TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 335, Number 1, January 1993

SUBVARIETIES OF MODULI SPACE DETERMINED BY FINITE GROUPS ACTING ON SURFACES

JOHN F. X. RIES

ABSTRACT. Suppose the finite group G acts as orientation preserving homeomorphisms of the oriented surface S of genus g. This determines an irreducible subvariety $\mathcal{M}_g^{[G]}$ of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g consisting of all surfaces with a group G_1 of holomorphic homeomorphisms of the same topological type as G. This family is determined by an equivalence class of epimorphisms ψ from a Fuchsian group Γ to G whose kernel is isomorphic to the fundamental group of S. To examine the singularity of $\mathcal{M}_g$ along this family one needs to know the full automorphism group of a generic element of $\mathcal{M}_g^{[G]}$. In §2 we show how to compute this from ψ. Let $\mathcal{M}_g^G$ denote the locus of all Riemann surfaces of genus g whose automorphism group contains a subgroup isomorphic to G. In §3 we show that the irreducible components of this subvariety do not necessarily correspond to the families above, that is, the components cannot be put into a one-to-one correspondence with the topological actions of G. In §4 we examine the actions of G on the spaces of holomorphic k-differentials and on the first homology. We show that when G is not cyclic, the characters of these actions do not necessarily determine the topological type of the action of G on S.

INTRODUCTION

Suppose the finite group G acts as orientation preserving homeomorphisms of the oriented surface S. By varying the complex structure of the quotient surface S/G and lifting to S, one should get all Riemann surfaces S_1 with a group G_1 of holomorphic homeomorphisms of the same topological type as the action of G on S. That is, these are actions by G on Riemann surfaces that are analytically deformable to each other. Such a family is determined by an equivalence class of epimorphisms $\psi : \Gamma \to G$, where Γ is a group of Fuchsian type and the kernel of ψ is isomorphic to the fundamental group of S. These families were studied in [3–8, 10–12]. In §1 we review the relevant facts from Teichmüller space theory which are needed to describe these families.

There are questions about these families that can be answered given ψ. For instance, is G the automorphism group of the generic element of this family? If not, what is the automorphism group of the generic element? This is important to know if one is interested in the equisingularity strata of the singular set of the moduli space of genus g [14]. Using a result of Singerman [18], we characterize in §2 those G-actions for which the normalizer of G in the automorphism group

Received by the editors May 9, 1990 and, in revised form, October 15, 1990.
1991 *Mathematics Subject Classification*. Primary 14H15, 14H30, 14H45; Secondary 30F10.

of any surface in the corresponding family is strictly larger than G. We also note that if generically the normalizer of G is not the entire automorphism group, then the family consists of one point. Next we consider the set of all Riemann surfaces of genus g with a subgroup of its automorphism group isomorphic to G. In §3 we see that for certain G the components of this subvariety do not correspond in a one-to-one fashion with the topological actions of G, that is, there may be a G-action which determines a family completely contained in the family for another action of G.

Another set of questions involves invariants associated to group actions. If G is contained in the automorphism group of the Riemann surface S, then there is an induced action of G on the vector spaces of k-differentials on S for all $k \geq 1$. Note that the representation on quadratic differentials of the full automorphism group of S determines the singularity of the moduli space at S. The sequence of characters of these representations of G is determined up to composition with automorphisms of G. These invariants could be used to decide whether or not two epimorphisms $\psi_1, \psi_2 : \Gamma \to G$ determine equivalent actions of G. For cyclic groups these questions have been addressed by Guerrero [6], Harvey [7], A. Kuribayashi [10], and I. Kuribayashi [11]. In fact I. Kuribayashi proves that the sequence of characters determines the action of G for G cyclic. In §4 we offer some negative examples. We first review the relationship between the characters of the representations of G on differentials and the rotation data at fixed points. We prove that the character of the representation of G on integral homology determines the signature of Γ. Using the results of §2, we give two unramified S_7 covers of the surface of genus 2, one of which inherits the hyperelliptic involution and the other does not. Since both covers are unramified, the Eichler trace formula [4] implies that the corresponding representations of S_7 on k-differentials are equivalent for all k. We also give two examples of unramified D_{2n} covers for even n. To show they are not equivalent, we calculate the representations on integral homology. We see that these representations are equivalent in $\mathrm{Sp}(2g, \mathbb{Q})$, but not in $\mathrm{Sp}(2g, \mathbb{Z})$. These examples should be contrasted with cyclic group actions yielding representations on integral homology that are equivalent in $\mathrm{SL}(2g, \mathbb{Z})$, but not in $\mathrm{Sp}(2g, \mathbb{Z})$ [5].

1

We state some results from [3, 7–10, 12]. We will include some details, since we will be making explicit calculations in later sections.

Fix a closed, oriented surface S of genus g, with $g \geq 2$. Let $K = K_g$ denote $\pi_1(S, *)$, the fundamental group of S, $\mathrm{Aut}^+(K)$ the group of orientation preserving automorphisms of K, $\mathrm{Inn}(K)$ the subgroup of inner automorphisms of K, and $\mathrm{Mod}_g = \mathrm{Out}^+(K) = \mathrm{Aut}^+(K)/\mathrm{Inn}(K)$ the modular group or mapping class group of genus g. By a theorem of Nielsen this is isomorphic to the group of orientation preserving homeomorphisms of S modulo isotopy. Suppose θ is an orientation preserving homeomorphism of S. It may not fix the basepoint, and so it may not induce a well-defined automorphism of K. But θ is isotopic to homeomorphisms that do fix the basepoint, and so θ induces a well-defined element of $\mathrm{Out}^+(K)$, denoted by $\overline{\theta}$. We use the same notation for groups of homeomorphisms of S.

Suppose S_1 and S_2 are two Riemann surfaces of genus g and for $i = 1, 2$, $G_i < \mathrm{Aut}(S_i)$, the group of holomorphic homeomorphisms of S_i. We say (S_1, G_1) is topologically equivalent to (S_2, G_2) if there is an orientation preserving homeomorphism $\theta : S_1 \to S_2$ and an isomorphism $\beta : G_1 \to G_2$ such that for all $g \in G_1$, $\theta \circ g = \beta(g) \circ \theta$. Suppose $\pi_i : S_i \to S_i/G_i$ is the quotient map. Then this is equivalent to the existence of orientation preserving homeomorphisms $\theta : S_1 \to S_2$ and $\theta_0 : S_1/G_1 \to S_2/G_2$ such that $\pi_2 \circ \theta = \theta_0 \circ \pi_1$. If θ can be chosen to be biholomorphic, then we say (S_1, G_1) is isomorphic to (S_2, G_2).

Again suppose $G_i < \mathrm{Aut}(S_i)$ for $i = 1, 2$. Suppose $\theta_i : S \to S_i$, $i = 1, 2$, are any orientation preserving homeomorphisms. Then $G_i' = \theta_i^{-1} G_i \theta_i$, $i = 1, 2$, are two groups of homeomorphisms of S, which determine subgroups $\overline{G_1'}$ and $\overline{G_2'}$ of Mod_g. If (S_1, G_1) is topologically equivalent to (S_2, G_2) by θ, and $\alpha = \theta_2^{-1} \circ \theta \circ \theta_1$, then $\overline{G_2'} = \overline{\alpha}\overline{G_1'}\overline{\alpha}^{-1}$ and so $\overline{G_1'}$ and $\overline{G_2'}$ are conjugate subgroups of Mod_g.

On the other hand, suppose G is a finite subgroup of Mod_g. This determines, up to isomorphism, a unique extension of K via

$$\begin{array}{lccccccccc} & 1 & \longrightarrow & \mathrm{Inn}(K) & \longrightarrow & \mathrm{Aut}^+(K) & \longrightarrow & \mathrm{Out}^+(K) & \longrightarrow & 1 \\ (1) & & & \| & & \vee & & \vee & & \\ & 1 & \longrightarrow & K & \xrightarrow{i} & \Gamma & \xrightarrow{\psi} & G & \longrightarrow & 1. \end{array}$$

By Kerckhoff's solution of the Nielsen realization problem [9], Γ is a group of Fuchsian type, say with signature $[p; m_1, \dots, m_t]$. Let U be the upper half-plane. Then $\mathrm{PSL}(2, \mathbb{R})$ acts on U as its group of holomorphic homeomorphisms. Let $T(\Gamma)$, the Teichmüller space of Γ, be one of the two components of the set of equivalence classes $[r]$ modulo conjugation by $\mathrm{PSL}(2, \mathbb{R})$ of injections $r : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ such that $r(\Gamma)$ is discrete. Let $\mathrm{Mod}(\Gamma) = \mathrm{Aut}^+(\Gamma)/\mathrm{Inn}(\Gamma)$, where again $\mathrm{Aut}^+(\Gamma)$ is the group of orientation preserving type preserving automorphisms of Γ. Suppose $\alpha \in \mathrm{Aut}^+(\Gamma)$. Then $\overline{\alpha} \in \mathrm{Mod}(\Gamma)$ acts on $T(\Gamma)$ (on the right) by $\overline{\alpha}[r] = [r \circ \alpha]$. Then $\mathrm{Mod}(\Gamma)$ acts discontinuously on $T(\Gamma)$ and $\mathscr{M}(\Gamma) = T(\Gamma)/\mathrm{Mod}(\Gamma)$ is the space of conjugacy classes of discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$ isomorphic to Γ. The injection i induces a map $\overline{i} : T(\Gamma) \to T(K) = T_g$, defined by $\overline{i}[r] = [r \circ i]$, which is a holomorphic homeomorphism onto T_g^G, the locus of fixed points of G in T_g.

Suppose $[r] \in T(\Gamma)$ and let $\rho = r \circ i$. Let S_ρ denote the Riemann surface $U/\rho(K)$ of genus g. Then G acts on S_ρ as follows. Let $z \in U$ and $g \in G$. Suppose $\gamma \in \Gamma$ is any element that projects to g. Then

$$g\rho(K)z = r(\gamma)\rho(K)z.$$

This gives us an injection $\phi_r : G \to \mathrm{Aut}(S_\rho)$. Let $G_r = \phi_r(G)$. By covering space theory there is a homeomorphism $\theta : S \to S_\rho$ inducing $\pi_1(S, *) = K \xrightarrow{r} r(K) \simeq \pi_1(S_\rho, *)$, and the subgroup of $\mathrm{Out}^+(K)$ determined by G_r and θ as above is G.

Let $S_r = U/r(\Gamma)$ and $\pi : S_\rho \to S_r$ be the quotient map by the action of G_r. Then S_r is a Riemann surface of genus p, with t distinguished points $p_1, \dots, p_t$, which are the ramification points for the map π. If Γ has presentation

$$(2) \qquad \begin{aligned} \langle x_1, y_1, \dots, y_p, z_1, \dots, z_t \mid [x_1, y_1] \cdots [x_p, y_p] z_1 \cdots z_t \\ = 1 = z_1^{m_1} = \cdots = z_t^{m_t} \rangle, \end{aligned}$$

then

(a) $\psi(x_1), \psi(y_1), \ldots, \psi(z_t)$ generate G,
(b) $\psi(z_i)$ has order m_i in G,
(c) the elements of G_r which fix points of S_ρ over p_i are conjugate to powers of $\phi_r(\psi(z_i))$,
(d) (the Riemann-Hurwitz relation)

$$2g-2 = |G|\left(2p-2+\sum_{i=1}^{t}\left(1-\frac{1}{m_i}\right)\right).$$

Now suppose $[r]$ and $[s]$ are two elements of $T(\Gamma)$. Since $r(\Gamma)$ is isomorphic to $s(\Gamma)$ and they have compact quotient, it is known that there is a homeomorphism θ of U such that for all $\gamma \in \Gamma$,

$$s(\gamma)\circ\theta = \theta\circ r(\gamma).$$

Since $[r]$ and $[s]$ lie in the same component of $T(\Gamma)$, θ is orientation preserving [12]. Let $\rho = r\circ i$ and $\sigma = s\circ i$. Then θ descends to an orientation preserving homeomorphism of S_ρ and S_σ intertwining the actions of G_r and G_s. Thus (S_ρ, G_r) is topologically equivalent to (S_σ, G_s).

Suppose $\alpha \in \mathrm{Aut}^+(K)$ and consider $\overline{\alpha}G\overline{\alpha}^{-1}$. Then $\alpha\Gamma\alpha^{-1}$ is the subgroup of $\mathrm{Aut}^+(K)$ containing $\mathrm{Inn}(K)$ which projects to $\overline{\alpha}G\overline{\alpha}^{-1}$. We denote by ι_α the isomorphism of conjugation by α, $\iota_\alpha(\gamma) = \alpha\gamma\alpha^{-1}$. Then we have the followng commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K & \xrightarrow{j} & \alpha\Gamma\alpha^{-1} & \longrightarrow & \overline{\alpha}G\overline{\alpha}^{-1} & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\iota_\alpha} & & \uparrow{\scriptstyle\iota_{\overline{\alpha}}} & & \\
1 & \longrightarrow & K & \xrightarrow{i} & \Gamma & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

Then $\overline{\iota}_\alpha : T(\alpha\Gamma\alpha^{-1}) \to T(\Gamma)$ is an analytic homeomorphism and we have $\overline{j}T(\alpha\Gamma\alpha^{-1}) = T_g^{\overline{\alpha}G\overline{\alpha}^{-1}} = \overline{\alpha}^{-1}(T_g^G)$. Note that if $\alpha \in \Gamma$, then $\overline{\iota}_\alpha$ is the identity map on $T(\Gamma)$.

We can draw several conclusions from this. First let $N(G)$ be the normalizer of G in Mod_g. Then $N(G)$ is contained in the stabilizer of $\overline{i}T(\Gamma) = T_g^G$ in Mod_g and the homomorphism which sends α to ι_α induces an injection of $N(G)/G$ into $\mathrm{Mod}(\Gamma)$. Since $K \lhd \Gamma$, we can define the relative mapping class group of Γ and K as

$$\mathrm{Mod}(\Gamma, K) = \{\overline{\alpha} \in \mathrm{Mod}(\Gamma) \mid \alpha(K) = K\}.$$

Then $\mathrm{Mod}(\Gamma, K)$ is a subgroup of finite index in $\mathrm{Mod}(\Gamma)$ and

$$N(G)/G \simeq \mathrm{Mod}(\Gamma, K). \tag{3}$$

On the other hand, if $\overline{\alpha}(T_g^G) = T_g^G$, then $T_g^{\overline{\alpha}G\overline{\alpha}^{-1}} = T_g^G$. Let H be the group generated by $\overline{\alpha}G\overline{\alpha}^{-1}$ for $\overline{\alpha}$ in the stabilizer of T_g^G. Then $T_g^G = T_g^H$ and so H is a finite group. Thus $N(G)$ is a subgroup of finite index in $N(H)$, which is the stabilizer of T_g^G in Mod_g.

Suppose $\alpha \in \mathrm{Aut}^+(K)$ and $[r] \in T(\alpha\Gamma\alpha^{-1})$. Then $\overline{\iota}_\alpha[r] = [r\circ\iota_\alpha] \in T(\Gamma)$. Let $\rho_1 = r\circ j$ and $\rho_2 = r\circ\iota_\alpha\circ i$. Then $\rho_1\circ\alpha = \rho_2$ and so $\rho_1(K) = \rho_2(K)$, $S_{\rho_1} = S_{\rho_2}$, and $\phi_r(\overline{\alpha}G\overline{\alpha}^{-1}) = \phi_{r\circ\iota_\alpha}(G)$ in $\mathrm{Aut}(S_{\rho_1})$.

We can summarize the above remarks as follows. Suppose $\mathscr{G}$ is a conjugacy class of finite subgroups of Mod_g and let $G \in \mathscr{G}$. We also denote this conjugacy class by $[G]$. Let Γ be the subgroup of $\mathrm{Aut}^+(K)$ which projects to G. Then Γ is of Fuchsian type and

$$\mathscr{M}(\mathscr{G}) = T(\Gamma)/\mathrm{Mod}(\Gamma, K)$$

is the moduli space of isomorphism classes of pairs (S_1, G_1), where S_1 is a Riemann surface of genus g, $G_1 < \mathrm{Aut}(S_1)$, and $\overline{G}_1$, any subgroup of Mod_g induced by G_1, lies in $\mathscr{G}$. The map $\bar{i} : T(\Gamma) \to T_g$ covers the natural forgetful map $\mathscr{M}(\mathscr{G}) \to \mathscr{M}_g$ which sends (S_1, G_1) to S_1. Let $\mathscr{M}_g^{\mathscr{G}}$ denote the image of this map. This map is finite-to-one, but may not be injective since the submanifolds T_g^G and $T_g^{\overline{\alpha}G\overline{\alpha}^{-1}}$ of T_g may not be disjoint, that is, there may be a finite subgroup H of Mod_g which contains G and $\overline{\alpha}G\overline{\alpha}^{-1}$ but these are not conjugate in H. For example, using the results of the next section, we can see that there is only one conjugacy class $\mathscr{G}$ of subgroups of Mod_5 isomorphic to S_4 and the map from $\mathscr{M}(\mathscr{G})$ to $\mathscr{M}_5^{\mathscr{G}}$ is generically two-to-one. If Γ has signature $[p; m_1, \dots, m_t]$, then

$$\dim \mathscr{M}(\mathscr{G}) = \dim T(\Gamma) = 3p - 3 + t.$$

Finally suppose Γ is a group of Fuchsian type with signature $[p; m_1, \dots, m_t]$ and $\psi : \Gamma \to G$ is a homomorphism to the finite group G having the above properties (a), (b), and (d) for some $g \geq 2$. Then the kernel of ψ is torsion free and isomorphic to K. Let $i : K \to \Gamma$ be an isomorphism from K to $\ker \psi$. Since the center of Γ is trivial, the action of Γ on K induced by conjugation yields injections $\Gamma \to \mathrm{Aut}^+(K)$ and $G \to \mathrm{Mod}_g$ and we are in the situation above. More precisely, suppose $\gamma \in \Gamma$ and let α_γ denote the automorphism of K induced by conjugation by γ, that is, $\iota_\gamma \circ i = i \circ \alpha_\gamma$. Note that if $\gamma = i(k)$ for $k \in K$, then $\alpha_\gamma = \iota_k$. For $a \in G$, let $\sigma(a)$ denote the induced element of Mod_g, that is, if $\gamma \in \Gamma$ and $\psi(\gamma) = a$, then $\sigma(a) = \overline{\alpha}_\gamma$. We say that σ is induced by the pair (i, ψ). Then the following can easily be shown:

(a) If $\delta \in \mathrm{Aut}(G)$, then $(i, \delta \circ \psi)$ induces $\sigma \circ \delta^{-1}$.
(b) If $\beta \in \mathrm{Aut}^+(\Gamma)$, then $(\beta^{-1} \circ i, \psi \circ \beta)$ induces σ.
(c) If $\alpha \in \mathrm{Aut}^+(K)$, then $(i \circ \alpha^{-1}, \psi)$ induces $\iota_{\overline{\alpha}} \circ \sigma$.
(d) If $\beta \in \mathrm{Aut}^+(\Gamma)$ and preserves $i(K)$, then there is a $\delta \in \mathrm{Aut}(G)$ so that $\delta \circ \psi \circ \beta = \psi$ and there is an $\alpha \in \mathrm{Aut}^+(K)$ so that $\beta \circ i \circ \alpha = i$. Then $(i, \psi \circ \beta)$ and $(\beta \circ i, \psi)$ induce $\sigma \circ \delta = \iota_{\overline{\alpha}} \circ \sigma$ and hence $\overline{\alpha} \in N(\sigma(G))$.

Thus $\sigma(G)$, the image of G in Mod_g, depends on both i and ψ, but the conjugacy class $\mathscr{G}_\psi = [\sigma(G)]$ is determined by ψ. Note that two such epimorphisms $\psi_1, \psi_2 : \Gamma \to G$ determine isomorphic extensions of K and hence topologically equivalent G-actions iff there are $\beta \in \mathrm{Aut}^+(\Gamma)$ and $\lambda \in \mathrm{Aut}(G)$ so that $\psi_2 = \lambda \circ \psi_1 \circ \beta$.

Notation. Suppose $\mathscr{G}$ and $\mathscr{H}$ are conjugacy classes of subgroups of Mod_g. Then we will write $\mathscr{G} < \mathscr{H}$ (resp. $\mathscr{G} \lhd \mathscr{H}$) if there are $G \in \mathscr{G}$ and $H \in \mathscr{H}$ such that $G < H$ (resp. $G \lhd H$).

2

In this paper we want to study the subvarieties $\mathscr{M}_g^{\mathscr{G}}$ of $\mathscr{M}_g$. We have already noted their dimensions. Their singularities would have similar description as for

$\mathscr{M}_g$ itself, possible quotient singularities at S_1 when $G_1 \neq N_{\mathrm{Aut}(S_1)}(G_1)$, but also possible identification singularities when $\mathrm{Aut}(S_1)$ contains two subgroups G_1 and G_2 from $\mathscr{G}$ which are not conjugate in $\mathrm{Aut}(S_1)$.

It is clear that if $G < H$ are finite subgroups of Mod_g, then $\mathscr{M}_g^{[H]} \subseteq \mathscr{M}_g^{[G]}$. We will first examine cases when these can be equal even though $G \neq H$. If $\psi : \Gamma \to G$ and Γ has presentation (2), let $a_i = \psi(x_i)$, $b_i = \psi(y_i)$, $1 \leq i \leq p$, and $c_i = \psi(z_i)$, $1 \leq i \leq t$.

Theorem. *Let $\psi : \Gamma \to G$ and $i : K_g \to \Gamma$ induce the inclusion of G as a subgroup of Mod_g. Suppose one of the following is true:*

1. *Γ has signature $[2; -]$ and there is an automorphism α of G such that, if $d = a_1^{-1}b_1^{-1}a_2b_2$,*

$$\alpha(a_1) = a_1^{-1}, \quad \alpha(b_1) = b_1^{-1}, \quad \alpha(a_2) = da_2^{-1}d^{-1}, \quad \alpha(b_2) = db_2^{-1}d^{-1}.$$

2. *Γ has signature $[1; k, k]$ and there is an $\alpha \in \mathrm{Aut}(G)$ such that, if $d = a_1^{-1}b_1^{-1}c_1$,*

$$\alpha(a_1) = a_1^{-1}, \quad \alpha(b_1) = b_1^{-1}, \quad \alpha(c_1) = dc_2d^{-1}, \quad \alpha(c_2) = dc_1d^{-1}.$$

3. *Γ has signature $[1; k]$ and there is an $\alpha \in \mathrm{Aut}(G)$ such that*

$$\alpha(a_1) = a_1^{-1}, \quad \alpha(b_1) = b_1^{-1}, \quad \alpha(c_1) = a_1^{-1}b_1^{-1}c_1b_1a_1.$$

4. *Γ has signature $[0; k, l, k, l]$ for some k, l with $2 \leq k$ and $3 \leq l$. If $k \neq l$, there is an $\alpha \in \mathrm{Aut}(G)$ such that*

$$\alpha(c_1) = c_3, \quad \alpha(c_2) = c_4, \quad \alpha(c_3) = c_1, \quad \alpha(c_4) = c_2.$$

If $k = l$, then α exists after replacing ψ by $\psi \circ \mu$ for some $\mu \in \mathrm{Aut}^+(\Gamma)$.

5. *Γ has signature $[0; k, k, k, k]$ for some k with $3 \leq k$ and there are $\alpha, \beta \in \mathrm{Aut}(G)$ such that*

$$\alpha(c_1) = c_3, \quad \alpha(c_2) = c_4, \quad \alpha(c_3) = c_1, \quad \alpha(c_4) = c_2,$$
$$\beta(c_1) = c_2, \quad \beta(c_2) = c_1, \quad \beta(c_3) = c_1^{-1}c_4c_1, \quad \beta(c_4) = c_2c_3c_2^{-1}.$$

6. *Γ has signature $[0; l, l, k]$ for some k, l with $3 \leq l$, $2 \leq k$ and at least one of the inequalities is strict. If $k \neq l$, there is an $\alpha \in \mathrm{Aut}(G)$ such that*

$$\alpha(c_1) = c_2, \quad \alpha(c_2) = c_1, \quad \alpha(c_3) = c_2c_3c_2^{-1}.$$

If $k = l$, then α exists after replacing ψ by $\psi \circ \mu$ for some $\mu \in \mathrm{Aut}^+(\Gamma)$.

7. *Γ has signature $[0; k, k, k]$ for some $k \geq 4$ and there is a $\beta \in \mathrm{Aut}(G)$ such that*

$$\beta(c_1) = c_2, \quad \beta(c_2) = c_3, \quad \beta(c_3) = c_1.$$

8. *Γ has signature $[0; k, k, k]$ for some $k \geq 4$ and there are $\alpha, \beta \in \mathrm{Aut}(G)$ such that*

$$\alpha(c_1) = c_2, \quad \alpha(c_2) = c_1, \quad \alpha(c_3) = c_2c_3c_2^{-1},$$
$$\beta(c_1) = c_2, \quad \beta(c_2) = c_3, \quad \beta(c_3) = c_1.$$

If H is the corresponding group with presentation

1, 2, 3, 4, 6. $H = \langle G, a \mid \cdots, a^2 = 1, aga = \alpha(g), \text{for } g \in G \rangle$,

5. $H = \langle G, a, b \mid \cdots, a^2 = b^2 = 1, aga = \alpha(g), bgb = \beta(g), \text{for } g \in G, abab = (c_2c_3)^{-1} \rangle$,

7. $H = \langle G, b \mid \cdots, \ b^3 = 1, bgb^2 = \beta(g), for \ g \in G\rangle$,
8. $H = \langle G, a, b \mid \cdots, \ a^2 = b^3 = 1, aga = \alpha(g), bgb^2 = \beta(g),$ $for \ g \in G, abab = c_1^{-1}\rangle$,

where the dots denote the relations of G, *then there exist* Γ_0, *an inclusion* $j : \Gamma \to \Gamma_0$, *and an epimorphism* $\psi_0 : \Gamma_0 \to H$ *such that* $\psi = \psi_0 \circ j$ *and* $(j \circ i, \psi_0)$ *induces the inclusion of* H *as a subgroup of* Mod_g *with the property that* $G \neq H$, $G \lhd H$ *and* $\mathscr{M}_g^{[G]} = \mathscr{M}_g^{[H]}$.

Conversely, any subgroup H *of* Mod_g *with the above property arises in this way.*

Proof. By the above remarks Γ is identified with a subgroup of $\mathrm{Aut}^+(K)$ and ψ becomes the projection in (1). Suppose there is a finite subgroup H of Mod_g such that $G \lhd H$, $G \neq H$, and $\mathscr{M}_g^{[G]} = \mathscr{M}_g^{[H]}$. Let Γ_0 be the subgroup of $\mathrm{Aut}^+(K)$ containing $\mathrm{Inn}(K)$ which projects to H and let $\psi_0 : \Gamma_0 \to H$ be the projection. Then $\Gamma \lhd \Gamma_0$ and, if $j : \Gamma \to \Gamma_0$ denotes the inclusion, then $\bar{j}T(\Gamma_0) = T(\Gamma)$, since they have the same dimension.

Assume Γ_0 has signature $[p_0; n_1, \dots, n_s]$ and presentation

$$\langle X_1, Y_1, \dots, Y_{p_0}, Z_1, \dots, Z_s \mid [X_1, Y_1]\cdots[X_{p_0}, Y_{p_0}]Z_1\cdots Z_s = 1 = Z_1^{n_1} = \cdots = Z_s^{n_s}\rangle,$$

and Γ has signature $[p; n_1, \dots, n_t]$ and presentation (2). Singerman lists all possible pairs Γ, Γ_0 with the above properties in [18]. Let $H_0 = \Gamma_0/\Gamma$ and let $\theta_0 : \Gamma_0 \to H_0$ be the quotient homomorphism. Then (j, θ_0) induces $H_0 < \mathrm{Mod}(\Gamma)$. Given ψ so that $\ker\psi = i(K)$, then $i(K)$ will be a normal subgroup of Γ_0 iff $H_0 < \mathrm{Mod}(\Gamma, K)$ iff for all $\gamma \in \Gamma_0$ there is an $\alpha_\gamma \in \mathrm{Aut}(G)$ so that $\psi \circ \iota_\gamma = \alpha_\gamma \circ \psi$. If this is true, then $H_0 \simeq H/G$. So, for each generator of H_0, we choose a particular coset representative $\gamma \in \Gamma_0$ projecting to it. Then $\psi_0(\gamma)$ gives a generator of H over G so that $\iota_{\psi_0(\gamma)} = \alpha_\gamma \in \mathrm{Aut}(G)$ and any word in these γ projecting to the identity in H_0 must lie in Γ and so project via ψ into G. The following generators and relations determine an inclusion j of Γ as a subgroup of Γ_0 and can be verified using a standard Reidemeister-Schreier rewriting process [13].

1. Γ has signature $[2; -]$, Γ_0 has signature $[0; 2, 2, 2, 2, 2, 2]$, $H_0 = \mathbb{Z}_2 = \{\pm 1\}$, and $\theta_0(Z_i) = -1$, $1 \le i \le 6$. We identify $\ker\theta_0$ with Γ by

$$x_1 = Z_1Z_2, \quad y_1 = Z_3Z_2, \quad x_2 = Z_4Z_5, \quad y_2 = Z_6Z_5.$$

If we take $\gamma = Z_2$, then $\gamma^2 = 1$ and $\alpha = \alpha_\gamma$. Since x_1, y_1, x_2, y_2, Z_2 generate Γ_0, we can calculate ψ_0. Setting $a = \psi_0(Z_2)$ and $d = a_1^{-1}b_1^{-1}a_2b_2$, we get

$$\psi_0(Z_1) = a_1a, \quad \psi_0(Z_2) = a, \quad \psi_0(Z_3) = b_1a,$$
$$\psi_0(Z_4) = a_2d^{-1}a, \quad \psi_0(Z_5) = d^{-1}a, \quad \psi_0(Z_6) = b_2d^{-1}a.$$

2. Γ has signature $[1; k, k]$, Γ_0 has signature $[0; 2, 2, 2, 2, k]$, $H_0 = \mathbb{Z}_2$, $\theta_0(Z_i) = -1$, $1 \le i \le 4$, and $\theta_0(Z_5) = 1$. We identify $\ker\theta_0$ with Γ by

$$x_1 = Z_2Z_3, \quad y_1 = Z_4Z_3, \quad z_1 = Z_5, \quad z_2 = Z_1Z_5Z_1.$$

If we take $\gamma = Z_3$, then $\gamma^2 = 1$ and $\alpha = \alpha_\gamma$. Setting $a = \psi_0(Z_3)$ and

$d = a_1^{-1} b_1^{-1} c_1$, we get

$$\psi_0(Z_1) = d^{-1}a, \quad \psi_0(Z_2) = a_1 a, \quad \psi_0(Z_3) = a,$$
$$\psi_0(Z_4) = b_1 a, \quad \psi_0(Z_5) = c_1.$$

3. Γ has signature $[1; k]$, Γ_0 has signature $[0; 2, 2, 2, 2k]$, $H_0 = \mathbb{Z}_2$, and $\theta_0(Z_i) = -1$, $1 \le i \le 4$. We identify $\ker\theta_0$ with Γ by

$$x_1 = Z_1 Z_2, \quad y_1 = Z_3 Z_1, \quad z_1 = Z_4^2.$$

If we take $\gamma = Z_1$, then $\gamma^2 = 1$ and $\alpha = \alpha_\gamma$. Setting $a = \psi_0(Z_1)$, we get

$$\psi_0(Z_1) = a, \quad \psi_0(Z_2) = a a_1, \quad \psi_0(Z_3) = a b_1^{-1}, \quad \psi_0(Z_4) = a b_1^{-1} a_1^{-1}.$$

4. Γ has signature $[0; k, l, k, l]$, Γ_0 has signature $[0; 2, 2, k, l]$, $H_0 = \mathbb{Z}_2$, $\theta_0(Z_i) = -1$, $1 \le i \le 2$, and $\theta_0(Z_i) = 1$, $3 \le i \le 4$. We identify $\ker\theta_0$ with Γ by

$$z_1 = Z_3, \quad z_2 = Z_4, \quad z_3 = Z_1 Z_3 Z_1, \quad z_4 = Z_1 Z_4 Z_1.$$

If we take $\gamma = Z_1$, then $\gamma^2 = 1$ and $\alpha = \alpha_\gamma$. Setting $a = \psi_0(Z_1)$, we get

$$\psi_0(Z_1) = a, \quad \psi_0(Z_2) = a c_2^{-1} c_1^{-1}, \quad \psi_0(Z_3) = c_1, \quad \psi_0(Z_4) = c_2.$$

5. Γ has signature $[0; k, k, k, k]$, Γ_0 has signature $[0; 2, 2, 2, k]$, $H_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2\}$, $\theta_0(Z_1) = \varepsilon_1$, $\theta_0(Z_2) = \varepsilon_2$, $\theta_0(Z_3) = \varepsilon_1\varepsilon_2$, and $\theta_0(Z_4) = 1$. We identify $\ker\theta_0$ with Γ by

$$z_1 = Z_4, \quad z_2 = Z_1 Z_4 Z_1, \quad z_3 = Z_3 Z_4 Z_3, \quad z_4 = Z_3 Z_1 Z_4 Z_1 Z_3.$$

If we take $\gamma_1 = Z_3$ and $\gamma_2 = Z_1$, then $\gamma_1^2 = \gamma_2^2 = 1$, $\gamma_1\gamma_2\gamma_1\gamma_2 = (z_2 z_3)^{-1}$, $\alpha = \alpha_{\gamma_1}$, and $\beta = \alpha_{\gamma_2}$. Setting $a = \psi_0(Z_3)$ and $b = \psi_0(Z_1)$, we get

$$\psi_0(Z_1) = b, \quad \psi_0(Z_2) = b c_1^{-1} a, \quad \psi_0(Z_3) = a, \quad \psi(Z_4) = c_1.$$

6. Γ has signature $[0; l, l, k]$, Γ_0 has signature $[0; 2, l, 2k]$, $H_0 = \mathbb{Z}_2$, $\theta_0(Z_1) = -1$, $\theta_0(Z_2) = 1$, and $\theta_0(Z_3) = -1$. We identify $\ker\theta_0$ with Γ by

$$z_1 = Z_1 Z_2 Z_1, \quad z_2 = Z_2, \quad z_3 = Z_3^2.$$

If we take $\gamma = Z_1$, then $\gamma^2 = 1$ and $\alpha = \alpha_\gamma$. Setting $a = \psi_0(Z_1)$, we get

$$\psi_0(Z_1) = a, \quad \psi_0(Z_2) = c_2, \quad \psi_0(Z_3) = a c_1^{-1}.$$

7. Γ has signature $[0; k, k, k]$, Γ_0 has signature $[0; 3, 3, k]$, $H_0 = \mathbb{Z}_3 = \{1, \varepsilon, \varepsilon^2\}$, $\theta_0(Z_1) = \varepsilon$, $\theta_0(Z_2) = \varepsilon^2$, and $\theta_0(Z_3) = 1$. We identify $\ker\theta_0$ with Γ by

$$z_1 = Z_3, \quad z_2 = Z_2 Z_3 Z_2^2, \quad z_3 = Z_2^2 Z_3 Z_2.$$

If we take $\gamma = Z_2$, then $\gamma^3 = 1$ and $\beta = \alpha_\gamma$. Setting $b = \psi_0(Z_2)$, we get

$$\psi_0(Z_1) = c_1^{-1} b^2, \quad \psi_0(Z_2) = b, \quad \psi_0(Z_3) = c_1.$$

8. Γ has signature $[0; k, k, k]$, Γ_0 has signature $[0; 2, 3, 2k]$, $H_0 = S_3$, $\theta_0(Z_1) = (12)$, $\theta_0(Z_2) = (123)$, and $\theta_0(Z_3) = (13)$. We identify $\ker\theta_0$ with Γ by

$$z_1 = Z_3^2, \quad z_2 = Z_2 Z_3^2 Z_2^2, \quad z_3 = Z_2^2 Z_3^2 Z_2.$$

If we take $\gamma_1 = Z_1$ and $\gamma_2 = Z_2$, then $\gamma_1^2 = \gamma_2^3 = 1$, $\gamma_1\gamma_2\gamma_1\gamma_2 = z_1^{-1}$, $\alpha = \alpha_{\gamma_1}$, and $\beta = \alpha_{\gamma_2}$. Setting $a = \psi_0(Z_1)$ and $b = \psi_0(Z_2)$, we get

$$\psi_0(Z_1) = a, \quad \psi_0(Z_2) = b, \quad \psi(Z_3) = b^2 a.$$

Moreover, with this particular j and a presentation of the corresponding group H_0, the presentation of H is that given in the theorem.

The remaining problem is that the generators of Γ singled out above may not be the initially chosen ones, that is, suppose j is replaced by $j \circ \mu^{-1}$ for some $\mu \in \mathrm{Aut}^+(\Gamma)$. Then by the remarks in §1, H_0 is replaced by $\overline{\mu} H_0 \overline{\mu}^{-1}$. But $T(\Gamma)^{H_0} = \overline{j}T(\Gamma_0) = T(\Gamma)$ and so also $T(\Gamma)^{\overline{\mu}H_0\overline{\mu}^{-1}} = T(\Gamma)$. Let H_1 be the subgroup of $\mathrm{Mod}(\Gamma)$ generated by $\overline{\mu} H_0 \overline{\mu}^{-1}$ for all $\overline{\mu} \in \mathrm{Mod}(\Gamma)$. Then $H_0 < H_1 \triangleleft \mathrm{Mod}(\Gamma)$ and $T(\Gamma)^{H_1} = T(\Gamma)$. Thus H_1 is a finite group and if Γ_1 is the subgroup of $\mathrm{Aut}^+(\Gamma)$ containing Γ (i.e., $\mathrm{Inn}(\Gamma)$) which projects to H_1, then $\Gamma \triangleleft \Gamma_1$ and $\dim T(\Gamma) = \dim T(\Gamma_1)$. Therefore the pair Γ, Γ_1 must be on Singerman's list. Thus in all cases except 4 and 6 with $k = l$, we have $\Gamma_0 = \Gamma_1$ and $H_0 = H_1$. This shows that the results above are actually independent of j for these cases. But in case 4 (resp. case 6) when $k = l$, the pair Γ, Γ_1 comes from case 5 (resp. case 8) and $H_1 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (resp. S_3). The three subgroups of H_1 isomorphic to H_0 are conjugate in $\mathrm{Mod}(\Gamma)$ and so there exists μ transforming our given j from the beginning of the proof into the one of the above computations. This will replace (i, ψ) with $(\mu^{-1} \circ i, \psi \circ \mu)$ which induces the same inclusion of G into Mod_g. Then $\psi \circ \mu$ has the property stated in the theorem.

Conversely, suppose ψ satisfies any of the cases of the theorem. If we form the indicated group H, then by the calculations in the first part of the proof, there is an inclusion j of Γ as a normal subgroup of the corresponding Γ_0 and an epimorphism $\psi_0 : \Gamma_0 \to H$ such that $\psi_0 \circ j = \psi$. Then $(j \circ i, \psi_0)$ induces an inclusion of H as a subgroup of Mod_g such that $G \triangleleft H$, $G \neq H$, and, since $\overline{j}T(\Gamma_0) = T(\Gamma)$, we have $\mathscr{M}_g^{[G]} = \mathscr{M}_g^{[H]}$. $\square$

Remark. Suppose G is a finite subgroup of Mod_g and suppose there is a subgroup H of Mod_g with the properties

$$G \triangleleft H, \quad G \neq H, \quad \text{and} \quad \mathscr{M}_g^{[G]} = \mathscr{M}_g^{[H]}.$$

If H_1 is the subgroup of Mod_g generated by all such H, then H_1 also has the above property. It follows from the theorem that if H_1 corresponds to case 1, 2, 3, 4, 6, or 7, then H_1 is the only subgroup of Mod_g with the above property. On the other hand, if H_1 corresponds to case 5 (resp. case 8), then there are also three subgroups of H_1 corresponding to case 4 with $k = l$ (resp. three subgroups of H_1 corresponding to case 6 with $k = l$ and a subgroup corresponding to case 7) with the above property. Note that their conjugacy classes may or may not be distinct.

As was noted earlier, $\mathscr{M}(\Gamma)$ is the space of isomorphism classes of Riemann surfaces of genus p with t distinguished points $p_1, \ldots, p_t$ of orders $m_1, \ldots, m_t$ resp. For most Γ the generic pointed Riemann surface corresponding to an element of $\mathscr{M}(\Gamma)$ has trivial automorphism group. But for the Γ listed in the theorem, the automorphism groups of all elements of $\mathscr{M}(\Gamma)$ contain the corresponding group H_0.

1. All Riemann surfaces of genus 2 are hyperelliptic.

2. All Riemann surfaces of genus 1 with distinguished points $p_1 \neq p_2$ have the automorphism $\alpha(p) = p_1 + p_2 - p$.

3. All Riemann surfaces of genus 1 with distinguished point p_1 have the automorphism $\alpha(p) = 2p_1 - p$.

4, 5. The automorphism group of $\mathbb{P}^1$ with any four distinct distinguished points contains $\mathbb{Z}_2 \times \mathbb{Z}_2$.

6, 7, 8. $\mathrm{Aut}(\mathbb{P}^1 \sim \{0, 1, \infty\}) = S_3$.

Thus the theorem gives the conditions for some or all of these automorphisms to lift to a regular cover branched over the distinguished points.

This also shows that we can say more in cases 4 and 6 when $k = l$. Suppose Γ has signature $[0; k, k, k, k]$, corresponding to $\mathbb{P}^1$ with four distinguished points p_1, p_2, p_3, p_4. Every orientation preserving automorphism of Γ sends z_i to a conjugate of some z_j and $\mathrm{Mod}(\Gamma)$ maps onto S_4. The automorphism group noted above permutes p_1, p_2, p_3, p_4 as the normal $\mathbb{Z}_2 \times \mathbb{Z}_2$ in S_4. There are three elements of order 2 and any one could lift to the cover of $\mathbb{P}^1$ determined by ψ. Thus case 4 says that, up to composing ψ with one of three coset representatives of $N(\Gamma_0)$ in $\mathrm{Aut}^+(\Gamma)$, ι_{Z_1} induces an automorphism of G, or, equivalently, in the notation of case 5, one of ι_{Z_1}, ι_{Z_2}, or ι_{Z_3} induces an automorphism of G. Similar remarks hold for case 6 with $k = l$, since $\mathrm{Mod}(\Gamma) = S_3$, which has three elements of order 2.

Remark. Note that the dimension of $\mathscr{M}_g^{[G]}$ is 3 in case 1, 2 in case 2, 1 in cases 3, 4, 5, and 0 in cases 6, 7, 8.

Remark. Again suppose ψ and i are as in (1). The above considerations could be applied to any finite subgroup H_0 of $\mathrm{Mod}(\Gamma)$ to determine whether the automorphisms determined by H_0 of surfaces uniformized by Γ lift to the cover determined by i. This is equivalent to asking if $H_0 < \mathrm{Mod}(\Gamma, K)$, that is, is there a subgroup H of Mod_g so that $G \lhd H$, $H/G \simeq H_0$, and $\bar{i}(T(\Gamma)^{H_0}) = T_g^H$? Again, one needs a group Γ_0 of Fuchsian type, an inclusion $j : \Gamma \to \Gamma_0$, and an epimorphism $\theta_0 : \Gamma_0 \to H_0$ such that $\ker\theta_0 \simeq j(\Gamma)$ and (j, θ_0) determines H_0 as a subgroup of $\mathrm{Mod}(\Gamma)$. Thus the reason that the above results are so easily computable is that in those cases H_0 is (or is very nearly) a normal subgroup of $\mathrm{Mod}(\Gamma)$.

It is also possible that $G < H$ are finite subgroups of Mod_g, G not normal in H, and $\mathscr{M}_g^{[G]} = \mathscr{M}_g^{[H]}$. Then, as in the theorem, we have Fuchsian groups $\Gamma < \Gamma_0 < \mathrm{Aut}^+(K_g)$, Γ not normal in Γ_0, K_g normal in Γ, and Γ_0 with $\Gamma/K_g \simeq G$ and $\Gamma_0/K_g \simeq H$, and if $j : \Gamma \to \Gamma_0$ is the inclusion, then $\bar{j}T(\Gamma_0) = T(\Gamma)$. Singerman lists all the possibilities in [18]; all such pairs are triangle groups and so $\mathscr{M}_g^{[G]}$ is a single point. Thus if $\mathscr{M}_g^{[G]}$ is not a single point and is not a case of the above theorem, then the generic element of $\mathscr{M}_g^{[G]}$ has automorphism group isomorphic to G.

Let Λ be the largest normal subgroup of Γ_0 contained in Γ. Then, if $T(\Gamma_0) = \{[r]\}$, the map $U/r(\Lambda) \to U/r(\Gamma_0)$ is the Galois closure of the map $U/r(\Gamma) \to U/r(\Gamma_0)$. Let $H_0 = \Gamma_0/\Lambda$, $G_0 = \Gamma/\Lambda$, and $n = [\Gamma_0 : \Gamma] = [H_0 : G_0]$. Singerman gives in [18] the permutation representation, $\theta_0 : \Gamma_0 \to S_n$, of Γ_0 on the set of cosets Γ_0/Γ. Then $\ker\theta_0 = \Lambda$ and $\theta_0(\Gamma) = \theta_0(\Gamma_0) \cap S_{n-1} \simeq G_0$. One can easily see that the signatures of Γ and Γ_0 (and hence the index n) determine the cycle structure of the elements $\theta_0(Z_1)$, $\theta_0(Z_2)$, and

$\theta_0(Z_3)$ of S_n, and consequently θ_0 is uniquely determined up to composition with an inner automorphism of S_n. On the other hand, in each case given Γ_0 and H_0, there is a unique epimorphism $\theta_0 : \Gamma_0 \to H_0$ up to composition with $\mathrm{Aut}(H_0)$, and then $\Gamma \simeq \theta_0^{-1}(G_0)$ and $\Lambda = \ker \theta_0$. Finally suppose H_0 and Λ are given and let Λ have signature $[h; k_1, \ldots, k_r]$. Then the signature of Γ_0 is determined and in each case there is a unique suitably pointed Riemann surface of genus h whose automorphism group contains (and hence equals) H_0.

In Table 1 we reproduce Singerman's list and identify H_0, G_0, and Λ. Again suppose $T(\Gamma_0) = \{[r]\}$ and let $S_1 = U/r(\Lambda)$. In cases A and C, S_1 is the Klein-Hurwitz surface of genus 3. In B, S_1 is Macbeath's surface of genus 7. In D, S_1 is the Fermat quartic of genus 3 and hence $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4 \rtimes S_3$ of order 96, where S_3 acts as in the nontrivial component of the permutation representation. G_1 is any cyclic subgroup of order 8. In F, $H_2 = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \rtimes A_4$ of order 324, where again A_4 acts as in the nontrivial component of the permutation representation. G_2 has order 27 and is generated by any two elements g_1, g_2 of order 9 such that $\langle g_1 \rangle \cap \langle g_2 \rangle$ has order 3. In G, S_1 is Bring's surface of genus 4.

Suppose Γ is from the table, $\psi : \Gamma \to G$ is an epimorphism, and i is an isomorphism from K_g to $\ker \psi$. Then the pair (i, ψ) allows us to identify G with a subgroup of Mod_g. Suppose S is the surface of genus g with this G-action. Then there is a group $H < \mathrm{Mod}_g$ with $G < H$, G not normal in H, and $\mathcal{M}_g^{[G]} = \mathcal{M}_g^{[H]}$ if and only if there is an inclusion $j : \Gamma \to \Gamma_0$ for some Γ_0 corresponding to Γ in the table and an epimorphism $\psi_0 : \Gamma_0 \to H$ such that $\psi_0 \circ j = \psi$ and $\ker \psi_0 = j(i(K))$. We will not go through the details of each case, but make the following remarks.

TABLE 1

	Γ	Γ_0	H_0	G_0	Λ	n
A	$[0; 7, 7, 7]$	$[0; 2, 3, 7]$	$\mathrm{PSL}(2, 7)$	$\mathbb{Z}_7$	$[3; -]$	24
B	$[0; 2, 7, 7]$	$[0; 2, 3, 7]$	$\mathrm{PSL}(2, 2^3)$	$\mathrm{Aff}(1, 2^3)$	$[7; -]$	9
C	$[0; 3, 3, 7]$	$[0; 2, 3, 7]$	$\mathrm{PSL}(2, 7)$	$\mathrm{Aff}^+(1, 7)$	$[3; -]$	8
D	$[0; 4, 8, 8]$	$[0; 2, 3, 8]$	H_1	G_1	$[3; -]$	12
E	$[0; 3, 8, 8]$	$[0; 2, 3, 8]$	$\mathrm{PGL}(2, 3^2)$	$\mathrm{Aff}(1, 3^2)$	$[16; -]$	10
F	$[0; 9, 9, 9]$	$[0; 2, 3, 9]$	H_2	G_2	$[10; -]$	12
G	$[0; 4, 4, 5]$	$[0; 2, 4, 5]$	$\mathrm{PGL}(2, 5)$	$\mathrm{Aff}(1, 5)$	$[4; -]$	6
H	$[0; m, 4m, 4m]$ $(m \geq 2)$	$[0; 2, 3, 4m]$	S_4	$\mathbb{Z}_4$	$[0; m, m, m, m, m, m]$	6
I	$[0; m, 2m, 2m]$ $(m \geq 3)$	$[0; 2, 4, 2m]$	D_8	$\mathbb{Z}_2$	$[0; m, m, m, m]$	4
J	$[0; m, 3, 3m]$ $(m \geq 3)$	$[0; 2, 3, 3m]$	A_4	$\mathbb{Z}_3$	$[0; m, m, m, m]$	4
K	$[0; 2, m, 2m]$ $(m \geq 4)$	$[0; 2, 3, 2m]$	S_3	$\mathbb{Z}_2$	$[0; m, m, m]$	3

If j and ψ_0 exist, then (a) there is a homomorphism $\pi : G \to H_0$ such that $\pi \circ \psi = \theta_0 \circ j$, and then $\pi(G)$ is (conjugate to) G_0. This says that the quotient map $S \to S/G$ factors through the surface S_1 mentioned above. If $k : \Lambda \to \Gamma$ is an isomorphism from Λ to $\ker(\pi \circ \psi)$ and $F = \ker \pi$, then let $\psi_1 = \psi \circ k$ be the epimorphism from Λ onto F. If conjugation by Γ_0 acts on Λ via $j \circ k$, then (b) for every $\gamma_0 \in \Gamma_0$ there is an $\alpha_{\gamma_0} \in \mathrm{Aut}(F)$ so that $\psi_1 \circ \iota_{\gamma_0} = \alpha_{\gamma_0} \circ \psi_1$. This says that $H_0 = \mathrm{Aut}(S_1)$ lifts to S. Conversely if (a) and (b) hold for a given j, then ψ_0 exists.

Conversely we could also ask if a given finite subgroup H of Mod_g contains a subgroup G such that $\mathscr{M}_g^{[G]} = \mathscr{M}_g^{[H]}$. Given all the information above we can give an easily computable condition in the next theorem. We continue the above notation and let $G_0 = \langle 1 \rangle$ in the context of the previous theorem. Note that θ_0 depends on the particular case of that theorem and not just on Γ_0 and H_0. In cases 2, 4, and 6 the epimorphism θ_0 given above is not the only one having a group of the "type" of Γ as kernel. In cases 2 and 4 when $k = 2$ and in case 6 when $l = 2k$, Γ is not a characteristic subgroup of Γ_0. In addition, in case 6 when l is even we may switch the roles of Z_2 and Z_3.

Theorem. *Suppose $\psi_0 : \Gamma_0 \to H$ determines the conjugacy class $\mathscr{G}_{\psi_0}$ of finite subgroups of* Mod_g. *There is a conjugacy class $\mathscr{G}$ of finite subgroups of* Mod_g *such that $\mathscr{G} < \mathscr{G}_{\psi_0}$, $\mathscr{G} \neq \mathscr{G}_{\psi_0}$, and $\mathscr{M}_g^{\mathscr{G}} = \mathscr{M}_g^{\mathscr{G}_{\psi_0}}$ iff Γ_0 is from the previous theorem or Table* 1 *and there is an epimorphism $\pi : H \to H_0$ such that $\pi \circ \psi_0 = \theta_0$, except in the following cases of the theorem*:

2. *When $k = 2$, $\pi \circ \psi_0(Z_i) = -1$ for all but one generator of Γ_0.*
4. *When $k = 2$, $\pi \circ \psi_0(Z_i) = -1$ for any two of Z_1, Z_2, Z_3.*
6. *When l is even, $\pi \circ \psi_0(Z_i) = -1$ for Z_1 and any one of Z_2, Z_3.*

In each case $\mathscr{G} = \mathscr{G}_\psi$, where $\psi : \Gamma \to G$ is defined by letting $G = \pi^{-1}(G_0)$, j be an isomorphism of Γ with $\psi_0^{-1}(G)$, and $\psi = \psi_0 \circ j$.

We mention one consequence of the above.

Corollary. *Suppose S is a Hurwitz surface with Hurwitz group H and further suppose there is no epimorphism from H to either* $\mathrm{PSL}(2, 7)$ *or* $\mathrm{PSL}(2, 2^3)$. *Then no proper subgroup of H "determines" S, that is, for every proper subgroup G of H, $\mathscr{M}_g^{[G]}$ has positive dimension.*

Finally suppose that $\mathscr{G}_1$ and $\mathscr{G}_2$ are conjugacy classes of finite subgroups of Mod_g such that $\mathscr{M}_g^{\mathscr{G}_1} = \mathscr{M}_g^{\mathscr{G}_2}$. Suppose $G_i \in \mathscr{G}_i$ are chosen so that $T_g^{G_1} = T_g^{G_2}$ and neither G_1 nor G_2 contains the other. Let H be the subgroup of Mod_g generated by G_1 and G_2. Then $T_g^H = T_g^{G_1}$ and H is a finite group. Note that it need not be true that $T_g^{G_1 \cap G_2} = T_g^{G_1}$. Let Γ_0, Γ_1, Γ_2, and Γ_3 be the subgroups of $\mathrm{Aut}^+(K_g)$ projecting to H, G_1, G_2, and $G_1 \cap G_2$, resp. Then the pairs Γ_1, Γ_0 and Γ_2, Γ_0 must come from the first theorem or from Table 1.

We will consider the case where the dimension of $\mathscr{M}_g^{[H]}$ is greater than or equal to 1. Then G_1 and G_2 are normal subgroups of H. Let $F_3 = H/(G_1 \cap G_2) = \Gamma_0/\Gamma_3$ and $F_i = H/G_i = \Gamma_0/\Gamma_i$. Up to switching Γ_1 and Γ_2 there are only four possibilities which we record in Table 2. This should be compared to the theorem in §2 of [6].

TABLE 2

Γ_0	Γ_1	Γ_2	Γ_3	F_1	F_2	F_3
$[0; 2, 2, 2, 2, 2]$	$[1; 2, 2]$	$[1; 2, 2]$	$[2; -]$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$[0; 2, 2, 2, 2k]$ $(k \geq 2)$	$[0; 2, 2k, 2, 2k]$	$[1; k]$	$[1; k, k]$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$[0; 2, 2, 2, k]$ $(k \geq 3)$	$[0; 2, k, 2, k]$	$[0; 2, k, 2, k]$	$[0; k, k, k, k]$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$[0; 2, 2, 2, 2k]$ $(k \geq 2)$	$[0; 2k, 2k, 2k, 2k]$	$[1; k]$	$[1; k, k, k, k]$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Notation. We introduce the following notation for the sake of brevity. Given Γ with signature $[p; m_1, \dots, m_t]$ and presentation (2), we specify homomorphisms $\psi: \Gamma \to G$ by listing the images of the generators and we will write

$$\psi \sim (\psi(x_1), \psi(y_1), \dots, \psi(z_t)) \in G^{2p+t}.$$

3

Suppose $g \geq 2$ and the finite group G are fixed. Let $[G_1], \dots, [G_m]$ be all the conjugacy classes of finite subgroups of Mod_g isomorphic to G. Then

$$\mathscr{M}_g^G = \bigcup_{i=1}^m \mathscr{M}_g^{[G_i]}$$

is the space of all Riemann surfaces S of genus g having a subgroup of its automorphism group isomorphic to G. It is a finite union of subvarieties of $\mathscr{M}_g$, but in general the number of components of $\mathscr{M}_g^G$ is less than m. That is, there may be classes $[G_1]$ and $[G_2]$ such that $G_1 \simeq G_2 \simeq G$ and $\mathscr{M}_g^{[G_1]} \subset \mathscr{M}_g^{[G_2]}$.

We consider the case where $G = \langle \delta \mid \delta^n = 1 \rangle$ is cyclic of order n for some $n \geq 3$. Suppose $\psi: \Gamma \to G$ determines the family $\mathscr{M}_g^{\mathscr{G}_\psi}$. We refer to the cases of the first theorem in §2.

In cases 1 and 2, $H \simeq D_{2n}$, the dihedral group of order $2n$, and is the automorphism group of the generic element S of $\mathscr{M}_g^{\mathscr{G}_\psi}$. In case 1, $g = n+1$, and in case 2, $g = n - n/k + 1$, where $1 < k \mid n$. Hence, generically, $\mathrm{Aut}(S)$ contains only one subgroup isomorphic to G. In case 3, Γ has no abelian quotient with torsion free kernel.

Suppose in case 4 that $c_1 = \psi(z_1)$ is a generator of G. By composing ψ with an automorphism of G we may assume $c_1 = \delta$. We also assume that H is abelian, that is, that α is the identity automorphism of G. Then $H \simeq \mathbb{Z}_n \times \mathbb{Z}_2$ and $\psi \sim (\delta, \delta^e, \delta, \delta^e)$ for some e such that $1 \leq e \leq n-1$ and $2 + 2e \equiv 0 \pmod{n}$. Hence $g = n - (n, e)$. If n is odd, then $e = n-1$ and if $n = 2n_1$ is even, then there are two solutions $e = n-1$ and $e = n_1 - 1$.

First suppose e is odd. Note that this implies $n = 2n_1$ and $g = n-1$. Then Γ_0 has signature $[0; 2, 2, 2n_1, 2n_1]$ and

$$\psi_0 \sim (a, a\delta^{-e-1}, \delta, \delta^e) = (a, a\delta_1^{-e-1}, a\delta_1, a\delta_1^e), \tag{4}$$

where $\delta_1 = a\delta$ also has order n. Suppose $\pi: H \to \mathbb{Z}_2$ is the epimorphism with kernel generated by δ_1. If $\Gamma_1 = \ker(\pi \circ \psi_0)$, then $i(K_g) \lhd \Gamma_1 \lhd \Gamma_0$. From

(4) we see that each generator of Γ_0 is sent to the generator of $\mathbb{Z}_2$. Again an easy computation shows that, if

$$x_1 = Z_1Z_2, \quad y_1 = Z_3Z_2, \quad z_1 = Z_3^2, \quad z_2 = Z_4^2,$$

then x_1, y_1, z_1, z_2 generate Γ_1 with relations $[x_1, y_1]z_1z_2 = 1 = z_1^{n_1} = z_2^{n_1}$. Thus Γ_1 has signature $[1; n_1, n_1]$ and if $\psi_1 = \psi_0|_{\Gamma_1}: \Gamma_1 \to G_1 = \langle\delta_1\rangle$, then $\psi_1 \sim (\delta_1^{-e-1}, \delta_1^e; \delta_1^2, \delta_1^{-2})$. Thus $\mathcal{M}_g^{\mathcal{G}_\psi} \subset \mathcal{M}_g^{\mathcal{G}_{\psi_1}}$.

Remark. If the Reimann surface S corresponds to an element of $\mathcal{M}_g^{\mathcal{G}_\psi}$, then the projection from S to S/G_1, the quotient by the action of G_1, is a map to a Riemann surface of genus 1 with two branch points. Since

$$\iota_{Z_2}(x_1) = x_1^{-1}, \qquad \iota_{Z_2}(y_1) = y_1^{-1}z_1,$$
$$\iota_{Z_2}(z_1) = y_1^{-1}z_1y_1, \qquad \iota_{Z_2}(z_2) = (x_1y_1)^{-1}z_2(x_1y_1),$$

we see that the automorphism of order two of the quotient surface S/G_1 which lifts to S to give the automorphism a in H fixes both ramification points, and so is not the automorphism of order 2 possessed by all elements of $\mathcal{M}_g^{\mathcal{G}_{\psi_1}}$ as in case 2.

Remark. If $n = 4n_0 + 2$, then only $e = 4n_0 + 1$ is odd and corresponds to $g = n - 1$.

Remark. If $n = 4n_0$, then $e = 4n_0 - 1$ and $e' = 2n_0 - 1$ are both odd and correspond to $g = 4n_0 - 1$. Then by the construction above we get two homomorphisms ψ and ψ' from Γ to G which determine two homomorphisms $\psi_1 \sim (1, \delta_1^{-1}; \delta_1^2, \delta_1^{-2})$ and $\psi_1' \sim (\delta_1^{2n_0}, \delta_1^{2n_0-1}; \delta_1^2, \delta_1^{-2})$ from Γ_1 with signature $[1; 2n_0, 2n_0]$ to G_1. Without supplying details [1, 7], we claim that there is a $\beta \in \mathrm{Aut}^+(\Gamma_1)$ so that $\psi_1 \circ \beta = \psi_1'$. Since ψ and ψ' are not equivalent, we have three distinct conjugacy classes $\mathcal{G}_\psi$, $\mathcal{G}_{\psi'}$, and $\mathcal{G}_{\psi_1}$ of cyclic groups of order $4n_0$ such that $\mathcal{M}_g^{\mathcal{G}_\psi} \cup \mathcal{M}_g^{\mathcal{G}_{\psi'}} \subset \mathcal{M}_g^{\mathcal{G}_{\psi_1}}$.

Now suppose n and e are even. Then $n = 4n_0 + 2$, $e = 2n_0$, and $g = n - 2 = 4n_0$. Then Γ_0 has signature $[0; 2, 2, 4n_0 + 2, 2n_0 + 1]$ and

$$\psi_0 \sim (a, a\delta^{2n_0+1}, \delta, \delta^{2n_0}) = (a, \delta_1^{2n_0+1}, a\delta_1, \delta_1^{2n_0}),$$

where $\delta_1 = a\delta$ also has order n. Suppose $\pi : H \to \mathbb{Z}_2$ is the epimorphism with kernel generated by δ_1. If $\Gamma_1 = \ker(\pi \circ \psi_0)$, then $i(K_g) \triangleleft \Gamma_1 \triangleleft \Gamma_0$ and $\pi \circ \psi_0$ sends the generators Z_1 and Z_3 of Γ_0 to the generator of $\mathbb{Z}_2$, and Z_2 and Z_4 to the identity. If

$$z_1 = Z_1Z_2Z_1, \quad z_2 = Z_2, \quad z_3 = Z_3^2, \quad z_4 = Z_3^{-1}Z_4Z_3, \quad z_5 = Z_4,$$

then $z_1, \dots, z_5$ generate Γ_1 and it is easy to see that Γ_1 has signature $[0; 2, 2, 2n_0 + 1, 2n_0 + 1, 2n_0 + 1]$. If $\psi_1 = \psi_0|_{\Gamma_1}: \Gamma_1 \to G_1 = \langle\delta_1\rangle$, then

$$\psi_1 \sim (\delta_1^{2n_0+1}, \delta_1^{2n_0+1}, \delta_1^2, \delta_1^{2n_0}, \delta_1^{2n_0})$$

and we have $\mathcal{M}_g^{\mathcal{G}_\psi} \subset \mathcal{M}_g^{\mathcal{G}_{\psi_1}}$.

Remark. If n is odd, then $H \simeq \mathbb{Z}_{2n}$ and does not contain a second subgroup isomorphic to $\mathbb{Z}_n$.

Remark. If $e = n-1$, then ψ satisfies the hypotheses of case 5 with $\beta(\delta) = \delta^{-1}$ and the automorphism group of the generic element of $\mathcal{M}_g^{\mathcal{G}_\psi}$ is isomorphic to $H = D_{2n} \times \mathbb{Z}_2$. If Γ_0 has signature $[0; 2, 2, 2, n]$, then $\psi_0 : \Gamma_0 \to H$ is given by $\psi_0 \sim (b, b\delta^{-1}a, a, \delta)$. From the form of ψ_0 we can see that a has $2n$ fixed points and so must be the hyperelliptic involution. Thus all elements of $\mathcal{M}_g^{\mathcal{G}_\psi}$ are hyperelliptic. If in addition $n = 2n_1$, then ψ_0 satisfies case 3. If Γ_2 has signature $[1; n_1]$ and $G_2 = \langle a\delta, ab\rangle \simeq D_{2n}$, we get $\psi_2 : \Gamma_2 \to G_2$ defined by $\psi_2 \sim (\delta^{-1}a, ab; \delta^2)$ and $\mathcal{M}_g^{\mathcal{G}_\psi} = \mathcal{M}_g^{\mathcal{G}_{\psi_2}}$, where $g = n-1$. Thus ψ and ψ_2 give us an example of the fourth case of Table 2. Note that $G \cap G_2 = \langle \delta^2 \rangle \simeq \mathbb{Z}_{n_1}$ and G is not conjugate to a subgroup of G_2 in Mod_g.

If $n = 4n_0$ and $e = 2n_0 - 1$, then $e^2 \equiv 1 \pmod n$ and again ψ satisfies the hypotheses of case 5 with $\beta(\delta) = \delta^e$. It can be seen from this that the general element of this $\mathcal{M}_g^{\mathcal{G}_\psi}$ is not hyperelliptic. Note that for $n_0 = e = 1$, this family of genus 3 Riemann surfaces contains the Fermat quartic.

In [16] the Jacobi varieties of the hyperelliptic examples above were studied. Some other examples of genus 3 with automorphism group containing S_4 were also studied. Suppose Γ has signature $[0; 4, 4, 3]$ and $\psi : \Gamma \to S_4$ is the epimorphism defined by $\psi \sim ((1243), (1234), (123))$. Then $\alpha = \iota_{(34)}$ satisfies the hypothesis of case 6. Let $a_0 = a(34)$. Then $a_0^2 = 1$ and a_0 commutes with G and so $H = G \times \langle a_0 \rangle$. From the proof of case 6, we have Γ_0 has signature $[0; 2, 4, 6]$ and $\psi_0 : \Gamma_0 \to H$ is determined by

$$\psi_0 \sim (a, (1234), a(1342)) = (a_0(34), (1234), a_0(132)).$$

There is a second subgroup of H isomorphic to S_4; we identify G with this subgroup by composing all elements of $S_4 - A_4$ with a_0. Then ψ_0 becomes

$$\psi_0 \sim ((34), a_0(1234), a_0(132)).$$

Now suppose $\pi : H \to \mathbb{Z}_2$ is the epimorphism whose kernel is the new G. If $\Gamma_1 = \ker(\pi \circ \psi_0)$, then $i(K_3) \lhd \Gamma_1 \lhd \Gamma_0$ and $\pi \circ \psi_0$ sends the generators Z_2 and Z_3 of Γ_0 to the generator of $\mathbb{Z}_2$, and sends Z_1 to the identity. If

$$z_1 = Z_1, \quad z_2 = Z_2 Z_1 Z_2^{-1}, \quad z_3 = Z_2^2, \quad z_4 = Z_3^2,$$

then z_1, z_2, z_3, z_4 generate Γ_1, and Γ_1 has signature $[0; 2, 2, 2, 3]$. If $\psi_1 = \psi_0|_{\Gamma_1} : \Gamma_1 \to G$, then

$$\psi_1 \sim ((34), (23), (13)(24), (123)),$$

which is equivalent to the example in [16]. Hence $\mathcal{M}_3^{\mathcal{G}_\psi} \subset \mathcal{M}_3^{\mathcal{G}_{\psi_1}} = \mathcal{M}_3^G$.

Remark. Suppose Γ_2 has signature $[0; 2, 2, 2, 2, 2, 2]$ and $\psi_2 : \Gamma_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is defined by $\psi_2 \sim (\varepsilon_1, \varepsilon_1, \varepsilon_2, \varepsilon_2, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_2)$. Then $\ker \psi_2 \simeq K_3$ and ψ_2 determines a conjugacy class $\mathcal{G}_{\psi_2}$ in Mod_3. Using ψ_1 above we have both $\mathcal{G}_{\psi_2} < \mathcal{G}_{\psi_1}$ (nonnormally) and $\mathcal{G}_{\psi_2} \lhd \mathcal{G}_{\psi_1}$. This is true since all the induced actions from ψ_1 of subgroups $\mathbb{Z}_2 \times \mathbb{Z}_2$ of S_4 are topologically equivalent to ψ_2.

Since there are unique S_4 actions in genera 4, 5, and 6, the next example with $G = S_4$ occurs when $g = 7$. Here Γ has signature $[0; 2, 4, 2, 4]$ and $\psi : \Gamma \to S_4$ is defined by $\psi \sim ((13), (1432), (14), (1342))$. Then $\alpha = \iota_{(34)}$ satisfies the hypothesis of case 4. Again letting $a_0 = a(34)$ we have $a_0^2 = 1$ and

$H = G \times \langle a_0 \rangle$; Γ_0 has signature $[0; 2, 2, 2, 4]$ and $\psi_0 : \Gamma_0 \to H$ is defined by $\psi_0 \sim (a, a(12)(34), (13), (1432)) = (a_0(34), a_0(12), (13), (1432))$. Again we identify G with the other subgroup of H isomorphic to S_4 by composing all elements of $S_4 - A_4$ with a_0. Then ψ_0 becomes

$$\psi_0 \sim ((34), (12), a_0(13), a_0(1432)).$$

Again suppose $\pi : H \to \mathbb{Z}_2$ is the epimorphism with the new G as kernel and $\Gamma_1 = \ker(\pi \circ \psi_0)$. Then $\pi \circ \psi_0$ sends the generators Z_1 and Z_2 of Γ_0 to the identity of $\mathbb{Z}_2$, and sends Z_3 and Z_4 to the generator of $\mathbb{Z}_2$. If

$$z_1 = Z_1, \quad z_2 = Z_2, \quad z_3 = Z_3 Z_1 Z_3, \quad z_4 = Z_3 Z_2 Z_3, \quad z_5 = Z_4^2,$$

then $z_1, \dots, z_5$ generate Γ_1, and Γ_1 has signature $[0; 2, 2, 2, 2, 2]$. If $\psi_1 = \psi_0 |_{\Gamma_1}: \Gamma_1 \to G$, then $\psi_1 \sim ((34), (12), (14), (23), (13)(24))$, and $\mathscr{M}_7^{\mathscr{G}_\psi} \subset \mathscr{M}_7^{\mathscr{G}_{\psi_1}} = \mathscr{M}_7^G$.

4

We have seen that the action of G on a collection of Riemann surfaces of genus g is determined by an epimorphism $\psi : \Gamma \to G$ whose kernel is isomorphic to K_g. In this section we will study some topological and analytic invariants associated to an action of G. In particular, we will examine to what extent these invariants determine the (equivalence class of) ψ or can distinguish between two such.

Suppose S is a Riemann surface of genus $g \geq 2$ and $G < \operatorname{Aut}(S)$. Suppose the action of G is determined by ψ and i as in (1) and $S = S_r$, where $r = \rho \circ i$ and $[\rho] \in T(\Gamma)$. The action of Γ on K via conjugation yields an action of G on $K/K' \simeq H_1(S, \mathbb{Z})$, where K' is the commutator subgroup of K. This action preserves the intersection pairing. Choosing a canonical basis [4] in $H_1(M, \mathbb{Z})$, we have a representation $R : G \to \operatorname{Sp}(2g, \mathbb{Z})$, determined up to conjugation in $\operatorname{Sp}(2g, \mathbb{Z})$. Let χ denote the character of R.

Let $H^0(S, \Omega^q)$ denote the space of holomorphic q-differentials on S, which we can identify with the space of holomorphic q-differentials on U invariant with respect to the (pull-back) action of $r(K)$. Then $\rho(\Gamma)$ preserves this space and induces a (right) action of G. The character χ_q of this representation of G is independent of the choice of ρ and of the choice of $[\rho]$ in $T(\Gamma)$.

Remark. Note that $\chi = \chi_1 + \overline{\chi}_1$.

Suppose $p \in S$, $a \in G$, $a \neq 1$, and $a(p) = p$. Then there is a local coordinate z centered at p and an integer u, $1 \leq u < o(a)$, where $o(a)$ denotes the order of a, so that u is relatively prime to $o(a)$ and $z \circ a = e^{\frac{2\pi i u}{o(a)}} z$. That is, the transformation a rotates a disk neighborhood around p by the angle $\frac{2\pi u}{o(a)}$. The integer u is called the rotation number of a at p. For $a \in G - \{1\}$ and $u \in \mathbb{Z}$, let $\lambda_a(u)$ be the number of points p in S at which a has rotation number congruent to u modulo $o(a)$. Note that $\lambda_a(u) = 0$ if $(u, o(a)) \neq 1$. The collection $\{\lambda_a | a \in G, a \neq 1\}$ is called the rotation data of the action of G on S. It can be calculated directly from ψ.

Remark. For all $a, b \in G$ and $u, k \in \mathbb{Z}$ with $a \neq 1$, we have $\lambda_a = \lambda_{bab^{-1}}$. If $(k, o(a)) = 1$, then $\lambda_{a^k}(ku) = \lambda_a(u)$, and if $k | o(a)$, so that $o(a) = k o(a^k)$, then

$$\lambda_{a^k}(u) \geq \lambda_a(u) + \lambda_a(u + o(a^k)) + \cdots + \lambda_a(u + (k-1) o(a^k)).$$

Remark. If ψ is replaced by $\beta^{-1} \circ \psi$ for some β in $\mathrm{Aut}(G)$, then χ, χ_q and λ_a are replaced by $\chi \circ \beta$, $\chi_q \circ \beta$, and $\lambda_{\beta(a)}$ respectively.

Eichler's trace formula [2, 4, 11] states that, for $1 \leq q$ and $a \in G$, $a \neq 1$,

$$\chi_q(a) = \sum_{u=1}^{o(a)-1} \lambda_a(u) \frac{\zeta^{qu}}{1 - \zeta^u} + \delta_{q,1}, \tag{5}$$

where $\zeta = e^{\frac{2\pi i}{o(a)}}$. Since $\chi_q(1) = (2q-1)(g-1) + \delta_{q,1}$, this shows that the rotation data determines the sequence of characters χ_q. We will see that the converse of this statement is also true.

For $F < G$, let $\chi_F = (1_F)^G$ be the character of the permutation representation of G on the set of cosets G/F. In particular, $\chi_{\langle 1 \rangle}$ is the character of the regular representation of G and χ_G is the trivial character of G. The following are easy to verify. For $b \in G$, let $C(b)$ denote the conjugacy class of b in G and let $\mathrm{Cent}(b)$ be the centralizer of b.

(a) $\chi_F(b) = |\mathrm{Cent}(b)|\ |C(b) \cap F|/|F|$ and so $\chi_F(b) = 0$ if and only if no conjugate of $\langle b \rangle$ is contained in F.
(b) $\chi_{bFb^{-1}} = \chi_F$.
(c) If $(k, o(b)) = 1$, then $\chi_F(b^k) = \chi_F(b)$.
(d) Suppose $\langle 1 \rangle, \langle a_1 \rangle, \ldots, \langle a_s \rangle$ is a list of representatives from the set of conjugacy classes of cyclic subgroups of G. Then $\chi_{\langle 1 \rangle}, \ldots, \chi_{\langle a_s \rangle}$ form a basis for the space of class functions on G with property (c).

Proposition. *The character χ determines the signature of Γ.*

Proof. Suppose the action of G is determined by the epimorphism $\psi : \Gamma \to G$ as in (1), where Γ has signature $[p; m_1, \ldots, m_t]$ and presentation (2). Let $c_i = \psi(z_i)$ for $i = 1, \ldots, t$. Then we have [2]

$$\chi = 2\chi_G + (2p - 2 + t)\chi_{\langle 1 \rangle} - \sum_{i=1}^{t} \chi_{\langle c_i \rangle}. \tag{6}$$

First note that p is determined, since $2p = \langle \chi, \chi_G \rangle$. For $l = 1, \ldots, s$, let n_l be the number of the c_i so that $\langle c_i \rangle$ is conjugate to $\langle a_l \rangle$. Then (6) can be rewritten as

$$\chi = 2\chi_G + (2p - 2 + t)\chi_{\langle 1 \rangle} - \sum_{l=1}^{s} n_l \chi_{\langle a_l \rangle}. \tag{7}$$

Suppose no conjugate of $\langle a_r \rangle$ is contained in any other of the $\langle a_l \rangle$. Then $\chi_{\langle a_l \rangle}(a_r) = 0$ if and only if $l \neq r$. Evaluating (7) at a_r, we obtain

$$\chi(a_r) = 2 - n_r \frac{|N(\langle a_r \rangle)|}{o(a_r)},$$

which determines n_r. Suppose that n_l is known for all $\langle a_l \rangle$ which contain a conjugate of $\langle a_r \rangle$. Then evaluating (7) at a_r yields an equation involving n_r and only known n_l's. In this way the n_l can be found. Thus the signature of Γ is $[p; o(a_1)^{n_1}, \ldots, o(a_s)^{n_s}]$, where m^n means that m is repeated n times. $\square$

Remark. The proof of the above proposition could be based on the monodromy description of covers of Riemann surfaces (see, for example, Lemma 1.2 of

[16]). We have the following expressions for the number of fixed points of a_l:

$$2 - \chi(a_l) = \sum_{u=1}^{o(a_l)-1} \lambda_{a_l}(u) = \sum_{\substack{k=1 \\ C(a_l)\cap\langle a_k\rangle\neq\varnothing}}^{s} n_k \frac{|N(\langle a_k\rangle)|}{o(a_k)}. \tag{8}$$

Let $L = 1$ if $t = 0$, and $L = \operatorname{lcm}\{m_1, \dots, m_t\}$ otherwise.

Proposition. *For $q \geq 1$ we have*

$$\chi_{q+L} = \chi_q + \frac{2L(g-1)}{|G|}\chi_{\langle 1\rangle} - \delta_{q,1}\chi_G.$$

In particular, if $t = 0$ we have

$$\chi_q = (2q-1)(p-1)\chi_{\langle 1\rangle} + \delta_{q,1}\chi_G.$$

Proof. Since $o(c_i^k)$ divides L for any k, the Eichler trace formula (5) implies that for any $a \in G$, $a \neq 1$,

$$\chi_{q+L}(a) = \chi_q(a) - \delta_{q,1}.$$

The result follows easily from this. Note that the Riemann-Hurwitz relation implies that

$$\frac{2L(g-1)}{|G|} = L\left(2p - 2 + \sum_{i=1}^{t}(1 - \frac{1}{o(c_i)})\right)$$

is a positive integer. □

Proposition. *The sequence of characters χ_q determines the rotation data λ.*

Proof. This follows directly from the Eichler trace formula. Fix $a \in G$, $a \neq 1$. Equation (5) with $q = 1, \dots, \phi(o(a))$ gives a system of linear equations in the unknowns $\lambda_a(u)\frac{\zeta^u}{1-\zeta^u}$, $(u, o(a)) = 1$, whose coefficient matrix is Vandermonde. □

Combining the above results we see that knowing the sequence of characters χ_q is equivalent to knowing the signature of Γ and the rotation data λ. Note that I. Kuribayashi proves the above result in [11], where he obtains the best estimate on the number of χ_q needed to determine λ.

We now examine some applications of the above to the questions raised at the beginning of this section. We will consider fixed point free actions of G on a Riemann surface S of genus g. Then the rotation data λ is identically zero and all the characters χ_q are determined by G and g. Let $p = \frac{g-1}{|G|} + 1$ be the genus of S/G and $\pi : S \to S/G$ the quotient map.

Suppose first that G is abelian. Then the cover π is determined by an injection j of $\widehat{G}$, the character group of G, into $H_1(S/G, \mathbb{R})/H_1(S/G, \mathbb{Z})$, the real form of the Jacobi variety of S/G (see, for example, [15, 16]). Let $\mathscr{L}$ be the sublattice of $H_1(S/G, \mathbb{R})$ such that $j(\widehat{G}) = \mathscr{L}/H_1(S/G, \mathbb{Z})$. The map π induces an injection $\tilde{\pi} : H_1(S/G, \mathbb{R}) \to H_1(S, \mathbb{R})$. The image of $\tilde{\pi}$ is $H_1(S, \mathbb{R})^{R(G)}$ and $\tilde{\pi}(\mathscr{L}) = H_1(S, \mathbb{Z})^{R(G)}$. Furthermore, for all γ_1 and γ_2 in $\mathscr{L}$, we have

$$\langle\tilde{\pi}(\gamma_1), \tilde{\pi}(\gamma_2)\rangle_S = |G|\langle\gamma_1, \gamma_2\rangle_{S/G},$$

where $\langle\,,\,\rangle_S$ and $\langle\,,\,\rangle_{S/G}$ denote the intersection pairings on S and S/G respectively. Then $\tilde{\pi}$ covers the induced map $\tilde{\pi} : J(S/G) \to J(S)$ between the

Jacobi varieties of S/G and S, the kernel of $\tilde{\pi}$ is $j(\widehat{G}) = \mathscr{L}/H_1(S/G, \mathbb{Z})$, and $\mathscr{L}^{\perp}/\mathscr{L}$ is the kernel of the polarization induced on $\tilde{\pi}(J(S/G))$ by $J(S)$, where

$$\mathscr{L}^{\perp} = \{\gamma \in H_1(S/G, \mathbb{R}) \mid |G|\langle \gamma, \delta \rangle_{S/G} \in \mathbb{Z} \text{ for all } \delta \in \mathscr{L}\}.$$

Thus the group $\mathscr{L}^{\perp}/\mathscr{L}$ is an invariant of the fixed point free action of G on S and by the above can be calculated from R.

For example, suppose $G = \mathbb{Z}_n \times \mathbb{Z}_n = \langle \varepsilon_1 \rangle \times \langle \varepsilon_2 \rangle$ and Γ has signature $[p; -]$ and presentation

$$\langle X_1, Y_1, \dots, X_p, Y_p \mid [X_1, Y_1] \cdots [X_p, Y_p] = 1 \rangle. \tag{9}$$

Then using automorphisms of Γ as in [1, 7] it is easy to show that any epimorphism $\psi : \Gamma \to G$ is equivalent to

$$\psi_k \sim (\varepsilon_1, \varepsilon_2^k, \varepsilon_2, 1, \dots, 1)$$

for some positive $k, k \mid n$. The problem now is to show that the ψ_k are inequivalent. But an easy calculation shows that for ψ_k,

$$\mathscr{L}^{\perp}/\mathscr{L} \simeq (\mathbb{Z}_k)^2 \times (\mathbb{Z}_{n^2/k})^2 \times (\mathbb{Z}_{n^2})^{2p-4}.$$

Thus we can say that the representation R of G distinguishes the fixed point free actions of G on S.

If S/G has nontrivial automorphisms, then these may or may not lift to S for different G-actions. For example, suppose S/G is hyperelliptic. It is known that the hyperelliptic involution does not lift to every Galois cover. If G has fixed points, then a necessary condition is that the hyperelliptic involution preserves the image of the set of branch points in S/G. However, if G is abelian and acts without fixed points, then it always lifts [15]. We will give some examples to show that this is not the case if G is not abelian.

Suppose Γ has signature $[p; -]$ and $G = \mathrm{Aff}^{+}(1, 7)$. Then there is one equivalence class of epimorphisms $\psi : \Gamma \to G$. Since G is solvable, one can see that a lift of the hyperelliptic involution must conjugate an element of order three to its square times an element of order seven. But G has no such automorphism. Thus the hyperelliptic involution cannot lift to any unramified G cover of a surface of genus p. The action of G on $\mathbb{F}_7$ yields an embedding of G into S_7. Let $p = 2$ and define two homomorphisms ψ_1 and ψ_2 from Γ to S_7 by

$$\psi_1 \sim \big((12), (1234567), (1234567), (12)\big),$$

$$\psi_2 \sim \big((142)(356), (1234567), (147)(26)(35), (1234)\big).$$

It is clear that ψ_1 is an epimorphism. Let a, b, c, d denote the four entries in ψ_2. The subgroup of S_7 generated by a, b, c, and d is not contained in A_7. On the other hand $d^3c^2d = (127)$ and $d^3c^4db = (34567)$ and so this subgroup contains A_7. Thus ψ_2 is also an epimorphism. It is clear that ψ_1 satisfies case 1 of the theorem of §2. But a and b generate a copy of $\mathrm{Aff}^{+}(1, 7)$ in S_7 and so ψ_2 cannot satisfy case 1. Thus ψ_1 and ψ_2 determine inequivalent unramified S_7 covers of surfaces of genus 2.

We now examine the case of fixed point free actions by D_{2n}, the dihedral group of order $2n$. Suppose Γ has signature $[p; -]$ and presentation (9) and

$D_{2n} = \langle a, b \mid a^n = b^2 = abab = 1\rangle$. Using the results in [1, 7], we can show that any epimorphism $\psi : \Gamma \to D_{2n}$ is equivalent to either

$$\psi_1 \sim (b, 1, a, 1, \dots, 1)$$

or, if $n = 2n_0$,

$$\psi_2 \sim (b, a^{n_0}, a, 1, \dots, 1).$$

Again the problem is to show that these are inequivalent when $n = 2n_0$. Since they determine fixed point free actions of D_{2n} on surfaces of genus $g = 2n(p-1)+1$, all the characters χ_q agree. We will examine the (symplectic equivalence classes of the) representations R_1 and R_2. For simplicity, we assume now that $p = 2$.

For ψ_1 and ψ_2 we will give canonical bases $\{x_1, \dots, x_{2n+1}, y_1, \dots, y_{2n+1}\}$ of $H_1(S, \mathbb{Z})$ and determine the corresponding actions of D_{2n}. Recall that a canonical basis is one with the property

$$\langle x_i, x_j\rangle_S = \langle y_i, y_j\rangle_S = 0, \quad \langle x_i, y_j\rangle_S = \delta_{ij}, \qquad 1 \le i, j \le 2n+1.$$

We will use the same letter to denote a curve, its homotopy class, or its homology class. We will write the operation in homology as addition, while still writing concatenation of curves multiplicatively, read from left to right.

Suppose $\pi_1 : S \to S/D_{2n}$ is the quotient map for the action of D_{2n} on S determined by ψ_1. Let $p_0 \in S/D_{2n}$ and label the points of $\pi_1^{-1}(p_0)$ with the elements of D_{2n} so that for $c, d \in D_{2n}$, $c \cdot p_d = p_{cd}$. Suppose $X \in \Gamma \simeq \pi_1(S/D_{2n}, p_0)$ and $c \in D_{2n}$. Let $X(c)$ denote the lift of X to S starting at p_c. Then the other endpoint of $X(c)$ is $p_{c\psi_1(X)}$. Also, if $d \in D_{2n}$, then $d \cdot X(c) = X(dc)$. For example, since $\psi_1(X_1) = b$, $X_1(1)X_1(b)$ is a closed curve in S. Now define

$$\begin{aligned}
x_1 &= X_1(1)X_1(b), & y_1 &= Y_1(b) - Y_2(1),\\
x_k &= a^{k-1}\cdot x_1, & y_k &= a^{k-1}\cdot y_1, \ 2 \le k \le n,\\
x_{n+1} &= -X_1(1)X_2(b)X_1(ba)X_2(a^{n-1}), & y_{n+1} &= -Y_2(1),\\
x_{n+k} &= a^{k-1}\cdot x_{n+1}, & y_{n+k} &= a^{k-1}\cdot y_{n+1}, \ 2 \le k \le n,\\
x_{2n+1} &= X_2(b)X_2(ba)\cdots X_2(ba^{n-1}), & y_{2n+1} &= Y_2(ba) - Y_2(1).
\end{aligned}$$

Since the closed curves which appear on this list either share a segment or intersect only over p_0, it is not hard to show that the homology classes have the right intersections.

We identify D_{2n} with a subgroup of S_n by setting $a = (12\cdots n)^{-1}$ and $b = (2\,n)(3\,n-1)\cdots(n_0\,n_0+2)$. To any permutation t in S_n we associate the n by n permutation matrix defined by $M(t) = (\delta_{it,j})_{1\le i,j\le n}$. Let ε be the row vector of size n all of whose entries are 1. Define the two $2n+1$ by $2n+1$ matrices A and B by

$$A = \begin{pmatrix} M(a) & 0 & 0\\ 0 & M(a) & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad B = \begin{pmatrix} M(b) & 0 & -\varepsilon^t\\ 0 & M(ab) & -\varepsilon^t\\ 0 & 0 & -1\end{pmatrix}.$$

Then we have

$$R_1(a) = \begin{pmatrix} A & 0\\ 0 & A\end{pmatrix} \quad\text{and}\quad R_1(b) = \begin{pmatrix} B & 0\\ 0 & B^t\end{pmatrix}.$$

Suppose $\pi_2 : S \to S/D_{2n}$ is the quotient map for the action of D_{2n} on S determined by ψ_2. We repeat the same constructions as before. Define $x_1, \dots, x_{2n+1}$ as above for ψ_1 and

$$\begin{aligned} y_1 &= Y_1(ba^{n_0})X_2(b)\cdots X_2(ba^{n_0-1}) - Y_2(a^{n_0}) + x_{n_0+1} + \cdots + x_n, \\ y_k &= a^{k-1}\cdot y_1, \qquad 2 \le k \le n, \\ y_{n+1} &= -Y_2(1), \\ y_{n+k} &= a^{k-1}\cdot y_{n+1}, \qquad 2 \le k \le n, \\ y_{2n+1} &= Y_2(ba) - Y_2(1) + x_1 + \cdots + x_{n_0} + x_{2n+1}. \end{aligned}$$

Again $\{x_1, \dots, y_{2n+1}\}$ is a canonical basis of $H_1(S, \mathbb{Z})$. By construction $R_2(a) = R_1(a)$. Also, the action of b on $x_1, \dots, x_{2n+1}$ is the same as for ψ_1. An easy but tedious calculation yields

$$R_2(b) = \begin{pmatrix} B & -BD \\ 0 & B^t \end{pmatrix},$$

where

$$D = \begin{pmatrix} 0 & D_0 & 0 \\ D_0^t & \varepsilon^t\varepsilon & \varepsilon^t \\ 0 & \varepsilon & 0 \end{pmatrix}$$

and

$$D_0 = I + M(a) + \cdots + M(a)^{n_0-1}.$$

Note that $\varepsilon^t\varepsilon = M(a)D_0 + D_0^t = I + M(a) + \cdots + M(a)^{n-1}$, $D = D^t$, and $-BD = DB^t$.

We now want to show that, even after composing with an automorphism of D_{2n}, R_1 cannot be conjugated to R_2 in $\mathrm{Sp}(4n+2, \mathbb{Z})$. In fact we can show that $R_1(D_{2n})$ cannot be conjugated to $R_2(D_{2n})$ in $\mathrm{SL}(4n+2, \mathbb{Z})$. Consider the induced actions of D_{2n} on $H_1(S, \mathbb{R})/H_1(S, \mathbb{Z})$, the real form of the Jacobi variety $J(S)$ of S. Let $\nu \in \mathbb{R}^{4n+2}$. Then $R_1(a)\nu = R_2(a)\nu \equiv \nu \bmod \mathbb{Z}$ if and only if $\nu \equiv (r_1\varepsilon, r_2\varepsilon, r_3, s_1\varepsilon, s_2\varepsilon, s_3)^t \bmod \mathbb{Z}$, for some $r_1, r_2, r_3, s_1, s_2, s_3 \in \mathbb{R}$. Assuming ν has this form we see that

$$R_1(b)\nu \equiv \nu \bmod \mathbb{Z} \quad \text{iff} \quad r_3 \equiv ns_1 + ns_2 + 2s_3 \equiv 0 \bmod \mathbb{Z},$$

while

$$R_2(b)\nu \equiv \nu \bmod \mathbb{Z} \quad \text{iff} \quad r_3 - n_0s_2 \equiv n_0s_1 + n_0s_2 + s_3 \equiv 0 \bmod \mathbb{Z}.$$

In particular, if we consider elements of order 2 in $J(S)$, that is, we restrict r_i and s_i to be 0 or $\frac{1}{2}$, then we find that $R_1(D_{2n})$ fixes $(\mathbb{Z}_2)^5$, while $R_2(D_{2n})$ fixes $(\mathbb{Z}_2)^4$. Thus $R_1(D_{2n})$ cannot be conjugate to $R_2(D_{2n})$ in $\mathrm{SL}(4n+2, \mathbb{Z})$.

Remark. Suppose the matrix $E \in \mathrm{Sp}(4n+2, \mathbb{Q})$ has the form

$$E = \begin{pmatrix} I & E_0 \\ 0 & I \end{pmatrix},$$

where $E_0^t = E_0$. Then for all c in D_{2n}, $E^{-1}R_1(c)E = R_2(c)$ if and only if

$$AE_0 - E_0A = 0 \quad \text{and} \quad BE_0 - E_0B^t = -BD.$$

We have a solution to these equations readily available, that is, $E_0 = -\frac{1}{2}D$. Thus $R_1(D_{2n})$ is conjugate to $R_2(D_{2n})$ in $\mathrm{Sp}(4n+2, \mathbb{Q})$.

This also shows that 2 is the only "bad" prime. Suppose $m = 2k + 1$ is any odd positive integer. Taking $E_0 = kD$ and reducing modulo m we see that $R_1(D_{2n})$ is conjugate to $R_2(D_{2n})$ in $\mathrm{Sp}(4n + 2, \mathbb{Z}_m)$. This example is related to Theorem 4 in [3].

Remark. This example should be compared to those in [5]. There Gilman and Patterson examine the action of $\mathbb{Z}_p < \mathrm{Aut}(S)$, for p prime, first on a non-canonical basis of $H_1(S, \mathbb{Z})$. The result depends only on g, p, and the number of fixed points. It is only when finding the action on a canonical basis that the rotation numbers play a role. Thus actions of $\mathbb{Z}_p$ having the same number of fixed points yield representations conjugate in $\mathrm{SL}(2g, \mathbb{Z})$, but not conjugate in $\mathrm{Sp}(2g, \mathbb{Z})$.

References

1. I. Bernstein and A. Edmonds, *On the classification of generic branched coverings of surfaces*, Illinois J. Math. **28** (1984), 64–82.
2. S. A. Broughton, *The homology and higher representations of the automorphism group of a Riemann surface*, Trans. Amer. Math. Soc. **300** (1987), 153–158.
3. C. J. Earle, *On the moduli of closed Riemann surfaces with symmetries*, Advances in the Theory of Riemann Surfaces, Ann. of Math. Stud., no. 66, Princeton Univ. Press, Princeton, N.J., pp. 119–130.
4. H. Farkas and I. Kra, *Riemann surfaces*, Graduate Texts in Math., vol. 71, Springer-Verlag, Berlin and New York, 1980.
5. J. Gilman and D. Patterson, *Intersection matrices for bases adapted to automorphisms of a compact Riemann surface*, Riemann Surfaces and Related Topics, Ann. of Math. Stud., no. 97, Princeton Univ. Press, Princeton, N.J., 1981, pp. 149–166.
6. I. Guerrero, *Holomorphic families of compact Riemann surfaces with automorphisms*, Illinois J. Math. **26** (1982), 212–225.
7. W. J. Harvey, *On branch loci in Teichmüller space*, Trans. Amer. Math. Soc. **153** (1971), 387–399.
8. W. J. Harvey and C. Machlachlan, *On mapping class groups and Teichmüller spaces*, Proc. London Math. Soc. **30** (1975), 496–512.
9. S. Kerckhoff, *The Nielsen realization problem*, Ann. of Math. (2) **117** (1983), 235–265.
10. A. Kuribayashi, *On analytic families of compact Riemann surfaces with non-trivial automorphism*, Nagoya Math. J. **28** (1966), 119–165.
11. I. Kuribayashi, *On automorphism groups of a curve as linear groups*, J. Math. Soc. Japan **39** (1987), 51–77.
12. A. M. Macbeath and D. Singerman, *Spaces of subgroups and Teichmüller space*, Proc. London Math. Soc. **31** (1975), 211–256.
13. W. Magnus, A. Karass, and D. Solitar, *Combinatorial group theory*, Dover, New York, 1976.
14. H. Popp, *Stratifikation von Quotientenmannigfaltigkeiten* (*in Characteristik* 0) *und insbesondere der Modulmannigfaltigkeiten für Kurven*, J. Reine Angew. Math. **250** (1971), 12–41.
15. J. F. X. Ries, *The Prym variety for a cyclic unramified cover of a hyperelliptic surface*, J. Reine Angew. Math. **340** (1983), 59–69.
16. ——, *The splitting of some Jacobi varieties using their automorphism groups*, preprint.
17. D. Singerman, *Subgroups of Fuchsian groups and finite permutation groups*, Bull. London Math. Soc. **2** (1970), 319–323.
18. ——, *Finitely maximal Fuchsian groups*, J. London Math. Soc. **6** (1972), 29–38.

Department of Mathematical Sciences, State University of New York at Binghamton, Binghamton, New York 13902

Current address: Department of Mathematics, East Carolina University, Greenville, North Carolina 27858-4353