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A DYNAMICAL PROOF OF THE MULTIPLICATIVE ERGODIC THEOREM

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ABSTRACT. We shall give a proof of the following result of Oseledec, in which $GL(d)$ denotes the space of invertible $d \times d$ real matrices, $\|\cdot\|$ denotes any norm on the space of $d \times d$ matrices, and $\log^+(t) = \max(0, \log(t))$ for $t \in [0, \infty)$.

Oseledec's Multiplicative Ergodic Theorem [7]. *Let T be a measure-preserving transformation of a Lebesgue probability space $(X, \mathscr{B}, m)$ and let $x \to A_x$ be a measurable map of X into $GL(d)$ such that the real-valued functions $x \to \log^+ \|A_x\|$, $x \to \log^+ \|(A_x)^{-1}\|$ are integrable. There exists $Y \in \mathscr{B}$ with $TY \subset Y$ and $m(Y) = 1$ such that*

(i) *there is a measurable function $s: Y \to \mathbb{N}$ with $s \circ T = s$;*
(ii) *for each $x \in Y$ there are real numbers $\lambda^{(s(x))}(x) < \lambda^{(s(x)-1)}(x) < \cdots < \lambda^{(2)}(x) < \lambda^{(1)}(x)$ with $\lambda^{(i)}(Tx) = \lambda^{(i)}(x)$ when $1 \leq i \leq s(x)$, and $x \to \lambda^{(i)}(x)$ is measurable on $\{x \in Y | s(x) \geq i\}$;*
(iii) *for each $x \in Y$ there are linear subspaces,*
$$\{0\} \equiv V_x^{(s(x)+1)} \subset V_x^{(s(x))} \subset \cdots \subset V_x^{(2)} \subset V_x^{(1)} = R^d,$$
of R^d with $A_x V_x^{(i)} = V_{Tx}^{(i)}$, and $x \to V_x^{(i)}$ is a measurable map from $\{x \in Y | s(x) \geq i\}$ into the Grassmannian of R^d;
(iv) *$\forall x \in Y\ \forall v \in V_x^{(i)} \backslash V_x^{(i+1)}$, $\frac{1}{n} \log \|A_{T^{n-1}x} \cdots A_{Tx} \cdot A_x v\| \to \lambda^{(i)}(x)$ where $\|\cdot\|$ denotes any norm on R^d.*

The proof we give is a dynamical proof free of most of the matrix calculations of previous proofs [7, 9, 10, 6, 3]. As indicated in [8] it is quite straightforward to obtain (i), (ii), (iii) above and to get (iv) with the limit replaced by limit superior (see §1). The work comes in showing the limit exists in (iv) and this is done by using two results, Theorems 11 and 13, the second of which considers the ergodic theory of a transformation on $X \times P(R^d)$ where $P(R^d)$ is the projective space obtained from R^d. We have only been able to use the second result under the assumptions that both $x \to \log^+ \|A_x\|$, $x \to \log^+ \|(A_x)^{-1}\|$ are integrable (which is equivalent to requiring $x \to \log \|A_x\|$, $x \to \log \|(A_x)^{-1}\|$ are integrable), whereas Oseledec's theorem is true under only the first assumption but in this case $\lambda^{(s(x))}(x)$ can be $-\infty$. We have indicated where the second

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assumption enters the proof. The numbers $\lambda^{(i)}(x)$ are called the Lyapunov characteristic exponents of (T, A) at x.

We shall use $\mathscr{M}(d)$ to denote the space of all real $d \times d$ matrices and $\|.\|$ to denote any norm on this vector space. The natural basis in R^d is denoted by $\{e_1, e_2, \dots, e_d\}$, and we always consider the Euclidean inner-product on R^d. From now on $(X, \mathscr{B}, m)$ will be a Lebesgue probability space and $T: X \to X$ a measure-preserving transformation. Hence m is a complete measure and the ergodic decomposition holds. Also $A: X \to \mathscr{M}(d)$ $(x \to A_x)$ will be a measurable map with $x \to \log^+ \|A_x\|$ integrable. For $n > 0$, $(A^n)_x$ denotes $A_{T^{n-1}x} \cdots A_{Tx}.A_x$ (matrix multiplication).

If P is a compact metric space equipped with its Borel σ-algebra $\mathscr{B}(P)$, then $M_m(X \times P)$ denotes the space of probability measures on $X \times P$ that project to m on X, and if $S: X \times P \to X \times P$ is a measurable map covering T then $M_m(X \times P, S)$ denotes those members of $M_m(X \times P)$ which are invariant under S.

1. Proof of (i), (ii), (iii) and most of (iv)

We shall use the following simple result about sequences of real numbers.

Lemma 1. *If* $a_n, b_n \geq 0$ *for* $n \geq 1$ *then*

$$\limsup_{n \to +\infty} \frac{1}{n} \log(a_n + b_n) = \max\left(\limsup_{n \to +\infty} \frac{1}{n} \log a_n, \limsup_{n \to +\infty} \frac{1}{n} \log b_n\right)$$

and

$$\liminf_{n \to +\infty} \frac{1}{n} \log(a_n + b_n) \geq \max\left(\liminf_{n \to +\infty} \frac{1}{n} \log a_n, \liminf_{n \to +\infty} \frac{1}{n} \log b_n\right).$$

Lemma 2. *Suppose* $T: X \to X$ *is a measure-preserving transformation of* $(X, \mathscr{B}, m)$ *and let* $A: X \to \mathscr{M}(d)$ *be such that* $x \to \log^+ \|A_x\|$ *is integrable. Define* $\chi: X \times R^d \to R \cup \{\pm\infty\}$ *by* $\chi(x, v) = \limsup_{n \to +\infty} \frac{1}{n} \log \|(A^n)_x v\|$. *Then*

(i) *there exists* $X_1 \in \mathscr{B}$ *with* $m(X_1) = 1$ *and* $TX_1 \subset X_1$ *such that* $\chi(x, v) \in R \cup \{-\infty\}$ $\forall x \in X_1$, $\forall v \in R^d$;

(ii) $\chi(x, 0) = -\infty$ $\forall x \in X$;

(iii) $\chi(x, v) = \chi(x, av)$ $\forall a \in R \backslash \{0\}$ $\forall v \in R^d$ $\forall x \in X$;

(iv) $\chi(x, v_1 + v_2) \leq \max(\chi(x, v_1), \chi(x, v_2))$ $\forall v_1, v_2 \in R^d$ $\forall x \in X$;

(v) $\chi(Tx, A_x v) = \chi(x, v)$ $\forall v \in R^d$ $\forall x \in X$.

Proof. (i) $\|(A^n)_x v\| \leq \|(A^n)_x\| \cdot \|v\| \leq (\prod_{i=0}^{n-1} \|A_{T^i x}\|)\|v\|$ and since $x \to \log^+ \|A_x\|$ is integrable $\chi(x, v) \in R \cup \{-\infty\}$ by Birkhoff's ergodic theorem.

The proofs of (ii) and (iii) are clear and (iv) follows from Lemma 1. Part (v) follows since

$$\begin{aligned} \chi(Tx, A_x v) &= \limsup_{n \to +\infty} \frac{1}{n} \log \|(A^{n+1})_x v\| \\ &= \limsup_{n \to +\infty} \frac{n+1}{n} \cdot \frac{1}{n+1} \log \|(A^{n+1})_x v\| \\ &= \chi(x, v). \quad \square \end{aligned}$$

Corollary 3. *With the notation of Lemma 2 we have*

(i) $\forall x \in X_1$ $\forall t \in R$ *the set* $V_x(t) = \{v \in R^d | \chi(x, v) \le t\}$ *is a linear subspace of* R^d. *We have* $A_x V_x(t) \subset V_{Tx}(t)$. *Also* $s \le t$ *implies* $V_x(s) \subset V_x(t)$.

(ii) $\forall x \in X_1$, $\chi(x, .): R^d \to R \cup \{-\infty\}$ *takes only a finite number,* $s(x)$, *of different values* $\lambda^{(s(x))}(x) < \lambda^{(s(x)-1)}(x) < \cdots < \lambda^{(1)}(x)$, *where* $\lambda^{(s(x))}(x)$ *could be* $-\infty$. *We have* $s(Tx) \ge s(x)$, *and* $\lambda^{(s(x))}(x), \ldots, \lambda^{(1)}(x)$ *are among the values* $\{\lambda^{(j)}(Tx): 1 \le j \le s(Tx)\}$.

(iii) *If, for* $x \in X_1$, *we define* $V_x^{(i)}$ *to be* $V_x(\lambda^{(i)}(x))$, $1 \le i \le s(x)$, *then*

$$\{0\} \equiv V_x^{(s(x)+1)} \subset V_x^{(s(x))} \subset \cdots \subset V_x^{(2)} \subset V_x^{(1)} = R^d$$

and

$$v \in V_x^{(i)} \backslash V_x^{(i+1)} \Leftrightarrow \limsup_{n\to+\infty} \frac{1}{n} \log \|(A^n)_x v\| = \lambda^{(i)}(x), \qquad 1 \le i \le s(x).$$

Proof.

(i) Each $V_x(t)$ is a linear subspace of R^d by Lemma 2(iii) and (iv). Part (v) of Lemma 2 gives $A_x V_x(t) \subset V_{Tx}(t)$. The last claim is clear.

(ii) Fix $x \in X_1$. Since $s < t$ implies $V_x(s) \subset V_x(t)$ and hence $\dim(V_x(s)) \le \dim(V_x(t))$, we can let $\lambda^{(s(x))} < \cdots < \lambda^{(1)}(x)$ be the values of t where $t \to \dim(V_x(t))$ changes. Therefore $V_x(\lambda^{(i)}(x)) = \{v \in R^d | \chi(x, v) \le \lambda^{(i)}(x)\}$, and $\chi(x, .)$ can only take the values $\lambda^{(s(x))}(x), \ldots, \lambda^{(1)}(x)$. The last statement follows from Lemma 2(v).

(iii) This part is now clear from the proof of (ii). □

Lemma 4. *With the notation of Lemma 2,*

(i) $s: X_1 \to \mathbb{N}$ *is measurable and hence* $\exists X_2 \in \mathscr{B}$ *with* $m(X_2) = 1$, $TX_2 \subset X_2$, *and* $s \circ T = s$ *on* X_2;

(ii) $\lambda^{(i)}: \{x \in X_1 | s(x) \ge i\} \to R \cup \{-\infty\}$ *is measurable and* $\lambda^{(i)}(Tx) = \lambda^{(i)}(x)$ $\forall x \in X_2 \cap \{x | s(x) \ge i\}$;

(iii) $x \to V_x^{(i)}$ *is a measurable map from* $\{x \in X_1 | s(x) \ge i\}$ *into the Grassmannian of* R^d *and* $A_x V_x^{(i)} \subset V_{Tx}^{(i)}$.

Proof. We shall deal with the measurability questions in the next section. From Corollary 3(ii) we have $s \circ T \ge s$ so $s \circ T = s$ a.e. We get $\lambda^{(i)} \circ T = \lambda^{(i)}$ a.e. by Corollary 3(ii). □

To prove Oseledec's theorem we need to show that for a.e. $x \in X$ $\lim_{n\to\infty} \frac{1}{n} \log \|(A^n)_x v\|$ exists $\forall v \in R^d$. The rest of the paper, after §2, is devoted to this.

2. Measurability questions

If Y is a complete separable metric space let $\mathscr{P}_K(Y)$ denote the collection of all nonempty compact subsets of Y. We can equip $\mathscr{P}_K(Y)$ with the Hausdorff metric and it becomes a complete separable metric space. The following result is from [2, pp. 80 and 62].

Theorem 5 [2]. *Let $(X, \mathscr{B}, m)$ be a complete probability space and let Y be a complete separable metric space. Let $\Gamma: X \to \mathscr{P}_K(Y)$ be a map. The following statements are equivalent:*

(a) *Γ is measurable (using the Borel σ-algebra on the metric space $\mathscr{P}_K(Y)$).*
(b) *$\forall$ open subset U of Y, $\{x \in X | \Gamma(x) \cap U \neq \varnothing\} \in \mathscr{B}$.*
(c) *There is a sequence $\{\sigma_n\}$ of measurable maps $\sigma_n: X \to Y$ with $\sigma_n(x) \in \Gamma(x)$ $\forall x \in X$ and $\Gamma(x) = \overline{\{\sigma_n(x)\}_{n=1}^{\infty}}$.*
(d) $\text{graph}(\Gamma) \equiv \{(x, y) | y \in \Gamma(x)\} \in \mathscr{B} \times \mathscr{B}(Y)$.

We also have

Theorem 6 [2, p. 75]. *If $(X, \mathscr{B}, m)$ is a complete probability space and Y is a complete separable metric space consider the natural projection from $X \times Y$ to X. If $D \in \mathscr{B} \times \mathscr{B}(Y)$ the projection of D to X is in $\mathscr{B}$.*

If V is a real vector space we let $P(V)$ denote the corresponding projective space obtained by collapsing each line through the origin of V to a point. The topology on $P(R^d)$ is the quotient topology. Let $G(R^d)$ denote the Grassmannian of R^d, which consists of all linear subspaces of R^d suitably topologized, and let $G_k(R^d)$ denote the space of all k-dimensional linear subspaces of R^d [5]. We have

Theorem 7. *Let $(X, \mathscr{B}, m)$ be a complete probability space and let a map $x \to V_x$ of X into $G(R^d)$ be given. Define $r: X \to \mathbb{N}$ by $r(x) = \dim V_x$. The following are equivalent.*

1. *$x \to V_x$ is a measurable map of X into $G(R^d)$.*
2. $\{(x, v) | x \in X, v \in V_x\} \in \mathscr{B} \times \mathscr{B}(R^d)$.
3. $\{(x, y) | x \in X, y \in P(V_x)\} \in \mathscr{B} \times \mathscr{B}(P(R^d))$.
4. *$x \to P(V_x)$ is a measurable map of X into $\mathscr{P}_K(P(R^d))$.*
5. *$r: X \to R$ is measurable and for each k, $x \to V_x$ is a measurable map of $r^{-1}(k)$ into $G_k(R^d)$.*
6. *$r: X \to R$ is measurable and for each k there are measurable maps $v_1, \ldots, v_k: r^{-1}(k) \to R^d$ such that $\forall x \in r^{-1}(k)$, $\{v_1(x), \ldots, v_k(x)\}$ is an orthonormal basis for V_x.*
7. *$r: X \to R$ is measurable and for each k there is a bimeasurable bijection from $\{(x, v) | x \in r^{-1}(k), v \in V_x\}$ onto $r^{-1}(k) \times R^k$ which is linear on fibres and covers the identity map of X.*
8. *$r: X \to R$ is measurable and for each k there are measurable maps $u_1, \ldots, u_k: r^{-1}(k) \to R^d$ such that $\forall x \in r^{-1}(k)$, $\{u_1(x), \ldots, u_k(x)\}$ is a basis for V_x.*

Proof. Clearly (6) and (8) are equivalent by using the Gram-Schmidt process. Also (7) and (8) are clearly equivalent. Since the map $E \to P(E)$ of $G(R^d)$ into $\mathscr{P}_K(P(R^d))$ is injective and continuous it is a homeomorphism onto its image and hence (1) is equivalent to (4). By Theorem 5, (3) and (4) are equivalent. Clearly (3) implies (2). To show that (2) implies (3) it suffices to show, by Theorem 5, that for each open subset U of $P(R^d)$ we have $\{x \in X | P(V_x) \cap U \neq \varnothing\} \in \mathscr{B}$. If $q: X \times R^d \to X$ denotes projection to the first factor and $\pi: R^d \to P(R^d)$ denotes the natural projection then $\{x \in X | P(V_x) \cap U \neq \varnothing\} = q(\{(x, v) | x \in X, v \in V_x\} \cap (X \times \pi^{-1}U))$ which belongs to $\mathscr{B}$ by Theorem 6. Therefore (2) and (3) are equivalent. Since the dimension function is continuous on $G(R^d)$ we get (1) is equivalent to (5).

It remains to show that (5) and (8) are equivalent. Let $E \in G_k(R^d)$. There is a neighbourhood $\mathscr{V}(E)$ of E in $G_k(R^d)$ and continuous maps $\xi_1, \dots, \xi_k$: $\mathscr{V}(E) \to R^d$ so that for each $V \in \mathscr{V}(E)$, $\{\xi_1(V), \xi_2(V), \dots, \xi_k(V)\}$ is a basis for V. To do this let $R^d = E \oplus E^\perp$, and then there is a neighbourhood $\mathscr{V}(E)$ of E such that each $V \in \mathscr{V}(E)$ is the graph of a unique linear map $L_V\colon E \to E^\perp$. Let $\{a_1, \dots, a_k\}$ be a basis for E and then $\{(a_1, L_V(a_1)), (a_2, L_V(a_2)), \dots, (a_k, L_V(a_k))\}$ is a basis for V. Since $G_k(R^d)$ is compact we can choose a finite collection $\mathscr{V}(E_1), \dots, \mathscr{V}(E_r)$ of such neighbourhoods that cover $G_k(R^d)$, and by disjointifying them we get $v_1, \dots, v_k$: $r^{-1}(k) \to R^d$ defined with the properties in (8). Hence (5) implies (8). If (8) holds then let $x_0 \in r^{-1}(k)$. If $y \in r^{-1}(k)$, the linear map

$$L_y\colon \operatorname{span}\{u_1(x_1), \dots, u_k(x_0)\} \to (\operatorname{span}\{u_1(x_0), \dots, u_k(x_0)\})^\perp$$

with graph $\operatorname{span}\{u_1(y), \dots, u_k(y)\}$ depends measurably on y, and hence $x \to V_x$ is a measurable map of $r^{-1}(k)$ into $G_k(R^d)$. □

We say that $\{V_x\}_{x\in X}$ is a measurable subbundle of $X \times R^d$ when one, and hence all, of the statements in Theorem 7 hold. Note that if $\{U_x\}_{x\in X}$, $\{V_x\}_{x\in X}$ are measurable subbundles of $X \times R^d$ with $U_x \subset V_x$, and W_x is defined by $V_x = U_x \overset{\perp}{\oplus} W_x$, using the Euclidean inner-product, then $\{W_x\}_{x\in X}$ is a measurable subbundle.

We can now complete the proof of Lemma 4.

Proof of Lemma 4. We know $\chi\colon X \times R^d \to R$ is measurable so $\lambda^{(1)}(x) = \sup_v \chi(x, v) = \max_i \chi(x, e_i)$ is measurable. Consider $\Lambda_2 = \{(x, v) \in X \times R^d | \chi(x, v) < \lambda^{(1)}(x)\} \in \mathscr{B} \times \mathscr{B}(R^d)$. If $\pi_1\colon X \times R^d \to X$ is the natural projection then by Theorem 6 $\pi_1(\Lambda_2) \in \mathscr{B}$. Also $\pi_1(\Lambda_2) = \{x | s(x) > 1\}$. By Theorem 7 applied to $x \to \pi_1^{-1}(x) \cap \Lambda_2$ on $\pi_1(\Lambda_2)$ we have measurable maps $r\colon \pi_1(\Lambda_2) \to \mathbb{N}$ and $u_1, \dots, u_k\colon \pi_1(\Lambda_2) \cap r^{-1}(k) \to R^d$ with $u_1(x), \dots, u_{r(x)}(x)$ a basis for $\Lambda_2 \cap \pi_1^{-1}(x)$. Hence $\lambda^{(2)}(x) = \sup\{\chi(x, v) | (x, v) \in \Lambda_2\} = \max_i \chi(x, u_i(x))$ is measurable. Let $\Lambda_3 = \{(x, v) \in X \times R^d | \chi(x, v) < \lambda^{(2)}(x)\}$. As above, we get $\pi_1(\Lambda_3) \in \mathscr{B}$, $\pi_1(\Lambda_3) = \{x | s(x) > 2\}$, and $\lambda^{(3)}$ is measurable. In this way we get that $s\colon X \to \mathbb{N}$ is measurable and $\lambda^{(i)}\colon \{x | s(x) \geq i\} \to R$ is measurable for each i. Now

$$\{(x, v) | x \in X,\ v \in V_x^{(j)}\} = \{(x, v) \in X \times R^d | \chi(x, v) \leq \lambda^{(j)}(x)\} \in \mathscr{B} \times \mathscr{B}(R^d)$$

since χ and $\lambda^{(j)}$ are measurable. By Theorem 7, $x \to V_x^{(j)}$ is measurable. □

3. Existence of the limit in (iv)

We first note that it suffices to prove part (iv) of Oseledec's theorem when m is an ergodic T-invariant measure. To see this suppose (iv) holds whenever m is an ergodic measure and consider

$$\Delta = \left\{(x, v) \in X \times R^d \,\Big|\, \lim_{n\to\infty} \frac{1}{n} \log \|(A^n)_x v\| \text{ exists}\right\} \in \mathscr{B} \times \mathscr{B}(R^d).$$

Let $X' = \pi\Delta\backslash\pi(\pi^{-1}\pi\Delta\backslash\Delta) = \{x \in X | \{x\} \times R^d \subset \Delta\}$. Then $X' \in \mathscr{B}$ by Theorem 6 and $m(X') = 1$ for every ergodic T-invariant measure. Therefore $m(X') = 1$ for every T-invariant measure by the ergodic decomposition.

Our first aim is to reduce the problem to a special case by proving Theorem 11. We use the next two simple lemmas to obtain Corollary 10.

Lemma 8. *If T is a measure-preserving transformation of a probability space $(X, \mathscr{B}, m)$ and $h: X \to [0, \infty)$ is measurable then $\liminf_{n\to\infty} \frac{1}{n}h(T^n x) = 0$ a.e.*

Proof. Let $A_k = \{x | h(x) \leq k\}$. THen $\bigcup_{k=1}^{\infty} A_k = X$. If $m(A_k) > 0$ then, by the recurrence theorem, for a.e. $x \in A_k$ there exist $n_1(x) < n_2(x) < \cdots$ with $T^{n_i(x)}(x) \in A_k \ \forall i \geq 1$. Hence $h(T^{n_i(x)}(x)) \leq k$ and so $\liminf_{n\to\infty} \frac{1}{n}h(T^n x) = 0$. This holds for a.e. $x \in \bigcup_{k=1}^{\infty} A_k = X$. □

For $f: X \to R$, $f^+: X \to [0, \infty)$ denotes the positive part of f defined by $f^+(x) = \max(0, f(x))$.

Lemma 9. *If T is a measure-preserving transformation of a probability space $(X, \mathscr{B}, m)$ and $h: X \to R$ is measurable and $(h - h \circ T)^+ \in L^1(m)$ then $\frac{1}{n}h(T^n x) \to 0$ a.e.*

Proof. We have $h(T^n x) = h(x) - \sum_{i=0}^{n-1}(h - h \circ T)(T^i x)$. Since $(h - h \circ T)^+ \in L^1(m)$ the ergodic theorem gives $\lim \frac{1}{n}h(T^n x)$ exists a.e. but could take on the value of ∞. However the limit is zero by Lemma 8. □

Corollary 10. *Suppose T is an ergodic measure-preserving transformation of $(X, \mathscr{B}, m)$ and that $x \to A_x$ is a measurable map of X into $\mathscr{M}(d)$ with $x \to \log^+ \|A_x\|$ integrable. Suppose $\{U_x\}_{x\in X}$ is a measurable subbundle of $X \times R^d$ with $A_x U_x \subset U_{Tx}$. By the subadditive ergodic theorem we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n|_{U_x}\|$$

exists and is constant a.e., but could be $-\infty$, and suppose the value of this limit is less than or equal to $\rho \in R$. For $\varepsilon > 0$ define

$$a_\varepsilon(x) = \sup_{n\geq 0}(\|A^n|_{U_x}\| e^{-n(\rho+\varepsilon)}).$$

Then $\frac{1}{n}\log a_\varepsilon(T^n x) \to 0$ a.e.

Proof. By the choice of ρ we have $1 \leq a_\varepsilon(x) < \infty$. Also

$$\frac{a_\varepsilon(x)}{a_\varepsilon(Tx)} \leq \max(\|A|_{U_x}\| e^{-(\rho+\varepsilon)}, 1)$$

so that

$$\log a_\varepsilon(x) - \log a_\varepsilon(Tx) \leq \max(\log^+ \|A_x\| - (\rho + \varepsilon), 0).$$

Hence $x \to (\log a_\varepsilon(x) - \log a_\varepsilon(Tx))^+$ is integrable and we can apply Lemma 9. □

We shall mostly use the next result with $U_x = V_x^{(i+1)}$, $V_x = V_x^{(i)}$ for some i, but we consider a more general situation in order to prove Corollary 12.

Theorem 11. *Let T be an ergodic measure-preserving transformation of the probability space $(X, \mathscr{B}, m)$ and let $A\colon X \to \mathscr{M}(d)$ be such that $x \to \log^+ \|A_x\|$ is integrable. Assume $\{U_x\}_{x\in X}$, $\{V_x\}_{x\in X}$ are measurable subbundles of $X \times R^d$ with $U_x \subset V_x$, $A_x U_x \subset U_{Tx}$, and $A_x V_x \subset V_{Tx}$ $\forall x \in X$. Let $Y \in \mathscr{B}$ be a set of full measure with $TY \subset Y$ and let ρ, λ be real numbers where*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(A^n)_x u\| \le \rho \quad \forall u \in U_x\backslash\{0\} \quad \forall x \in Y$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(A^n)_x v\| = \lambda \quad \forall v \in V_x\backslash U_x \quad \forall x \in Y.$$

Assume $\rho < \lambda$. Define the measurable subbundle $\{W_x\}_{x\in X}$ by $V_x = U_x \overset{\perp}{\oplus} W_x$ using the Euclidean inner-product on R^d. Let $A_x|_{V_x}\colon V_x \to V_{Tx}$ induce the linear maps $C_x\colon W_x \to W_{Tx}$, $B_x\colon W_x \to U_{Tx}$ by $A_x(w) = B_x(w) \overset{\perp}{\oplus} C_x(w)$. Then there exists $Y' \in \mathscr{B}$ with $m(Y') = 1$ and $TY' \subset Y'$ with the following properties: We have

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\| = \limsup_{n\to\infty} \frac{1}{n} \log \|(A^n)_x (u \oplus w)\| \qquad \forall w \in W_x\backslash\{0\},\ \forall u \in U_x,\ \forall x \in Y'.$$

Moreover, if $\lim_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\|$ exists for some $w \in W_x\backslash\{0\}$ and some $x \in Y'$ then $\lim_{n\to\infty} \frac{1}{n} \log \|(A^n)_x (u \oplus w)\|$ exists $\forall u \in U_x$ and equals $\lim_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\|$.

Proof. We use u to denote a general element of some U_x, and w to denote a general element of some W_x. We have $(A^n)_x(u \oplus w) = [(A^n)_x u + D_x^{(n)} w] \oplus (C^n)_x w$, where $D_x^{(n)}\colon W_x \to U_{T^n x}$ is $\sum_{i=0}^{n-1} (A^{n-i-1})_{T^{i+1}x} B_{T^i x} (C^i)_x$. Taking the square of the Euclidean norm of each side and using Lemma 1 we get

$$(*)\quad \begin{aligned}&\limsup_{n\to\infty} \frac{1}{n} \log \|(A^n)_x (u \oplus w)\| \\ &\quad = \max\left(\limsup_{n\to\infty} \frac{1}{n} \log \|(A^n)_x u + D_x^{(n)} w\|,\ \limsup_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\| \right).\end{aligned}$$

Putting $u = 0$ and $w \neq 0$ gives

$$(**)\quad \begin{aligned}&\limsup_{n\to\infty} \frac{1}{n} \log \|(A^n)_x w\| \\ &\quad = \max\left(\limsup \frac{1}{n} \log \|D_x^{(n)} w\|,\ \limsup \frac{1}{n} \log \|(C^n)_x w\| \right).\end{aligned}$$

For $\varepsilon > 0$ let $a_\varepsilon(x) = \sup_{n\ge 0}(\|A^n|_{U_x}\| e^{-n(\rho+\varepsilon)})$. Let $\{\varepsilon_p\}_{p=1}^\infty$ be a sequence with $\varepsilon_p \searrow 0$ and, using Corollaries 9 and 10, let Y_p be the subset of Y of full measure on which $\frac{1}{n} \log a_{\varepsilon_p}(T^n x) \to 0$ and $\frac{1}{n} \log^+ \|A_{T^n x}\| \to 0$ $\forall x \in Y_p$. We shall show $\forall x \in \bigcap_{p=1}^\infty Y_p$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\| = \limsup_{n\to\infty} \frac{1}{n} \log \|(A^n)_x w\|, \qquad \forall w \neq 0.$$

If not, choose $x \in \bigcap_{p=1}^{\infty} Y_p$ and $w \neq 0$ with

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\| = \tau < \lambda' = \limsup_{n\to\infty} \frac{1}{n} \log \|(A^n)_x w\|.$$

Choose p so that $\max(\tau, \rho) + \varepsilon_p < \lambda'$. There exists N, depending on x, w, and p, such that $n \geq N$ implies $\|(C^n)_x w\| < e^{n(\tau+\varepsilon_p)}$. If we write $L_x \colon U_x \to U_{Tx}$ instead of $A_x|_{U_x}$ then

$$\begin{aligned}
\|D_x^{(n)} w\| &\leq \sum_{i=0}^{n-1} \|(L^{n-i-1})_{T^{i+1}x}\| \cdot \|B_{T^i x}\| \cdot \|(C^i)_x w\| \\
&\leq n \max_{0\leq i\leq n-1} \|(L^{n-i-1})_{T^{i+1}x}\| \cdot \|B_{T^i x}\| \cdot \|(C^i)_x w\| \\
&= n \|(L^{n-i_n-1})_{T^{i_n+1}x}\| \cdot \|B_{T^{i_n} x}\| \cdot \|(C^{i_n})_x w\|
\end{aligned}$$

for some $0 \leq i_n \leq n-1$, which depends on x and w. Note that $\{i_n\}$ is an increasing sequence. If $\{i_n\}$ is unbounded then $i_n \geq N$ eventually and

$$\begin{aligned}
\frac{1}{n} \log \|D_x^{(n)} w\| \leq \frac{1}{n} \log n &+ \frac{1}{n} \log a_{\varepsilon_p}(T^{1+i_n} x) + \frac{(n-1-i_n)}{n}(\rho + \varepsilon_p) \\
&+ \frac{1}{n} \log^+ \|B_{T^{i_n} x}\| + \frac{i_n}{n}(\tau + \varepsilon_p)
\end{aligned}$$

so $\limsup_{n\to\infty} \frac{1}{n} \log \|D_x^{(n)} w\| \leq \max(\tau, \rho) + \varepsilon_p < \lambda'$, which contradicts $(**)$. Here we used

$$\frac{1}{n} \log^+ \|B_{T^{i_n} x}\| \leq \frac{i_n}{n} \frac{1}{i_n} \log^+ \|A_{T^{i_n} x}\| \to 0.$$

If $\{i_n\}$ is bounded, say $i_n \leq M$ for all n, then

$$\begin{aligned}
\frac{1}{n} \log \|D_x^{(n)} w\| \leq \frac{1}{n} \log n &+ \max_{0\leq i\leq M} \frac{1}{n} \log \|(L^{n-i-1})_{T^{i+1}x}\| \\
&+ \max_{0\leq i\leq M} \frac{1}{n} \log \|B_{T^i x}\| + \max_{0\leq i\leq M} \frac{1}{n} \log \|(C^i)_x w\|,
\end{aligned}$$

so that $\limsup_{n\to\infty} \frac{1}{n} \log \|D_x^{(n)} w\| \leq \rho$, which contradicts $(**)$. Hence

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\| = \limsup_{n\to\infty} \frac{1}{n} \log \|(A^n)_x w\|$$

$$\forall x \in \bigcap_{p=1}^{\infty} Y_p \equiv Y' \ \forall w \in W_x \backslash \{0\}.$$

By $(*)$ and the above reasoning this also equals

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(A^n)_x (u \oplus w)\| \quad \forall u \in U_x.$$

Now suppose $\lim_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\|$ exists for some $w \in W_x \backslash \{0\}$ and some $x \in Y'$. By Lemma 1 we have, using the Euclidean norm,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log \|(A^n)_x (u \oplus w)\| \\
&\quad \geq \max\left(\liminf_{n\to\infty} \frac{1}{n} \log \|(A^n)_x u + D_x^{(n)} w\|, \liminf_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\|\right) \\
&\quad \geq \lim_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\|.
\end{aligned}$$

With the first part of the proof this gives that $\lim_{n\to\infty} \frac{1}{n} \log \|(A^n)_x(u \oplus w)\|$ exists $\forall u \in U_x$ and equals $\lim_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\|$. □

If we consider Theorem 11 with $U_x = V_x^{(i+1)}$, $V_x = V_x^{(i)}$ for some i then we can consider the induced map $C_x\colon W_x \to W_{T_x}$ where $V_x^{(i)} = V_x^{(i+1)} \overset{\perp}{\oplus} W_x$ and we have $\limsup_{n\to\infty} \frac{1}{n} \log \|(C^n)_x w\| = \lambda^{(i)}\ \forall w \in W_x \backslash \{0\}$. Therefore under $\{C_x\}$ every nonzero vector gives the same value $\lambda^{(i)}$ for the limit supremum. If we could replace "lim sup" by "lim" for $\{C_x\}$ then, by the last part of Theorem 11, we could conclude $\lim_{n\to\infty} \frac{1}{n} \log \|(A^n)_x v\|$ exists $\forall v \in V_x^{(i)} \backslash V_x^{(i+1)}$. So we can reduce our problem to considering a family $\{C_x\}$ for which the lim sup takes the same value for all nonzero vectors. We shall deal with this case by considering the map induced on $X \times P(R^d)$ and using some "relative" ergodic theory. However we consider the ergodic theory of the map induced on $X \times P(R^d)$ in the general case $\{A_x\}$ as this throws light on this situation. To define the map on $X \times P(R^d)$ we need each A_x to be invertible:

Corollary 12. *Let T be ergodic. To prove Oseledec's theorem for the case when each $A_x \in \mathscr{M}(d)$ and $x \to \log^+ \|A_x\|$ is integrable it suffices to prove it for the case when $A_x \in GL(d)$ a.e. and $x \to \log^+ \|A_x\|$ is integrable.*

Proof. If $A_x \notin GL(d)$ then $A_x v = 0$ for some $v \neq 0$ so $\lambda^{(s(x))}(x) = -\infty$. If this occurs on a set of positive measure then $\lambda^{(s(x))}(x) = -\infty$ a.e. Define the measurable subbundle $\{W_x\}_{x\in X}$ by $W_x = (V_x^{(s(x))})^\perp$ and consider the linear map $C_x\colon W_x \to W_{Tx}$ defined by letting $C_x w$ be the orthogonal projection of $A_x w$ onto W_{Tx}. Then $x \to \log^+ \|C_x\|$ is integrable and each C_x is invertible since

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(C^n x) w\| \neq -\infty \quad \text{a.e.}$$

by Theorem 11. The Lyapunov exponents for $x \to C_x$ are $\lambda^{(s(x)-1)}(x), \ldots,$ $\lambda^{(1)}(x)$ and the corresponding filtration is $V_x^{(s(x)-1)} \cap W_x, \ldots, V_x^{(1)} \cap W_x = W_x$, by Theorem 11. Suppose we have proved Oseledec's theorem for $x \to C_x$, so that $\lim_{n\to\infty} \frac{1}{n} \log \|(C^n)_w w\| = \lambda^{(j)}(x)$ for a.e. x and all

$$w \in (V_x^{(j)} \cap W_x) \backslash (V_x^{(j+1)} \cap W_x) = (V_x^{(j)} \backslash V_x^{(j+1)}) \cap W_x.$$

Then by the second part of Theorem 11 $\lim_{n\to\infty} \frac{1}{n} \log \|(A^n)_x v\| = \lambda^{(j)}(x)$ for a.e. x and all $v \in V_x^{(j)} \backslash V_x^{(j+1)}$ and Oseledec's theorem holds for $x \to A_x$. □

From now on we assume $A_x \in GL(d)\ \forall x \in X$. We can then define $\varphi\colon X \times P(R^d) \to R$ by $\varphi(x, u) = \log(\|A_x \tilde{u}\| / \|\tilde{u}\|)$ where $\tilde{u}$ is any nonzero element of R^d on the line in R^d representing $u \in P(R^d)$. The function φ is measurable and for each $x \in X$ the map $u \to \varphi(x, u)$ is a real-valued continuous map of $P(R^d)$. If we define $\Phi\colon X \to R$ by $\Phi(x) = \sup_{u\in P(R^d)} \varphi(x, u)$ then $\Phi(x) = \log \|A_x\|$ so the condition that $x \to \log^+ \|A_x\|$ be integrable is equivalent to $x \to \Phi^+(x) \equiv \max(0, \Phi(x))$ being integrable. Define $S\colon X \times P(R^d) \to X \times P(R^d)$ by $S(x, u) = (Tx, A_x u)$ where $A_x\colon P(R^d) \to P(R^d)$ is the homeomorphism induced by $A_x\colon R^d \to R^d$. If $\tilde{u}$ is any nonzero element of R^d on the line given by $u \in P(R^d)$ we have

$$\sum_{i=0}^{n-1} \varphi(S^i(x, u)) = \log \|(A^n)_x \tilde{u}\| - \log \|\tilde{u}\|$$

so

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(S^i(x, u)) = \limsup_{n\to\infty} \frac{1}{n} \log \|(A_x^n)\tilde{u}\|$$

and if one side has a limit so does the other.

In the proof of the next result we shall use the following fact about dual spaces. Let $(X, \mathscr{B}, m)$ be a probability space and let E be a separable Banach space with dual space E^*. Let $L_m^1(E)$ be the space of all measurable functions $f: X \to E$ with $x \to \|f(x)\|$ integrable where two such functions are identified if they are equal a.e. This is a Banach space with norm $\|f\| = \int \|f(x)\| \, dm(x)$. Let $L_m^\infty(E^*, E)$ be the space of all maps $\gamma: X \to E^*$ for which $x \to \gamma_x(v)$ is bounded and measurable for each $v \in E$, where two such functions $\gamma^{(1)}, \gamma^{(2)}$ are identified if $x \to \gamma_x^{(1)}(v)$ is equal a.e. to $x \to \gamma_x^{(2)}(v)$ for every $v \in E$. This is a Banach space with norm $\|\gamma\|_\infty =$ ess.sup.$\|\gamma(x)\|$, which is finite by the principle of uniform boundedness. Then the map $\Psi: L_m^\infty(E^*, E) \to (L_m^1(E))^*$, given by $(\Psi\gamma)(f) = \int \gamma_x(f_x) \, dm(x)$ where $\gamma: X \to E^*$ $(x \to \gamma_x)$ is in $L_m^\infty(E^*, E)$ and $f: X \to E$ $(x \to f_x)$ is in $L_m^1(E)$, is an isometric isomorphism of Banach spaces [1, p. 47; 11, p. 95]. We shall be interested in the case when $E = C(P, R)$ for a compact metric space P. The set $L_m(M(P))$ of all measurable maps from X to $M(P)$ is a subset of the unit ball in $L_m^\infty(E^*, E)$, with $E = C(P, R)$, and is closed with respect to the weak*-topology on $L_m^\infty(E^*, E)$. Hence $L_m(M(P))$ is compact with respect to this topology. The set $L_m(M(P))$ can be identified with $M_m(X \times P) = \{\nu \in M(X \times P) | \nu$ projects to m on $X\}$ via the map $\alpha \to \overline{\alpha} \in M_m(X \times P)$ where for $x \to f_x$ in $L_m^1(C(P, R))$ we have

$$\int_{X\times P} f_x(y) \, d\overline{\alpha}(x, y) = \int_X \left(\int_P f_x(y) \, d\alpha_x(y) \right) dm(x).$$

The map $\varphi: X \times P(R^d) \to R$, defined above, gives a map $x \to \varphi(x, .)$ of X into $C(P(R^d), R)$. We noted above that $\sup\{\varphi(x, u) | u \in P(R^d)\} = \log \|A_x\|$ and an easy calculation gives $\inf\{\varphi(x, u) | u \in P(R^d)\} = -\log \|(A_x)^{-1}\|$. Hence $\log^+ \|(A_x)^{-1}\| = -\min(0, \inf\{\varphi(x, u) | u \in P(R^d)\}$. Therefore $x \to \varphi(x, .)$ is in $L_m^1(C(P(R^d), R))$ iff both $x \to \log^+ \|A_x\|$ and $x \to \log^+ \|(A_x)^{-1}\|$ are integrable iff both $x \to \log \|A_x\|$ and $x \to \log \|(A_x)^{-1}\|$ are integrable. It is for this reason we have to assume both of these integrability conditions in the last stage of the proof.

A version of the following result appears in [4].

Theorem 13. *Let T be an ergodic measure-preserving transformation of a probability space $(X, \mathscr{B}, m)$. Let P be a compact metric space and let $S: X \times P \to X \times P$ be a measurable map of the form $S(x, u) = (Tx, S_x u)$ where, for each $x \in X$, the map $S_x: P \to P$ is continuous. Let $\varphi: X \times P \to R$ be measurable and for every $x \in X$ let $\varphi(x, .): P \to R$ be continuous. If $\Phi: X \to R$ is defined* *by* $\Phi(x) = \sup_{u\in P} \varphi(x, u)$ *then suppose* $\Phi^+ \in L^1_m$. *Then for* m *a.e.* $x \in X$,

$$\lim_{n\to\infty} \sup_{u\in P} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(S^i(x, u)) = \sup\left\{ \int \varphi \, d\nu | \nu \in M_m(X \times P, S) \right\}.$$

Proof. Note that the extreme points of the convex set $M_m(X \times P, S)$, of all S-invariant members of $M_m(X \times P)$, are exactly the ergodic members of $M_m(X \times P, S)$.

Note that if $\psi: X \times P \to R$ is measurable and $\psi(x, .): P \to R$ is continuous for each $x \in X$ then $x \to \sup_{u\in P} \psi(x, u)$ is measurable since if $\{u_n\}_{n=1}^{\infty}$ is a dense subset of P then $\sup_{u\in P} \psi(x, u) = \sup_{n\geq 1} \psi(x, u_n)$. Let

$$M_n(x) = \sup_{u\in P} \sum_{i=0}^{n-1} \varphi(S^i(x, u)).$$

This is measurable and $M_1^+ \in L^1_m$ so the subadditive ergodic theorem gives that $\lim_{n\to\infty} \frac{1}{n} M_n(x)$ exists a.e. and is equal to a constant c a.e. If we put $b = \sup\{\int \varphi \, d\nu | \nu \in M_m(X \times P, S)\}$ then we have to show $c = b$. Note that c, b could equal $-\infty$.

To show $c \geq b$ is easy. Suppose $b \neq -\infty$, or there is nothing to prove. For each $\varepsilon > 0$ we can choose an ergodic $\nu_\varepsilon \in M_m(X \times P, S)$ with $\int \varphi \, d\nu_\varepsilon > b - \varepsilon$. By Birkhoff's ergodic theorem

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi(S^i(x, u)) \to \int \varphi \, d\nu_\varepsilon$$

for ν_ε a.e. $(x, u) \in X \times P$. But $M_n(x) \geq \sum_{i=0}^{n-1} \varphi(S^i(x, u))$ so $c \geq b - \varepsilon$.

In order to show $c \leq b$ consider, for each $n \geq 1$, the set

$$\Delta_n = \left\{ (x, u) \in X \times P | \sum_{i=0}^{n-1} \varphi(S^i(x, u)) = M_n(x) \right\} \in \mathscr{B} \times \mathscr{B}(P).$$

We have $\pi\Delta_n = X$, where $\pi: X \times P \to X$ is the natural projection. Since for each x, $\{u \in P | (x, u) \in \Delta_n\}$ is closed we can choose a measurable map $w_n: X \to P$ with $(x, w_n(x)) \in \Delta_n \ \forall x \in X$ (Theorem 5). Hence $M_n(x) = \sum_{i=0}^{n-1} \varphi(S^i(x, w_n(x))$ and $x \to \delta_{w_n(x)}$ is in $L_m(M(P))$. For each $n \geq 1$ the linear functional on $L^1_m(C(P, R))$ given by

$$\psi \to \frac{1}{n} \sum_{i=0}^{n-1} \int \psi S^i(x, w_n(x)) \, dm(x)$$

gives an element $\alpha^{(n)}$ of $L_m(M(P))$. The sequence $\{\alpha^{(n)}\}$ has a convergent subsequence in the weak*-topology. Therefore there is $n_j \nearrow \infty$ and $\alpha \in L_m(M(P))$, which corresponds to some $\overline{\alpha} \in M_m(X \times P)$, with

$$\int \frac{1}{n_j} \sum_{i=0}^{n_j - 1} \psi S^i(x, w_{n_j}(x)) \, dm(x) \to \int \psi \, d\overline{\alpha} \quad \forall \psi \in L^1_m(C(P, R)).$$

If, for each $N \geq 1$, we let $\varphi_N = \max(\varphi, -N)$ then $\varphi_N \in L^1_m(C(P, R))$ and $M_{n_j}(x) \leq \sum_{i=0}^{n_j - 1} \varphi_N S^i(x, w_{n_j}(x))$. Hence $c \leq \int \varphi_N \, d\overline{\alpha}$ for each N and hence

$c \leq \int \varphi \, d\overline{\alpha}$. We now show $\overline{\alpha} \in M_m(X \times P, S)$ and this gives $c \leq b$. If $\psi \in L^1_m(C(P, R))$ then

$$\left|\int \psi \circ S \, d\overline{\alpha} - \int \psi \, d\overline{\alpha}\right|$$
$$= \lim_{j\to\infty} \frac{1}{n_j} \left|\int [\psi(S^{n_j}(x, w_{n_j}(x))) - \psi(x, w_{n_j}(x))] \, dm(x)\right|$$
$$\leq \lim_{j\to\infty} \frac{1}{n_j} 2 \int \|\psi(x, .)\| \, dm(x) = 0.$$

Hence $\overline{\alpha}$ is S-invariant. □

Corollary 14. *Suppose in addition to the assumptions of Theorem* 13 *that* $\int \varphi \, d\nu$ *takes the same value* b *for all* $\nu \in M_m(X \times P, S)$ *and that* $x \to \inf_{u\in P} \varphi(x, u)$ *is also in* L^1_m. *Then for* m *a.e.* $x \in X$, $\frac{1}{n}\sum_{i=0}^{n-1} \varphi(S^i(x, u)) \to b$ *uniformly in* $u \in P$.

Proof. By Theorem 13 we have

$$\lim_{n\to\infty} \sup_{u\in P} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(S^i(x, u)) \to b \quad \forall x \in X'$$

with $m(X') = 1$. By applying Theorem 13 to $-\varphi$ instead of φ we get

$$\lim_{n\to\infty} \inf_{u\in P} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(S^i(x, u)) \to b \quad \forall x \in X''$$

with $m(X'') = 1$. Hence for $x \in X' \cap X''$, $\frac{1}{n}\sum_{i=0}^{n-1} \varphi(S^i(x, u)) \to b$ uniformly in $u \in P$. □

We now complete the proof of Osedelec's theorem. Because

$$x \to \log^+ \|(A_x)^{-1}\|$$

is integrable we have $\lambda^{(s(x))}(x) \neq -\infty$ a.e. By Lemma 4 and the ergodicity of T there is a set $X_3 \in \mathscr{B}$ with $m(X_3) = 1$, $TX_3 \subset X_3$ on which $x \to s(x)$ is constant, each $x \to \lambda^{(i)}(x)$ is constant and each $x \to \dim(V_x^{(i)})$ is constant. By Lemma 4(iii) and Theorem 7 there is a bimeasurable bijection between $\{V_x^{(s)}\}_{x\in X_3}$ and $X_3 \times R^{r_s}$ covering the identity and which is linear on fibres, where $r_s = \dim(V_x^{(s)})$. Using this bijection we can define $\varphi \colon X_3 \times R^{r_s} \to R$ by $\varphi(x, u) = \log(\|A_x\tilde{u}\|/\|\tilde{u}\|$ where $\tilde{u}$ is any nonzero element of R^{r_s} on the line given by u. We have $\limsup_{n\to\infty} \frac{1}{n} \log \|(A^n)_x v\| = \lambda^{(s)}$ $\forall v \in V_x^{(s)} \backslash \{0\}$ so we have

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(S^i(x, u)) = \lambda^{(s)} \quad \forall u \in P(R^{r_s}).$$

By Birkhoff's ergodic theorem we have $\int \varphi \, d\nu = \lambda^{(s)}$ $\forall \nu \in M_m(X_3 \times P(R^{r_s}), S)$. Because we are assuming $x \to \log^+ \|(A_x)^{-1}\|$ is integrable, the conditions of Corollary 14 hold and we get for all $x \in X_4$, where $m(X_4) = 1$ and $TX_4 \subset X_4$, $\frac{1}{n}\sum_{i=0}^{n-1} \varphi(S^i(x, u)) = \lambda^{(s)}$ uniformly in $u \in P(R^{r_s})$. This says $\frac{1}{n} \log \|(A^n)_x v\| \to \lambda^{(s)}$ $\forall v \in V_x^{(s)} \backslash \{0\}$ (and uniformly over $\{v \in V_x^s \mid \|v\| = 1\}$).

Now consider $\{V_x^{(s-1)}\}_{x\in X_4}$. Write $V_x^{(s-1)} = V_x^s \overset{\perp}{\oplus} W_x$. By Theorem 11 we have for a.e. x,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(C_x^n)w\| = \lambda^{(s-1)} \quad \forall w \in W_x \backslash \{0\},$$

where $C_x: W_x \to W_{Tx}$ is induced from A_x. Since $\{W_x\}$ is a measurable subbundle and $\|C_x\| \le \|A_x\|$, $\|(C_x)^{-1}\| \le \|(A_x)^{-1}\|$ we can apply the above reasoning to $x \to C_x$ on $\{W_x\}$. This gives for a.e. x that $\frac{1}{n}\log\|(C^n)_x w\| \to \lambda^{(s-1)}$ $\forall w \in W_x\backslash\{0\}$. Then by the second part of Theorem 11 we get for a.e. x,

$$\frac{1}{n}\log\|(A^n)_x v\| \to \lambda^{(s-1)} \quad \forall v \in V_x^{(s-1)} \backslash V_x^s.$$

The proof follows by repeating the above reasoning. □

Note that for $\varphi: X \to R^d \to R$ defined by $\varphi(x, u) = \log(\|A_x\tilde{u}\|/\|\tilde{u}\|)$ we have for almost every $x \in X$,

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi(S^i(x, u)) \to \lambda^{(i)} \quad \forall u \in P(V_x^{(i)})\backslash P(V_x^{(i+1)}).$$

For each ergodic $\nu \in M_m(X\times P, S)$ there is a largest i with $\nu(\bigcup_x P(V_x^{(i)})) = 1$, and for this i, $\int \varphi\, d\nu = \lambda^{(i)}$. Moreover, for each i there exists an ergodic measure corresponding to i in the above way. Hence the exponents are the values of $\int \varphi\, d\nu$ as ν runs over ergodic measures in $M_m(X \times P, S)$. This was first pointed out by Ledrappier [6].

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