TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 335, Number 1, January 1993

K-THEORY OF EILENBERG-MAC LANE SPACES AND CELL-LIKE MAPPING PROBLEM

A. N. DRANISHNIKOV

ABSTRACT. There exist cell-like dimension raising maps of 6-dimensional manifolds. The existence of such maps is proved by using K-theory of Eilenberg-Mac Lane complexes.

1. INTRODUCTION

One of the main notions of geometric topology is the notion of cell-like map. The reason is that the cell-like maps between closed manifolds of dimension $\neq 3$ can be approximated by homeomorphisms [Si, Q]. This statement in dimension 3 implies the Poincaré conjecture and, of course, it is not proved. In dimension 3 a weaker statement is true [Ar]. Cell-like maps of manifolds often are obtained as decompositions maps. In that case the image is not necessarily a manifold. It is only a homology manifold. R. D. Edwards proved [E1] that for $n \geq 5$ if the decomposition space X is finite dimensional and has additionally the "Disjoint Disk Property" introduced by Cannon [C], then X is a manifold and more than that the quotient map can be approximated by homeomorphisms. Then R. D. Daverman [Da] derived from Edwards' theorem that if the decomposition space X of cell-like decompositions of manifolds is a finite-dimensional one then $X \times \mathbb{R}^2$ is a manifold. Now the following problem is natural.

Cell-like mapping problem. Is the image of a cell-like map of an n-manifold always finite dimensional?

Recall that a map between compacta $f: Y \to X$ is called cell-like if the preimage of each point, $f^{-1}(x)$, can be embedded in a manifold as a cellular subset = intersection of a decreasing system of closed topological cells. Note that a cell-like map is always surjective. The cell-like problem arose after Bing's works on decompositions of manifolds appeared [B1, B2] and it turned out to be that [E2] the cell-like problem is equivalent to a very old problem of Alexandroff in homological dimension theory [A, W] (see also the surveys [D-S, D1, M-R]).

In [K-W] it was proved that the cell-like mapping problem has an affirmative answer for 3-dimensional manifolds. Then by using results of Edwards and Walsh and some computations in K-theory [B-M, A-H] an example of a

Received by the editors March 29, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55M10, 57Q55.
Key words and phrases. Cell-like map, cohomological dimension, Eilenberg-Mac Lane space.
The author was partially supported by the U. S. Army Research Office through the Mathematical Sciences Institute of Cornell University.

cell-like map $\phi: I^7 \to Y$ of the 7-dimensional cube with infinite dimensional range Y was constructed in [D2, D1]. Edwards' theorem [W] claims that for every compactum X with cohomological dimension $\text{c-dim}_{\mathbb{Z}} X \leq n$ with respect to the integers as coefficients there exists a cell-like map $f: Z \to X$ of an n-dimensional compactum Z onto X. Call such a map an Edwards-Walsh resolution. The map $\phi: I^7 \to Y$ was obtained as a quotient map generated by embedding an Edwards-Walsh resolution Z of some compactum X in 7-dimensional Euclidean space. The compactum X with the properties $\text{c-dim}_{\mathbb{Z}} X = 3$ and $\dim X = \infty$ was constructed by using complex K-theory with finite coefficients.

R. D. Daverman proved (oral communication) that any Edwards-Walsh resolution Z of that compactum X cannot be embedded in $\mathbb{R}^6$. The reason, roughly speaking, is that the 3-dimensional skeleton of high-dimensional simplex cannot be embedded in $\mathbb{R}^6$ and X contains such skeletons by the construction.

In this paper by using K-theory of Eilenberg-Mac Lane spaces $K(\pi, 2)$ for $\pi = \mathbb{Z}_p$ and $\mathbb{Z}[\frac{1}{p}]$, and infinite-dimensional compactum X with $\text{c-dim}_{\mathbb{Z}} X \times X \leq 5$ is constructed. Then by improving Edwards' theorem, the Edwards-Walsh resolution $f: Z \to X$ with the additional property $\dim Z \times Z \leq 5$ is constructed. Then a recent result of [D-R-S, Sp] implies that such a compactum Z can be embedded in $\mathbb{R}^6$. All together, this produces a cell-like map of the 6-dimensional cube with infinite-dimensional image.[1]

2. An infinite-dimensional compactum with finite cohomological dimension

By $K(\pi, n)$ denote an Eilenberg-Mac Lane complex, i.e., an arbitrary CW-complex L with the properties $\pi_i(L) = 0$ if $i \neq n$ and $\pi_n(L) = \pi$. So a contractible CW-complex is regarded as an Eilenberg-Mac Lane complex $K(\{e\}, n)$ for arbitrary n where $\{e\}$ denoted the trivial group.

Recall that the cohomological dimension of a space X with group G as coefficients does not exceed n, $\text{c-dim}_G X \leq n$, if for any closed subset $A \subset X$ and for an arbitrary continuous map $\phi: A \to K(G, n)$ there exists an extension $\psi: X \to K(G, n)$ of ϕ [Ku, W, D1].

Definition [D1]. Let $f: X \to K$ be a map, and let K be a polyhedron with fixed triangulation τ. The formal inequality $\text{c-dim}_G(f, \tau) \leq n$ will denote the following statement:

> *For any subpolyhedron $A \subset K$ with respect to τ for an arbitrary map $\phi: A \to K(G, n)$ there exists an extension $\psi: X \to K(G, n)$ of the restriction $\phi \circ f|_{f^{-1}(A)}$.*

Recall that $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ and that $\mathbb{Z}[\frac{1}{p}] = \{m/p^k \in \mathbb{Q}\}$ is a localization of integers away from the prime p. The main result of this section is the following.

Theorem 1. *For arbitrary prime p there exists an infinite-dimensional compactum X with cohomological dimensions $\text{c-dim}_{\mathbb{Z}_p} X \leq 2$ and $\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]} X \leq 2$.*

---

[1]After this paper was submitted J. Dydak and J. Walsh solved negatively the cell-like mapping problem in dimension 5. Instead of calculation in K-theory they used the Sullivan conjecture proved by H. Miller.

Proposition 1. *The inequality* $\max\{\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]}X, \text{c-dim}_{\mathbb{Z}_p}X\} \le n$ *implies that* $\text{c-dim}_{\mathbb{Z}}X \le n+1$.

Proof. By Cohen's theorem [Ku] the inequality $\text{c-dim}_G X \le n$ for compact X is equivalent to the property: $H_c^{n+1}(U; G) = 0$ for every open set $U \subset X$, where H_c^* is Čech cohomology with compact support. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[\frac{1}{p}] \to \mathbb{Z}_{p^\infty} \to 0$ generates a long one (here $\mathbb{Z}_{p^\infty} = \varinjlim \mathbb{Z}_{p^k}$)

$$\cdots \to H_c^{n+1}(U; \mathbb{Z}_{p^\infty}) \to H_c^{n+2}(U; \mathbb{Z}) \to H_c^{n+2}(U; \mathbb{Z}[\tfrac{1}{p}]) \to \cdots$$

for arbitrary open sets $U \subset X$. By virtue of Bokshtein's inequality [Ku] $\text{c-dim}_{\mathbb{Z}_{p^\infty}} X \le \text{c-dim}_{\mathbb{Z}_p} X$, we have that $H_c^{n+1}(U; \mathbb{Z}_{p^\infty}) = 0$. Since $\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]}X \le n+1$ we have that $H_c^{n+2}(U; \mathbb{Z}[\frac{1}{p}]) = 0$. Hence $H_c^{n+1}(U; \mathbb{Z}) = 0$ and therefore $\text{c-dim}_{\mathbb{Z}}X \le n+1$.

The following lemma for $G = \mathbb{Z}$ actually was proved in [W].

Lemma 1. *Let* $n > 1$ *and* $G = \mathbb{Z}_p$ *or* $\mathbb{Z}[\frac{1}{p}]$ *than for arbitrary compact polyhedron* K *with fixed triangulation* τ *there exists a countable CW-complex* $W_\tau(G, n)$ *and a map* $\omega: W_\tau(G, n) \to K$ *with the following properties*:

(1) *for any simplex* $\sigma \in \tau$, $\omega^{-1}(\sigma) \simeq K(\bigoplus_1^{m_\sigma} G, n)$,

(2) $\text{c-dim}_G(\omega, \tau) \le n$,

(3) $W_\tau(G, n)$ *can be supplied with PL-structure compatible with the cellular one.*

We call the complex $W_\tau(G, n)$ *together with the map* ω *Edwards-Walsh construction for* τ, G, n.

Proof. If $\dim K \le n$ then define $W_\tau(G, n) = K$ and $\omega = \text{id}_K$.

If $\dim K = n+1$ then for every $(n+1)$-dimensional simplex $\sigma \in \tau$ replace σ by an Eilenberg-Mac Lane complex $K(G, n)$. In order to do this, fix an n-dimensional sphere S^n in $K(G, n)$ which generates the unit $1 \in \pi_n(K(G, n)) = G$ ($= \mathbb{Z}_p$ or $\mathbb{Z}[\frac{1}{p}]$) and identify that sphere with the boundary $\partial\sigma$ by some PL-homeomorphism. As a result, we will obtain a CW-complex $W_\tau(G, n)$. Define ω such that $\omega^{-1}|_{K^{(n)}}: K^{(n)} \to W_\tau(G, n)$ is an embedding. To achieve this property, send every attached Eilenberg-Mac Lane complex to the corresponding simplex σ and then move $K(G, n) - S^n$ off $\partial\sigma$ into $\dot{\sigma}$.

If $\dim K = n+2$ consider the Edwards-Walsh construction $\omega^1: W_{\tau_K^1}(G, n) \to K^{(n+1)}$ for the $(n+1)$-dimensional skeleton $K^{(n+1)}$ of K with restricted triangulation $\tau^1 = \tau|_{K^{(n+1)}}$. Consider an arbitrary $(n+2)$-simplex $\sigma \in \tau$. Denote by Y_σ the preimage $(\omega^1)^{-1}(\sigma^{(n+1)})$. In the case $G = \mathbb{Z}_p$ it is easy to see that $\pi_n(Y_\sigma) = \bigoplus \mathbb{Z}_p$. Then by attaching cells to Y_σ in the dimensions $> n+1$ it is possible to obtain a complex $K(\bigoplus \mathbb{Z}_p, n)$ which automatically will be glued to Y_σ. Do this for all $(n+2)$-dimensional simplexes to obtain $W_\tau(\mathbb{Z}_p, n)$ and define a map $\omega: W_\tau(\mathbb{Z}_p, n) \to K$ with the properties:

(1) $\omega|_{W_{\tau^1}(\mathbb{Z}_p, n)} = \omega^1$ and

(2) $\omega^{-1}|_{K^{(n+1)}} \equiv (\omega^1)^{-1}$.

In the case $G = \mathbb{Z}[\frac{1}{p}]$ we have $\pi_n(Y_\sigma)/\operatorname{Tor}\pi_n(Y_\sigma) = \bigoplus \mathbb{Z}[\frac{1}{p}]$. By attaching to Y_σ $n+1$-cells killing $\operatorname{Tor}\pi_n(Y_\sigma)$ and then cells in higher dimensions, it is possible to obtain a complex $K(\bigoplus \mathbb{Z}[\frac{1}{p}], n)$ glued to Y_σ. Similarly define the map $\omega: W_\tau(\mathbb{Z}[\frac{1}{p}], n) \to K$.

If the dimension of K is greater than $n+2$, the Edwards-Walsh construction can be produced by induction and for every m-dimensional simplex σ for $m > n+2$ we will have $\pi_n((\omega^1)^{-1}(\sigma^{(m-1)})) = \bigoplus G$. It is trivial for $G = \mathbb{Z}_p$ and can be easily computed for $G = \mathbb{Z}[\frac{1}{p}]$. Then build $K(\bigoplus G, n)$ by attaching to $(\omega^1)^{-1}(\sigma^{(m-1)})$ cells of dimension $> n+1$. Define $W_\tau(G, n)$ and ω as above.

The properties (1), (3) hold by the construction. In order to check (2) it is sufficient to prove that for every simplex $\sigma \in \tau$ the inclusion $j\colon \omega^{-1}(\partial\sigma) \to \omega^{-1}(\sigma)$ induces an epimorphism $j^*\colon H^n(\omega^{-1}(\sigma); G) \to H^n(\omega^{-1}(\partial\sigma); G)$. If $\dim\sigma > n+2$ this follows from the fact that $\omega^{-1}(\sigma)$ is obtained from $\omega^{-1}(\partial\sigma)$ by attaching cells only in the dimension $> n+1$. The same argument is valid for $\dim\sigma = n+2$ and $G = \mathbb{Z}_p$.

If $\dim\sigma = n+2$ and $G = \mathbb{Z}[\frac{1}{p}]$ the homomorphism j^* is an epimorphism because all $(n+1)$-cells attached to $\omega^{-1}(\partial\sigma)$ in the construction of $\omega^{-1}(\sigma)$ are attached by maps which generate elements of finite order in $\pi_n(\omega^{-1}(\partial\sigma))$ and $\operatorname{Tor}\mathbb{Z}[\frac{1}{p}] = 0$. Since the inclusion of the "unit" sphere $S^n \hookrightarrow K(G, n)$ induces an epimorphism of n-dimensional cohomology groups with G as coefficients, then j^* is an epimorphism in the case $\dim\sigma = n+1$. If $\dim\sigma \le n$ then there is no problem.

Let $\widetilde{K}^*_{\mathbb{C}}(X; \mathbb{Z}_p)$ denote the reduced complex K-theory with coefficients $\mathbb{Z}_p$ [B-M, A-H]. Recall that for connected X, $\widetilde{K}^{2i}_{\mathbb{C}}(X; \mathbb{Z}_p) = [X \wedge B^2_p, BU]$ and $\widetilde{K}^{2i+1}_{\mathbb{C}}(X; \mathbb{Z}_p) = [X \wedge B^2_p, U]$ where $B^2_p = S^1 \cup_p B^2$ is a Moore space.

Theorem 2 [A-H]. $\widetilde{K}^*_{\mathbb{C}}(K(\mathbb{Z}_p, 2); \mathbb{Z}_p) = 0$.

Corollary. $\widetilde{K}^*_{\mathbb{C}}(K(\bigoplus \mathbb{Z}_p, 2); \mathbb{Z}_p) = 0$.

Proposition 2. $\widetilde{K}^*_{\mathbb{C}}(K(\mathbb{Z}[\frac{1}{p}], 2); \mathbb{Z}_p) = 0$.

Proof. Since $\widetilde{K}^*_{\mathbb{C}}(K(\mathbb{Z}[\frac{1}{p}], 2))$ has a structure of $\mathbb{Z}[\frac{1}{p}]$-module the universal coefficient formula [A-H] implies the formula.

Corollary. $\widetilde{K}^*_{\mathbb{C}}(K(\bigoplus \mathbb{Z}[\frac{1}{p}], 2); \mathbb{Z}_p) = 0$.

Proposition 3. *Let $\omega\colon W_\tau(G, 2) \to K$ be a projection in Edwards-Walsh construction for $G = \mathbb{Z}_p$ or $\mathbb{Z}[\frac{1}{p}]$. Then ω induces an isomorphism $\omega^*\colon \widetilde{K}^*_{\mathbb{C}}(K; \mathbb{Z}_p) \to \widetilde{K}^*_{\mathbb{C}}(W_\tau(G, 2); \mathbb{Z}_p)$.*

Proof. By induction on $\dim K$. Apply the Mayer-Vietoris sequence and the Corollary of Theorem 2 in the case $G = \mathbb{Z}_p$ and the Corollary of Proposition 2 in the case $G = \mathbb{Z}[\frac{1}{p}]$.

Proposition 4 [B-M]. *Suppose that the CW-complex X is a direct limit $X = \varinjlim X_\alpha$, then $\widetilde{K}^*_{\mathbb{C}}(X; \mathbb{Z}_p) = \varprojlim \widetilde{K}^*_{\mathbb{C}}(X_\alpha; \mathbb{Z}_p)$.*

Proposition 5. *Let K be a polyhedron with triangulation τ and suppose the map $f\colon X \to K$ has the property* $\text{c-dim}_G(f, \tau) \le n$. *If $f' = f \circ g$ then* $\text{c-dim}_G(f', \tau) \le n$.

Corollary. *If $X' \subset X$ then* $\text{c-dim}_G(f_{|X'}, \tau) \le n$.

The proof is trivial.

Lemma 2. *For any prime p and for $G = \mathbb{Z}[\frac{1}{p}]$ or $\mathbb{Z}_p$, for an arbitrary compact polyhedron K with triangulation τ, and for arbitrary nontrivial $\alpha \in \widetilde{K}^*_{\mathbb{C}}(K; \mathbb{Z}_p)$ there exists a map $f: L \to K$ of a compact polyhedron L with the properties*

(1) $f^*(\alpha) \neq 0$,

(2) $\text{c-dim}_G(f, \tau) \leq 2$.

Proof. Consider the Edwards-Walsh construction $\omega: W_\tau(G, 2) \to K$. By property (3) of Lemma 1 there is a filtration of $W_\tau(G, 2)$ by compact polyhedra $L_1 \subset L_2 \subset \cdots \subset L_i \subset \cdots$. Denote by ε_i the inclusion $L_i \hookrightarrow W_\tau(G, 2)$. By Proposition 3 $\omega^*(\alpha) \neq 0$. By Proposition 4 there exists i such that $\varepsilon_i^*(\omega^*(\alpha)) \neq 0$. Consider $L = L_i$ and $f = \omega|_{L_i}$. We have $f^*(\alpha) = \varepsilon_i^*(\omega^*(\alpha)) \neq 0$. By property (2) of Lemma 1 and the Corollary of Proposition 5 we have $\text{c-dim}_G(f, \tau) \leq 2$.

Lemma 3 [D1]. *Let Z be the limit space of an inverse system of compact polyhedra $\{L_i, g_i^{i+1}\}$ with fixed triangulations τ_i and fixed metrics ρ_i. Suppose that for all k,*

$$\lim_{i\to\infty} \text{mesh}(g_k^{k+i}(\tau_{k+i})) = 0$$

and suppose that for infinitely many i, $\text{c-dim}_\pi(g_i^{i+1}, \tau_i) \leq n$. Then $\text{c-dim}_\pi Z \leq n$.

Proof of Theorem 1. We define X as a limit space of an inverse system $\{L_i, g_i^{i+1}\}$ and construct this system by induction. Define $L_1 = S^4$ and fix a metric ρ_1 on L_1 and triangulation τ_1 with $\text{mesh}\,\tau_1 < 1$. Fix $\alpha_1 \in \widetilde{K}^*_{\mathbb{C}}(S^4; \mathbb{Z}_p)$, $\alpha_1 \neq 0$, and apply Lemma 2 with $G = \mathbb{Z}[\frac{1}{p}]$ to obtain $g_1^2: L_2 \to L_1$. Define $\alpha_2 = (g_1^2)^*(\alpha_1) \neq 0$. Fix a metric ρ_2 on L_2 and choose a triangulation τ_2 with $\text{mesh}\,\tau_2 < \frac{1}{2}$ and $\text{mesh}\, g_1^2\tau_2 < \frac{1}{2}$. Then apply Lemma 2 with $G = \mathbb{Z}_p$ and so on.

Additionally we will obtain a sequence $\alpha_i \in \widetilde{K}^*_{\mathbb{C}}(L_i, \mathbb{Z}_p)$ such that $(g_1^i)^*(\alpha_1) = \alpha_i \neq 0$.

Lemma 3 implies that $\text{c-dim}_{\mathbb{Z}_p} X$, $\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]} X \leq 2$. The compact X is infinite dimensional. Then by Alexandroff's theorem [A, W] it follows that $\dim X = \text{c-dim}_{\mathbb{Z}} X$. By virtue of Proposition 1, $\dim X \leq 3$. Hence the projection $(g_1^\infty): X \to S^4$ is not essential. From the other side we have that $(g_1^\infty)^*(\alpha) \neq 0$. Therefore g_1^∞ is an essential map. Contradiction.

3. Edwards-Walsh resolution

Suppose that $\{(X_i, x_i), p_i^{i+1}\}$ is an inverse system of pointed spaces and base point preserving maps. Then there are natural embeddings of $\prod_{i=1}^m X_i$ into $\prod_{i=1}^\infty X_i$. The sequence $X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m$ defines an embedding of X_m into $\prod_{i=0}^m X_i$ and the inverse system $\{X_i, p_i^{i+1}\}$ defines an embedding of the limit space in $\prod_{i=1}^\infty X_i$ by the definition. So, for any pointed inverse system $\{X_i, p_i^{i+1}\}$ with limit space X there are natural embeddings $X_i \hookrightarrow \prod_{i=1}^\infty X_i$ and $X \hookrightarrow \prod_{i=1}^\infty X_i$. We will call this system of embeddings a realization of the inverse system in $\prod_{i=1}^\infty X_i$.

Suppose that ρ_i is a metric on X_i, and let $\overline{\rho}_i$ be the diameter of X_i. If $\sum_{i=1}^\infty \overline{\rho}_i < \infty$ then the formula $\rho(x, y) = \sum_{i=1}^\infty \rho_i(p_i^\infty(x), p_i^\infty(y))$ defines a metric on $\prod_{i=1}^\infty X_i$. Such a metric we will call a brick metric.

Let $\mathscr{M}$ be a finite covering of some space X with fixed metric. By $d(\mathscr{M})$ we denote $\max\{\operatorname{diam} M : M \in \mathscr{M}\}$ and by $\lambda(\mathscr{M})$ we denote the Lebesgue number of $\mathscr{M}$:

$$\lambda(\mathscr{M}) = \max\{r : \forall O_r(x) \exists M \in \mathscr{M} \text{ s.t. } O_r(x) \subset M\}.$$

Here $O_r(x)$ is the ball of radius r with x as a center. By M_x denote arbitrary sets $M \in \mathscr{M}$ with the property $x \in O_{\lambda(\mathscr{M})}(x) \subset M$.

The following lemma is a variation of M. Brown's lemma [Br, W].

Lemma 4. *Let* $X = \varprojlim\{K_i, f_i^{i+1}\}$ *be the limit space of an inverse system of compacta and suppose that the system* $\{K_i, f_i^{i+1}\}$ *is realized in* $\prod_{i=1}^{\infty} K_i$ *with brick metric* ρ *on it. Suppose* $Z = \varprojlim\{L_i, g_i^{i+1}\}$ *is the limit space of another inverse system of compacta and suppose that for all* i *a covering* $\mathscr{M}^i$ *of* K_i *and a map* $\alpha_i : L_i \to K_i$ *are defined such that*

(1) $\alpha_i(L_i) \cap M \neq \varnothing$ *for every* $M \in \mathscr{M}^i$,

(2) *the square diagram*

$$\begin{array}{ccc} L_{i+1} & \xrightarrow{\alpha_{i+1}} & K_{i+1} \\ \downarrow g_i^{i+1} & & \downarrow f_i^{i+1} \\ L_i & \xrightarrow{\alpha_i} & K_i \end{array}$$

is $(\lambda_i/4)$*-commutative, i.e.,* $\rho(\alpha_i \circ g_i^{i+1}, f_i^{i+1} \circ \alpha_{i+1}) < \lambda_i/4$ *where* $\lambda_i = \lambda(\mathscr{M}^i)$,

(3) $d_i = d(\mathscr{M}^i) < \lambda_{i-1}/4$.

Then there exists a map $\alpha : Z \to X$ *onto* X *such that a preimage of each point* $x \in X$ *is*

$$\alpha^{-1}(x) = \varprojlim\{\alpha_i^{-1}(M_{f_i^{\infty}(x)}), g_i^{i+1}|\dots\}.$$

Proof. (A) For any i, k, $\rho(\alpha_i \circ g_i^{i+k}, f_i^{i+k} \circ \alpha_{i+k}) < \lambda_i/2$. Prove it by induction on k. For $k = 1$ it follows by property (2). Suppose that $k > 1$. For an arbitrary point $z \in L_{i+k}$ the triangle inequality implies

$$\begin{aligned} &\rho(\alpha_i \circ g_i^{i+k}(z), f_i^{i+k} \circ \alpha_{i+k}(z)) \\ &\quad \le \rho(\alpha_i g_i^{i+1}(g_{i+1}^{i+k}(z)), f_i^{i+1}\alpha_{i+1}(g_{i+1}^{i+k}(z))) \\ &\qquad + \rho(f_i^{i+1}(\alpha_{i+1} g_{i+1}^{i+k}(z)), f_i^{i+1}(f_{i+1}^{i+k}\alpha_{i+k}(z))) \\ &\quad \le \lambda_i/4 + \rho(\alpha_{i+1} g_{i+1}^{i+k}(z), f_{i+1}^{i+k}\alpha_{i+k}(z)). \end{aligned}$$

The last inequality is due to (2) and a property of brick metric. Apply the induction assumption to conclude the proof.

(B) There exists a limit α of the sequence of maps $\alpha_i g_i^{\infty} : Z \to \prod_{i=1}^{\infty} K_i$.

Denote by s_k the sum $\sum_{i=k}^{\infty} \overline{\rho}_i$ where $\overline{\rho}_i = \operatorname{diam}_{\rho_i} K_i$. Then for any $z \in Z$,

$$\begin{aligned} &\rho(\alpha_i g_i^{\infty}(z), \alpha_{i+k} g_{i+k}^{\infty}(z)) \\ &\quad \le \rho(\alpha_i g_i^{i+k}(g_{i+k}^{\infty}(z)), f_i^{i+k}\alpha_{i+k}(g_{i+k}^{\infty}(z))) \\ &\qquad + \rho(f_i^{i+k}\alpha_{i+k}(g_{i+k}^{\infty}(z)), \alpha_{i+k} g_{i+k}^{\infty}(z)) < \lambda_i/2 + s_i. \end{aligned}$$

The Cauchy criterion implies the proof.

(C) $\alpha(Z) \subset X$. For arbitrary $z \in Z$, $\rho(\alpha_i g_i^{\infty}(z), (f_i^{\infty})^{-1}\alpha_i g_i^{\infty}(z)) < s_i$ and hence $\lim_{i\to\infty} \rho(\alpha_i g_i^{\infty}(z), X) = 0$.

(D) The sequence $\{\alpha_i^{-1}(M_{f_i^\infty(x)}),\ g_i^{i+1}|_{\dots}\}$ is well defined for any x and for arbitrary choice of $M_{f_i^\infty(x)} \in \mathscr{M}^i$.

Property (1) implies that $\alpha_i^{-1}M_{f_i^\infty(x)} \neq 0$ for all i. It suffices to show that

$$g_i^{i+1}(\alpha_{i+1}^{-1}(M_{f_{i+1}^\infty(x)})) \subset \alpha_i^{-1}(M_{f_i^\infty(x)}).$$

Let $y \in \alpha_{i+1}^{-1}M_{f_{i+1}^\infty(x)}$. We show that $\alpha_i g_i^{i+1}(y) \in M_{f_i^\infty(x)}$. By the triangle inequality we have

$$\begin{aligned}
&\rho(\alpha_i g_i^{i+1}(y),\ f_i^\infty(x)) \\
&\quad \le \rho(\alpha_i g_i^{i+1}(y),\ f_i^{i+1}\alpha_{i+1}(y)) + \rho(f_i^{i+1}\alpha_{i+1}(y),\ f_i^{i+1}f_{i+1}^\infty(x)) \\
&\quad \le \lambda_i/4 + d_{i+1} < \lambda_i/4 + \lambda_i/4 \quad \text{(by (3))} \\
&\quad \le \lambda_i/2.
\end{aligned}$$

Hence $\alpha_i g_i^{i+1}(y) \in O_{\lambda_i}(f_i^\infty(x)) \subset M_{f_i^\infty(x)}$.

(E) $\alpha(\varprojlim\{\alpha_i^{-1}(M_{f_i^\infty(x)})\}) = x$. Let $z \in \varprojlim\{\alpha_i^{-1}(M_{f_i^\infty(x)})\}$. Since

$$\rho(\alpha_i g_i^\infty(z),\ f_i^\infty(x)) < d_i$$

then $\rho(\alpha_i g_i^\infty(z),\ x) < d_i + s_i \to 0$.

(F) $\alpha^{-1}(x) \subset \varprojlim\{\alpha_i^{-1}(M_{f_i^\infty(x)})\}$.

Suppose that $z \notin \varprojlim\{\alpha_i^{-1}(M_{f_i^\infty(x)})\}$. Then there is a number i such that $g_i^\infty(z) \notin \alpha_i^{-1}M_{f_i^\infty(x)}$. Hence

$$(*) \qquad \rho(\alpha_i g_i^\infty(z),\ f_i^\infty(x)) > \lambda_i.$$

Brick metric properties and triangle inequality imply that

$$\begin{aligned}
&\rho(\alpha_{i+k} g_{i+k}^\infty(z),\ x) \ge \rho(f_i^{i+k}\alpha_{i+k} g_{i+k}^\infty(z),\ f_i^\infty(x)) \\
&\quad \ge \rho(\alpha_i g_i^\infty(z),\ f_i^\infty(x)) - \rho(f_i^{i+k}\alpha_{i+k}(g_{i+k}^\infty(z)),\ \alpha_i g_i^{i+k}(g_{i+k}^\infty(z))) \\
&\quad \ge \lambda_i - \lambda_i/2
\end{aligned}$$

(by $(*)$ and (A)). So $\rho(\alpha(z),\ x) \ge \lambda_i/2$.

Suppose that K is an n-dimensional polyhedron with fixed triangulation τ and p is a prime number. We define n-dimensional complexes $p\tau$ and $\frac{1}{p}\tau$ together with projections $\mu\colon p\tau \to K$ and $\nu = \frac{1}{p}\tau \to K$ and call them a *p-modification* of K and *$\frac{1}{p}$-modification* of K correspondingly. The complex $p\tau$ is obtained from K by replacement of n-dimensional simplexes by n-cells attached by maps of degree p. The projection μ is defined arbitrary with the property that $\mu^{-1}|K^{(n-1)}$ is an embedding. The complex $\frac{1}{p}\tau$ is obtained from K by replacement of n-simplices by infinite p-telescopes = the infinite union of mapping cylinders of maps of degree p (see [Su]), and define a map $\nu\colon \frac{1}{p}\tau \to K$ with the same property.

Proposition 6. *For arbitrary triangulation τ of an $(n+1)$-skeleton of m-simplex, $m > n \ge 2$, $\pi_n(p\tau) = \bigoplus \mathbb{Z}_p$ and $\pi_n(\frac{1}{p}\tau)$ is divisible by p.*

Proof. Since $|\tau^{(n)}|$ is $(n-1)$-connected, $\pi_n(\tau^{(n)})$ is a free module over $\mathbb{Z}$. It is generated by the family of boundaries of $(n+1)$-simplices in τ, say $a_1, \dots, a_m$. The relations are obtained from $(n+2)$-simplices of τ. Let it be $F_1, \dots, F_l$. So we know that the module $\mathbb{Z}[a_1, \dots, a_m]/\{F_i\}$ is free over $\mathbb{Z}$. By the construction of the p-modification we have

$$\pi_n(p\tau) = \mathbb{Z}_p[a_1, \dots, a_m]/\{F_i\}.$$

It is easy to check that every basis $e_1, \dots, e_k$ in $\mathbb{Z}[a_1, \dots, a_m]/\{F_i\}$ generates a basis $\bar{e}_1, \dots, \bar{e}_k$ in $\mathbb{Z}_p[a_1, \dots, a_m]/\{F_i\}$.

Since $\pi_n(\frac{1}{p}\tau) = \mathbb{Z}[\frac{1}{p}][a_1, \dots, a_m]/\{F_i\}$ then $\pi_n(\frac{1}{p}\tau)$ is divisible by p.

Proposition 7. *If G is divisible by p then*

$$\text{c-dim}_G Y \le \max\{\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]} Y, \text{c-dim}_{\mathbb{Z}_{p^\infty}} Y\}.$$

Proof. The short exact sequence $0 \to \operatorname{Tor} G \to G/\operatorname{Tor} G \to 0$ implies that $\text{c-dim}_G Y \le \max\{\text{c-dim}_{\operatorname{Tor} G} Y, \text{c-dim}_{G/\operatorname{Tor} G} Y\}$. The torsion part can be split as $\operatorname{Tor} G = \operatorname{Tor}' G \oplus p\text{-}\operatorname{Tor} G$ and $\operatorname{Tor}' G$ does not contain p-torsion. Bokshtein's inequalities [Ku] $\text{c-dim}_{\mathbb{Z}_{q^\infty}} \le \text{c-dim}_{\mathbb{Z}_q} \le \text{c-dim}_{\mathbb{Z}_{(q)}}$, where $\mathbb{Z}_{(q)}$ is a localization of the integers at some prime q, imply that $\text{c-dim}_{\operatorname{Tor}' G} \le \text{c-dim}_{\mathbb{Z}[\frac{1}{p}]}$. Since G is divisible by p then $p\text{-}\operatorname{Tor} G = \bigoplus \mathbb{Z}_{p^\infty}$ and hence $\text{c-dim}_{p\text{-}\operatorname{Tor} G} \le \text{c-dim}_{\mathbb{Z}_{p^\infty}}$. Since $G/\operatorname{Tor} G$ is divisible by p it follows [Ku] that $\text{c-dim}_{G/\operatorname{Tor} G} \le \text{c-dim}_{\mathbb{Z}[\frac{1}{p}]}$.

Proposition 8. *Let $\mu\colon p\tau \to |\tau|$ and $\nu\colon \frac{1}{p}\tau \to |\tau|$ be projections of the p-modification and $\frac{1}{p}$-modification correspondingly of an $(n+1)$-dimensional polyhedron. Then $\text{c-dim}_{\mathbb{Z}_p}(\mu, \tau) \le n$ and $\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]}(\nu, \tau) \le n$.*

The proof follows from the definition.

Lemma 5. *Suppose $w\colon R \to K$ is a map onto a polyhedron K. Let τ be a triangulation on K with $\operatorname{mesh}\tau < \varepsilon$ and assume that for every simplex $\sigma \in \tau$, $w^{-1}(\sigma) \simeq K(\bigoplus_1^{m_\sigma} \pi, n)$ for some fixed n. If $\text{c-dim}_\pi X \le n$ then for any map $f\colon X \to K$ there exists an ε-lifting $f'\colon X \to R$ (i.e., $\rho(wf', f) < \varepsilon$).*

Proof. Construct f' step by step defined on sets $f^{-1}(K^{(i)})$ where $K^{(i)}$ denotes the i-skeleton with respect to τ. Define f' on $f^{-1}(K^{(0)})$ by choosing some points in $w^{-1}(v)$ for all $v \in K^{(0)}$. Suppose that f' is defined on $f^{-1}(K^{(i)})$ with the property:

$$(*) \qquad \forall \sigma \in \tau\ \forall x \in X \text{ if } f(x) \in \sigma \text{ then } wf'(x) \in \sigma.$$

Consider an arbitrary $(i+1)$-dimensional simplex $\sigma \in \tau$ and extend the map

$$f'|_{\dots}\colon f^{-1}(\sigma^{(i)}) \to w^{-1}(\sigma) \simeq K\left(\bigoplus_1^{m_\sigma} \pi, n\right)$$

to a map of $f^{-1}(\sigma)$. Do this for all $(i+1)$-dimensional simplexes σ to define f' on $f^{-1}(K^{(i+1)})$. Property $(*)$ holds and implies the inequality $\rho(wf', f) < \varepsilon$.

By $|\tau|$ denote a geometric realization of a simplicial complex τ.

Lemma 6. *Let X be the limit space of an inverse system of compact polyhedra $\{N_k, q_k^{k+1}\}$ and suppose that $\text{c-dim}_{\mathbb{Z}_p} X \le n$ and $\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]} X \le n$ $(n \ge 2)$. Let the group G be equal to $\mathbb{Z}_p$ or $\mathbb{Z}[\frac{1}{p}]$, and let $N_1^{(n+1)}$ denote $(n+1)$-dimensional skeleton of N_1 with respect to some triangulation τ_1 with $\operatorname{mesh}\tau_1 < \varepsilon$. Then* *for any triangulation* γ *of* $N_1^{(n+1)}$ *there exists a number* k *such that for any triangulation* τ *of* N_k *there is a map* $g\colon |\tau^{(n+1)}| \to N_1^{(n+1)}$ *with the properties:*

(1) $\text{c-dim}_G(g, \tau) \le n$,

(2) $\rho(g, q_1^k|_{|\tau^{(n+1)}|}) < 3\varepsilon$.

Proof. $G = \mathbb{Z}_p$. There exists a CW-complex R and a map $\theta\colon R \to N_1$ with the properties:

(1) for any simplex $\sigma \in \tau_1$, $\theta^{-1}(\sigma) \simeq K(\pi_n(p(\gamma_{|\sigma^{(n+1)}})), n)$,

(2) the $(n+1)$-dimensional skeleton $R^{[n+1]}$ coincides with the p-modification $p\gamma$, and the restriction $\theta_{|p\gamma}$ coincides with $\mu\colon p\gamma \to |\gamma|$. We define R as a growing union of CW-complexes $R_{n+1} \subset R_{n+2} \subset \cdots \subset R_{\dim N_1} = R$ and define θ as a union of maps $\theta_i\colon R_i \to N_i^{(i)}$, $i \ge n+1$. First of all define R_{n+1} as an Edwards-Walsh construction $W_\gamma(\mathbb{Z}_p, n)$ and $\theta_{n+1} = \omega\colon R_{n+1} \to N_1^{(n+1)}$ such that $R_{n+1}^{[n+1]} = p\gamma$ and $\omega_{|p\gamma} = \mu\colon p\gamma \to N_1^{(n+1)}$. For every $(n+2)$-dimensional simplex $\sigma \in \tau_1$,

$$\pi_n(\theta_{n+1}^{-1}(\sigma^{(n+1)})) = \pi_n(\mu^{-1}(\sigma^{(n+1)})) = \pi_n(p(\gamma_{|\sigma^{(n+1)}})).$$

By virtue of Proposition 6 it is possible to obtain a CW-complex $K(\bigoplus_1^{m_\sigma} \mathbb{Z}_p, n)$ by attaching to $\theta_{n+1}^{-1}(\sigma^{(n+1)})$ cells of dimensions $\ge n+2$. Define a map θ_{n+2} on each newly attached cell such that θ_{n+2} sends an open cell into the interior of σ and

$$\theta_{n+2}|_{\theta_{n+1}^{-1}(\sigma^{(n+1)})} = \theta_{n+1}|_{\theta_{n+1}^{-1}(\sigma^{(n+1)})}.$$

Thus it is possible to define $\theta_{n+2}\colon R_{n+2} \to N_1^{(n+2)}$. By using Proposition 6 we may assume that for arbitrary $(n+3)$-dimensional simplex $\sigma \in \tau_1$, the nth homotopy group $\pi_n(\theta_{n+2}^{-1}(\sigma^{(n+2)}))$ coincides with the nth homotopy group of the $(n+1)$-skeleton $\equiv \pi_n(p(\gamma_{|\sigma^{(n+1)}}))$, and so on.

If $G = \mathbb{Z}[\frac{1}{p}]$ then there exists a CW-complex R and a map $\theta\colon R \to N_1$ such that

(1) for any simplex $\sigma \in \tau_1$, $\theta^{-1}(\sigma) \simeq K(\pi_n(\frac{1}{p}(\gamma_{|\sigma^{(n+1)}})), n)$,

(2) the $(n+1)$-dimensional skeleton $R^{[n+1]}$ coincides with the $\frac{1}{p}$-modification $\frac{1}{p}\gamma$ and the restriction $\theta|_{\frac{1}{p}\gamma}$ coincides with $\nu\colon \frac{1}{p}\gamma \to |\gamma|$.

The proof is the same.

By Proposition 6, $\pi_n(p(\gamma_{|\sigma^{(n+1)}})) = \bigoplus \mathbb{Z}_p$, and the group $\pi = \pi_n(\frac{1}{p}(\gamma_{|\sigma^{(n+1)}}))$ is divisible by p. Proposition 7 implies that $\text{c-dim}_\pi X \le n$. So, in both cases it is possible to apply Lemma 5 to the map $\theta\colon R \to N_1$. In both cases we will obtain an ε-lifting $f\colon X \to R$. Since $R \in \text{ANE}$ then there exists a number k and a map $f_k\colon N_k \to R$ such that $\rho(\theta \circ f, \theta \circ f_k \circ q_k^\infty) < \varepsilon$. Let τ be a triangulation of N_k. Denote by g' a cellular approximation of $f_k\colon |\tau^{(n+1)}| \to R$ into the $(n+1)$-skeleton $R^{[n+1]}$. We have $\rho(\theta \circ f_k, \theta \circ g) < \varepsilon$. For arbitrary $z \in |\tau^{(n+1)}|$ choose $x \in (q_\infty^k)^{-1}(z)$. Then

$$\rho(q_1^k(z), \theta \circ g'(z)) \le \rho(q_1^\infty(x), \theta \circ f(x)) + \rho(\theta \circ f(x), \theta \circ f_k \circ q_k^\infty(x)) + \rho(\theta \circ f_k(z), \theta \circ g'(z)) \le 3\varepsilon.$$

Denote by $g = \theta \circ g'\colon |\tau^{(n+1)}| \to N_1^{(n+1)}$. Property (2) has just been checked.

By Proposition 8, $\text{c-dim}_G(\theta_{|R^{[n+1]}}, \gamma) \le n$ and hence by virtue of Proposition 5, $\text{c-dim}_G(g, \tau) \le n$.

Lemma 7 [W]. *Let X be the limit space of an inverse system of compact polyhedra $\{N_k, q_i^{i+1}\}$ and suppose that* $\text{c-dim}_{\mathbb{Z}} X \leq m$. *Let $N_1^{(m)}$ be an $(n+1)$-skeleton of N_1 with respect to some triangulation τ_1 with* $\operatorname{mesh} \tau_1 < \varepsilon$. *Then there exists a number k such that for any triangulation τ of N_k there is a map $g: |\tau^{(m+1)}| \to N_1^{(m)}$ with $\rho(g, f_1^k) < 3\varepsilon$.*

Proof. Let $\omega: W_{\tau_1}(\mathbb{Z}, m) \to N_1$ be the Edwards-Walsh construction. By Lemma 5 there is an ε-lifting $f: X \to W_{\tau_1}(\mathbb{Z}, m)$ of q_1^∞. Apply the above argument to define $g': |\tau^{(m+1)}| \to W_{\tau_1}(\mathbb{Z}, m)^{[m+1]}$. Since $W_{\tau_1}(\mathbb{Z}, m)^{[m+1]} = W_{\tau_1}(\mathbb{Z}, m)^{[m]}$ the map $g = \omega \circ g'$ sends $|\tau^{(m+1)}|$ into $N_1^{(m)}$ and property $\rho(g, f_1^k) < 3\varepsilon$ holds.

The following is a generalization of Edwards' theorem [E2, W].

Theorem 3. *Suppose that the compactum X has cohomological dimensions* $\text{c-dim}_{\mathbb{Z}_p} X$ *and* $\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]} X \leq n$, $n \geq 2$, *for some p. Then there exist an $(n+1)$-dimensional compactum Z with* $\text{c-dim}_{\mathbb{Z}_p} Z \leq n$, $\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]} Z \leq n$ *and a cell-like map $\alpha: Z \to X$.*

Proof. We may assume that X is the limit space of an inverse system of compact polyhedra $\{N_k, q_k^{k+1}\}$. Proposition 1 implies that $\text{c-dim}_{\mathbb{Z}} X \leq n+1$.

We construct two inverse systems $\{K_i, f_i^{i+1}\}$, $\{L_i, g_i^{i+1}\}$, and a system of maps $\{\alpha_i: L_i \to K_i\}$ such that $X = \varprojlim \{K_i, f_i^{i+1}\}$, and properties (1)–(3) of Lemma 4 hold. We construct it by induction so that the step of induction from i to $i+1$ depends on the class of i mod 3. To demonstrate it consider in detail the cases $i = 1, 2$ and 3.

Define $K_i = N_1$. Consider a finite covering $\mathscr{M}^1$ of K_1 by contractible subpolyhedra with respect to some fixed triangulation of K_1 and with nontrivial Lebesgue number $\lambda_1 = \lambda(\mathscr{M}^1)$. Let us regard that for each N_k some metric ρ_k is fixed and $\sum_{k=1}^\infty \rho_k < \infty$. Let τ_1 be a triangulation of K_1 with $\operatorname{mesh} \tau_1 < \lambda_1/12$. Define $L_1 = |\tau_1^{(n+1)}|$ and let $\alpha_1: L_1 \hookrightarrow K_1$ be the natural embedding. Apply Lemma 6 for $G = \mathbb{Z}_p$ and for $\gamma_1 = \tau_1^{(n+1)}$ to obtain k. Then define $K_2 = N_k$ and consider a finite covering $\mathscr{M}^2$ of K_2 by contractible subpolyhedra with nontrivial Lebesgue number $\lambda_2 = \lambda(\mathscr{M}^2)$ with respect to a metric $\rho_1 + \rho_k$ on K_2 given by

$$(\rho_1 + \rho_k)(x_1, x_2) = \rho_1(q_1^k(x_1), q_1^k(x_2)) + \rho_k(x_1, x_2).$$

We can regard that $d_2 = \operatorname{diam} \mathscr{M}^2 < \lambda_1/4$. Choose a triangulation τ_2 of K_2 with $\operatorname{mesh} \tau_2 < \lambda_2/12$ (with respect to that metric $\rho_1 + \rho_k$). Define $L_2 = |\tau_2^{(n+1)}|$. By Lemma 6 there exists a map $g_1^2: L_2 \to L_1$ with $\text{c-dim}_{\mathbb{Z}_p}(g_1^2, \tau_1) \leq n$ and $\rho(g_1^2, q_1^k|_{|\tau_2^{(n+1)}|}) < 3\lambda_1/12 = \lambda_1/4$. Denote $f_1^2 = q_1^k$, and define $\alpha_2: L_2 \hookrightarrow K_2$ be the natural embedding. So, properties (1)–(3) of Lemma 4 for $i \leq 2$ hold. Choose a triangulation γ_2 of L_2 with $\max\{\operatorname{mesh} \gamma_2, \operatorname{mesh} g_1^2 \gamma_2\} < \lambda_2$ and apply Lemma 6 for $G = \mathbb{Z}[\frac{1}{p}]$ and the system $\{N_l, q_l^{l+1}\}_{l \geq k}$. We will obtain a number l such that for any triangulation τ_3 of N_l there is a map $g: |\tau^{(n+1)}| \to N_k^{(n+1)} = L_2$ with $\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]}(g, \tau_2) \leq n$ and

$$\rho(g, q_k^l|_{|\tau^{(n+1)}|}) < 3 \operatorname{mesh} \tau_2 < \lambda_3/4.$$

Choose τ_3 by the following routine way. Consider a finite covering M^3 of $K_3 = N_l$ by contractible subpolyhedra with respect to a nontrivial Lebesgue number $\lambda_3 = \lambda(M^3)$. Consider the metric $\rho_1 + \rho_k + \rho_l$ on K_3. We can regard that $d_2 = d(M^2) < \lambda_2/4$. At last choose τ_3 with $\operatorname{mesh}\tau_3 < \lambda_3/12$, and define $L_3 = |\tau_3^{(n+1)}|$. Let $\alpha_3\colon L_3 \hookrightarrow K_3$ be the embedding and let g_2^3 be a map $g\colon |\tau_3^{(n+1)}| \to L_2$ obtained by Lemma 6. Denote $f_2^3 = q_k^l$, and the properties (1)–(3) of Lemma 4 still hold.

Apply Lemma 7 to the sequence $\{N_r, q_r^{r+1}\}_{r\geq l}$, $m = n+1$ and triangulation τ_3 on $N_l = K_3$ to obtain a map $g\colon |\tau_2^{(n+2)}| \to L_3$ with $\rho(g, f_l^r) < 3\operatorname{mesh}\tau_3 < \lambda_3/4$ for some $r > l$ and arbitrary triangulation τ_4 of N_r. We choose τ_4 by using the above routine.

Define $L_4 = |\tau_4^{(n+1)}|$ and α_4 as the natural embedding. Denote q_l^r by f_3^4. The map g gives us a projection g_3^4. The properties (1)–(3) of Lemma 4 are satisfied.

Using this procedure we can construct two inverse systems $\{K_i, f_i^{i+1}\}$ and $\{L_i, g_i^{i+1}\}$ with a family of maps $\{\alpha_i\colon L_i \to K_i\}$ together with triangulations τ_i on K_i and γ_i on L_i and a family of coverings $\mathscr{M}^i$ of K_i by contractible subpolyhedra with respect to τ_i. We define a brick metric on $\prod_{i=1}^{\infty} K_i$ and some metric ρ_i' on L_i for each i with properties (1)–(3) from Lemma 4 and

(4) $\max\{\operatorname{mesh}\gamma_i, \operatorname{mesh} g_{i-1}^i\gamma_i, \dots, \operatorname{mesh} g_1^i\gamma_i\} < \lambda_i$,

(5) $L_i = |\tau_i^{(n+1)}|$,

(6) $\text{c-dim}_{\mathbb{Z}_p}(g_i^{i+1}, \gamma_i) \leq n$ if $i \equiv 1 \mod 3$,

(7) $\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]}(g_i^{i+1}, \gamma_i) \leq n$ if $i \equiv 2 \mod 3$,

(8) there exists an extension $\overline{g}_i\colon |\tau_{i+1}^{(n+2)}| \to L_i$ of the map g_i^{i+1} if $i \equiv 0 \mod 3$.

By Lemma 4 we have a map $\alpha\colon Z = \varprojlim\{L_i\} \to X = \varprojlim\{K_i\}$ with $\alpha^{-1}(x) = \varprojlim\{\alpha_i^{-1}M_{f_i^\infty(x)}, g_i^{i+1}|_{\dots}\}$. Denote $M_{f_i^\infty(x)}$ by M_i. By virtue of property (5), $\alpha_i^{-1}M_i = M^{(n+1)}$. Consider $i = 3k$. By property (8) the map $g_i^{i+1}|_{M_{i+1}}$ can be extended to a map $\overline{g}_i\colon M_{i+1}^{(n+2)} \to L_i^{(n+1)}$. Since M_i is contractible then there exists a retraction $r_i\colon L_i^{(n+1)} \to M_i^{(n+1)}$. Since $M_{i+1}^{(n+1)}$ is contractible in $M_{i+1}^{(n+2)}$ and $r_i \circ \overline{g}_i\colon M_{i+1}^{(n+2)} \to M_i^{(n+1)}$ and $r_i \circ \overline{g}_i|_{M_{i+1}^{(n+1)}} = g_i^{i+1}$ then g_i^{i+1} is homotopic to constant. Hence $\operatorname{Sh}\alpha^{-1}(x) = *$ for each $x \in X$. Therefore α is a cell-like map.

Properties (4) and (6) together with Lemma 3 imply that $\text{c-dim}_{\mathbb{Z}_p} Z \leq n$. Then properties (4), (7) and Lemma 3 imply that $\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]} \leq n$.

4. Main theorem

The aim of this paragraph is to prove the following.

Theorem 4. *There exists a cell-like map $f\colon I^6 \to Y$ of the 6-dimensional cube with infinite-dimensional image.*

The proof of Theorem 4 follows by Theorems 5 and 6 below.

Theorem 5. *There exists a compactum Z with $\dim Z \times Z = 5$ and a cell-like map $\phi\colon Z \to Y'$ with $\dim Y' = \infty$.*

Theorem 6 [D-R-S, Sp]. *If* $\dim Z \times Z < n$ *then every map* $\psi\colon Z \to \mathbb{R}^n$ *can be approximated by embeddings.*

Indeed, embed the compactum Z from Theorem 5 in the 6-dimensional cube and consider the quotient map $f\colon I^6 \to Y$ of the decomposition generated by $\{\phi^{-1}(x)\}$ and singletons.

Lemma 8. *Suppose that* $\text{c-dim}_{\mathbb{Z}_p} Z \le n$ *and* $\text{c-dim}_{\mathbb{Z}[\frac{1}{p}]} Z \le n$ *and* Z *is finite dimensional. Then* $\dim Z \times Z \le 2n + 1$.

Proof. The Bokshtein inequalities [Ku] imply that $\text{c-dim}_{\mathbb{Z}_q} Z \le n$ for all primes q and $\text{c-dim}_{\mathbb{Q}} Z \le n$ where $\mathbb{Q}$ is the rationals. The Künneth formula for fields implies that $\text{c-dim}_{\mathbb{Z}_q} Z \times Z \le 2n$ and $\text{c-dim}_{\mathbb{Q}} Z \times Z \le 2n$. By virtue of Bokshtein's theorem [Ku] $\text{c-dim}_{\mathbb{Z}}(Z \times Z) = \max \text{c-dim}_{\mathbb{Z}_{(q)}}(Z \times Z)$, where $\mathbb{Z}_{(q)}$ is the localization of the integers at q. By Bokshtein's inequalities [Ku, D1] $\text{c-dim}_{\mathbb{Z}_{q^\infty}} \le \text{c-dim}_{\mathbb{Z}_q}$ and $\text{c-dim}_{\mathbb{Z}_{(q)}} \le \max\{\text{c-dim}_{\mathbb{Q}}, \text{c-dim}_{\mathbb{Z}_q^\infty} + 1\}$ it follows that $\text{c-dim}_{\mathbb{Z}_{(q)}} Z \le 2n + 1$. Since Z is finite dimensional then Alexandroff's theorem [A, W] implies that $\dim Z \times Z \le 2n + 1$.

The proof of Theorem 5 follows by Theorem 3 and Lemma 8.

Problem. Suppose that $\dim Z \times Z = 2n - 1$ for some compactum Z. Is it possible to imbed Z in $\mathbb{R}^{2n-1}$?

References

[A] P. S. Alexandroff, *Dimension theorie, ein Beitrag zur Geometrie der abgesehlossenen Mengen*, Math. Ann. **106** (1932), 161–238.

[Ar] S. Armentrout, *Cellular decomposition of* 3*-manifolds that yield* 3*-manifolds*, Mem. Amer. Math. Soc. No. 107 (1971).

[A-H] D. W. Anderson and L. Hodgkin, *the K-theory of Eilenberg-Mac Lane complexes*, Topology 7 (1968), 317–329.

[B1] R. H. Bing, *A decomposition of* E^3 *into points and tame arcs such that the decomposition space is topologically different from* E^3, Ann. of Math. (2) **65** (1957), 484–500.

[B2] ——, *Decomposition of* E^3 *in topology of* 3*-manifolds and related topics*, M. K. Fort, ed., Prentice-Hall, 1962, pp. 5–21.

[Br] M. Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc. **11** (1960), 478–483.

[B-M] V. M. Bukshtaber and A. S. Mischenko, *K-theory for the category of infinite cellular complexes*, Math. USSR-Izv. **2** (1969), 515–556.

[C] J. W. Cannon, *Shrinking cell-like decompositions of manifolds. Codimensions three*, Ann. of Math. (2) **110** (1979), 83–112.

[Da] R. D. Daverman, *Detecting the disjoint disks property*, Pacific J. Math. **93:2** (1981), 277–298.

[D1] A. N. Dranishnikov, *Homological dimension theory*, Uspehi Mat. Nauk **43:4** (1988), 11–55 = Russian Math. Surveys **43** (1988), 11–63.

[D2] ——, *On Alexandroff problem*, Mat. Sb. **135:4** (1986), 59–111. (Russian)

[D-S] A. N. Dranishnikov and E. V. Schepin, *Cell-like maps. The problem of increase of the dimension*, Russian Math. Surveys **41:6** (1986), 59–111.

[D-R-S] A. N. Dranishnikov, D. Repovs and E. V. Schepin, *On intersection of compacta of complementary dimension in euclidean space*, Topology Appl. **38** (1991), 237–253.

[E1] R. D. Edwards, *Approximating certain cell-like maps by homeomorphisms*, Notices Amer. Math. Soc. **24** (1977), A-649.

[E2] ——, *A theorem and a question related to cohomological dimension and cell-like maps*, Notices Amer. Math. Soc. **25** (1978), A-259.

[Ku] V. I. Kuzminov, *Homological dimension theory*, Russian Math. Surveys **23:5** (1968), 1–45.

[K-W] G. Kozlowski and J. Walsh, *Cell-like mappings on* 3-*manifolds*, Topology **22** (1983), 147–153.

[M-R] W. J. R. Mitchel and D. Repovs, *The topology of cell-like mappings*, Proc. Workshop in Differential Geometry and Topology (Cala Gane, 1988) (to appear).

[Q] F. Quinn, *Ends of maps.* III: *dimensions* 4 *and* 5, J. Differential Geom. **17** (1982), 503–521.

[S] L. C. Sibenmann, *Approximating cellular maps by homeomorphisms*, Topology **11** (1972), 271–294.

[Sp] S. Spiez, *Imbeddings in* $\mathbb{R}^{2m}$ *of m-dimensional compacta with* $\dim(X \times X) < 2m$, Fund. Math. **134** (1990), 105–115.

[Su] D. Sullivan, *Geometric topology*, Part I: *Localization, periodicity and Galois symmetry*, M.I.T., Cambridge, Mass., 1970.

[W] J. J. Walsh, *Dimension, cohomological dimension, and cell-like mappings*, Lecture Notes in Math., vol. 870, Springer, 1981, pp. 105–118.

Steklov Mathematical Institute, Moscow 117966, Russia
E-mail address: dranish@top.mian.su

Department of Mathematics, Cornell University, Ithaca, New York 14853-7901