

## THE WEIL-PETERSSON SYMPLECTIC STRUCTURE AT THURSTON'S BOUNDARY

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**ABSTRACT.** The Weil-Petersson Kähler structure on the Teichmüller space  $\mathcal{T}$  of a punctured surface is shown to extend, in an appropriate sense, to Thurston's symplectic structure on the space  $\mathcal{MF}_0$  of measured foliations of compact support on the surface. We introduce a space  $\widetilde{\mathcal{MF}}_0$  of decorated measured foliations whose relationship to  $\mathcal{MF}_0$  is analogous to the relationship between the decorated Teichmüller space  $\widetilde{\mathcal{T}}$  and  $\mathcal{T}$ .  $\widetilde{\mathcal{MF}}_0$  is parametrized by a vector space, and there is a natural piecewise-linear embedding of  $\mathcal{MF}_0$  in  $\widetilde{\mathcal{MF}}_0$  which pulls back a global differential form to Thurston's symplectic form. We exhibit a homeomorphism between  $\widetilde{\mathcal{T}}$  and  $\widetilde{\mathcal{MF}}_0$  which preserves the natural two-forms on these spaces. Following Thurston, we finally consider the space  $\mathcal{Y}$  of all suitable classes of metrics of constant Gaussian curvature on the surface, form a natural completion  $\overline{\mathcal{Y}}$  of  $\mathcal{Y}$ , and identify  $\overline{\mathcal{Y}} - \mathcal{Y}$  with  $\mathcal{MF}_0$ . An extension of the Weil-Petersson Kähler form to  $\mathcal{Y}$  is found to extend continuously by Thurston's symplectic pairing on  $\mathcal{MF}_0$  to a two-form on  $\mathcal{Y}$  itself.

### INTRODUCTION

The aim of this note is to establish a relation between the Kähler structure of the Weil-Petersson metric (see [Wo] for instance) on the Teichmüller space  $\mathcal{T}$  of a punctured surface and Thurston's piecewise-linear symplectic structure (see [Pa1] for instance) on the space  $\mathcal{MF}_0$  of measured foliations of compact support on the surface. Roughly, the former is the imaginary part of the natural  $L^2$  pairing of harmonic Beltrami differentials on the surface, while the latter is connected to the algebraic intersection of homology cycles on the surface. That these two structures are related is strongly suggested by the work of [Go], and our main result is that the Kähler structure on  $\mathcal{T}$  limits, in an appropriate sense, to the symplectic structure on Thurston's boundary for Teichmüller space. That the Kähler structure so extends to the boundary has been the subject of some interest (see [Bo] for instance) for several years. We emphasize that our arguments apply only in the case of punctured surfaces.

In fact, we shall be dealing with the decorated Teichmüller space  $\widetilde{\mathcal{T}}$ , whose

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definition (to be recalled below) requires that the underlying surface have at least one puncture. We shall give a global parametrization of this space, where the set of parameters is the collection of all real (not necessarily nonnegative) weights on the edges of some branched one-submanifold in the surface. These parameters are intimately connected with both the “ $\lambda$ -length” coordinates [Pe1] on  $\widetilde{\mathcal{F}}$  and the “horocyclic foliation” coordinates [Th] on  $\mathcal{F}$ . Actually, the branched one-submanifolds we must consider are a slight generalization of train tracks, and the usual construction of a measured foliation from a measured train track leads naturally to a space of “decorated measured foliations”  $\widetilde{\mathcal{MF}}_0$  on the surface. Our parametrization will be seen to induce a homeomorphism between  $\widetilde{\mathcal{F}}$  and  $\widetilde{\mathcal{MF}}_0$ , and, by manipulating known formulas, we shall find that this homeomorphism preserves the natural two-forms.

Recall [FLP] that Thurston’s boundary of  $\mathcal{F}$  is the space  $\mathcal{PF}_0$  of projective classes of measured foliations of compact support on the surface. To understand the sense in which the Kähler structure on  $\mathcal{F}$  extends to the symplectic structure on Thurston’s boundary, we must consider a space  $\mathcal{Y}$ , which is roughly the collection of all classes of suitable metrics of constant Gaussian curvature on the surface. (A precise definition will be given in §5 below.) The space  $\mathcal{Y}$  is canonically homeomorphic to the product of  $\mathcal{F}$  with an open ray  $]0, \infty[$ , and rather than compactifying  $\mathcal{F}$  by adjoining  $\mathcal{PF}_0$ , we imagine adjoining  $\mathcal{MF}_0$  itself to  $\mathcal{F} \times ]0, \infty[$  along  $\mathcal{F} \times \{0\}$ . The Kähler form on  $\mathcal{F}$  extends to  $\mathcal{Y}$ , and known regularity properties [Pa2] of Thurston’s compactification in this setting allow us to conclude that this form extends naturally to the symplectic form on  $\mathcal{MF}_0$ .

We regard as basic the connections that have evolved here between the decorated Teichmüller theory and the Thurston theory. As a consequence, once the relations between various concepts and constructions are in place, the proofs of the results are not difficult. At the same time, this basic connection suggests several interesting future projects, some of which are discussed in §6 below.

## 1. NOTATIONS AND DEFINITIONS

Let  $F = F_g^s$  denote a fixed smooth genus  $g$  surface with  $s > 0$  distinct points removed, where we assume that  $F$  has negative Euler characteristic; let  $P$  denote the set of removed points. Let  $\mathcal{F} = \mathcal{F}(F_g^s)$  denote the Teichmüller space (see [Ab] for instance) of all complete finite-area hyperbolic structures on  $F$  so that about each puncture is a (deleted) neighborhood which is isometric to the quotient of  $\{z = x + \sqrt{-1}y : y > 1\}$  by a parabolic transformation of the upper half-plane  $\{z = x + \sqrt{-1}y : y > 0\}$  which fixes infinity. The *decorated Teichmüller space*  $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}(F_g^s)$  is the total space of the natural bundle over  $\mathcal{F}$ , where the fiber over a point is the collection of all  $s$ -tuples of horocycles in  $F$ , one horocycle about each puncture. The tuple of hyperbolic lengths of the horocycles give coordinates on the fibers, and provided that each of the distinguished horocycles is sufficiently small (in fact, see [Sh], it is sufficient that the hyperbolic length of each horocycle is at most unity), one imagines a point of  $\widetilde{\mathcal{F}}$  as the surface with boundary obtained from  $F$  by deleting the corresponding horoballs.

An *ideal triangulation*  $\Delta$  in  $F$  is by definition a CW decomposition of the

closed surface  $F \cup P$  into triangles whose zero-skeleton exactly coincides with  $P$ . Given any complete hyperbolic metric on  $F$  (which represents an element of  $\mathcal{F}$ ), we can associate to  $\Delta$  a well-defined collection of geodesics in  $F$  running between the punctures, where the collection of disjoint bi-infinite geodesics represents the homotopy class of  $\Delta$ . We shall fix an ideal triangulation  $\Delta$  of  $F$  once and for all for the entirety of this paper.

Given a point  $\tilde{\Gamma} \in \tilde{\mathcal{F}}$  and an edge  $\alpha$  of  $\Delta$ , we can associate a positive real number as follows. Straighten  $\alpha$  to a geodesic for the hyperbolic metric underlying  $\tilde{\Gamma}$  and lift it to the metric universal cover of  $F$ . Since  $\alpha$  runs between punctures of  $F$ , this lift is asymptotic to a pair of parabolic fixed points of the Fuchsian group underlying  $\tilde{\Gamma}$ , and there is a well-defined horocycle centered at each of these points corresponding to the distinguished horocycles on  $F$ . Let  $\delta$  denote the signed hyperbolic distance between these two horocycles taken with a positive sign if and only if the horocycles are disjoint. Finally, define the  $\lambda$ -length of  $\alpha$  for  $\tilde{\Gamma}$  to be

$$\lambda(\alpha; \tilde{\Gamma}) = \sqrt{2 \exp(\delta)}.$$

In fact, the tuple of  $\lambda$ -lengths of edges of  $\Delta$  give global coordinates of  $\tilde{\Gamma}$  [Pe1, Theorem 3.1], and the pull-back to  $\tilde{\mathcal{F}}$  of the Weil-Petersson Kähler two-form  $\omega$  on  $\mathcal{F}$  under the forgetful projection is given by [Pe2, Theorem A.2]

$$(\dagger) \quad \tilde{\omega} = -2 \sum d \log a \wedge d \log b + d \log b \wedge d \log c + d \log c \wedge d \log a,$$

where the sum is over the component triangles  $T$  of  $F - \Delta$ , and the  $\lambda$ -lengths of the edges of  $T$  are  $a, b, c$  in a (counterclockwise) order compatible with the orientation on  $T$  induced from the orientation on  $F$ .

Dual to the ideal triangulation  $\Delta \subset F$  is a graph embedded in  $F$  together with some extra structure. Indeed, the formal Poincaré dual of  $\Delta$  is a one-dimensional CW-complex  $G$  embedded as a strong deformation retract of  $F$ ; furthermore, the orientation of  $F$  induces a cyclic ordering in the natural way on the three hooks incident on each vertex of  $G$ . A one-dimensional CW-complex together with this extra structure is called a *fatgraph*, and for our purposes, we may consider the fatgraph  $G = G(\Delta)$  associated to our fixed ideal triangulation  $\Delta$  as embedded in the fixed surface  $F$ . It is often convenient to imagine  $\lambda$ -lengths as defined on the edges of  $G$ , where the  $\lambda$ -length of an edge of  $G$  is simply the  $\lambda$ -length of its dual edge of  $\Delta$ .

We adopt the standard notation and terminology of [FLP] for measured foliations; let  $\mathcal{MF}_0 = \mathcal{MF}_0(F_g^s)$  denote the space of all equivalence classes of measured foliations with compact support in  $F$ , and let  $\mathcal{PF}_0 = \mathcal{PF}_0(F_g^s)$  denote the quotient of  $\mathcal{MF}_0$  under the natural action of the positive real numbers  $\mathbb{R}_+$ . Furthermore, let  $\mathcal{MF}_0^+ = \mathcal{MF}_0^+(F_g^s)$  denote  $\mathcal{MF}_0$  together with the empty foliation topologized so that a neighborhood of the added point is homeomorphic to the cone from the empty foliation over  $\mathcal{PF}_0$ . Adding the empty foliation to  $\mathcal{MF}_0$  amounts to adding the zero functional to the image of  $\mathcal{MF} - \{0\}$  in the function space  $\mathbb{R}_+^{\mathcal{F}}$  (using the notations of [FLP], which we recall in §5 below), and taking the topology induced from the weak topology on the function space. We also adopt the standard terminology and notation of [PH] for train tracks in  $F$  and ask the reader to recall the construction

(see [PH, Construction 1.7.7] for instance) of a measured foliation or measured lamination in  $F$  from a suitable measured train track in  $F$ .

To close this section, we give a short exposition of Thurston’s symplectic form on  $\mathcal{MF}_0$ . Recall (see [Ph, §7] for instance) that the piecewise-linear manifold  $\mathcal{MF}_0$  admits an atlas whose charts are associated to maximal trivalent train tracks in  $F$ . If  $\tau \subset F$  is such a train track, let  $V(\tau)$  denote the vector space of assignments of real numbers to the branches of  $\tau$  which satisfy the switch conditions, and let  $E(\tau) \subset V(\tau)$  denote the collection of all (nonnegative and not identically zero) measures on  $\tau$ . Thus,  $E(\tau)$  is the image of a chart in the atlas mentioned above and is identified with a subset of  $\mathcal{MF}_0$  in the usual way;  $V(\tau)$  is furthermore naturally identified with the tangent space to  $\mathcal{MF}_0$  at any point in the interior of  $E(\tau)$ .

Relative to these identifications, Thurston’s symplectic form on  $\mathcal{MF}_0$  is given by the pairing

$$t: V(\tau) \times V(\tau) \rightarrow \mathbb{R}$$

$$u \times v \mapsto \frac{1}{2} \sum \det \begin{pmatrix} u(a_\sigma) & u(b_\sigma) \\ v(a_\sigma) & v(b_\sigma) \end{pmatrix},$$

where the sum is over all the switches  $\sigma$  of  $\tau$ ,  $\det$  denotes the two-by-two determinant,  $a_\sigma$  and  $b_\sigma$  denote the two branches of  $\tau$  incident on  $\sigma$  whose one-sided tangents at  $\sigma$  point in the same direction, and, near  $\sigma$ ,  $a_\sigma$  ( $b_\sigma$  respectively) lies to the right (left respectively) of the tangent line to  $\tau$  at  $\sigma$  oriented so that the common one-sided tangent vector to  $a_\sigma$  and  $b_\sigma$  points in the positive direction. Thus, given tangent vectors  $u, v \in V(\tau)$  based at a point  $w$  interior to  $E(\tau)$ , we have

$$t(u, v) = \frac{1}{2} \left[ \sum dw(a_\sigma) \wedge dw(b_\sigma) \right] (u, v),$$

where the sum is as before. Invariance under change of chart (namely, invariance under splitting, shifting, and isotopy), nondegeneracy, and other properties of this form are discussed in [Pa1, Pa2, Pa4, part 2 and PH, §3.2].

## 2. THE DUAL NULL-GON TRACK AND DECORATED MEASURED FOLIATIONS

Associated to the ideal triangulation  $\Delta$ , we now define the *dual null-gon track*  $\tau = \tau(\Delta)$ , which is closely related to the fatgraph  $G$  dual to  $\Delta$ . In each component  $T$  of  $F - \Delta$ , the track  $\tau \cap T$  is as in Figure 1, so there is one triangle complementary to  $\tau$  for each triangle complementary to  $\Delta$ .  $\tau$  is a train track in the usual sense except that it has exactly  $s$  (recall that  $s$  is the number of punctures of  $F = F_g^s$ ) complementary components which are once-punctured null-gons (that is, once-punctured topological disks in  $F$  with  $\mathcal{C}^1$  smooth frontier); such a complementary region is prohibited in the usual theory. The reader will notice that the null-gon track  $\tau$  is derived from the fatgraph  $G$  by “blowing up” each vertex of  $G$  into a triangle in the natural way.

**Lemma 2.1.** *Every measured foliation of compact support is carried by the null-gon track  $\tau$ .*

*Proof.* It is convenient to consider laminations (see [PH, §§1.6 and 1.7] for instance) instead of foliations, and we may choose for the purposes of this

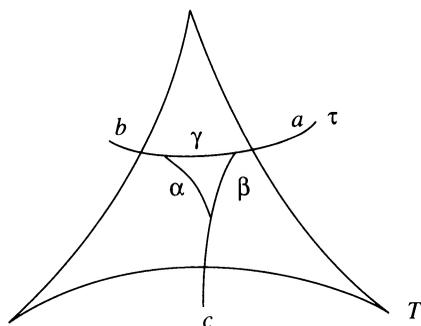


FIGURE 1



FIGURE 2

proof a hyperbolic structure on  $F$ . It suffices to show that any measured geodesic lamination  $L$  of compact support in  $F$  is carried by  $\tau$ . To see this, first observe that  $L$  must be transverse to (the geodesic representative of)  $\Delta$  in  $F$  since it has compact support. If  $T$  is a component of  $F - \Delta$ , it is therefore evident that  $L \cap T$  is carried by  $\tau \cap T$ , and one easily concludes that  $L$  itself is carried by  $\tau$ , as desired.  $\square$

The frontier of a punctured null-gon component of  $F - \tau$  is a puncture-parallel curve immersed in the track, and we call such a curve a *collar curve* of  $\tau$ ; there is one collar curve of  $\tau$  for each puncture of  $F$ . Since the homotopy class of a collar curve is independent of the choice of track  $\tau = \tau(\Delta)$ , we may simply refer to a curve in  $F$  which is homotopic to a collar curve of  $\tau$  as a collar curve in  $F_g^s$  itself. The assignment of a (not necessarily nonnegative) real number to each collar curve of  $\tau$  is called a *collar weight* on the track, and we may also refer simply to a collar weight on the surface itself.

Notice that the branches of  $\tau$  are of one of two types: a branch  $b$  of  $\tau$  either lies entirely inside some component of  $F - \Delta$  (and lies in the frontier of some triangle among the components of  $F - \tau$ ), or perhaps  $b$  meets some edge of  $\Delta$ . A branch of the former type is called *small* and of the latter type is called *large*; see Figure 2. A small branch is contained in exactly one collar curve, while a large branch may be contained in either one or two collar curves of  $\tau$ .

We refer to an assignment of (not necessarily nonnegative) real numbers to the edges of the null-gon track  $\tau$  as a *measure* on  $\tau$  provided that this assignment satisfies the switch conditions which are familiar from the train track theory, and we let  $V(\tau)$  denote the vector space of all measures on the

null-gon track  $\tau$ . Observe that  $\mu \in V(\tau)$  is uniquely determined by its values on the small branches alone, and the switch conditions are equivalent to a family of coupled linear constraints on these parameters. Explicitly, if  $c$  is a large branch whose closure contains the switches  $v_1 \neq v_2$ , and  $a_i$  and  $b_i$  are the small branches of  $\tau$  incident on  $v_i$  for  $i = 1, 2$ , then

$$\mu(a_1) + \mu(b_1) = \mu(c) = \mu(a_2) + \mu(b_2).$$

This constraint is called the *coupling equation* associated to the large branch  $c$ .

On the other hand, the values of the measure  $\mu$  on the large branches alone uniquely determine  $\mu$ ; in fact, these values of  $\mu$  give unconstrained coordinates on  $V(\tau)$ . Indeed, adopting the notation of Figure 1 for the various branches of  $\tau$  inside a component of  $F - \Delta$ , we find

$$\begin{aligned} \mu(\alpha) &= \frac{1}{2}\{\mu(b) + \mu(c) - \mu(a)\}, \\ (\dagger\dagger) \quad \mu(\beta) &= \frac{1}{2}\{\mu(a) + \mu(c) - \mu(b)\}, \\ \mu(\gamma) &= \frac{1}{2}\{\mu(a) + \mu(b) - \mu(c)\}. \end{aligned}$$

Now, if  $\mu$  is a nonnegative measure on  $\tau$ , then the usual construction of a (partial) measured foliation from  $(\tau, \mu)$  gives a well-defined equivalence class of measured foliations on  $F$ ; such a measured foliation will typically contain some collection of annuli foliated by collar curves of  $F$ , and deleting these foliated annuli then determines a well-defined (but possibly empty) class of measured foliation of compact support on  $F$ . Thus, a nonnegative measure on the null-gon track  $\tau$  canonically determines an element of  $\mathcal{MF}_0^+$  together with a nonnegative collar weight, where the weight associated to a collar curve is simply the total measure of a transverse arc connecting the boundary components of the corresponding foliated annulus.

More generally, suppose that  $\mu \in V(\tau)$  is not necessarily nonnegative, and let  $\alpha$  denote a collar curve of  $\tau$ . The set of switches of  $\tau$  decomposes  $\alpha$  into a collection of arcs, and the measure  $\mu$  associates to each such arc a real number; let  $\{c_1, \dots, c_n\}$  denote the collection of real numbers associated to the small branches of  $\tau$  that compose  $\alpha$ . We define the collar weight of  $\alpha$  to be

$$\mathcal{E}_\alpha = \min\{c_1, \dots, c_n\};$$

we make such an assignment of collar weight to each collar curve and so determine a collar weight on  $\tau$  itself.

Let us modify the original measure  $\mu$  on  $\tau$  by defining  $\mu'(b) = \mu(b) - \mathcal{E}_\alpha$  if  $b$  is contained in the collar curve  $\mathcal{E}_\alpha$  whenever  $b$  is a small branch of  $\tau$ . Since  $\mu$  satisfies the coupling equations,  $\mu'$  extends uniquely to a well-defined measure on  $\tau$ . Furthermore, the measure  $\mu' \in V(\tau)$  derived in this way from  $\mu$  is nonnegative, and the measured foliation associated, as before, to  $\mu'$  has identically vanishing collar weights, so the corresponding measured foliation is of compact support.

The discussion above determines a map  $\Pi: V(\tau) \rightarrow \mathcal{MF}_0^+$ , and it is easy to check that  $\Pi$  is continuous; moreover,  $\Pi$  is surjective by Lemma 2.1. Furthermore, we have found that a measure  $\mu \in V(\tau)$  determines both a (possibly empty) measured foliation  $\Pi(\mu) \in \mathcal{MF}_0^+$  and a collar weight on  $F$ ; we are led to define the space  $\widetilde{\mathcal{MF}}_0 = \widetilde{\mathcal{MF}}_0(F_g^s)$  of *decorated measured foliations* as

the space of pairs  $(\mathcal{F}, \mathcal{C})$ , where  $\mathcal{F} \in \mathcal{MF}_0^+$  and  $\mathcal{C}$  is a collar weight on  $F_g^s$ .

We summarize our discussion so far in

**Proposition 2.2.** *The space  $V(\tau)$  gives global coordinates on  $\widetilde{\mathcal{MF}}_0$ , and there is a canonical fiber bundle  $\Pi: \widetilde{\mathcal{MF}}_0 \rightarrow \mathcal{MF}_0^+$ , where the fiber above a point is given by the set of all collar weights on  $F$ .*

There is a well-defined diagonal action of the full mapping class group  $MC = MC(F_g^s)$  of  $F$  on  $\widetilde{\mathcal{MF}}_0$ , where the action on collar weights is induced by the permutation of punctures, and  $MC$  acts in this way as a group of bundle isomorphisms of  $\Pi$ . Furthermore, the bundle admits an  $MC$ -equivariant section  $\sigma: \mathcal{MF}_0^+ \rightarrow \widetilde{\mathcal{MF}}_0$  determined by the condition that each decorated measured foliation in the image has identically vanishing collar weights. The restriction  $\sigma$  of this section of  $\mathcal{MF}_0 \subset \mathcal{MF}_0^+$  may be thought of as a piecewise-linear embedding of the piecewise-linear manifold  $\mathcal{MF}_0$  into the linear manifold (vector-space)  $\widetilde{\mathcal{MF}}_0$ .

### 3. THE PAIRING ON DECORATED MEASURED FOLIATIONS

We identify  $\widetilde{\mathcal{MF}}_0$  with  $V(\tau)$  and define a differential two-form  $\iota$  in coordinates on  $\widetilde{\mathcal{MF}}_0$  as follows. If  $\mu \in V(\tau)$ , then we define

$$(iii) \quad \iota = -\frac{1}{2} \sum d\xi \wedge d\eta + d\eta \wedge d\zeta + d\zeta \wedge d\xi,$$

where the sum is over all triangles  $T$  complementary to  $\tau$  in  $F$ , and  $\xi, \eta$ , and  $\zeta$ , respectively, denote the  $\mu$ -values of the three small edges of  $\tau$  in the frontier of  $T$  in a (counterclockwise) order compatible with the orientation of  $T$  induced from the orientation of  $F$ .

To give a topological interpretation of  $\iota$ , fix a tie-neighborhood  $N(\tau)$  of  $\tau$  in  $F$ , and let  $u$  and  $v$  be (nonzero) tangent vectors to  $\mathcal{MF}_0$  based at a point interior to  $E(\tau)$ ; as before, we may regard  $u, v \in V(\tau)$ . Define a (null-gon) track  $\tau_u$  by removing the branches of  $\tau$  on which  $u$  vanishes (amalgamating into a single branch whenever a pair of branches is incident on a resulting bivalent vertex);  $u$  gives rise to an element of  $V(\tau_u)$  in the natural way. Similarly, derive the measured track  $\tau_v$  from  $v$ . Embed  $\tau_u$  and  $\tau_v$  in  $N(\tau)$  in such a way that these tracks are in general position with respect to one another and are furthermore transverse to the ties of  $N(\tau)$ . Arguing in analogy to [Pa2, §3], one finds that

$$\iota(u, v) = \sum \alpha_p(u, v),$$

where the sum is over all points  $p$  of  $\tau_u \cap \tau_v$ , and if  $p$  is contained in the branch  $b$  (respectively  $c$ ) of  $\tau_v$  (respectively  $\tau_u$ ), then  $\alpha_p(u, v) = \pm u(b)v(c)$ . To compute the sign of  $\alpha_p(u, v)$ , consider an open neighborhood  $U$  of  $p$  in  $N(\tau)$ . By construction, there are exactly two (opposite) distinguished components of  $U - (\tau_u \cup \tau_v)$  which meet the tie through  $p$ ; choose one such component  $U_*$ ; see Figure 3 (next page). Now consider small tangent vectors  $\nu_u$  (and  $\nu_v$  respectively) to  $\tau_u$  (and  $\tau_v$  respectively) at  $p$  whose exponentials lie in the frontier of  $U_*$ . The sign of  $\alpha_p(u, v)$  is taken to be positive if  $(\nu_u, \nu_v)$  is a

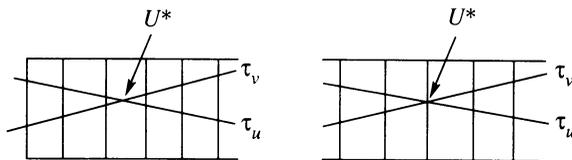


FIGURE 3

positive basis for the tangent plane to  $F$  at  $p$  and negative otherwise. Notice that this specification of sign is independent of the choice of distinguished component  $U_*$ .

*Remark.* Actually,  $\iota$  is induced locally by homology intersection numbers of cycles on a two-fold branched cover of  $F$ ; see [Pa4, Appendix 2] for details.

**Lemma 3.1.** *Suppose that  $u, v \in V(\tau)$  and  $\Pi(u) \in \mathcal{M}\mathcal{F}_0^+$  is the empty foliation. Then  $\iota(u, v) = 0$ .*

*Proof.* Since  $\Pi(u)$  is the empty foliation, the measured track  $\tau_u$  splits to a collection of weighted collar curves in  $F$ . Just as for  $t$ , one sees easily that  $\iota$  is invariant under splitting, shifting, and isotopy, and hence we may take  $\tau_u$  disjoint from  $\tau_v$ . By the topological interpretation of  $\iota$  given above, it follows that  $\iota(u, v) = 0$ , as was asserted.  $\square$

**Proposition 3.2.** *Thurston's symplectic form  $t$  on  $\mathcal{M}\mathcal{F}_0$  is the pull back  $\sigma^*(\iota)$  if  $\iota$  on  $\widetilde{\mathcal{M}\mathcal{F}_0}$ .*

*Proof.* To begin, we consider a family of train tracks in  $F$  derived from the null-gon track  $\tau$  as follows. Fix attention on some collar curve  $\gamma$  of  $\tau$ . This curve is the boundary of a null-gon component of the null-gon track. On this curve  $\gamma$ , there are a certain number of small branches of  $\tau$ . Remove from  $\tau$  exactly one small branch contained in  $\gamma$  (amalgamating branches as before), as indicated in Figure 4. Thus, the punctured null-gon component of  $F - \tau$  corresponding to  $\gamma$  becomes a once-punctured mono-gon complementary component of the resulting null-gon track. Performing this operation once on each collar curve of  $\tau$  produces an honest train track in  $F$ . Performing these operations in all possible ways for the various choices of one small branch from each collar curve of  $\tau$  produces a family of maximal train tracks  $\{\tau_k\}_1^K$  in  $F$ . Each  $\tau_k$  determines a cell  $E(\tau_k) \subset \mathcal{M}\mathcal{F}_0$  in the usual way. It follows directly from Lemma 2.1 that the union of the cells  $\bigcup_{k=1}^K E(\tau_k)$  covers  $\mathcal{M}\mathcal{F}_0$ .

By Lemma 3.1,  $t$  agrees with  $\sigma^*(\iota)$  on the union of the interiors of the  $E(\tau_k)$ . To finish the argument, we must show that for any  $\mathcal{F} \in \mathcal{M}\mathcal{F}_0$ , there is an ideal triangulation  $\Delta'$  and a train track  $\tau'$  derived as above from the null-gon track  $\tau(\Delta')$  with  $\mathcal{F}$  in the interior of  $E(\tau')$ . To see this, choose a pseudo-Anosov map (see [FLP] for instance)  $\psi: F \rightarrow F$  whose unstable foliation lies in the interior of  $E(\tau_k)$  for some  $k = 1, \dots, K$  (and so that  $\mathcal{F}$  is not in the class of the stable foliation of  $\psi$ ). By well-known properties (see [FLP] for instance) of the dynamics of the action of pseudo-Anosov maps on  $\mathcal{M}\mathcal{F}_0$ , there is some  $N$  (depending on  $\mathcal{F}$ ), so that  $\psi^N(\mathcal{F})$  lies interior to  $E(\tau_k)$ . Finally, take  $\tau' = \psi^{-N}(\tau_k)$ , which is a train track associated to the ideal triangulation  $\Delta' = \psi^{-N}(\Delta)$ , as desired.  $\square$

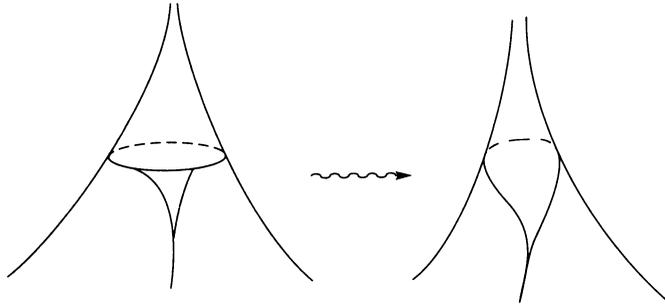


FIGURE 4

It is remarkable that Thurston’s symplectic form is actually the pull back of a smooth differential form under a piecewise-linear embedding.

4. THE HOMEOMORPHISM BETWEEN DECORATED SPACES

To begin, we recall [Th] Thurston’s parametrization

$$\mathcal{F}_\Delta: \mathcal{T} \rightarrow \mathcal{M}\mathcal{F}_0^+$$

of  $\mathcal{T}$  by horocyclic foliations. Suppose that  $g$  is the hyperbolic metric on  $F$  associated to some marked Riemann surface in  $\mathcal{T}$ . Define a measured foliation  $\mathcal{F}'_\Delta(g)$  of  $F$  as follows. Each component  $T$  of  $F - \Delta$  (with its metric induced from  $g$ ) is isometric to an ideal triangle, and we consider the three pencils of horocycles centered at the ideal vertices of  $T$ . There is a unique foliation of  $T$  whose leaves are segments of these horocycles and so that the complement of the support of the foliation is a small triangular region inside  $T$  whose boundary consists of three such horocyclic segments meeting tangentially at the frontier of  $T$ ; see Figure 5 (next page). These foliations of the components of  $F - \Delta$  combine in the natural way to produce a foliation  $\mathcal{F}'_\Delta(g)$  of  $F$  itself.

Of course,  $\mathcal{F}'_\Delta(g)$  is not compactly supported in  $F$ , but there is a well-defined compactly supported partial sub-measured foliation  $\mathcal{F}_\Delta(g)$  which is obtained from  $\mathcal{F}'_\Delta(g)$  by simply deleting all of the closed leaves of  $\mathcal{F}'_\Delta(g)$  which are puncture-parallel. Thurston proves [Th, §9] that the map  $\mathcal{F}_\Delta$  is in fact a homeomorphism. (This also follows from the fact that  $\lambda$ -lengths are coordinates on  $\tilde{\mathcal{T}}$ ; however, Thurston’s argument includes the more general case of arbitrary maximal laminations, not just ideal triangulations.)

*Remark.* In this construction, there is a leaf of  $\mathcal{F}'_\Delta(g) - \mathcal{F}_\Delta(g)$  of greatest hyperbolic length about each puncture, and we may think of this as giving a section of the decorated bundle  $\tilde{\mathcal{T}} \rightarrow \mathcal{T}$ . In the case of once-punctured surfaces, this section corresponds to taking the largest embedded horoball about the puncture. Also, notice that the empty foliation corresponds to the “center of the cell  $\mathcal{E}(\Delta)$ ,” which is an arithmetic Riemann surface; see [Pe1, §6].

From the definitions, we observe that the total transverse length with respect to  $\mathcal{F}_\Delta(g)$  of an edge of  $\Delta$  is just the  $g$ -length of a subarc of the edge running between certain horocycles. Recalling the definition of  $\lambda$ -lengths, we are led to define a map

$$\begin{aligned} \tilde{\mathcal{T}} &\rightarrow V(\tau) \\ \tilde{\Gamma} &\mapsto \mu, \end{aligned}$$

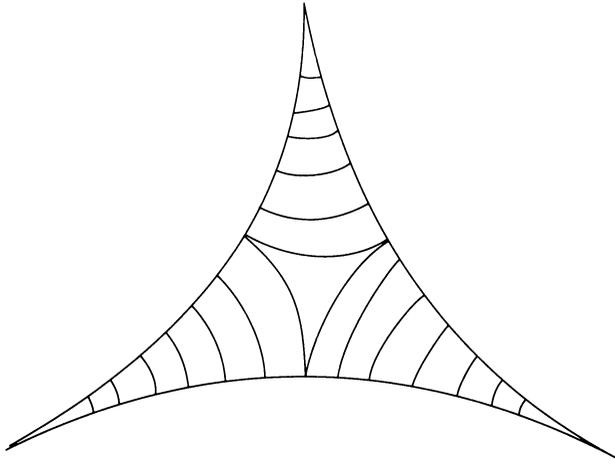


FIGURE 5

where  $\mu(b) = 2 \log \lambda(b^* ; \tilde{\Gamma})$  for each large branch of  $\tau$  with dual edge  $b^*$  in  $\Delta$ . Here we are tacitly using the fact observed before that the values of a measure on the large branches of  $\tau$  alone uniquely determine the measure.

*Remark.* The reader may notice the disappearance of a summand  $-\log 2$  from the natural definition of the measure on  $\tau$  associated to an element of  $\tilde{\mathcal{F}}$ ; here we are simply changing the lengths of all the horocycles by an overall factor  $\sqrt{2}$  to simplify computations. Furthermore, the reader familiar with the decorated Teichmüller theory will observe that the values of  $\mu$  on the small branches are simply the logarithms of corresponding “h-lengths.”

Identifying  $V(\tau)$  as before with  $\tilde{\mathcal{M}}\mathcal{F}_0$ , we have thus described a map  $\tilde{\mathcal{F}}_\Delta: \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{M}}\mathcal{F}_0$  which covers Thurston’s map  $\mathcal{F}_\Delta$  discussed above.

**Proposition 4.1.** *The homeomorphism  $\tilde{\mathcal{F}}_\Delta$  preserves the natural two-forms (where on the domain, the two-form is the pull back  $\tilde{\omega}$  of the Weil-Petersson Kähler two-forms  $\omega$  on  $\mathcal{F}$ , and on the range, the two-form is the form  $\iota$ ).*

*Proof.* It suffices to consider separately the contributions to the two-forms from a single component  $T$  of  $F - \Delta$ , and we adopt the notation of Figure 1 for the branches of  $\tau \cap T$ . Let us also adopt the convention that  $\lambda(x) = \lambda(x; \tilde{\Gamma})$  denotes the  $\lambda$ -length of the edge of  $\Delta$  dual to the large branch  $x$  of  $\tau$  for  $\tilde{\Gamma} \in \tilde{\mathcal{F}}$ , and set  $\tilde{\lambda}(x) = \text{dlog} \lambda(x)$  for convenience. Equations (††) and (†††) give that the contribution to  $\iota$  corresponding to  $T$  is

$$\begin{aligned} & -\frac{1}{2}\{[\tilde{\lambda}(b) + \tilde{\lambda}(c) - \tilde{\lambda}(a)] \wedge [\tilde{\lambda}(a) + \tilde{\lambda}(c) - \tilde{\lambda}(b)] \\ & \quad + [\tilde{\lambda}(a) + \tilde{\lambda}(c) - \tilde{\lambda}(b)] \wedge [\tilde{\lambda}(a) + \tilde{\lambda}(b) - \tilde{\lambda}(c)] \\ & \quad + [\tilde{\lambda}(a) + \tilde{\lambda}(b) - \tilde{\lambda}(c)] \wedge [\tilde{\lambda}(b) + \tilde{\lambda}(c) - \tilde{\lambda}(a)]\} \\ & = -2[\tilde{\lambda}(a) \wedge \tilde{\lambda}(b) + \tilde{\lambda}(b) \wedge \tilde{\lambda}(c) + \tilde{\lambda}(c) \wedge \tilde{\lambda}(a)], \end{aligned}$$

which is the contribution  $\tilde{\omega}$  corresponding to  $T$  according to equation (†).  $\square$

**Corollary 4.2.** *The homeomorphism  $\mathcal{F}_\Delta$  pulls back Thurston’s symplectic pairing on  $\mathcal{M}\mathcal{F}_0$  to the Weil-Petersson Kähler two-form  $\omega$  restricted to  $\mathcal{F} - \mathcal{F}_\Delta^{-1}$  (empty foliation).*

5. THE EXTENSION OF THE TWO-FORM

Let  $\mathcal{Y} = \mathcal{Y}(F_g^s)$  denote the Yamabe space of all complete finite-area metrics on  $F$  of constant Gaussian curvature, modulo push-forward by orientation-preserving diffeomorphisms of  $F$  which are isotopic to the identity. Each class of metrics in  $\mathcal{Y}$  is represented by the scalar multiple of an underlying conformal hyperbolic metric of constant curvature  $-1$ . Thus,  $\mathcal{Y}$  is canonically homeomorphic to  $\mathcal{T} \times ]0, \infty[$ , where, if  $g$  is the hyperbolic metric representing an element of  $\mathcal{T}$ , then  $(g, x) \in \mathcal{T} \times ]0, \infty[$  corresponds to the class of the metric  $xg$  (which has constant Gaussian curvature  $-x^2$ ). We henceforth identify  $\mathcal{Y}$  with  $\mathcal{T} \times ]0, \infty[$  in this way and let  $\pi: \mathcal{Y} \rightarrow \mathcal{T}$  denote the map induced by projection  $(g, x) \mapsto g$  onto the first factor.

Following [FLP], we let  $\mathcal{S} = \mathcal{S}(F_g^s)$  denote the set of all isotopy classes of simple closed curves in  $F$  which are neither null homotopic nor puncture-parallel, and define a map

$$I: \mathcal{Y} \rightarrow \mathbb{R}_+^{\mathcal{S}}$$

$$g \mapsto l_g(\cdot),$$

where  $l_g(\chi)$  denotes the  $g$ -length of the  $g$ -geodesic in the homotopy class  $\chi \in \mathcal{S}$  (note that  $\mathbb{R}_+$  denotes here the set of nonnegative real numbers). Thus, if  $g \in \mathcal{Y}$  corresponds to  $(\pi(g), x) \in \mathcal{T} \times ]0, \infty[$ , then  $I(g) = I((\pi(g), x)) = xI((\pi(g), 1))$ . Of course, one also has a map

$$J: \mathcal{M}\mathcal{F}_0^+ \rightarrow \mathbb{R}_+^{\mathcal{S}}$$

$$\mathcal{F} \mapsto i(\mathcal{F}, \cdot),$$

where  $i(\mathcal{F}, \chi)$  denotes the geometric intersection number of  $\mathcal{F}$  and  $\chi \in \mathcal{S}$ . The image of the empty foliation is, by definition, equal to the zero-functional.

The fundamental facts [FLP, Exposé 8, §2] are that each of  $I$  and  $J$  are embeddings with disjoint images, and a “convergence criterion” (which we will not need) is given in order that the image under  $I$  of a sequence in  $\mathcal{Y}$  converge to an element in the image of  $J$ . Define a “completion”  $\overline{\mathcal{Y}}$  of the Yamabe space  $\mathcal{Y}$  in  $\mathbb{R}_+^{\mathcal{S}}$  by setting

$$\overline{\mathcal{Y}} = I(\mathcal{Y}) \cup J(\mathcal{M}\mathcal{F}_0^+)$$

and identifying  $\mathcal{Y}$  with  $I(\mathcal{Y})$ . In fact, passing to the quotients by the respective homothetic actions of  $\mathbb{R}_+$  on  $\mathbb{R}_+^{\mathcal{S}} - \{0\}$ ,  $\mathcal{Y}$  and  $\mathcal{M}\mathcal{F}_0$ , one obtains Thurston’s compactification  $\mathcal{T} \cup \mathcal{P}\mathcal{F}_0 \approx \overline{\mathcal{Y}} - \{0\} / \mathbb{R}_+$  of  $\mathcal{T} \approx \mathcal{Y} / \mathbb{R}_+$  by  $\mathcal{P}\mathcal{F}_0 \approx \mathcal{M}\mathcal{F}_0 / \mathbb{R}_+$ .

We will rely below on an alternative convergence criterion (see [Pa3, Theorem 3.7]), as follows.

**Convergence criterion.** Fix an ideal triangulation  $\Delta$  of  $F$ , and suppose that  $(g_n, x_n)$  is a sequence in  $\mathcal{Y} \approx \mathcal{T} \times ]0, \infty[$  so that  $g_n$  is eventually disjoint from any compactum in  $\mathcal{T}$  and  $x_n$  tends to zero. then  $I((g_n, x_n))$  converges to  $J(\mathcal{F})$  in the topology of  $\mathbb{R}_+^{\mathcal{S}}$  if and only if the sequence  $x_n \mathcal{F}_\Delta(g_n)$  converges to  $\mathcal{F}$  in the topology of  $\mathcal{M}\mathcal{F}_0$ .

Now, define a map

$$h: \mathcal{Y} \rightarrow \mathcal{M}\mathcal{F}_0^+ \times [0, \infty[$$

$$(g, x) \mapsto (x\mathcal{F}_\Delta(g), x).$$

According to Thurston's horocyclic coordinatization of  $\mathcal{T}$ , the map  $h$  is an injection with image  $\mathcal{M}\mathcal{F}_0^+ \times ]0, \infty[$ , and by the convergence criterion above,  $h$  extends to a homeomorphism

$$\bar{h}: \overline{\mathcal{Y}} \rightarrow \mathcal{M}\mathcal{F}_0^+ \times [0, \infty[ ,$$

where the image of the set of ideal points  $\overline{\mathcal{Y}} - \mathcal{Y}$  is the subset  $\mathcal{M}\mathcal{F}_0^+ \times \{0\} \subset \mathcal{M}\mathcal{F}_0^+ \times [0, \infty[$ . In fact, if we let  $K: \mathcal{M}\mathcal{F}_0^+ \subset \mathcal{M}\mathcal{F}_0^+ \times [0, \infty[$  denote the natural inclusion identifying  $\mathcal{M}\mathcal{F}_0^+$  with  $\mathcal{M}\mathcal{F}_0^+ \times \{0\}$  and regard the embedding  $J$  above as an inclusion  $J: \mathcal{M}\mathcal{F}_0^+ \subset \overline{\mathcal{Y}}$ , then the convergence criterion moreover implies that  $K = \bar{h} \circ J$ .

Recall that  $\omega$  denotes the Weil-Petersson Kähler two-form on  $\mathcal{T}$ , and define a two-form

$$\omega' = x^{-2}\pi^*(\omega)$$

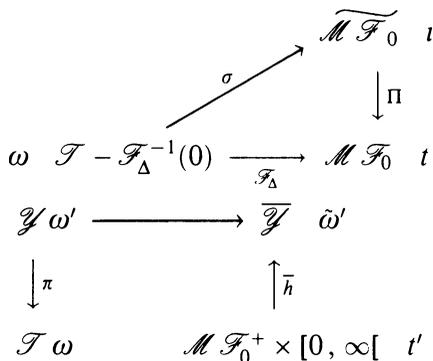
on  $\mathcal{Y}$ . Also, let  $t'$  denote the pull back of Thurston's symplectic form on  $\mathcal{M}\mathcal{F}_0$  to  $\mathcal{M}\mathcal{F}_0 \times ]0, \infty[$  under the projection onto the first factor. According to Corollary 4.2, the homeomorphism  $h$  respects the forms  $\omega'$  and  $t'$ .

Insofar as  $t'$  extends continuously to  $\mathcal{M}\mathcal{F}_0 \times [0, \infty[$  (again by pulling back  $t$  under projection onto the first factor), we find that  $\omega'$  extends continuously to a two-form  $\tilde{\omega}'$  on  $\overline{\mathcal{Y}}$  whose restriction  $\overline{\mathcal{Y}} - \mathcal{Y} - \{0\} \approx \mathcal{M}\mathcal{F}_0$  is exactly Thurston's symplectic pairing.

We summarize with

**Theorem 5.1.** *The Weil-Petersson Kähler two-form  $\omega$  on  $\mathcal{T}$  induces a two-form  $\omega'$  on  $\mathcal{Y}$  which extends continuously to  $\overline{\mathcal{Y}} \approx \mathcal{Y} \cup \mathcal{M}\mathcal{F}_0$ . This extension restricts to Thurston's symplectic form  $t$  on  $\mathcal{M}\mathcal{F}_0 \approx \overline{\mathcal{Y}} - \mathcal{Y} - \{0\}$ .*

For the purpose of helping the reader, we recollect in the following two diagrams the space that we were involved in, and the forms which were defined upon them.



6. CONCLUDING REMARKS

To close, we briefly discuss some open problems which arise from the investigations of this note. Of course, one might hope to prove a result analogous to Theorem 5.1 for closed surfaces using Thurston's general horocyclic foliation coordinates.

Given the connections that have arisen here between the decorated Teichmüller theory and the Thurston theory, the translation of ideas and constructions from one theory to the other should be fruitful.

For instance, a Whitehead move on an ideal triangulation  $\Delta$  corresponds to a composition of a split and a collapse (that is, the inverse of a split) on the dual null-gon track  $\tau(\Delta)$ . The effect of a Whitehead move on  $\lambda$ -lengths is given by a ‘‘Ptolemy transformation,’’ and one can derive a representation of the action of the mapping class group  $MC$  (see [Pe1, §7]) on  $\lambda$ -length coordinates for  $\tilde{\mathcal{F}}$  with respect to  $\Delta$  as a group of tuples of rational maps. Our identification here of  $\tilde{\mathcal{F}}$  with  $\tilde{\mathcal{M}\mathcal{F}}_0$  suggests that the determination of the type (namely, pseudo-Anosov, periodic, or reducible) could perhaps be discerned directly from the representing rational maps. A more enterprising project would be to discover the invariant projective foliations or the dilatation for pseudo-Anosov maps directly from the representing rational maps. This is in contrast to known solutions to these problems (see [Be] for instance) which take the form of algorithms which are shown to terminate.

Other problems which seem interesting are to relate the  $MC$ -invariant cell decomposition of  $\tilde{\mathcal{F}}$  (see [Pe1, §5]) to the train track polyhedral cover of  $\mathcal{P}\mathcal{F}_0$ , and to discover a sense in which the space of projective classes of decorated measured foliations provides a Thurston-like compactification of  $\tilde{\mathcal{F}}$  itself. We wonder also whether the train track theory might be useful in solving the fundamental arithmetic problems (see [Pe2, §7]) in the decorated Teichmüller theory and ask for a geometric interpretation of the ‘‘simplicial coordinates’’ in terms of train tracks. One also observes that cohomology classes (for instance, the Morita-Mumford classes [Mo]) on the moduli space  $\mathcal{F}/MC$  give rise to  $MC$ -invariant structures on  $\mathcal{M}\mathcal{F}_0$ , and we ask for topological interpretations of these invariant structures in analogy to the interpretation given here of the Weil-Petersson Kähler structure (which is essentially the first Morita-Mumford class) as induced by homology intersection numbers of cycles on the surface. Finally, it would be interesting to analyze the section of the decorated bundle  $\tilde{\mathcal{F}} \rightarrow \mathcal{F}$  which is associated to Thurston’s map  $\mathcal{F}_\Delta$  (see the first remark in §4); computing this section in  $\lambda$ -lengths could be interesting and might simplify certain computations in the decorated Teichmüller theory.

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