lambda^*"'] \arrow[dr, "\Delta"] & H^3(\Gamma, \mathbb{Z}) \arrow[r, "i_*"] \arrow[d, "\lambda^*"'] & H^3(\Gamma, \mathbb{R}) \arrow[d, "\lambda^*"'] \\
\check{H}^2(G/\Gamma, \mathbb{R}) \arrow[r, "\pi_*"] & \check{H}^2(G/\Gamma, \mathbb{T}) \arrow[r, "\delta_*"] & \check{H}^3(G/\Gamma, \mathbb{Z}) \arrow[r, "i_*"] & \check{H}^3(G/\Gamma, \mathbb{R}),
\end{tikzcd}$$

in which both rows are exact and the vertical arrows are isomorphisms. So the result will follow if we can identify the diagonal arrow $\Delta\colon H^2(\Gamma, \mathbb{T}) \to \check{H}^3(G/\Gamma, \mathbb{Z})$ with the homomorphism δ.

The existence of the isomorphisms λ^* is a celebrated theorem of Eilenberg and Mac Lane [13], but we shall require a detailed description of the action of λ^* on cocycles and we do not know where this has been explicity written down. It must be absolutely standard, however, and we warn experts that they may find our proof of the following lemma painful.

Lemma 2.6. *Suppose Γ, G are as above and $\lambda_{ij} : N_{ij} \to \Gamma$ are transition functions for the covering map $G \to G/\Gamma$. Let A be a discrete abelian group, and for $\omega \in Z^n(\Gamma, A)$ define a Čech cocycle $\lambda^*(\omega) \in Z^n(\{N_i\}, A)$ by*

$$\lambda^*(\omega)_{k_0k_1\cdots k_n}(t) = \omega(\lambda_{k_0k_1}(t), \lambda_{k_1k_2}(t), \dots, \lambda_{k_{n-1}k_n}(t)).$$

Then λ^* induces a natural isomorphism of $H^n(\Gamma, A)$ onto $\check{H}^n(G/\Gamma, A)$.

Proof. We shall show that λ^* is a composition

$$H^n(\Gamma, A) \xrightarrow{\theta^*} H^n_{\text{simp}}(G/\Gamma, A) \longrightarrow \check{H}^n(G/\Gamma, A),$$

where θ^* is the map described in [13, §11], which is a natural isomorphism because $\pi_p(G/\Gamma) \cong \pi_p(G) \cong 0$ for $p \geq 2$ (see [13, §11] and the references given there), and the second map is the standard isomorphism of simplicial and Čech cohomology. We may suppose $\{N_i\}$ is a good open cover of G/Γ and that there are points $P_i \in N_i$ which are the vertices of a triangulation of G/Γ satisfying $\overline{\text{St } P_i} \subset N_i$. Further, we shall fix paths τ_i joining the base point Γ of G/Γ to P_i, let σ_i denote the unique lift of τ_i to a path in G based at e, choose local sections $c_i\colon N_i \to G$ such that $c_i(P_i) = \sigma_i(e)$, and suppose the transition functions λ_{ij} are defined by $c_i\lambda_{ij} = c_j$.

The isomorphism θ^* is induced by the chain transformation $\theta\colon k(G/\Gamma) \to K(\Gamma) = K(\pi_1(G/\Gamma))$ which sends the oriented simplex $(P_{k_0}, P_{k_1} \cdots P_{k_n})$ to the element of $K(\Gamma)$ represented as in [13, §3] by the n-tuple $(d_{01}, d_{12}, \dots, d_{n-1,n})$, where $d_{i,i+1}$ is the class of the loop $\tau_{k_i}(P_{k_i}P_{k_{i+1}})\tau_{k_{i+1}}^{-1}$ based at Γ—in other words, $d_{ij} \in \Gamma$ is the inverse in Γ of the endpoint of the unique lift of $\tau_{k_i}(P_{k_i}P_{k_{i+1}})\tau_{k_{i+1}}^{-1}$ to a path in G starting at e. (It is the *inverse* of the endpoint because we are viewing G as a right Γ-space.) But $\sigma_{k_i}c_{k_i}(P_{k_i}P_{k_{i+1}})$ is a lift of $\tau_{k_i}(P_{k_i}P_{k_{i+1}})$ starting at e, and it finishes at $c_{k_i}(P_{k_{i+1}})$; thus the appropriate lift of $\tau_{k_{i+1}}^{-1}$ is $\sigma_{k_{i+1}}^{-1} \cdot \lambda_{k_ik_{i+1}}(P_{k_{i+1}})^{-1}$, and $d_{i,i+1}^{-1}$ is the endpoint $\lambda_{k_ik_{i+1}}(P_{k_{i+1}})^{-1}$ of this path. If $f\colon \Gamma^n \to A$ is an n-cochain, the corresponding cochain $\theta^*(f) \in D^n_{\text{simp}}(G/\Gamma, A)$ is defined by

$$\theta^*(f)(P_{k_0}P_{k_1} \cdots P_{k_n}) = f(\lambda_{k_0k_1}(P_{k_1}), \dots, \lambda_{k_{n-1}k_n}(P_{k_n})). \tag{2.2}$$

Finally, the isomorphism $H^*_{\text{simp}} \cong \check{H}^*$ is induced by the cochain map $\phi : C^n_{\text{simp}}(G/\Gamma, A) \to C^n(\{N_i\}, A)$ such that

$$\phi(g)_{k_0k_1\cdots k_n}(t) = g((P_{k_0}P_{k_1} \cdots P_{k_n})) \quad \text{for } t \in \bigcap_i \text{St } P_{k_i};$$

this makes sense because $\bigcap \text{St } P_{k_i} \neq \varnothing$ iff G/Γ contains a simplex with vertices P_{k_i}. Putting (2.2) and (2.3) together, and observing that each λ_{ij} is constant on the connected set N_{ij}, gives the lemma.

To finish off the proof of the theorem, we need to compute the map Δ in (2.1). However, the map $\delta : H^2(\Gamma, \mathbb{T}) \to \check{H}^3(G/\Gamma, \mathbb{Z})$ was defined in [44] as the composition

$$H^2(\Gamma, \mathbb{T}) \xrightarrow{\mu} H^2(G/\Gamma, \mathscr{S}) \xrightarrow{\delta_1} \check{H}^3(G/\Gamma, \mathbb{Z}),$$

where $\mathscr{S}$ is the sheaf of continuous $\mathbb{T}$-valued functions, μ was given on cocycles by the formula given for λ_* in the lemma, and δ_1 is the Bockstein homomorphism in sheaf cohomology. Since Γ is discrete, the range of μ is actually contained in the image of $\check{H}^2(G/\Gamma, \mathbb{T})$ in $H^2(G/\Gamma, \mathscr{S})$, and the diagram

$$\begin{tikzcd} \check{H}^2(G/\Gamma, \mathbb{T}) \arrow[d] \arrow[dr, "\delta"] & \\ H^2(G/\Gamma, \mathscr{S}) \arrow[r, "\delta^*"'] & \check{H}^3(G/\Gamma, \mathbb{Z}) \end{tikzcd}$$

commutes, so Δ does agree with δ, and the theorem is proved.

Remark 2.7. While the definition of $\delta(\omega) = \delta(\omega, \Omega)$ as $\delta(\mathrm{Ad}\, V, c)$ (i.e. in terms of Dixmier-Douady classes) is valid for general locally compact groups, the observation made in [44, §4] that $\delta(\omega, \Omega)$ is represented by the cocycle $\mu_{ijk}(t) = \omega(\lambda_{ij}(t), \lambda_{jk}(t))$ is only valid for continuous cocycles ω. (Unless ω is continuous, there is no reason to suppose μ_{ijk} is a section of the sheaf $\mathscr{S}$ of *continuous* $\mathbb{T}$-valued functions.) Here our group Γ is discrete, so this problem does not arise.

Corollary 2.8. *Let G, Γ be as in the theorem. Then the range of $\delta : H^2(\Gamma, \mathbb{T}) \to \check{H}^3(G/\Gamma, \mathbb{Z})$ is the torsion subgroup of $\check{H}^3(G/\Gamma, \mathbb{Z})$.*

Proof. This follows from the exact sequence and the universal coefficient theorem for $\check{H} \cong H^*_{\mathrm{simp}}$.

Remark 2.9. This corollary means that the twisted K-groups appearing in Theorem 2.3 are only those associated to torsion classes in $\check{H}^3(G/\Gamma, \mathbb{Z})$, and hence precisely the ones studied by Donovan and Karoubi [12]. In the original version, we guessed that these twisted K-groups should agree with the ordinary ones up to torsion, and the referee suggested that this should be true because the differentials in the Atiyah-Hirzebruch spectral sequence involve only functions of the twisting class and ordinary cohomology operations. Unfortunately, we have not yet been able to fill in the details of this argument.

Corollary 2.10. *Let G, Γ be as in the theorem, give $H^2(\Gamma, \mathbb{T})$ the quotient topology induced by the topology of pointwise convergence on $Z^2(\Gamma, \mathbb{T})$, and let $\omega_i \in H^2(\Gamma, \mathbb{T})$ for $i = 1, 2$. Then $\delta(\omega_1) = \delta(\omega_2)$ if and only if ω_1, ω_2 belong to the same path components of $H^2(\Gamma, \mathbb{T})$.*

Proof. Corollary 2.8 implies that the image of $H^2(\Gamma, \mathbb{T})$ under δ is finite, and since $H^2(\Gamma, \mathbb{T})$ is a compact abelian group, it follows that δ has compact kernel containing the connected component of the identity. On the other hand, $H^2(\Gamma, \mathbb{R})$ is a finite-dimensional vector space and hence its image is a path-connected subgroup of $H^2(\Gamma, \mathbb{T})$ (and in fact must be a torus). It therefore follows from the exact sequence of Theorem 2.5 that $\ker \delta$ is precisely the path-connected component of the identity in $H^2(\Gamma, \mathbb{T})$.

Our next corollary identifies some cases where $\ker \delta = \mathrm{im}\, \mathrm{Res}$. The appropriate condition is that Γ should be Ad-ample in G—that is, the algebraic

group hull of $\operatorname{Ad}\Gamma$ should contain $\operatorname{Ad} G$. Mostow has shown that this is automatic when G is a nilpotent simply-connected Lie group [33].

Corollary 2.11. *Suppose Γ is a discrete cocompact subgroup of a simply-connected solvable Lie group G, and that Γ is* Ad-*ample in G. Then we have an exact sequence*

$$H^2(G, \mathbb{T}) \xrightarrow{\operatorname{Res}} H^2(\Gamma, \mathbb{T}) \xrightarrow{\delta} \check{H}^3(G/\Gamma, \mathbb{Z}).$$

Proof. We shall need to compare the Moore cohomology groups $H^2(G, A)$ with the continuous cohomology groups $H^2_c(G, A)$. By [54, 10.30, 10.32] we have isomorphisms

$$H^2_c(G, \mathbb{R}) \cong H^2_c(G, \mathbb{T}) \cong H^2(\Gamma, \mathbb{T}),$$

where the first is induced by the exponential map π and the second is the natural inclusion. Mostow [33, 8.20] has shown that, when Γ is Ad-ample, restriction induces an isomorphism Res_c of $H^2_c(G, \mathbb{R})$ onto $H^2(\Gamma, \mathbb{R})$. Now the commutative diagram

$$\begin{array}{ccc} H^2_c(G, \mathbb{R}) & \xrightarrow{\quad \pi_* \quad} & H^2_c(G, \mathbb{T}) \cong H^2(G, \mathbb{T}) \\ \operatorname{Res}\downarrow & & \downarrow\operatorname{Res} \\ H^2(\Gamma, \mathbb{R}) & \xrightarrow{\quad \pi_* \quad} & H^2(\Gamma, \mathbb{T}) \end{array}$$

shows that $\operatorname{im}\operatorname{Res} = \operatorname{im}\pi_*$, and the result follows from the theorem.

Corollary 2.12. *Let Γ be a discrete cocompact subgroup of a 3-dimensional simply-connected Lie group G. Then $\delta\colon H^2(\Gamma, \mathbb{T}) \to \check{H}^3(G/\Gamma, \mathbb{Z})$ is zero, and*

$$K_*(C^*(\Gamma, \sigma)) \cong K^{*+1}(G/\Gamma) \cong K_*(C^*(\Gamma)) \quad \textit{for all } \sigma \in H^2(\Gamma, \mathbb{T}).$$

Proof. Since G/Γ is a 3-dimensional compact orientable manifold, Poincaré duality implies $\check{H}^3(G/\Gamma, \mathbb{Z}) \cong H_0(G/\Gamma, \mathbb{Z}) \cong \mathbb{Z}$, and Corollary 2.8 implies $\delta = 0$. The final statement follows from Theorem 2.3.

Example 2.13. (An induced C^*-algebra $\operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} B$ such that $K_*(\operatorname{Ind} B)$ is not isomorphic to $K_*(C_0(\mathbb{R}/\mathbb{Z}) \otimes B)$, in which the subgroup $\Gamma = \mathbb{Z}$ acts trivially on $\widehat{B}$.) We take $B = C_0(X, \mathscr{K})$ and an automorphism α and B which fixes $X = \widehat{B}$. Then $\operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}}(B, \alpha)$ is a continuous-trace C^*-algebra with spectrum $\mathbb{T} \times X$, and its Dixmier-Douady class $\delta(\operatorname{Ind} B) \in \check{H}^3(\mathbb{T} \times X, \mathbb{Z})$ is the extremal product of a generator 1 for $\check{H}^1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}$ with a class $\zeta(\alpha)$ in $\check{H}^2(X, \mathbb{Z})$ [43, Corollary 3.5]. Thus in this case we have $K_*(\operatorname{Ind} B) = K^*(\mathbb{T} \times X, 1 \times \zeta(\alpha))$, which we claim is not equal to $K^*(\mathbb{T} \times X)$ in general. In fact, we shall reverse the argument of this section to compute

$$K_*(\operatorname{Ind} B) \cong K_{*+1}((\operatorname{Ind} B) \rtimes \mathbb{R}) \cong K_{*+1}(B \rtimes_\alpha \mathbb{Z}).$$

Recall that, for $\pi \in (B \rtimes_\alpha \mathbb{Z})^\wedge$, the restriction $\operatorname{Res}\pi$ to B is also irreducible, and the resulting map $\operatorname{Res}\colon (B \rtimes_\alpha \mathbb{Z})^\wedge \to X$ is a principal $\mathbb{T}$-bundle whose class in $\check{H}^2(X, \mathbb{Z})$, is $\zeta(\alpha)$ [40, §2]; the crossed product $B \rtimes_\alpha \mathbb{Z}$ is isomorphic to the pullback $\operatorname{Res}^* B = C_0((B \rtimes_\alpha \mathbb{Z})^\wedge) \otimes_{C(X)} B$ [44, Proposition 1.4]. Since $B = C_0(X, \mathscr{K})$, we have $\operatorname{Res}^* B = C_0((B \rtimes_\alpha \mathbb{Z})^\wedge, \mathscr{K})$, and

$$K_*(\operatorname{Ind} B) \cong K_{*+1}(B \rtimes_\alpha \mathbb{Z}) \cong K^{*+1}((B \rtimes_\alpha \mathbb{Z})^\wedge).$$

If we choose X and α so that Res: $(B \rtimes_\alpha \mathbb{Z})^\frown \to X$ is the Hopf fibration $S^3 \to S^2$ ([40 Corollary 35] says this can be arranged), then

$$K_i(\operatorname{Ind} B) \cong K^{i+1}(S^3) \cong \mathbb{Z} \quad \text{for } i = 0, 1,$$

whereas

$$K_i(C_0(G/\Gamma) \otimes B) \cong K^i(\mathbb{T} \times S^2) \cong \mathbb{Z}^2 \quad \text{for } i = 0, 1.$$

3. Groups of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$

In this section we consider semidirect products of the form $\Gamma = \mathbb{Z}^n \rtimes_A \mathbb{Z}$, where $A \in \mathrm{GL}(n, \mathbb{Z})$ and $p \cdot m = A^p m$ for $p \in \mathbb{Z}$, $m \in \mathbb{Z}^n$. We begin by describing the multipliers of Γ, using the approach discussed in the appendix. We then give criteria for simplicity of the algebras $C^*(\Gamma, \sigma)$; these are consequences of our Theorem 1.5 and a theorem of Kishimoto on crossed products of simple C^*-algebras by outer actions. Our result is stronger than that given in [37, Example 1.10] (see Remark 3.3), and we given an example illustrating this.

When the matrix A has the form $\exp X$, we can embed Γ in the Lie group $G - \mathbb{R}^n \rtimes_{\exp rX} \mathbb{R}$, and try to compute $K_*(C^*(\Gamma, \sigma))$ by the method of §2. As we saw there, a crucial question is whether the multiplier σ is the exponential of a cocycle in $Z^2(\Gamma, \mathbb{R})$, and we discuss this point in Proposition 3.5. When $n = 2$, all multipliers are exponentials, but when $n = 3$ we can write down an $A = \exp X$ and $\sigma \in Z^2(\Gamma, \mathbb{T})$ which does not have the form $\exp 2\pi i\omega$. For this example we can compute $K_*(C^*(\gamma, \sigma))$ using the Pimsner-Voiculescu exact sequence, and it turns out to be different from $K_*(C^*(\Gamma))$.

To compute the multipliers of Γ, we use the Lyndon-Hochschild-Serre spectral sequence with $N = \mathbb{Z}^n$: since $G/N \cong \mathbb{Z}$, we have $E_2^{p,q} = 0$ for $p \geq 2$, and the sequence collapses at the E_2-level. To find specific realisations for these multipliers, we can use the approach of Appendix 2, and deduce from the exact sequence (A1)

$$0 \longrightarrow H^1(\mathbb{Z}, (\mathbb{Z}^n)^\frown) \longrightarrow H^2(\Gamma, \mathbb{T}) \xrightarrow{\text{Res}} H^2(\mathbb{Z}^n, \mathbb{T})^{\mathbb{Z}} \longrightarrow 0$$

that each is equivalent to one of the form $\sigma_{\alpha, f}$ (see equation (A4)). In fact, the exact sequence also splits in this case.

Proposition 3.1. *Let $\Gamma = \mathbb{Z}^n \rtimes_A \mathbb{Z}$. Then there is a homomorphism ϕ of $H^2(\mathbb{Z}^n, \Gamma)^{\mathbb{Z}}$ into $H^2(\Gamma, \mathbb{T})$ which is a splitting for* Res.

Proof. Recall that every multiplier $\alpha \in Z^2(\mathbb{Z}^n, \mathbb{T})$ is equivalent to one of the form $\alpha(v, w) = \exp(2\pi i v^t M w)$ for some skew-symmetric real matrix; two such matrices M_1, M_2 give equivalent multipliers if and only if $M_1 = M_2$ modulo $\frac{1}{2}$ [3, Theorem 3.3]. Now because the class of α is invariant,

$$(A^{-1})^*\alpha(v, w) = \alpha(Av, Aw) = \exp(2\pi i v^t (A^t M A) w)$$

is equivalent to α, and $A^t M A - M = 0 \mod \frac{1}{2}$. Since $A^t M A - M$ is skew-symmetric with entries in $\mathbb{Z} \cdot \frac{1}{2}$ there is a real *symmetric* matrix N with entries in $\mathbb{Z} \cdot \frac{1}{2}$ such that $A^t M A = M + N \mod 1$. Define $b: \mathbb{Z}^n \to \mathbb{T}$ by $b(v) =$

$\exp(\pi i v^t N v)$, and let $f\colon \mathbb{Z}\times\mathbb{Z}^n \to \mathbb{T}$ be the map satisfying condition (A2) and $f(1)(Av) = b(v)$—specially, we define

$$f(p)(v) = \begin{cases} 1 & \text{if } p = 0, \\ b(A^{-1}v)b(A^{-2}v)\cdots b(A^{-p}v) & \text{if } p > 0, \\ [b(v)b(Av)\cdots b(A^{-1-p}v)]^{-1} & \text{if } p < 0. \end{cases}$$

Then one can verify that the pair (α, f) satisfies the compatibility condition (A3) (observing that

$$\exp(2\pi i v^t N w) = b(v+w)[b(v)b(w)]^{-1}$$

makes the calculations easier).

We define $\eta(\alpha)$ to be the class of the multiplier σ defined by (α, f) and (A4). To see that this is independent of the choice of N, observe that if $N_1 = N_2 \bmod 1$, then $(v^t N_1 v)/2 = (v^t N_2 v)/2 \bmod 1$, and hence the corresponding b's and f's are exactly the same. To see that $\eta(\alpha)$ depends only on the class of α in $H^2(\mathbb{Z}^n, \mathbb{T})$, suppose β is given by a skew-symmetric matrix M' with $M' = M \bmod \frac{1}{2}$. Let P be a symmetric matrix with $M' = M + P \bmod 1$, and define $c(v) = \exp(2\pi i v^t P v)$. then

$$\beta(v, w) = \alpha(v, w)\exp(2\pi i v^t P w) = \alpha(v, w)c(v+w)[c(v)c(w)]^{-1},$$

so the multiplier $\sigma_{\alpha,f}$ is equivalent to $\sigma_{\beta,g}$, where

$$g(p)(v) = f(p)(v)c(v)c(A^{-p}v)^{-1}$$

(see equation (A5)). Note that this is determined by

$$g(1)(v) = \exp 2\pi i\{v^t((A^{-1})^t N A^{-1} + P - (A^{-1})^t P A^{-1})v/2\}.$$

Since we have

$$A^t M' A = M' + N - P + A^t P A \qquad \bmod 1,$$

$\eta(\beta)$ is by definition the class of the cocycle $\sigma_{\beta,f'}$, where

$$f'(1)(v) = b'(A^{-1}v) = \exp 2\pi i\{(A^{-1}v)^t(N - P + A^t P A)(A^{-1}v)/2\}.$$

But $f' = g$, so $\sigma_{\beta,f'}$ is equivalent to $\sigma_{\alpha,f}$, and $\eta(\alpha) = \eta(\beta)$. Finally, because $\eta(\alpha)$ is independent of N it is easy to see η is a homomorphism, and $\operatorname{Res}\eta(\alpha) = \alpha$ because $f(0) \equiv 1$.

Theorem 3.2. *Let $A \in \mathrm{GL}(n, \mathbb{Z})$, let $\Gamma = \mathbb{Z}^n \rtimes_A \mathbb{Z}$ and let $\sigma = \sigma_{\alpha,f}$ be the multiplier of Γ corresponding via* (A4) *to the pair (α, f) satisfying* (A2), (A3). *Let S be the symmetriser subgroup (in $\mathbb{Z}^n$) of $\alpha \in Z^2(\mathbb{Z}^n, \mathbb{T})$, and suppose that $\alpha|_{S\times S} = 1$. (This can be arranged without changing the class of α in $H^2(\mathbb{Z}^n, \mathbb{T})$, but the corresponding f may have to be changed too—see the appendix.)*

(1) *If $S \neq \{0\}$, then $C^*(\Gamma, \sigma)$ is simple if and only if $\Gamma/\mathbb{Z}^n = \mathbb{Z}$ acts minimally on $\widehat{S}$ for the action given by*

$$(p\cdot\gamma)(v) = f(p)(v)^{-1}\gamma(A^{-p}v).$$

(2) *If $S = \{0\}$, let N be the normal subgroup*

$$N = \{(v, p) \in \mathbb{Z}^n \rtimes \mathbb{Z} : A^p = I\}$$

of Γ. *Then* $C^*(\Gamma, \sigma)$ *is simple if and only if the symmetriser of* $\sigma|_{N\times N}$ *is* $\{0\}$.

Remark 3.3. Since S is a subgroup of $\mathbb{Z}^n$, S will be isomorphic to $\mathbb{Z}^m$ for some $m \leq n$. Now condition (A3) implies that S is invariant under A, and that each $f(p)(\cdot)$ is a homomorphism on S; thus by (A2), $f\colon \mathbb{Z} \to \widehat{S}$ is a cocycle for the action of $\mathbb{Z}$ on $\widehat{S}$ given by $p \cdot \gamma(v) = \gamma(A^{-p}v)$. The dual group $\widehat{S}$ is isomorphic to $\mathbb{T}^m$, and the action described in (1) is then an affine action of $\mathbb{Z}$ on $\mathbb{T}^m$ in the sense of [37]. By results of Hahn [19] and Hoare and Parry [22, Corollary 1 to Theorem 3], such an action can be minimal only if $A - I$ is nilpotent, and then it is minimal if and only if for every $v \in S$ such that $\{A^p v : p \in \mathbb{Z}\}$ is finite, there exists $p \in \mathbb{Z}$ with $A^p v = v$ and $f(p)(v) \neq 1$ (see [37, Corollary 2.5]). Now if A is of infinite order, condition (2) reduces to the trivial case of (1) and we recover the necessary and sufficient condition for simplicity of $C^*(\Gamma, \sigma)$ given in [37, Example 1.10]; if A has finite order, condition (2) is the same as that given in [37, 1.10]. Note that we do *not* need to assume $A - I$ is nilpotent on all of $\mathbb{Z}^n$, just on S, and thus the present result is stronger; indeed, we give an example below in which $C^*(\mathbb{Z} \rtimes_A \mathbb{Z}, \sigma)$ is simple even though $A - I$ is not nilpotent.

Proof of Theorem 3.2. We apply Theorem 1.5 to the subgroup $N = \mathbb{Z}^n$. Using (A4), we can calculate that the action of $\mathbb{Z} = \Gamma/N$ on $\widehat{S}$ is given by

$$(p \cdot \gamma)(v) = \tilde{\sigma}((v, 0), (0, p))\gamma((-p) \cdot v) = f(p)(v)^{-1}\gamma(A^{-p}v).$$

If $S \neq \{0\}$, then $S \cong \mathbb{Z}^k$ and $\widehat{S} \cong \mathbb{T}^k$ for some $k > 0$; thus $\mathbb{Z}$ can only act minimally on $\widehat{S}$ if it also acts freely, and (1) therefore follows from Theorem 1.5. To establish the necessity in (2), we suppose $\sigma|_{N\times N}$ has nontrivial symmetriser S. In this case, N must be strictly larger than $\mathbb{Z}^n$, and A must have finite order p, say. Then $N \cong \mathbb{Z}^{n+1}$, $\Gamma/N \cong \mathbb{Z}/p\mathbb{Z}$, and $S \cong \mathbb{Z}^k$ for some $k > 0$. But there is no way a finite group can act minimally on a torus, so Theorem 1.5 implies that $C^*(\Gamma, \sigma)$ is not simple.

For the other direction in (2), we use a theorem of Kishimoto [25] which says that if a countable discrete group acts on a simple C^*-algebra by outer automorphisms, then the reduced crossed product is simple. To apply this, we decompose

$$C^*(\Gamma, \sigma) \cong C^*(\mathbb{Z}^n, \operatorname{Res} \sigma = \alpha) \rtimes \mathbb{Z}$$

(by, for example, [38, 4.1]), and suppose the symmetriser of $\sigma|_{N\times N}$ is trivial. We claim that the von Neumann algebra $W^*(\Gamma, \sigma)$ generated by the left σ-representation of Γ on $L^2(\Gamma)$ is a factor. For by [37, Proposition 1.3], it is enough to show that every σ-regular conjugacy class of Γ is infinite (see [37, §1] for the definition). If $A^p \neq I$, then the conjugacy class of (v, p) is infinite, so we suppose $A^p = I$. Then $(v, p) \in N$, and because $\sigma|_{N\times N}$ has trivial symmetriser, there exists $(w, q) \in N$ such that $\tilde{\sigma}((v, p), (w, q)) \neq 1$. Since N is contained in the centraliser of (v, p), this shows the conjugacy class of (v, p) is not σ-regular, and $W^*(\Gamma, \sigma)$ is a factor, as claimed. Because we can decompose $W^*(\Gamma, \sigma)$ as a von Neumann algebra crossed product $W^*(\mathbb{Z}^n, \alpha) \rtimes \mathbb{Z}$, and $W^*(\mathbb{Z}^n, \alpha)$ is also a factor, this implies that $\mathbb{Z}$ acts on $W^*(\mathbb{Z}^n, \alpha)$ by outer automorphisms. But the action of $\mathbb{Z}$ on $C^*(\mathbb{Z}^n, \alpha)$ is the restriction of this action on $W^*(\mathbb{Z}^n, \alpha)$, so it too must consist of outer automorphisms. Thus Kishimoto's theorem applies, and $C^*(\Gamma, \sigma)$ is simple.

Example 3.4 (where $C^*(\mathbb{Z}^2 \rtimes_A \mathbb{Z}, \sigma)$ is simple but $A - I$ is not nilpotent). Let $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$. We have

$$H^1(\mathbb{Z}, (\mathbb{Z}^2)^\wedge) = (\mathbb{Z}^2)^\wedge / \{\gamma(A^*\gamma)^{-1} : \gamma \in (\mathbb{Z}^2)^\wedge\}.$$

Since the endomorphism $\gamma \to \gamma(A^*\gamma)^{-1}$ of $(\mathbb{Z}^2)^\wedge \cong \mathbb{R}^2/\mathbb{Z}^2$ is given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$, it is onto, and $H^1(\mathbb{Z}, (\mathbb{Z}^2)^\wedge) = 0$. On the other hand, every multiplier of $\mathbb{Z}^2$ is equivalent to one of the form $\alpha_\theta(v, w) = \exp(2\pi i v^t M w)$, where $M = \left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right)$ for some $\theta \in [0, \frac{1}{2})$, and we have $A^*\alpha_\theta \sim \alpha_\theta$ if and only if $A^t M A = M \bmod \frac{1}{2}$. But

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \quad \text{for all } \theta,$$

so $\mathbb{Z}$ acts trivially on $H^2(\mathbb{Z}^2, \mathbb{T})$, and $(\alpha_\theta, 1)$ satisfies (A2), (A3). Thus every multiplier of $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is equivalent to one of the form

$$\sigma_\theta((v, p), (w, q)) = \exp 2\pi i \left(v^t \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^p w \right).$$

If θ is irrational, the symmetriser of $\alpha_\theta = \mathrm{Res}\, \sigma_\theta$ in $\mathbb{Z}^2$ is $\{0\}$, and since A has infinite order in $\mathrm{GL}(2, \mathbb{Z})$, Theorem 3.2 implies that $C^*(\mathbb{Z}^2 \rtimes_A \mathbb{Z}, \alpha_\theta)$ is simple. Of course,

$$A - I = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

is not nilpotent, so this could not be deduced from [37].

Proposition 3.5. *Let $A \in \mathrm{SL}(n, \mathbb{Z})$ and $\Gamma = \mathbb{Z}^n \rtimes_A \mathbb{Z}$.*

(a) *Suppose $A = \exp X$ for some $X \in M_n(\mathbb{R})$ with no nonzero eigenvalues in $2\pi i\mathbb{Z}$, and $G = \mathbb{R}^n \rtimes_{\exp rX} \mathbb{R}$. Then $\sigma \in Z^2(\Gamma, \mathbb{T})$ is equivalent to* $\mathrm{Res}\, \omega$ *for some $\omega \in Z^2(G, \mathbb{T})$ if and only if there exists τ equivalent to σ and a skew-symmetric matrix $L \in M_n(\mathbb{R})$ such that $\tau(m, n) = \exp(2\pi i m^t L n)$ for $m, n \in \mathbb{Z}^n \subset \Gamma$, and*

$$(\exp rX)^t L (\exp rX) = L \quad \textit{for all } r \in \mathbb{R}.$$

(b) *A multiplier $\sigma \in Z^2(\Gamma, \mathbb{T})$ is equivalent to one of the form* $\exp 2\pi i\omega$ *for $\omega \in Z^2(\Gamma, \mathbb{R})$ if and only if there exists τ equivalent to σ and a skew-symmetric matrix $L \in M_n(\mathbb{R})$ such that $\tau(m, n) = \exp(2\pi i m^t L n)$ for $m, n \in \mathbb{Z}^n$ and $A^t L A = L$.*

Proof. (a) Suppose σ is equivalent to $\mathrm{Res}\, \omega$ for some $\omega \in Z^2(G, \mathbb{T})$. The restriction of ω to $\mathbb{R}^n$ is equivalent to a multiplier of the form $\omega_L(v, w) = \exp(2\pi i v^t L w)$ where L is a skew-symmetric real matrix, and two such multipliers ω_K, ω_L are equivalent only if $K = L$. Since ω_L is equivalent to $\alpha = \mathrm{Res}\, \sigma$ on $\mathbb{Z}^n$, we have $L = M \bmod \frac{1}{2}$. On the other hand, since the range of $\mathrm{Res}\colon H^2(G, \mathbb{T}) \to H^2(\mathbb{R}^n, \mathbb{T})$ is fixed by the natural action of $\mathbb{R} = G/\mathbb{R}^n$, the multiplier

$$\omega_K(v, w) = \omega_L((\exp rX)v, (\exp rX)w) = \omega_{(\exp rX)^t L (\exp rX)}(v, w)$$

is equivalent to ω_L, and therefore we have

$$(\exp rX)^t L (\exp rX) = L \quad \text{for } r \in \mathbb{R}.$$

Now suppose we have such an L and X does not have eigenvalues of the form $2\pi in$ for $n \neq 0$. Without changing the class of $\sigma_{\alpha,f}$ we may suppose that $M = L$, and then conditions (A2), (A3) say that f is a cocycle in $Z^1(\mathbb{Z}, (\mathbb{Z}^n)^\wedge)$. Since the cocycle α extends to a cocycle $\beta \in Z^2(\mathbb{R}^n, \mathbb{T})$ such that

$$\beta((\exp rX)v, (\exp rX)w) = \beta(v, w) \quad \text{for all } r,$$

we have to show how to construct a cocycle $g \in Z^1(\mathbb{R}, (\mathbb{R}^n)^\wedge)$ whose restriction to $\mathbb{Z} \times \mathbb{Z}^n$ is equivalent to f. The cocycle f is determined by $f(1) \in (\mathbb{Z}^n)^\wedge$, which is the restriction of a character $\gamma_v : w \to \exp(2\pi i v^t w)$ of $\mathbb{R}^n$. The action $p \cdot v = (\exp pX)v$ restricts to the given action of $\mathbb{Z}$ on $\mathbb{Z}^n \subset \mathbb{R}^n$, and we can therefore extend f to a cocycle $h \in Z^1(\mathbb{Z}, (\mathbb{R}^n)^\wedge)$ by setting $h(1) = \gamma_v$ and using the cocycle identity to define $h(p)$. We want to show that, under our hypothesis on X, the class of h must belong to the range of the restriction map $\mathrm{Res}: H^1(\mathbb{R}, (\mathbb{R}^n)^\wedge) \to H^1(\mathbb{Z}, (\mathbb{R}^n)^\wedge)$. According to Moore's spectral sequence, the range of Res is the kernel of a homomorphism

$$d_2 : H^1(\mathbb{Z}, (\mathbb{R}^n)^\wedge)^{\mathbb{R}} \to H^2(\mathbb{T}, H^0(\mathbb{R}, (\mathbb{R}^n)^\wedge)) = H^2(\mathbb{T}, ((\mathbb{R}^n)^\wedge)^{\mathbb{R}});$$

since $((\mathbb{R}^n)^\wedge)^{\mathbb{R}}$ is a vector space, this last group is 0 [30, Theorem 1.1], and we have only to show that $\mathbb{R}$ acts trivially on $H^1(\mathbb{Z}, (\mathbb{R}^n)^\wedge)$.

As we saw above, $Z^1(\mathbb{Z}, (\mathbb{R}^n)^\wedge)$ is isomorphic to $(\mathbb{R}^n)^\wedge$ and hence to $\mathbb{R}^n$; the action of $\mathbb{R}$ on Z^1 is given pointwise by the dual action of $\mathbb{R}$ on $(\mathbb{R}^n)^\wedge$, and hence is given on $\mathbb{R}^n$ by $r \cdot v = \exp(-rX)^t(v)$. The coboundaries in Z^1 are carried by this isomorphism into the vectors of the form $(I - (A^{-1})^t)(v)$, and we therefore have

$$H^1(\mathbb{Z}, (\mathbb{R}^n)^\wedge) \cong \mathbb{R}^n / \operatorname{range}(I - (A^{-1})^t).$$

Observe that $\mathbb{R}$ will act trivially on this space provided there is a neighbourhood of 0 which acts trivially. Now $r \cdot [v] = [v]$ if and only if the vector $[I - \exp(-rX)^t](v)$ belongs to the range of $I - (A^{-1})^t$, so r acts trivially if and only if

$$\operatorname{range}(I - \exp(-rX)^t) \subset \operatorname{range}(I - (A^{-1})^t). \tag{4.1}$$

But

$$I - \exp(-rX)^t = I - \exp(-rX^t) = rX^t \left(I - \frac{rX}{2!} + \frac{r^2X^2}{3!} - \cdots \right)^t,$$

and for small r, $(I - rX/2 + \cdots)$ is invertible, so the range of $I - \exp(-rX)^t$ is just the range of X^t. Since $A^{-1} = \exp(-X)$, the range of $I - (A^{-1})^t$ is contained in the range of X^t, and (4.1) will hold if and only if

$$\dim \ker(I - (A^{-1})^t) = \dim \ker(I - \exp(-rX)^t). \tag{4.2}$$

The kernel of $I - (A^{-1})^t$ is the eigenspace of A^t corresponding to the eigenvalue 1, and since $A^t = \exp(X^t)$, this is the span of the eigenspaces of X^t corresponding to the eigenvalues $2\pi in$. Similarly the kernel of $I - \exp(-rX)^t$ is the eigenspace of X^t corresponding to the eigenvalues $2\pi in/r$, and for small r this is just the kernel of X^t. Hence provided $2\pi n$ is not an eigenvalue of X for $n \neq 0$, these two kernels coincide, and (4.2) holds. Thus all sufficiently small r's act trivially on $H^1(Z, (\mathbb{R}^n)^\wedge)$, and (a) follows.

(b) Each cocycle in $Z^2(\mathbb{Z}^n, \mathbb{R})$ is equivalent to one of the form $\omega_L(m, n) = m^t L n$, where L is skew-symmetric, and two such cocycles ω_L, ω_M are equivalent only if $L = M$. Thus if $\sigma = \exp 2\pi i\omega$ we may suppose $\omega|_{\mathbb{Z}^n \times \mathbb{Z}^n} = \omega_L$, and the invariance of ω_L under $\mathbb{Z}$ forces $A^t L A = L$. Conversely, suppose $A^t A L = L$ and $\sigma = \exp 2\pi i\omega_L$ on $\mathbb{Z}^n$. We view σ as $\sigma_{\alpha, f}$ as in the appendix, where $\alpha = \exp 2\pi i\omega_L$ and $f\colon \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{T}$ satisfy (A2) and (A3). Since $A^t L A = L$, these equations say $f \in Z^1(\mathbb{Z}, (\mathbb{Z}^n)^\wedge)$, which is completely determined by $f(1) \in (\mathbb{Z}^n)^\wedge$. Choose $v \in \mathbb{R}^n$ such that $f(1)(n) = \exp(2\pi i v^t n)$. Then we can define $g(1) \in \operatorname{Hom}(\mathbb{Z}^n, \mathbb{R})$ by $g(1)(w) = v^t w$, extend g to an element of $Z^1(\mathbb{Z}, \operatorname{Hom}(\mathbb{Z}^n, \mathbb{R}))$ using the cocycle identity, and use the additive version of (A4) to define a cocycle $\omega = \omega_{\omega_L, g} \in Z^2(\mathbb{Z}^n \rtimes \mathbb{Z}, \mathbb{R})$ such that $\exp 2\pi i\omega = \sigma$.

Corollary 3.6. *Suppose* $A \in \mathrm{SL}(2, \mathbb{Z})$ *and* $\Gamma = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$.

(a) *If* $A = \exp X$ *and* X *has no nonzero eigenvalues of the form* $2\pi i n$, *then every multiplier of* Γ *is equivalent to the restriction of a multiplier of* $\mathbb{R}^2 \rtimes_{\exp rX} \mathbb{R}$.

(b) *Every multiplier in* $Z^2(\Gamma, \mathbb{T})$ *is equivalent to* $\exp 2\pi i\omega$ *for some* $\omega \in Z^2(\Gamma, \mathbb{R})$.

Proof. If L is a skew-symmetric 2×2 matrix and $A \in \mathrm{SL}(2, \mathbb{R})$, then $A^t L A = L$.

Remark 3.7. Part (b) of this corollary has already been proved indirectly in Corollary 2.11. However, the example in Remark 3.8(3) below shows that the hypothesis on X in part (a) cannot be lifted from the corollary or the theorem.

Remark 3.8. (1) The proof of the theorem shows that the elements f of $Z^1(\mathbb{Z}, (\mathbb{Z}^n)^\wedge)$ for which $\sigma_{1, f}$ does not lift to G are those where $f(1) = \gamma_v$ implies

$$[I - \exp(-rX)^t](v) \notin \operatorname{range}(I - (A^{-1})^t) \quad \text{for some } r.$$

We shall use this observation in (3) below.

(2) Although it cannot be lifted, the hypothesis on X in part (a) is fairly innocuous—indeed, we claim that if A has the form $\exp X$ for $X \in M_n(\mathbb{R})$, then we can choose such an X which has no eigenvalues of the form $2\pi i n$ for $n \neq 0$. To see this, we recall that X has a real normal form

$$X = T \begin{pmatrix} J_1 & 0 & \\ 0 & J_2 & \\ & & \ddots \end{pmatrix} T^{-1}, \qquad T \in \mathrm{GL}(n, \mathbb{R}),$$

in which the blocks J correspond to complex conjugate pairs of eigenvalues λ; if λ is real, J is a usual Jordan matrix, and if $\lambda = a \pm i\beta$, then J consists of 2×2 blocks $\left(\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right)$ along the diagonal, possibly with some 2×2 identity matrices sitting immediately above the diagonal. We shall show that if J corresponds to $\lambda = \pm 2\pi i n$, then replacing the $2\pi n$'s by 0 (to give J', say) will not change $\exp X$. Since $\exp(TYT^{-1}) = T(\exp Y)T^{-1}$, we can exponentiate block by block, and it is enough to show $\exp J = \exp J'$. But then we can write $J = J' + D$, where D is a block diagonal matrix. One can easily verify that J' commutes with D (since all the diagonal blocks in J are the same), and hence $\exp J = (\exp J')(\exp D)$. Since

$$\exp \begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} = \begin{pmatrix} \cos r & \sin r \\ -\sin r & \cos r \end{pmatrix},$$

we have $\exp D = I$, and the claim follows.

(3) Different choices of the matrix X with $\exp X = A$ give rise to different actions of $\mathbb{R}$ on $\mathbb{R}^n$, and hence give different solvable Lie groups in which Γ can be embedded as a discrete cocompact subgroup. The proof of Remark (2) above shows that, for $n = 2$, a different choice is only possible when $A = I$—when Γ is the integer Heisenberg group, for example, $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and we must take $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, which leads to the usual embedding of Γ in the real Heisenberg group. When $A = I$, the obvious choice $X = 0$ gives the usual embedding of $\mathbb{Z}^2 \rtimes_A \mathbb{Z} = \mathbb{Z}^3$ in $\mathbb{R}^3$, and the other possible choices are all conjugate in $\mathrm{GL}(2, \mathbb{R})$ to $\left(\begin{smallmatrix} 0 & 2\pi n \\ -2\pi n & 0 \end{smallmatrix}\right)$. We then have

$$\exp rX = \begin{pmatrix} \cos 2\pi nr & \sin 2\pi nr \\ -\sin 2\pi nr & \cos 2\pi nr \end{pmatrix},$$

and we obtain the embeddings of $\mathbb{Z}^3$ in the semidirect products $\mathbb{R}^2 \rtimes \mathbb{R}$, in which $r \in \mathbb{R}$ acts on $\mathbb{R}^2$ by rotation through $2\pi nr$. For these examples, $I - (A^{-1})^t = 0$ and $I - \exp(-rX)^t$ is invertible for small r, so no nonzero vector v satisfies the condition of (1), and no cocycle $\sigma_{\alpha, f}$ with f nontrivial lifts to G. It follows from Corollary 3.6 and Theorem 2.5 that $\delta(\sigma_{\alpha, f}) = 0$, so this provides examples where $\ker \delta \colon H^2(\Gamma, \mathbb{T}) \to \check{H}^3(G/\Gamma, \mathbb{T})$ strictly contains the image of $\mathrm{Res} : H^2(G, \mathbb{T}) \to H^2(\Gamma, \mathbb{T})$ (see §2).

Example 3.9. (A multiplier σ of a semidirect product $\Gamma = \mathbb{Z}^3 \rtimes_A \mathbb{Z}$ which is not equivalent to one of the form $\exp 2\pi i w$.) We take

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & 0 \\ 0 & 0 & \end{pmatrix},$$

and define $\alpha(m, n) = \exp(2\pi i m^t M n)$ for $m, n \in \mathbb{Z}^3$. Since

$$A^t M A = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} = M \mod \frac{1}{2},$$

$\alpha \in H^2(\mathbb{Z}^3, \mathbb{T})$, is invariant under the action of $\mathbb{Z}$, and by Proposition 3.1 there is a multiplier $\sigma \in H^2(\Gamma, \mathbb{T})$ such that $\mathrm{Res}\, \sigma = \alpha$. We claim σ does not have the form $\exp 2\pi i w$. For if

$$L = \begin{pmatrix} 0 & p & q \\ -p & 0 & s \\ -q & -s & 0 \end{pmatrix}$$

then

$$A^t L A = \begin{pmatrix} 0 & p & q \\ -p & 0 & -2p + s \\ -q & 2p - s & 0 \end{pmatrix},$$

so $A^t L A = L$ implies $p = 0$ and L cannot satisfy $L = M \bmod \frac{1}{2}$. Thus there is no L satisfying the conditions of Proposition 3.5(b), and σ does not have the form $\exp 2\pi i w$.

If we want to write down a specific formula for such a multiplier σ, we need to find a function $f : \mathbb{Z} \times \mathbb{Z}^3 \to \mathbb{T}$ such that the pair (α, f) satisfies (A2), (A3),

and we actually showed how to do this in the proof of Proposition 3.1: we take

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix},$$

define $f(1)(Av) = \exp(\pi i v^t N v)$, and use equation (A2) to find $f(p)$ for $p \neq 1$. We could then work out a formula for $\sigma_{\alpha, f}$ using (A4). However, we can find a multiplier equivalent to $\sigma_{\alpha, f}$ with a much neater formula by observing that α is equivalent to $\beta(v, w) = \exp(\pi i v_2 w_1)$; indeed, if $\rho(v) = \exp(-\pi i v_1 v_2/2)$, then

$$\beta(v, w) = \alpha(v, w)\rho(v + w)[\rho(v)\rho(w)]^{-1}.$$

According to formula (A5), if

$$g(p)(v) = f(p)(v)\rho(v)\rho(A^{-p}v)^{-1},$$

then (β, g) satisfies (A2), (A3); but multiplying this out shows $g(1)(Av) = 1$ for all v, so $(\beta, 1)$ satisfies (A2), (A3) and

$$\sigma((v, p), (w, q)) = \beta(v, A^p w) = \exp(\pi i v_2 w_1) = (-1)^{v_2 w_1}$$

defines a multiplier σ with $\operatorname{Res} \sigma$ equivalent to α.

We shall finish by showing that, with Γ and σ as in this example, the K-theory of $C^*(\Gamma, \sigma)$ is different from that of $C^*(\Gamma)$. For the rest of this section, then, A, $\Gamma = \mathbb{Z}^3 \rtimes_A \mathbb{Z}$, and σ will be as in Example 3.9, and Γ_0 will be the group $\mathbb{Z}^3 \rtimes_B \mathbb{Z}$ where

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We shall show that $C^*(\Gamma, \sigma)$ is Morita equivalent to the ordinary group algebra $C^*(\Gamma_0)$, and then apply the Pimsner-Voiculescu exact sequence to compute $K_*(C^*(\Gamma_0))$ and $K_*(C^*(\Gamma))$.

Lemma 3.10. *With the above notation, $C^*(\Gamma, \sigma)$ is strongly Morita equivalent to $C^*(\Gamma_0)$.*

Proof. We begin by observing that $C^*(\Gamma, \sigma)$ has unitary generators U, V, W, Z satisfying

$$UV = -VU, \quad UW = WU, \quad VW = WV,$$
$$\operatorname{Ad} Z(U) = U, \quad \operatorname{Ad} Z(V) = V, \quad \operatorname{Ad} Z(W) = U^2 W;$$

the algebra generated by U, V, W is isomorphic to $C^*(\mathbb{Z}^3, \operatorname{Res} \sigma)$, and $C^*(\Gamma, \sigma)$ is isomorphic to $C^*(\mathbb{Z}^3, \operatorname{Res} \sigma) \rtimes_{\operatorname{Ad} Z} \mathbb{Z}$. Similarly, we can view $C^*(\Gamma_0)$ as the crossed product $C^*(\mathbb{Z}^3) \rtimes_{\operatorname{Ad} Z_0} \mathbb{Z}$ generated by three commuting unitaries U_0, V_0, W_0 and a fourth Z_0 satisfying

$$\operatorname{Ad} Z_0(U) = U, \quad \operatorname{Ad} Z_0(V) = V, \quad \operatorname{Ad} Z_0(W) = UW.$$

Using work of Rieffel [48], we shall show that $C^*(\mathbb{Z}^3)$ is strongly Morita equivalent to $C^*(\mathbb{Z}^3, \operatorname{Res} \sigma)$, via an imprimitivity bimodule X which carries an action of $\mathbb{Z}$ compatible with the actions $\operatorname{Ad} Z_0$ and $\operatorname{Ad} Z$.

In [48], Rieffel constructs a $C(\mathbb{T}^2) - A_{1/2}$ imprimitivity bimodule $X(2, 1)$; since $C^*(\mathbb{Z}^3) \cong C(T^2) \otimes C(\mathbb{T})$ and $C^*(\mathbb{Z}^3, \operatorname{Res} \sigma) \cong A_{1/2} \otimes C(\mathbb{T})$ (via $U \to$

$U \otimes 1$, $V \to V \otimes 1$, $W \to 1 \otimes W$), we can take as our bimodule X the tensor product $X(2, 1) \otimes C(\mathbb{T})$. By [48, Proposition 3.8], $X(2, 1)$ consists of the continuous functions $h : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$ satisfying

$$h(s, t-2) = \exp(2\pi i s) h(s, t) \quad \text{for } s \in \mathbb{R}/\mathbb{Z},\ t \in \mathbb{R},$$

and, as observed in [48, p. 597], the norm on $X(2, 1)$ is equivalent to the usual sup norm. Thus we may view our bimodule X as the continuous functions on $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$ satisfying

$$h(w, t-2, z) = w h(w, t, z) \quad \text{for } (w, t, z) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}.$$

Since the left action of $C(\mathbb{T}^2)$ on $X(2, 1)$ is by pointwise multiplication, $C^*(\mathbb{Z}^3) \cong C(\mathbb{T}^3)$ also acts by pointwise multiplication on X, and we may take

$$\begin{aligned}(U_0 h)(w, t, z) &= \exp(2\pi i t) h(w, t, z),\\ (V_0 h)(w, t, z) &= w h(w, t, z),\\ (W_0 h)(w, t, z) &= z h(w, t, z).\end{aligned}$$

The $C(\mathbb{T}^2) \otimes C(\mathbb{T})$-valued inner product on $X(2, 1) \otimes C(\mathbb{T})$ is given by

$$\langle x \otimes f, y \otimes g \rangle = \langle x, y \rangle_{C(\mathbb{T}^2)} \otimes f\overline{g},$$

and $\langle h, k \rangle_{C(\mathbb{T}^2)}$ is described in [48, Lemma 3.6], so the $C(\mathbb{T}^3)$-valued inner product on X is given by

$$\langle h, k \rangle(w, \exp(2\pi i t), z) = \sum_{j=0}^{1} h(w, t+j, z)\overline{k(w, t+j, z)}.$$

To compute the right action of $C^*(\mathbb{Z}^3, \operatorname{Res} \sigma)$ on X, we have first to compute the action of $A_{1/2} = C^*(U, V)$ on $X(2, 1)$. In his construction, Rieffel views $A_{1/2}$ as the transformation group algebra $C^*(H, G/K)$, where $G = \mathbb{R}$, $H = \mathbb{Z}$ and $K = 2\mathbb{Z}$, and $X(2, 1)$ as a partial Fourier transform of $C_c(G)$. Identifying G/K with $\mathbb{T}$ via $t \to \exp(\pi i t)$, the actions of $\mathbb{Z} = H$ and $C(\mathbb{T}) = C(G/K)$ on $C_c(G)$ are given by

$$(x \cdot n)(t) = x(t-n), \qquad (x \cdot f) = x(t) f(\exp(\pi i t));$$

if we view $A_{1/2}$ as $C^*(U, V)$ where U, V generate $C(\mathbb{T})$, $\mathbb{Z}$ respectively, we have

$$(xU)(t) = x(t)\exp(\pi i t), \qquad (xV)(t) = x(t-1).$$

The partial Fourier transform is given by

$$\hat{x}(w, t) = \sum_{n \in \mathbb{Z}} w^n x(t+2n),$$

and now a simple calculation shows $A_{1/2} = C^*(U, V)$ acts on $X(2, 1)$ via

$$(hV)(w, t) = h(w, t-1), \qquad (hU)(w, t) = h(w, t)\exp(\pi i t).$$

Thus when we view X as functions on $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$, the algebra $C^*(\mathbb{Z}^3, \operatorname{Res} \sigma) = C^*(U, V, W)$ acts via

$$\begin{aligned}(hU)(w, t, z) &= h(w, t, z)\exp(\pi i t),\\ (hV)(w, t, z) &= h(w, t-1, z),\\ (hW)(w, t, z) &= h(w, t-1), z) z.\end{aligned}$$

We now define $Q : X \to X$ by

$$Qh(w, t, z) = h(w, t, \exp(\pi i t)z).$$

We can verify easily that

$$Q(U_0 h) = U_0 Q(h), \; Q(V_0 h) = V_0 Q(h), \; Q(W_0 h) = U_0 W_0 Q(h),$$

and

$$Q(hU) = Q(h)U, \qquad Q(hV) = Q(h)V, \qquad Q(hW) = Q(h)U^2 W,$$

so Q is compatible with the action $\mathrm{Ad}\, Z_0$ of $\mathbb{Z}$ on $C^*(\mathbb{Z}^3)$ and the action $\mathrm{Ad}\, Z$ of $\mathbb{Z}$ on $C^*(\mathbb{Z}^3, \mathrm{Res}\, \sigma)$. When we identify $C^*(\mathbb{Z}^3)$ with $C(\mathbb{T}^3)$, the action $\mathrm{Ad}\, Z_0$ is given by

$$\mathrm{Ad}\, Z_0(f)(w, \zeta, z) = f(w, \zeta, \zeta z),$$

and it is then straightforward to verify that

$$\langle Qh, Qk\rangle_{C(\mathbb{T}^3)} = \mathrm{Ad}\, Z_0(\langle h, k\rangle_{C(\mathbb{T}^3)}).$$

Thus Q is an isometric automorphism of the C^*-module X and the pair (X, Q) is a Morita equivalence between the actions $\mathrm{Ad}\, Z_0$ and $\mathrm{Ad}\, Z$ in the sense of Combes [8]; thus by the theorem on p. 299 of [8], the crossed products $C^*(\mathbb{Z}^3) \rtimes_{\mathrm{Ad}\, Z_0} \mathbb{Z}$ and $C^*(\mathbb{Z}^3, \mathrm{Res}\, \sigma) \rtimes_{\mathrm{Ad}\, Z} \mathbb{Z}$ are Morita equivalent, as claimed.

Proposition 3.11. *Let A, Γ and σ be as in Example* 3.9. *Then*

$$K_0(C^*(\Gamma, \sigma)) \cong K_1(C^*(\Gamma, \sigma)) \cong \mathbb{Z}^6$$

and

$$K_0(C^*(\Gamma)) \cong K_1(C^*(\Gamma)) \cong \mathbb{Z}^6 \oplus (\mathbb{Z}/2\mathbb{Z}).$$

Proof. To compute $K_*(C^*(\Gamma)) = K_*(C^*(\mathbb{Z}^3) \rtimes \mathbb{Z})$, we use the sequence of Pimsner-Voiculescu [41]

$$\begin{array}{ccccc}
K_0(C^*(\mathbb{Z}^3)) & \xrightarrow{\mathrm{id}-(\alpha_*)^{-1}} & K_0(C^*(\mathbb{Z}^3)) & \longrightarrow & K_0(C^*(\Gamma)) \\
\uparrow & & & & \downarrow \\
K_1(C^*(\Gamma)) & \longleftarrow & K_1(C^*(\mathbb{Z}^3)) & \xrightarrow{\mathrm{id}-(\alpha_*)^{-1}} & K_1 * (C^*(\mathbb{Z}^3))
\end{array}$$

and we therefore need to know the effect of the generating automorphism α on

$$K_*(C^*(\mathbb{Z}^3)) \cong K_*(C(\mathbb{T}^3)) \cong K^*(\mathbb{T}^3) \cong H^*(\mathbb{T}^3).$$

Now $H^*(\mathbb{T}^3)$ is isomorphic to the exterior algebra $\Lambda(\mathbb{Z}^3)$, where the generators e_1, e_2, e_3 for $\mathbb{Z}^3 = H^1(\mathbb{T}^3) = [\mathbb{T}^3, \mathbb{T}]$ can be taken to be the co-ordinate functions on $\mathbb{T}^3$. The automorphism α of $C^*(\mathbb{Z}^3)$ induces the homeomorphism of $\mathbb{T}^3 = C^*(\mathbb{Z}^3)^\wedge$ given by the matrix $(A^{-1})^t$ acting on $\mathbb{R}^3/\mathbb{Z}^3$, and the induced action on $H^1(\mathbb{T}^3)$ is given by A^{-1}. We can choose as generators for $H^2(\mathbb{T}^3) \cong \mathbb{Z}^3$ the elements $f_1 = e_2 \wedge e_3$, $f_2 = e_3 \wedge e_1$, $f_3 = e_1 \wedge e_2$, and then we can easily verify that α_* is given by A^t. Of course, α_* acts trivially on H^0 and H^3. It therefore follows that on both $K_0(\mathbb{T}^3)$ and $K_1(\mathbb{T}^3)$, we have

$$\ker(\mathrm{id}-(\alpha_*)^{-1}) \cong \mathbb{Z}^3, \qquad \mathrm{coker}(\mathrm{id}-(\alpha_*)^{-1}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

The formula for $K_*(C^*(\Gamma))$ now follows from the exact sequence. A similar computation using B instead of A gives $K_*(C^*(\Gamma_0))$ —replacing the 2 in A by

a 1 changes the cokernel of $\mathrm{id} - (\alpha_*)^{-1}$ to $\mathbb{Z}^3$—and we obtain $K_j(C^*(\Gamma_0)) \cong \mathbb{Z}^6$. The result now follows from the lemma.

Appendix 1. Twisted group algebras and twisted covariance algebras

Let G be a second countable locally compact group and $\sigma \in Z^2(G, \mathbb{T})$ a (Borel) multiplier on it. The *twisted group C^*-algebra* $C^*(G, \sigma)$ is by definition the enveloping algebra of twisted L^1-algebra $L^1(G, \sigma)$—that is, $L^1(G)$ with multiplication and involution defined by

$$f * g(s) = \int f(t) g(t^{-1}s) \sigma(t, t^{-1}s)\, dt,$$
$$f^*(s) = \Delta(s)^{-1} \overline{\sigma(s, s^{-1}) f(s^{-1})}.$$

This enveloping algebra has a universal property with respect to the σ-representations of the group G, and can therefore also be viewed as a crossed product for the twisted dynamical system $(\mathbb{C}, G, \mathrm{id}, \sigma)$ in the sense of [38].

A twisted covariant system $(A, H, \alpha, \mathscr{T})$ is a C^*-dynamical system (A, H, α), together with a strictly continuous homomorphism $\mathscr{T}$ of a normal subgroup N of H into $UM(A)$ such that $\mathrm{Ad}\, \mathscr{T} = \alpha|_N$ and $\alpha_s(\mathscr{T}(n)) = T(sns^{-1})$ for $n \in N$, $s \in H$ ($\mathscr{T}$ is called the *twist* or *twisting map* of the system). The *twisted covariance algebra* $C^*(H, A, \mathscr{T})$ is the enveloping C^*-algebra of the normed $*$-algebra $C_c(H, A, \mathscr{T})$ of continuous functions $f : H \to A$ such that $f(ns) = f(s)\mathscr{T}(n)^{-1}$ for $s \in H$, $n \in N$, and such that $sN \to \|f(s)\|$ has compact support in H/N (see [17, §1]). It is shown in [38, §5] that if H is second countable, then every $C^*(H, A, \mathscr{T})$ is isomorphic to the twisted crossed product of A by a twisted action (β, u) of H/N: given a Borel section $c : H/N \to H$, we can take $\beta_{sN} = \alpha_{c(s)}$ and $u(sN, tN) = \mathscr{T}(c(s)c(t)c(st)^{-1})$. If $A = \mathbb{C}$, H must act trivially, the Borel cocycle u is $\mathbb{T} = U(\mathbb{C})$-valued, and we obtain an isomorphism of $C^*(H, \mathbb{C}, \mathscr{T})$ with the twisted group algebra $\mathbb{C} \rtimes_{\mathrm{id}, u} (H/N) = C^*(H/N, u)$. More precisely, we have:

Proposition A1. *Let H be a second countable locally compact group, and $(\mathbb{C}, H, \mathrm{id}, \mathscr{T})$ a twisted covariant system with twist $\mathscr{T}$ defined on N. Let $c : H/N \to H$ be a Borel section, and set*

$$\sigma(sN, tN) = \mathscr{T}(c(s)c(t)c(st)^{-1}) \qquad \text{for } sN, tN \in H/N.$$

Then $\sigma \in Z^2(H/N, \mathbb{T})$, and the map $f \to f \circ c$ from $C_c(H, \mathbb{C}, \mathscr{T})$ to $L^1(H/N)$ extends an isomorphism of $C^*(H, \mathscr{C}, T)$ onto $C^*(H/N, \sigma)$.

Proof. All that remains is for us to verify that the isomorphism ϕ provided by [38, Proposition 5.1] satisfies $\phi(f) = f \circ c$. By [38], $C^*(H, \mathbb{C}, \mathscr{T})$ is generated by unitaries $j_H(s)$, and the element of $C^*(H, \mathbb{C}, \mathscr{T})$ corresponding to $f \in C_c(H, \mathbb{C}, \mathscr{T})$ is $\int_{H/N} f(s) j_H(s) d(sN)$. Now according to [38],

$$\phi(j_H(s)) = i_{\mathbb{C}}(\mathscr{T}(sc(s)^{-1})) i_{H/N}(sN) = \mathscr{T}(sc(s)^{-1}) i_{H/N}(sN),$$

and the condition $f(ns) = f(s)\mathscr{T}(n)^{-1}$ therefore implies

$$\phi(f) = \int_{H/N} f(s)\phi(j_H(s)) d(sN) = \int_{H/N} f(c(s)) i_{H/N}(sN) d(sN),$$

which is the element of $\mathbb{C} \times_{\mathrm{id},\sigma} (H/N)$ corresponding to $f \circ c \in L^1(H/N)$.

Remark. We can also use this result to view a twisted group algebra $C^*(G, \sigma)$ as a twisted covariance algebra. We let G_σ denote the locally compact group extension corresponding to $\sigma \in Z^2(G, \mathbb{T})$: that is, $G_\sigma = \mathbb{T} \times G$ setwise, with multiplication given by

$$(z, s)(w, t) = (zw\sigma(s, t), st),$$

and the unique locally compact topology for which the product of Haar measures is a Haar measure. Then the identity map $\mathscr{T}_\sigma$ of $\mathbb{T} \subset G_\sigma$ into $\mathbb{T} \subset U(\mathbb{C})$ is a twisting map for the trivial action of G_σ on $\mathbb{C}$, and the proposition asserts that $f \to f|_{\{1\}\times G}$ extends to an isomorphism of $C^*(G_\sigma, \mathbb{C}, \mathscr{T}_\sigma)$ onto $C^*(G, \sigma)$: to see this, we just have to observe that $c(s) = (1, s)$ is a Borel section, and that we then have

$$\mathscr{T}_\sigma(c(s)c(t)c(st)^{-1}) = \mathscr{T}_\sigma(\sigma(s, t), e) = \sigma(s, t).$$

We shall use this isomorphism repeatedly to enable us to apply the results of [17] to twisted group algebras.

Appendix 2. On the multipliers of group extensions

Let G be a second countable locally compact group and N a closed normal subgroup. Any multiplier $\omega \in \mathbb{Z}^2(G/N, \mathbb{T})$ can be inflated to a multiplier $\operatorname{Inf} \omega \in Z^2(G, \mathbb{T})$, and we want to characterise those multipliers of G which arise this way. When G is discrete, the Lyndon-Hochschild-Serre spectral sequence [23] tells us what to look for: if $\operatorname{Res}: H^2(G, \mathbb{T}) \to H^2(N, \mathbb{T})$ is the usual restriction map, then there is a homomorphism

$$d: \operatorname{Ker}\operatorname{Res} \to H^1(G/N, H^1(N, \mathbb{T})) \quad \text{such that range } \operatorname{Inf} = \ker .$$

Therefore what we need is a specific version of this result for Moore cohomology. There are two minor problems to overcome: first of all, the terms in the analogous spectral sequence for Moore cohomology are not always easily identified (see [31]), and secondly, we do not know where the required characterisation has been explicitly written down even in the discrete case, although it must be well known.

Suppose, then, that $\sigma \in H^2(G, \mathbb{T})$ has $\operatorname{Res}\sigma$ trivial in $H^2(N, \mathbb{T})$. Then there is a Borel map $b : G \to \mathbb{T}$ such that the multiplier $\tau = (\partial b)\sigma$ defined by

$$\tau(s, t) = [(\partial b)\sigma](s, t) = b(s)b(t)b(st)^{-1}\sigma(s, t)$$

is identically one on $N \times N$. We now define $\phi(\tau) : G \to H^1(N, \mathbb{T}) = \widehat{N}$ by

$$\phi(\tau)(t)(n) = \tilde{\tau}(n, t)\tau = \tau(n, t)(t, t^{-1}nt)^{-1};$$

because $\tau|_{N\times N} \equiv 1$, equations (1.1) and (1.2) imply that $\phi(\tau)$ is constant on N cosets in G, and that $\phi(\tau)(t)$ is a homomorphism for all t. The map $t \to \phi(\tau)(t)$ is Borel from G to $Z^1(N, \mathbb{T}) = H^1(N, \mathbb{T})$ (see the first paragraph of the proof of [31, Theorem 1]), and hence induces a Borel map, also denoted $\phi(\tau)$, from G/N to $\widehat{N} = H^1(N, \mathbb{T})$. For $s, t \in G$, equation (1.1) gives

$$\phi(\tau)(st)(n) = \tilde{\tau}(n, st) = \tilde{\tau}(n, s)\tilde{\tau}(s^{-1}ns, t) = (\phi(\tau)(s)s \cdot [\phi(\tau)(t)])(n),$$

and $\phi(\tau)$ therefore belongs to $Z^1(G/N, \widehat{N})$. Further, the class of $\phi(\tau)$ in $H^1(G/N, \widehat{N})$ is independent of the choice of multiplier τ. For if $\mu \in Z^2(G, \mathbb{T})$ is equivalent to σ and satisfies $\mu|_{N\times N} \equiv 1$ there is a Borel map $c: G \to \mathbb{T}$ such that $\tau = (\partial c)\mu$; since both μ and τ are $\equiv 1$ on $N \times N$, this last equation implies that $c|_N$ is a homomorphism, and a calculation shows

$$\begin{aligned}\phi(\tau)(t)(n) &= c(n)c(t^{-1}nt)^{-1}\phi(\mu)(t)(n)\\ &= [(c|_N)t \cdot (c|_N)^{-1}\phi(\mu)(t)](n).\end{aligned}$$

The class $d(\sigma)$ of $\phi(\tau)$ in $H^1(G/N, \widehat{N})$ therefore depends only on the class of σ in $H^2(G, \mathbb{T})$, and the resulting map d of $\ker \mathrm{Res}$ into $H^1(G/N, \widehat{N})$ is easily seen to be a homomorphism.

Proposition A2. *Let G, N be as above, and suppose $\sigma \in H^2(G, \mathbb{T})$ satisfies* $\mathrm{Res}\,\sigma = 1$ *in $H^2(N, \mathbb{T})$. Then $\sigma = \mathrm{Inf}\,\omega$ for some $\omega \in H^2(G/N, \mathbb{T})$ if and only if $d(\sigma) = 1$ in $H^1(G/N, \widehat{N})$.*

Proof. If $\sigma \in \mathrm{range}\,\mathrm{Inf}$, we can take the multiplier τ used in the construction of $d(\sigma)$ to be the inflation of a multiplier $\omega \in Z^2(G/N, \mathbb{T})$, and then we have $\phi(\mathrm{Inf}\,\omega) \equiv 1$. So suppose $d(\sigma) = 1$ in $H^1(G/N, \widehat{N})$. Without loss of generality we may suppose $\sigma|_{N\times N} \equiv 1$, and then $d(\sigma) = 1$ means there exists $\eta \in \widehat{N}$ such that

$$\phi(\sigma)(t)(n) = \tilde{\sigma}(n, t) = \eta(n)\eta(t^{-1}nt)^{-1} \quad \text{for } n \in N,\ t \in G.$$

We choose a Borel section $c: G/N \to G$ with $c(e) = e$, define $a: G \to \mathbb{T}$ by

$$a(t) = \sigma(c(tN), c(tN)^{-1}t)\eta(c(tN)^{-1}t)^{-1},$$

and set $\mu = (\partial a)\sigma$. A simple calculation using $\sigma|_{N\times N} \equiv 1$ shows that

$$a(tn) = \sigma(t, n)\eta(n)^{-1}a(t) \quad \text{for } n \in N,\ t \in G,$$

and it follows from this and more calculations that μ is constant on N-cosets. Thus μ is inflated from a multiplier ω on G/N, and the result follows.

When the group G is a semidirect product, these ideas can be extended to give a complete description of $H^2(G, \mathbb{T})$: this was originally done by Mackey ([27; §9]; see also [3]), although we have used different notation to make the result compatible with Moore cohomology [31]. We suppose that $G = N \rtimes H$—in other words, that we have a continuous action of H as automorphisms of N, and that G is the topological product space $N \times H$ with multiplication given by

$$(m, h)(n, k) = (m(h \cdot n), hk).$$

For each $f \in Z^1(H, \widehat{N})$, we can define a cocycle $\psi(f) \in Z^2(G, \mathbb{T})$ by

$$\psi(f)((m, h), (n, k)) = f(h)(h \cdot n);$$

one can easily check that this is a cocycle, whose class in $H^2(G, \mathbb{T})$ depends only on the class of f in $H^1(H, \widehat{N})$, and which satisfies $d(\psi(f)) = f$. The resulting homomorphism $\psi: H^1(H, \widehat{N}) \to H^2(G, \mathbb{T})$ is therefore a splitting for d: $\ker \mathrm{Res} \to H^1(H, \widehat{N})$. As above, the kernel of d is the image of $\mathrm{Inf}: H^2(H, \mathbb{T}) \to H^2(G, \mathbb{T})$, and restricting cocycles to $\{e\} \times H$ gives a splitting

for this map. Therefore $\ker \mathrm{Res}$ decomposes as a direct product and we have an exact sequence

$$\text{(A1)} \qquad 0 \longrightarrow H^2(H, \mathbb{T}) \times H^1(H, \widehat{N}) \xrightarrow{\zeta} H^2(N \rtimes H, \mathbb{T}) \xrightarrow{\mathrm{Res}} H^2(N, \mathbb{T})^H,$$

where for $\omega \in Z^2(H, \mathbb{T})$, $f \in Z^1(H, \widehat{N})$ we have

$$\zeta(\omega, f)((m, h), (n, k)) = \omega(h, k) f(h)(h \cdot n).$$

At least in the discrete case, the range of Res is described via the Lyndon-Hochschild Serre spectral sequence as the kernel of a pair of homomorphisms

$$d_2 \colon H^2(N, \mathbb{T})^H \to H^2(H, \widehat{N}),$$
$$d_3 \colon \ker d_2 \to (\text{a quotient of } H^3(H, \mathbb{T})).$$

The exact sequence (A1) will not in general split. In order to construct a 2-cocycle over $N \rtimes H$ with given restriction $\alpha \in Z^2(N, \mathbb{T})$, we need a Borel function $f \colon H \times N \to \mathbb{T}$ satisfying the compatibility conditions

$$\text{(A2)} \qquad f(kh)(n) = f(h)(n) f(k)(h^{-1} \cdot n),$$

$$\text{(A3)} \qquad \alpha(h \cdot m, h \cdot n) = \alpha(m, n) f(h)(h \cdot (mn))[f(h)(h \cdot m) f(h)(h \cdot n)]^{-1};$$

we can then define $\sigma = \sigma_{\alpha, f} \in Z^2(N \rtimes H, \mathbb{T})$ by

$$\text{(A4)} \qquad \sigma((m, h), (n, k)) = \alpha(m, h \cdot n) f(h)(h \cdot n).$$

Of course, it is not at all obvious when such a function f exists (this amounts to knowing whether α belongs to the range of Res), but if one does, multiplying by elements of $Z^1(H, \widehat{N})$ gives the others; two f's give equivalent multipliers σ if and only if they differ by a coboundary $\partial\gamma \in Z^1(H, \widehat{N})$. Now suppose $\beta \in Z^2(N, \mathbb{T})$ is equivalent to α, say

$$\beta(m, n) = \alpha(m, n) \delta(mn) [\delta(m) \delta(n)]^{-1},$$

and $f \colon H \times N \to \mathbb{T}$ is compatible with α. Then

$$\text{(A5)} \qquad g(h)(n) = f(h)(n) \delta(m) \delta(h^{-1} \cdot n)^{-1},$$

gives a function $g : H \times N \to \mathbb{T}$ such that (β, g) satisfies (A2) and (A3), and the corresponding multiplier $\sigma_{\beta, g}$ is equivalent to $\sigma_{\alpha, f}$: with $\mu(m, h) = \delta(m)$, we have

$$\begin{aligned} &\sigma_{\beta, g}((m, h), (n, k)) \\ &\qquad = \sigma_{\alpha, f}((m, h), (n, k)) \mu(m(h \cdot n), hk) [\mu(m, h) \mu(n, k)]^{-1}. \end{aligned}$$

Finally, we can complete the description of $H^2(N \rtimes H, \mathbb{T})$ by observing that every cocycle of $N \rtimes H$ is equivalent to a product of the form $\sigma_{\alpha, f} \cdot \mathrm{Inf}\, \omega$, where $\sigma_{\alpha, f}$ is given by (A4) and $\omega \in Z^2(H, \mathbb{T})$; the equivalence class of $\sigma \cdot \mathrm{Inf}\, \omega$ depends only on the class of ω in $H^2(H, \mathbb{T})$ and the class of σ, which was described above in terms of α and f.

Acknowledgments

This work was started when the first author visited the University of New South Wales in October 1986, and continued while she was at the Institute for Advanced Study in 1987, and she wishes to thank these institutions for their support. It was completed while the second author was visiting the University of Edinburgh, and he is grateful to Allan Sinclair and his colleagues for their hospitality.

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