SUPERCUSPIDAL REPRESENTATIONS
AND THE THETA CORRESPONDENCE. II:
SL(2) AND THE ANISOTROPIC O(3)

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Abstract. A parametrization is given of the local theta correspondence attached to the reductive dual pair \((\text{SL}_2(F), O(F))\) where \(F\) is a nonarchimedean local field of odd residual characteristic and \(O\) is the orthogonal group of a ternary quadratic form which is anisotropic over \(F\). The parametrization is in terms of inducing data. Various lattice models of the oscillator representation are used.

In this paper we examine Howe's local theta correspondence for the reductive dual pair \((\text{SL}_2(F), O(F))\) where \(F\) is a nonarchimedean local field of odd residual characteristic and \(O\) is the orthogonal group of a quadratic form in three variables which is anisotropic over \(F\). In particular we determine if a representation of \(O(F)\) occurs in the theta correspondence and determine the corresponding representation if that representation is supercuspidal. The determination is in terms of inducing data from compact open subgroups.

The correspondence we study here has been studied by Rallis and Schiffman [RS] and Waldspurger [W] among others. What is new here is that we provide an explicit parametrization of the correspondence. The two principal techniques used here are the lattice model of the oscillator (Weil) representation and the parametrization of supercuspidal representations via induction from compact open subgroups. These two techniques were also the primary techniques in our paper [M1] to which this paper is a sequel (also see [M2] where we treat the split case). In future sequels, we plan to further evidence the power of using these techniques in tandem to study the theta correspondence.

This paper is organized as follows. First, as this paper is a sequel we rely heavily on the notation, motivation, and results of [M1]. Assuming this material (while also providing references), the first section of this paper is devoted to a brief recounting of the theory of the irreducible admissible representations of \(O(F)\). Since this group is isomorphic to the direct product of the projectivization of the multiplicative group of the nonsplit quaternion algebra over \(F\) and \(\mathbb{Z}/2\mathbb{Z}\) this is straightforward if not well known. We present it in a manner suitable for our parametrization of the correspondence. In §2 of the paper we explicitly parametrize the correspondence in terms of inducing data for the representations of \(O(F)\) and the supercuspidal representations of the nontrivial

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two-fold cover of $SL_2(F)$. The key here is the appropriate choices of various lattice models and the methods and results of our paper [M1].

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1. SOME PARAMETERS

In this section we establish notation and parametrize the admissible dual of the orthogonal group associated to an anisotropic ternary quadratic form over a $p$-adic field. Since most of this material is known or easily derived from the literature we will be quite brief in our discussion. For unexplained terminology or notation see [M1].

Let $F$ be a nonarchimedean local field of residual characteristic $p$ with $p$ odd. Let $O = O_F$ be the ring of integers of $F$ and let $\omega = \omega_F$ be a generator of the maximal ideal $P = P_F$ in $O$. Let $k = k_F$ denote the residue class field $O/P$ and let $q = q_F$ be the cardinality of $k$. Finally let $\nu(x) = \nu_F(x)$ denote the order of an element $x$ in $F$ and normalize the absolute value $| \cdot |$ on $F$ so that $|\omega_F| = q^{-1}$.

Let $V_2$ be a three-dimensional vector space equipped with a symmetric non-degenerate bilinear form $\langle , \rangle_2$ which does not (nontrivially) represent zero. Then we may identify $V_2$ and $\langle , \rangle_2$ with the (reduced) trace zero elements, $D^0$ say, in a nonsplit quaternion algebra, $D$ say, over $F$ equipped with the nondegenerate symmetric bilinear form defined by the (reduced) norm $N_{D/F}$ (see [D, p. 57]); let $\text{tr}_{D/F}$ denote the trace map from $D$ to $F$ and let $\sigma_D$ denote the involution on $D$ such that $N_{D/F}(x) = x\sigma_D(x)$ and $\text{tr}_{D/F}(x) = x + \sigma_D(x)$. Also let $O = O_D$ denote the ring of integers in $D$, let $P = P_D$ denote the prime ideal in $O_D$ and set $k = k_D = O_D/P_D$. Let $v_D(x)$ denote the order of an element $x$ in $D$ and normalize the absolute value $| \cdot |_D$ on $D$ so that $|x|_D = q^{-v_D(x)}$.

Now if $E$ is a quadratic subfield of $D$, then we may realize $D$ as the cyclic algebra $(E/F, \sigma, a)$ where $\sigma$ is the nontrivial element of the Galois group $\Gamma(E/F)$ of $E/F$ and $a$ is an element of $F^\times$ which is not in the image of the norm map $N_{E/F}$ from $E$ to $F$ (see, e.g., [R, §31]). In particular given a generator $x$ for $E/F$ there exists an element in $z$ in $D^\times$ such that $zz^{-1} = \sigma(x)$ and $z^2 = a$. Note that if, in addition, $N_{E/F}(x)$ is not a square in $F^\times$, then $D^\times$ is generated by $D^1$, $x$, and $z$ where $D^1$ is the subgroup of $D^\times$ consisting of elements of norm one.

Let $\sigma_2$ denote the involution of $AF(D^0) (= \text{End}_F(D^0)$ as in [M1]) associated to $\langle , \rangle_2$. If we let an element $x$ in $D^\times$ act on $D^0$ via conjugation, then the associated element $d_x$ say in $AF(D^0)$ is in $G_2$—the isometry group of $D^0$ with respect to $\langle , \rangle_2$. Then if we set $PD^1 = D^1/\{\pm 1\}$ we may identify $PD^1$ with a subgroup of $G_2$; in fact with $E = F[x]$ and $z$ as above with $\text{tr}x = 0$ we have $G_2 = (PD^1 \times \langle d_x, d_z \rangle) \times \langle \sigma_D \rangle$ where $\langle \sigma_D \rangle$ denotes the subgroup of $G_2$ generated by $\sigma_D$ (restricted to $D^0$ where it is just $-I$). Note that $\langle d_x, d_z \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\langle \sigma_D \rangle \cong \mathbb{Z}/2\mathbb{Z}$. In what follows we will not distinguish between an element of $D^1$ and its image in $PD^1$ when considering $G_2$; the resultant ambiguity in notation will either not matter or be resolvable from context.

Now consider the Lie algebra $Af(D^0)$ of $G_2$. For $y$ in $D^0$, let $ady$ denote the $F$-endomorphism of $D$ defined by $x \mapsto yx - xy$. Then the restriction
of \( ady \) maps \( D^0 \) into \( D^0 \) and we may identify (as is well known or easily checked) \( D^0 \) with \( A(D^0)_- \) via the map \( y \mapsto ady \). Now let \( A^*(D^0) \) denote the set of elements \( x \) in \( A(D^0) \) such that \( 1 + x \) is invertible; it is straightforward to check that \( A^*(D^0) \) contains \( A(D^0)_- \). Then recall that the Cayley transform is the well-defined map \( c_A = c \) from \( A^*(D^0) \) to itself defined by \( c(x) = (1-x)(1+x)^{-1} \). Recall also that \( c \) maps \( A(D^0)_- \) into \( G_2 \) injectively with inverse \( c \) itself.

Now recall that a lattice chain \( L = \{L_i\} \) in \( D^0 \) is said to be self-dual if for each \( i \) the dual lattice \( L_i^* = \{x \in D^0|^\langle x, l \rangle \in \mathcal{O} \ \forall l \in L\} \) to \( L_i \) is in \( L \). Recall further that the only self-dual lattice chain in \( V_2 \) (up to equivalence) is the chain \( L^D = \{L_i^D = P_i^D\}_i \in \mathcal{Z} \) so that the only \( \sigma_2 \)-stable hereditary order in \( A_F(D^0) \) is \( \mathcal{A}(L^D) \) (see, for example [Mo2, §1]). Since the associated hereditary order is unique, we set \( \mathcal{A}(D^0) = \mathcal{A}(L^D) \) and similarly define \( \mathcal{P}(D^0) \), \( \mathcal{P}^+_1(D^0) \), \( \mathcal{P}_1(D^0) \), and \( U^n(D^0) \) for \( n \) positive. With the identifications, we have \( \mathcal{P}^+_1(D^0) = P^+_1 \), where \( P^+_1 = P^+_D \cap D^0 \), \( U^n(D^0) = U^n_B \) for \( n \) a positive integer where \( U^n_B = \{x \in D^0|x-1 \text{ is in } P^n\} \) and \( U^0(D^0) = U(D^0) = G_2 \). Finally, we note that the nonstandard filtrations constructed by Morris [Mo2, §2] collapse in this case to the standard filtration by powers of the radical.

**Lemma 1.1.** The Cayley transform yields a bijection between \( P^0_- \) and \( PD^1 \). Further, for \( n \) a positive integer this bijection takes \( P^n_- \) to \( U^n_B \).

**Proof.** If not well known, this is certainly straightforward.

Now let \( D^* = D - \{-1\} \) and let \( c_D: D^* \to D^* \) be the map defined by \( y \mapsto (1-y)(1+y)^{-1} \). Then the following two lemmas are also straightforward.

**Lemma 1.2.** With notation as above,

(i) \( c_D \) is a bijection onto \( D^* \) with inverse \( c_D \) itself.

(ii) The restriction of \( c_D \) to \( D^0 \) yields a bijection onto \( D^* \cap D^0 \).

(iii) For \( n \) a positive integer, the restriction of \( c_D \) to \( P^n_- \) yields a bijection onto \( U^n_B \).

**Lemma 1.3.** Let \( y \) be an element of \( D^0 \). Then \( c_A(y/2)^2 = c_D(y) \).

We now turn to the representation theory of \( G_2 \). First, since \( G_2 \) is compact, its irreducible representations are finite dimensional, admissible, and supercuspidal. Second, since \( G_2 \) is \( PD^\times \times (\sigma_D) \) a parametrization of its (admissible) dual can be easily derived from known parametrizations of the admissible dual of \( D^\times \) (see, e.g., [C]) or derived from scratch. Thus we provide below, without proof, such a parametrization convenient for our purposes.

To begin, suppose that \( E \) is a nontrivial subfield (hence quadratic) of \( D \). Then under our identifications we may view \( PE^1 \) as a subgroup of \( G_2 \) where \( E^1 \) denotes the set of \( x \) in \( E \) such that \( N_{E/F}(x) = 1 \) and \( PE^1 = E^1/\{±1\} \). Now set \( \Lambda^2 = \Lambda^2(E) = \{\lambda^2|\lambda \text{ a character of } E^1\} \). One can check that \( (PE^1)^{-} \cong \Lambda^2 \). Also let \( \Lambda^2_1 \) denote the subgroup of \( \Lambda^2 \) consisting of characters trivial on \( PE^1 \cap U^1_E \) where, for a positive integer \( k \), \( U^k_E = \{u \in E|u-1 \in P^k_E\} \) and where \( U^k_E \) is viewed as a subgroup of \( PE^1 \). Note that the centralizer of \( PE^1 \) in \( G_2 \) is \( (PE^1 \times \langle d_\beta \rangle) \times (\sigma_D) \) where \( \beta \) is some generator of \( E/F \) of nonsquare norm. Now suppose that \( \alpha \) is a nonzero element in \( E \cap D^0 \) such that \( \nu_D(\alpha) < 0 \). (Note that \( \alpha \) being nonzero and in \( D^0 \) implies that \( F[\alpha] = E \) and, moreover, that \( \alpha \) is in fact \( E/F \)-minimal.) Then, setting \( n = -\nu_D(\alpha) \) and \( m' = \).
\[ [(n+2)/2], \text{let } \Lambda^2_\alpha \text{ denote the set of } \lambda \text{ in } \Lambda^2 \text{ which agree with the character } \psi_\alpha \text{ of } U^m(D^0) \text{ upon restriction to } U^m(D^0) \cap PE = U_E^m. \text{ Then to each } \lambda \text{ in } \Lambda^2_\alpha \text{ we associate a character } \rho'(\alpha, \lambda) \text{ of } (PE^1)U^m(D^0) \text{ in the obvious manner.} \]

Let \( m = [(n+1)/2] \). If \( n \) is odd, then \( m = m' \) and \( E/F \) is ramified; in this case set \( \rho(\alpha, \lambda) = \rho'(\alpha, \lambda) \). If \( n \) is even, then \( m = m' - 1 \) and \( E/F \) is unramified. If further \( m \) is even, set \( \rho(\alpha, \lambda) = \rho'(\alpha, \lambda) \) as a representation of \( (PE^1)U^m(D^0) = (PE^1)U^m(D^0) \). Finally, if \( n \) is even while \( m \) is odd, let \( \rho(\alpha, \lambda) \) denote the unique irreducible \( q \)-dimensional representation occurring in \( \text{Ind}(PE^1)U^m(D^0), (PE^1)U^m(D^0) ; \rho'(\alpha, \lambda) \) with multiplicity one. We note that in this last case the induced representation decomposes as a sum of representations of the form \( \rho(\alpha, \lambda') \) with \( \lambda' \) in \( \Lambda^2_\alpha \) and \( \lambda' \lambda^{-1} \) in \( \Lambda^2_\alpha \); indeed, all such representations occur and occur with multiplicity two with the exception of \( \rho(\alpha, \lambda) \) itself which occurs with multiplicity one.

In all the above cases the representation

\[ \pi(\alpha, \lambda) = \text{Ind}(G_2, (PE^1)U^m(D^0) ; \rho(\alpha, \lambda)) \]

decomposes as the sum of 4 distinct irreducible representations \( \pi(\alpha, \lambda, \gamma_1, \gamma_2) \), where \( \gamma_i = \pm 1 \) and \( \pi(\alpha, \lambda, \gamma_1, \gamma_2) \) corresponds to the subspace of the space of \( \pi(\alpha, \lambda) \) where \( \pi(\alpha, \lambda)(d^*_\beta) \) (\( \beta \) as above) acts by multiplication by \( \gamma_1 \) and \( \pi(\alpha, \lambda)(\sigma_D) \) acts by multiplication by \( \gamma_2 \).

We now construct the remaining representations of \( G_2 \). If \( \lambda \) is in \( \Lambda^2_\alpha \), set \( \pi(\lambda) = \text{Ind}(G_2, (PE^1)U^1(D^0) ; \lambda) \) where \( \lambda \) is the character of \( (PE^1)U^1(D^0) \) which is trivial on \( U^1(D^0) \) and is \( \lambda \) on \( PE^1 \). Let \( E = F[\beta] \) such that \( N_{E/F}(\beta) \) is not a square. Then, if \( \lambda \) is not trivial, \( \pi(\lambda) \) decomposes as the sum of four distinct irreducible two-dimensional representations \( \pi(\lambda, \gamma_1, \gamma_2) \), where \( \gamma_1 = \pm 1 \) and \( \pi(\lambda, \gamma_1, \gamma_2) \) corresponds to the subspace of the space of \( \pi(\lambda) \) where \( \pi(\lambda)(d^*_\beta) \) acts by multiplication by \( \gamma_1 \) and \( \pi(\lambda)(\sigma_D) \) acts by multiplication by \( \gamma_2 \). On the other hand, \( \pi(1) \) decomposes as the sum of eight distinct one-dimensional representations \( \pi^\pm(1, \gamma_1, \gamma_2) \) with \( \gamma_1 \) and \( \gamma_2 \) as above and \( \pm \) referring to the action of \( d_\omega \) in the obvious manner.

**Proposition 1.4.** The representations \( \pi(\alpha, \lambda, \gamma_1, \gamma_2) \), \( \pi(\lambda, \gamma_1, \gamma_2) \), and \( \pi^\pm(1, \gamma_1, \gamma_2) \) constructed above exhaust the admissible dual of \( G_2 \). These representations enjoy the following equivalences.

(i) A representation of the form \( \pi(\alpha, \lambda, \gamma_1, \gamma_2) \) is never equivalent to a representation of the form \( \pi(\lambda', \gamma_1', \gamma_2') \) or to a representation of the form \( \pi^\pm(1, \gamma_1', \gamma_2') \).

Likewise a representation of the form \( \pi(\lambda, \gamma_1, \gamma_2) \) is never equivalent to a representation of the form \( \pi^\pm(1, \gamma_1', \gamma_2') \).

(ii) Representations \( \pi(\alpha, \lambda, \gamma_1, \gamma_2) \) and \( \pi(\alpha', \lambda', \gamma_1', \gamma_2') \) are equivalent if and only if \( \gamma_1 = \gamma_1', \gamma_2 = \gamma_2', \) and there exists a \( z \) in \( PD^1 \) such that

(a) \( \alpha' = \alpha z \) is in \( p_D^{[[n+1]/2]} \) where \( n = \nu_D(\alpha); \)
(b) \( F[\alpha]^2 = F[\alpha']; \)
(c) \( \lambda^2 = \lambda'. \)

(iii) Representations \( \pi(\lambda, \gamma_1, \gamma_2) \) and \( \pi(\lambda', \gamma_1', \gamma_2') \) are equivalent if and only if \( \gamma_1 = \gamma_1', \gamma_2 = \gamma_2', \) and there exists a \( z \) in \( PD^1 \) such that \( E = (E')^2 \) where \( \lambda \) is a character of \( E \) and \( \lambda' \) a character of \( E' \) such that \( \lambda = (\lambda')^2 \) or \( \lambda^{-1} = (\lambda')^2 \).

(iv) There are eight distinct representations of the form \( \pi^\pm(1, \gamma_1, \gamma_2) \).
Remark 1.5. (i) We note that $a$ and $b$ of (ii) above imply that we may parameterize a representation of the form $\pi(\alpha, \lambda, \gamma_1, \gamma_2)$ by $\pi(a, \lambda, \gamma_1, \gamma_2)$ where $N_D(\alpha) = -a$. Indeed an $\alpha$ may be recovered from $a$ and $\lambda$ as the unique element of $E \cap D^0$ (where $\lambda$ is a quasicharacter of $PE^1$) of norm $-a$ (this determines $\alpha$ up to sign) such that $\psi_\alpha$ and $\lambda$ agree on $U^{m'}(D^0)$ where $m' = \left[(\nu_D(\alpha) + 2)/2\right]$. Note further that $a$ is a nonsquare and that any nonsquare of negative valuation can occur.

(ii) With notation as in the above proposition, we note for further reference that in the case $n$ even $m$ odd the representations occurring in the restriction of $\rho(\lambda, \alpha)$ to $PE^1$ are the characters $\lambda'$ of $PE_1$ such that $\lambda'^2 = \lambda^2 - 1$ in $A_1$ and these representations occur with multiplicity two with the exception of $\lambda$ which occurs with multiplicity one.

(iii) As in [RS], we say that a representation $\pi$ of $G_2$ is spherical if it occurs in the decomposition of the natural action of $G_2$ on $L^2(D^0)$. Then one can verify directly (using a realization of $D$ as an appropriate cyclic algebra) that if a representation of the form $\pi(\alpha, \lambda^2, \gamma_1, \gamma_2)$ or $\pi(\lambda^2, \gamma_1, \gamma_2)$ is spherical then $\gamma_2 = \lambda(-1)$ if $E = F[\alpha]/F$ is unramified and $-1$ is not a square in $F^\times$, and $\gamma_1 = \gamma_2 = \lambda(-1)$ otherwise. Similarly, one can show that if a representation of the form $\pi(1, \gamma_1, \gamma_2)$ is spherical and $-1$ is a square in $F^\times$ then $\gamma_1 = \gamma_2$, while if $-1$ is not a square in $F^\times$ then $\gamma_2 = 1$. Later, independent of [RS, W], we will show that these necessary conditions for a representation to be spherical are also sufficient. For the moment we call a representation satisfying the above necessary conditions pseudospherical.

(iv) We note that the four pseudospherical representations of the form $\pi(1, \gamma_1, \gamma_2)$ are the unique pseudospherical extensions of the characters $\psi_a \circ N$ of $PD^\times$ as $a$ ranges over a set of equivalence classes for $F^\times/(F^\times)^2$.

2. THE CORRESPONDENCE

We continue with the notation of [M1] and the previous section. In particular, $G_2$ is the isometry group of an anisotropic form $(\cdot, \cdot)_2$ in three variables. In keeping with the notation of [M1], we let $V_1$ be a two-dimensional $F$-vector space equipped with a nondegenerate skew-symmetric bilinear form $(\cdot, \cdot)$; let $G_1 \cong SL_2(F)$ denote the isometry group of $V_1$. Further, we set $W = \text{Hom}(V_1, V_2)$ and equip $W$ with the nondegenerate skew-symmetric bilinear form $(\cdot, \cdot)$ defined by $(w, w') = \text{tr} w \lambda(w')$ (with $\lambda$ as in [M1]). Let $G$ denote the isometry group of $(\cdot, \cdot)$ and identify $G_1$ and $G_2$ with subgroups of $G$ via their respective actions of premultiplications by inverses and postmultiplication. Now suppose $\chi$ is a (continuous) nontrivial additive character of $F$ and let $\omega_\chi$ denote the oscillator representation of $\tilde{G}$ attached to $\chi$ where $\tilde{G}$ is the unique nontrivial two-fold cover of $G$. Also for $H$ a closed subgroup of $G$, let $\tilde{H}$ denote the inverse image of $H$ in $G$ under the covering map. Note that $\tilde{G}_1$ is the nontrivial two-fold cover of $G_1$ and $\tilde{G}_2 \cong G_2 \times \{\pm 1\}$. Then, using the Schrödinger model of the oscillator representation, Rallis and Schiffman have shown that

$$\omega_\chi|_{\tilde{G}_1 \cap G_2} \cong \bigoplus_\rho \pi_\rho \otimes \rho,$$

where $\rho$ runs over the set of spherical representations of $G_2$ (see Remark 1.5), the $\pi_\rho$ are distinct and each $\pi_\rho$ is an irreducible unitary representation of

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\( \tilde{G}_1 \) which is supercuspidal unless \( \rho \) is trivial (but is otherwise undetermined) and is \( L^2 \) if \( \rho \) is trivial [RS]. This is then a special case of the local theta correspondence. Now let \( \omega_x^\infty \) denote the restriction of \( \omega_x \) to smooth vectors so that we are in the more general setting for the theta correspondence (this is not necessary here but is used when both \( G_1 \) and \( G_2 \) are noncompact). If \( H \) is a closed subgroup of \( G_1 \) let \( \mathcal{R}_x(H) \) denote the set of irreducible admissible representations of \( H \) which are quotients of the restriction of \( \omega_x^\infty \) to \( H \). Then in this section, independent of [RS and W], we will use various lattice models of the oscillator representation to determine the supercuspidal representations in \( \mathcal{R}_x(G_1) \) and \( \mathcal{R}_x(G_2) \) and the correspondence afforded by \( \mathcal{R}_x(G_1G_2) \). What is new here is that we explicitly parametrize the correspondence in terms of inducing data.

**Remark 2.1.** Before beginning the argument we note that it is easy to show that \( \mathcal{R}_x(G_2) \) consists of the spherical representations (see, e.g., [RS]). The more difficult task is the determination of the correspondence.

To be specific as to the correspondence we first fix \( \chi = \psi_w \) where \( \psi \) is fixed as in [M1]. In particular, if \( k' \) is the subfield of \( k = \mathcal{O}_F/P_F \) of cardinality \( P \) and \( \psi_{k'} \) is the additive character of \( k' \), such that \( \psi_{k'}(1) = e^{2\pi i/p} \), then we require that \( \psi \) be an additive character of \( F \) that factors to the character \( \psi_k \) of \( k_k \) defined by \( \psi_{k'} \circ \text{Tr}_{k'/k} \) and we set \( \psi_w(a) = \psi(\varpi a) \).

Now let \( L_1 \) be a self-dual lattice in \( V_1 \) (with respect to \( \chi \) and \( V_1 \)) and let \( L_2 = P_0 \). Then \( L = \text{Hom}_{\mathcal{O}_F}(L_1, L_2) \) is not a self-dual lattice in \( W \) but we may identify \( L^* \) with \( \text{Hom}_{\mathcal{O}_F}(L_1, P_0^{-1}) \) and we have \( P_k L^* \subset L \subset L^* \). Thus we may realize \( \omega_x \) in a non-self-dual lattice model associated to \( L \); note that \( \dim_{k_k}(L^*/L) = 4 \).

For \( k \) an integer, set \( L_k = \text{Hom}_{\mathcal{O}_F}(L_1, P_k) \) and set \( L_k^* = P_k L_1 \). Further set \( \mathcal{L} = \{L_k \}_{k \in \mathbb{Z}} \) and \( \mathcal{L}_1 = \{L_k^* \}_{k \in \mathbb{Z}} \) and note that \( \mathcal{L} \) and \( \mathcal{L}_1 \) are self-dual lattice chains in \( W \) and \( V_1 \) respectively. Finally set \( \mathcal{A}_1 = \mathcal{A}(\mathcal{L}_1) \), \( \mathcal{A}_2 = \mathcal{A}(D^0) \), and \( \mathcal{A} = \mathcal{A}(L) \). Then one checks that for \( l \) a nonnegative integer \( G_2 \cap U^l(\mathcal{A}_1) = U^l(\mathcal{A}_2) \) and \( G_1 \cap U^l(\mathcal{A}) = U^l(l+1)/2(\mathcal{A}_1) \).

**Lemma 2.2.** With notation as above, let \( k \) be a positive integer and let \( Y_{2k} \) be the set of functions in \( Y \) (the space of \( \omega_x \) in our lattice model) supported on \( L \). Then the following hold.

(i) \( U_{2k+1}(\mathcal{A}_1) \) and \( U_{4k+1}(\mathcal{A}_2) \) fix \( Y_{2k} \) pointwise.

(ii) If \( f \) is in \( Y_{2k} \) and \( h \) is in \( U_k(\mathcal{A}_1) \) or \( U_{2k}(\mathcal{A}_2) \), then

\[
\omega_x(h)f(w) = \rho_L(2c(h)w)\chi(w, c(h)w)f(w),
\]

where \( c(h) = (1-h)(1+h)^{-1} \) is the Cayley transform.

(iii) If \( f \) in \( Y_{2k} \) is supported on \(-w + L^* \), then \( f \) transforms according to \( \psi_{b_1} \) and \( \psi_{b_2} \) under the actions of \( U_{k+1}(\mathcal{A}_1) \) and \( U_{2k+1}(\mathcal{A}_2) \) respectively where \( b_1 = \varpi \lambda(w)/2 \) and \( b_2 = \varpi w \lambda(w)/2 \). Moreover, \( b_1 \) is an element of \( (\mathcal{P}_1)_{-}^{-2k} \) and \( b_2 \) is an element of \( (\mathcal{P}_2)_{-}^{-4k} \).

**Proof.** Argue as in the proof of [M1, Lemma 4.5] using \( M = L^{2k} \).

**Lemma 2.3.** With notation as above, under our identification of \( A(D^0) \) with \( D^0 \), \( N_D(b_2) = \det(b_1)/4 \). Further, \( \nu_D(b_2) = -4k \) if and only if the \( k \)-vector space homomorphism \( \overline{w} : L_1^0/L_1^1 \to P_{-2k-1}^{-1}/P_{-2k}^{-1} \) induced by \( w \) has rank 2.
Proof. One checks that if \( y \) in \( D^0 \) is nonzero, then \( ady \) has minimal polynomial \( X(X^2 + 4ND(y)) \). On the other hand one can check that \( b_2 \) satisfies the equation \( X(X^2 + \det b_1) = 0 \) since \( b_1 \) satisfies \( X^2 + \det b_1 = 0 \). Thus, if \( b_2 \) is nonzero, \( ND(b_2) = \det(b_1)/4 \). If \( b_2 \) is zero, then, since \( b_2 = \omega w\lambda(w)/2 \), \( w \) cannot have rank two as an element of \( W \) but then \( b_1 = -\omega \lambda(w)w/2 \) must have rank 0 or 1 as an element of \( A_F(V_1) \) so that \( \det(b_1) = 0 \) whence the first part of the lemma.

Now if \( \bar{w} \) has rank 2 then \( w \) maps \( L^0_1 \) onto \( P_{-2k-1} \) since
\[
\dim_k P_{-2k-1}/P_{-2k} = 2.
\]
Then the equality \( \langle wv_1, v_2 \rangle_2 = \langle v_1, \lambda(w)v_2 \rangle_1 \) for all \( v_1 \) in \( V_1 \) and \( v_2 \) in \( V_2 \) implies that \( b_1 \) maps \( L^0_1 \) onto \( L_{-2k} \) and thus \( \nu_F(\det b_1) = -4k \) so that \( \nu_D(b_2) = -4k \) also. If \( \bar{w} \) does not have rank 2, then \( w \) does not map \( L^0_1 \) onto \( P_{-2k-1} \) and then \( b_1 \) maps \( L^0_1 \) properly into \( L_{-2k} \) whence the result.

Lemma 2.4. Let \( a \) be a nonsquare element of \( F^\times \) such that \( \nu_F(a) = -4k \) with \( k \) a positive integer. Then, a representation of the form, \( \pi(a, \eta, \gamma_1, \gamma_2) \) occurs in \( \mathcal{R}_F(G_2) \) if and only if it is pseudospherical.

Proof. By Remark 1.5 it suffices to show that if the representation is pseudospherical then it occurs in \( \mathcal{R}_F(G_2) \). Let \( w \) be an element of \( L_{-2k-1} \) such that \( \bar{w} \) (as in the previous lemma) has rank 2. Then, by the previous lemma, \( \nu_D(b_2) = -4k \). Let \( b \) be an element of \( \mathcal{O}_F^\times \) such that \( b^2 ND(b_2) = -a \) (note that \( \text{tr}(b_2) = 0 \) so \( -ND(b_2) \) is not a square) and let \( g \) be an element of \( \mathcal{O}_F^\times \) such that \( \det g = b \). Then one checks that for any \( x \) in \( X \) (the space of \( \rho_L \) \( yw,x \) transforms according to \( \varphi_x \) under the action of \( U^{2k+1} \) where \( \alpha \) is an element of \( P_{-4k} \) such that \( ND(\alpha) = -a \). Without loss of generality (changing \( w \) if necessary) we may assume \( g = 1 \). Then, by, for example, [M1, 2.12],
\[
\omega_x(h)yw,x = yw, \rho_L(2c(h)w)(x(w,c(h)w)x)
\]
for \( h \) in \( U^{2k}(\mathcal{O}_2^\times) = (U^{2k}(\mathcal{O}_2^\times) \cap P E^1)U^{2k+1}(\mathcal{O}_2^\times) \). Now the images of the vectors \( c(h)w \) in \( L^*/L \) lie in an isotropic (1-dimensional!) subspace so that we may realize \( \rho_L \) in a Schrödinger model (see [M1, 2.10]) where, for each \( h \) in \( U^{2k}(\mathcal{O}_2^\times) \), \( \chi((w, c(h)w)) = 1 \) and \( \rho_L(2c(h)w) \) acts by translation on \( x \). Thus, changing \( x \) if necessary, we have that the \( U^{2k}(\mathcal{O}_2^\times) \)-span of \( yw,x \) is isomorphic to \( \text{Ind}(U^{2k}(\mathcal{O}_2^\times), U^{2k+1}(\mathcal{O}_2^\times); \psi_\alpha) \). Now one checks that since \( \bar{w} \) has rank 2 the stabilizer in \( PD^\times \) of \( -w + L^* \) in \( W/L^* \) is \( U^{2k}(\mathcal{O}_2^\times) \). Thus the \( PD^\times \)-span of \( yw,x \) is isomorphic to \( \text{Ind}(PD^\times, U^{2k}(\mathcal{O}_2^\times); \psi_\alpha) \). But then since a spherical representation is uniquely determined by its restriction to \( PD^\times \), the result follows.

Remark 2.5. That only spherical representations can occur in general can be proved using the lattice model. We will not prove this here since the Schrödinger model proof (as in [RS], see Remarks 2.1 and 1.5) is patently more simple. See the portion of the proof of Theorem 4.8 in [M1] concerning the nonoccurrence of the determinant representation of the unramified anisotropic \( O(2) \) for a lattice model argument similar in style.

Theorem 2.6. Let \( \pi(\mathcal{A}_1, \alpha, \eta) \) be a representation of \( \tilde{G}_1 \) such that \( \nu_{\mathcal{A}_1}(\alpha) \) is even. Then \( \pi(\mathcal{A}_1, \alpha, \eta) \) occurs in \( \mathcal{R}_F(\tilde{G}_1) \) and pairs with \( \pi(-\det(\alpha)/4, \eta^2, \gamma_1, \eta(-1)) \) where \( E = F[\alpha] \) has been identified with a subfield of \( D \) and

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\( \gamma_1 = \eta(-1) \) if \(-1\) is a square in \( F^\times \) and \( \gamma_1 = \eta(\mu) \) where \( \mu \) is a fourth root of unity in \( E^1 \) if \(-1\) is not a square in \( F^\times \).

**Proof.** Since \( \nu_{\mathcal{A}}(\alpha) \) is even (note that this implies that \( F[\alpha]/F \) is unramified), by Lemma 2.3 there exists a \( w \) in \( W \) such that, for each \( x \) in \( X \), \( y_{w,x} \) transforms according to \( \psi_{\alpha/2} \) under the action of \( U^{2k+1}(\mathcal{A}_2) \) where \( \nu_{\mathcal{A}}(\alpha) = -2k \). By Lemma 2.1 and Lemma 2.2, \( y_{w,x} \) transforms according to some \( \psi_b \) under the action of \( U^{k+1}(\mathcal{A}_1) \) where \( \det(b) = \det \alpha \) and \( \nu_{\mathcal{F}}(\det \alpha) = -4k \). It follows that there exists a \( g \) in \( \mathcal{A}_1^\times \) such that \( b^g = \alpha \). Then, changing \( w \) if necessary (to \( wg^{-1} \)), we may assume \( y_{w,x} \) transforms according to \( \psi_a \) under the action of \( U^{k+1}(\mathcal{A}_1) \) and according to \( \psi_{\alpha/2} \) under the action of \( U^{2k+1}(\mathcal{A}_2) \).

For \( k \) a nonnegative integer set, \( U^k(\mathcal{A}_1, E) = U^k(\mathcal{A}_1) \cap E^1 \) and set \( U(\mathcal{A}_1, E) = U^0(\mathcal{A}_1, E) \cap E^1 \). Now let \( a \) be an element of \( E^1 \) such that either \( a \) is in \( U^1(\mathcal{A}_1, E) \) or \( a^2 \) is not in \( U^1(\mathcal{A}_1, E) \). Then there exists a \( b \) in \( F \) such that \( a = c_1(ba) \) where \( c_1(x) = (1-x)(1+x)^{-1} \). Now using that \( aw/2 = -wa \) one can check that \( c_1(ba/2)w = wc_1(ba) \). Thus, by Lemma 1.3,

\[
(2.6.1) \quad aw = wa^2.
\]

It follows that if \( y \) is a nonzero vector in the \( G_2 \)-span of \( y_{w,x} \) transforming according to \( \pi(\alpha/2, \eta^2, \gamma_1, \eta(-1)) \) as in the proof of the previous lemma, then \( y \) transforms to \( \rho'(\mathcal{A}_1, \alpha, \eta) \) under the action of \( U^1(\mathcal{A}_1, E)U^{k+1}(\mathcal{A}_1) \) where \( \rho'(\mathcal{A}_1, \alpha, \eta) \) denotes the restriction of the representation \( \rho'(\mathcal{A}_1, \alpha, \eta) \) of \( U(\mathcal{A}_1, E)U^{k+1}(\mathcal{A}_1) \) (as defined in [M1, §1.3]) to \( U^1(\mathcal{A}_1, E)U^{k+1}(\mathcal{A}_1) \). Indeed, \( \eta^2 \) determines \( \eta \) up to the nontrivial real-valued character \( \theta \) say of \( E^1 \) and \( \theta \) is trivial on \( U^1(\mathcal{A}_1, E) \). Now using the representation theory of the Heisenberg group (for an entirely similar argument see the proof of [M1, Theorem 4.8(i)]) one can show that we may choose a nonzero \( y \) in the \( U^1(\mathcal{A}_1, E)U^k(\mathcal{A}_1) \cdot G_2 \)-span of \( y_{w,x} \) such that under the action of \( G_2 \) \( y \) transforms according to \( \pi(\alpha/2, \eta^2, \gamma_1, \eta(-1)) \) and under the action of \( U(\mathcal{A}_1, E)U^k(\mathcal{A}_1) \) it transforms according to some extension \( \hat{\rho} \) say of \( \rho_1(\mathcal{A}_1, \alpha, \eta) \) to \( U(\mathcal{A}_1, E)U^k(\mathcal{A}_1) \). Write \( \hat{\rho} = \rho(\mathcal{A}_1, \alpha, \eta') \) for some \( \eta' \) such that \( (\eta')^{-1} \eta \) is in \( \Lambda_1 \). Now the proposition will follow from Frobenius reciprocity if we can show that \( \eta = \eta' \). To this end, it suffices (see, e.g., [M1, 1.7]) to determine the characters and their multiplicities occurring in the restriction of the \( U(\mathcal{A}_1, E)U^k(\mathcal{A}_1) \)-span of \( y \) to \( E^1 \). This however is a straightforward calculation involving (2.6.1), the properties of the lattice model (see, e.g., [M1, 2.11]), Remark 2.7 below, and Tanaka’s determination [T, §9] of the theta correspondence for \( (\text{SL}_2(k_F), O_2(k_F)) \) where \( O_2 \) is the orthogonal group of an anisotropic binary quadratic form over \( k_F \).

**Remark 2.7.** If not well known, the following is easily checked. In the notation of [M1], the representations occurring in the restriction to \( E^1 \) of the \( (q-1) \)-dimensional irreducible representation \( \rho(\mathcal{A}_1, \eta) \) of \( U(\mathcal{A}_1) \) (the inflation of the cuspidal representation of \( \text{SL}_2(k_F) \) associated to \( \eta \)) associated to a non-real-valued character \( \eta \) in \( \Lambda_1 \) are the characters \( \mu \) in \( \Lambda_1 \) such that \( \mu \neq \eta \) but \( \mu(-1) = \eta(-1) \). Further, these characters occur with multiplicity two. Similarly, each of the non-real-valued characters in \( \Lambda_1 \) occurs and occurs with multiplicity one in the restriction of the \( (q-1)/2 \) dimensional irreducible cuspidal representations \( \rho(\mathcal{A}_1, +) \) and \( \rho(\mathcal{A}_1, -) \) to \( E^1 \) (notation also as in [M1]).
Theorem 2.8. All pseudospherical representations of the form \( \pi(\eta, \gamma_1, \gamma_2) \), \( \pi^+(1, \gamma_1, \gamma_2) \), and \( \pi^-(1, -1, \gamma_2) \) occur in \( \mathcal{R}_\chi(G_2) \). These representations pair with representations of \( G_1 \) as follows where \( \mathcal{A}_1 \) is the hereditary order in \( \text{End}_F(V_1) \) associated to the self-dual lattice \( L_1 \) as above and \( E \) is such that \( \mathcal{E}_E = L_1 \).

(i) If \( \eta \) is not real-valued, then \( \pi(\mathcal{A}_1, \eta) \) pairs with \( \pi(\eta^2, \gamma_1, \eta(-1)) \) where \( \gamma_1 = \eta(-1) \) if \(-1\) is a square in \( F \) and \( \gamma_1 = \eta(\mu) \) where \( \mu \) is a fourth root of unity in \( E \) if \(-1\) is not a square in \( F \).

(ii) The trivial representation \( \pi^+(1, 1, 1) \) does not pair with a supercuspidal representation.

(iii) The unique nontrivial pseudospherical representations of the forms \( \pi^+(1, \gamma_1, \gamma_2) \) and \( \pi^-(1, -1, \gamma_2) \) pair with \( \pi(\mathcal{A}_1, \text{sgn}(A)) \) and \( \pi(\mathcal{A}_1, -\text{sgn}(A)) \) respectively where

\[
A = (-1)^{[(f+1)/2][p-1]/2}(-1)^{f+1}(-1)\frac{(p^2-1)f}{8}
\]

with \( f = \log p \).

Proof. Consider the functions in \( Y \) supported on \( L^{-1} \). Then a straightforward computation, Theorem 3.1 of [M1] and [T, §9] imply that all the pseudospherical representations listed do occur. Then (i) follows from a straightforward computation and [T, §9] while (iii) follows as in Theorem 4.8 and Remark 4.10 of [M1]. Finally, to show that (ii) holds it seems best to use a Schrödinger model. In particular, realize \( \omega_\chi \) in the Schrödinger model attached to a polarization \((X, Y)\) say of \( W \) arising from a polarization \((X_1, Y_1)\) say of \( V_1 \). Let \( f \) be the function in the Schwartz space on \( X \) supported at 0 and taking the value 1 there. Identifying \( X \) with \( D^0 \) the action of \( G_2 \) becomes linear and thus \( f \) is fixed by \( G_2 \). On the other hand let \( N \) be the unipotent radical of the ("upper") parabolic subgroup of \( G_1 \) associated to the polarization \((X_1, Y_1)\) of \( V_1 \). Then \( N \) imbeds as a subgroup of \( \tilde{G}_1 \) and fixes \( f \). It follows then that the representation corresponding to the trivial representation could not be supercuspidal.

Remark 2.9. (i) The argument for (ii) above is standard. We only include it for completeness. For a determination of the corresponding representation (a discrete series representation) see [RS] or [Wa].

(ii) We note that (iii) also follows from Theorem 3.1 of [M1] and either [RS] or [Wa].

Now to detect other representations of \( G_2 \) attached to unramified tori and the corresponding representations of \( \tilde{G}_1 \) we consider another lattice model. To be precise, let \( L_1 \) be a lattice in \( V_1 \) such that \( L_1^* = P^{-1}L_1 \) and let \( L_2 = P_1^* \). Then \( L = \text{Hom}_{\mathcal{F}}(L_1, L_2) \) is not a self-dual lattice in \( W \) but we may identify \( L^* \) with \( P^{-1}\text{Hom}_{\mathcal{F}}(L_1, P_2) \) and we have \( P_F L^* \subseteq L \subseteq L^* \). We now realize \( \omega_\chi \) is a non-self-dual lattice model associated to \( L \); note that \( \dim_{\mathcal{F}}(L^*/L) = 2 \).

For \( k \) an integer, set \( L^k = \text{Hom}_{\mathcal{F}}(L_1, P_1^{k+1}) \) and set \( L_1^k = P_k L_1 \); further set \( \mathcal{L} = \{L^k\}_{k \in \mathbb{Z}} \) and \( \mathcal{L}_1 = \{L_1^k\}_{k \in \mathbb{Z}} \) and note that \( \mathcal{L} \) and \( \mathcal{L}_1 \) are self-dual lattices in \( W \) and \( V_1 \) respectively. Set \( \mathcal{A}_1 = \mathcal{A}(\mathcal{L}) \), \( \mathcal{A}_2 = \mathcal{A}(D^0) \), and \( \mathcal{A} = \mathcal{A}(\mathcal{L}) \). Then one checks that for \( l \) a nonnegative integer \( G_2 \cap U^l(\mathcal{A}) = U^l(\mathcal{A}_2) \) and \( G_1 \cap U^l(\mathcal{A}) = U^l(\mathcal{A}_2) \).
Lemma 2.10. With notation as above, let $k$ be a positive integer and let $Y_{2k}$ be the set of functions in $Y$ (the space of $\omega_x$ in our lattice model) supported on $L^{-2k}$. Then the following hold.

(i) $U_k(\mathcal{A}_1)$ and $U_{k-1}(\mathcal{A}_2)$ fix $Y_{2k}$ pointwise.

(ii) If $f$ is in $Y_{2k}$ and $h$ is in $U_k(\mathcal{A}_1)$ or $U_{k-1}(\mathcal{A}_2)$ then

$$\omega_x(h)f(w) = \rho_L(2c(h)w_\chi((w, c(h)w))f(w),$$

where $c(h) = (1 - h)(1 + h)^{-1}$ is the Cayley transform.

(iii) If $f$ in $Y_{2k}$ is supported on $-w + L^*$ then $f$ transforms according to $\psi_{b_1}$ and $\psi_{b_2}$ under the actions of $U_k(\mathcal{A}_1)$ and $U_{k-1}(\mathcal{A}_2)$ respectively where $b_1 = -\omega \lambda(w)w/2$ and $b_2 = \omega w\lambda(w)/2$. Moreover $b_1$ is an element of $(\mathcal{P}_1, -)^{-2k+1}$ and $b_2$ is an element of $(\mathcal{P}_2, -)^{-4k+2}$.

Proof. Argue similarly to the proof of Lemma 2.2 with $M = L^{2k-1}$.

Lemma 2.11. With notation as above, under our identification of $A(D^0)_-$ with $D^0$, $N_D(b_2) = \det(b_1)/4$. Further, $\nu_D(b_2) = -4k + 2$ if and only if the $k$-vector space homomorphism $\overline{w}: L_1/L_1 \to P_{-2k+1}^*/P_{-2k+2}^*$ induced by $w$ has rank 2.

Proof. The proof of this lemma is similar to that of Lemma 2.3.

Proposition 2.12. Let $\pi(\mathcal{A}_1, \alpha, \eta)$ be an element of $\tilde{G}_1$ such that $E = F[\alpha]/F$ is unramified and $\nu_{\mathcal{A}_1}(\alpha)$ is odd. Then $\pi(\mathcal{A}_1, \alpha, \eta)$ occurs in $\mathcal{R}_x(G_1)$.

Proof. With notation as above, let $\nu_{\mathcal{A}_1}(\alpha) = -2k + 1$ with $k$ a positive integer and let $w$ be an element of $L^{-2k}$ such that $\overline{w}$ has rank 2. Then by the previous lemmas $\nu_D(b_1) = -2k + 1$ and $-\det(b_1)$ is nonsquare. Now let $b$ be an element of $\mathcal{O}_F^*$ such that $b^2\det(b_1) = \det(\alpha)$ and let $g$ be an element of $\mathcal{A}_1^\times$ such that $\det g = b$. Then one checks that for any $x$ in $X$ (the space of $\rho_L$) $y_{gw,x}$ transforms according to $\psi_{\alpha'}$ under the action of $U_k(\mathcal{A}_1)$ where $\alpha' = \alpha^h$ for some $h$ in $U(\mathcal{A}_1)$. Then one checks that $y_{gw,x}$ transforms according to $\psi_{\alpha}$ under the action of $U_k(\mathcal{A}_1)$. Without loss of generality assume $hg = 1$. Finally since $\overline{w}$ has rank 2 the stabilizer in $G_1$ of $-w + L^*$ in $W/L^*$ is $U_k(\mathcal{A}_1)$ (recall that $G_1 \cap U_{2k-1}(\mathcal{A}_2) = U_{2k}(\mathcal{A}_2)$) so that the lemma now follows from Frobenius reciprocity.

Theorem 2.13. With notation as above, let $\pi(\mathcal{A}_1, \alpha, \eta)$ be a representation of $\tilde{G}_1$ such that $F[\alpha]/F$ is unramified and $\nu_{\mathcal{A}_1}(\alpha)$ is odd. Then $\pi(\mathcal{A}_1, \alpha, \eta)$ occurs in $\mathcal{R}_x(\tilde{G}_1)$ and pairs with $\pi(-\det(\alpha)/4, \eta^2, \gamma_1, \eta(-1))$ where $E = F[\alpha]$ is identified with a subfield of $D$ and $\gamma_1 = \eta(-1)$ is a square in $F$ and $\gamma_1 = \eta(\mu)$ where $\mu$ is a fourth root of unity in $E$ if $1$ is not a square in $F$.

Proof. Since $\nu_{\mathcal{A}_1}(\alpha)$ is odd, by Lemma 2.12 there exists a $w$ in $W$ such that for each $x$ in $X$, $y_{w,x}$ transforms according to $\psi_\alpha$ under the action of $U_k(\mathcal{A}_1)$ where $\nu_{\mathcal{A}_1}(\alpha) = -2k + 1$. By Lemma 2.10 and Lemma 2.11, $y_{w,x}$ transforms according to some $\psi_\beta$ under the action of $U_{2k}(A_2)$, where $N_D(b) = \det(\alpha)/4$. Using $b$ to identify $E$ with a subfield of $D$ we may assume $y_{w,x}$ transforms according to $\psi_\alpha$ under the action of $U_k(\mathcal{A}_1)$ and $\psi_{\alpha/2}$ under the action of $U_{2k}(\mathcal{A}_2)$ for any $x$ in $X$. Arguing as in the proof of Lemma 2.6 one can show that if $y$ is a nonzero vector in the $U(\mathcal{A}_1)$-span of $y_{w,x}$ transforming according to $\rho(\mathcal{A}_1, \alpha, \eta)$ then $y$ transforms according to $\rho_1(\alpha/2, \eta^2)$ under the action of $U_{1}(\mathcal{A}_2,E)U_{2k}(\mathcal{A}_2)$ where $\rho_1(\alpha/2, \eta^2)$
denotes the restriction of the representation \( \rho'(\alpha/2, \eta^2) \) of \( U(\mathfrak{g}_2, E)U^{2k}(\mathfrak{g}_2) \) to \( U^1(\mathfrak{g}_2, E)U^{2k}(\mathfrak{g}_2) \). Then one checks that, for \( x \) appropriately chosen, the \( U^{2k-1}(\mathfrak{g}_2) \) span of \( y_{w,x} \) is isomorphic to the restriction of \( \rho(\alpha/2, \sigma^2) \) to \( U^{2k-1}(\mathfrak{g}_2) \) for any \( \sigma \) (recall that \( U^{2k-1}(\mathfrak{g}_2) \cap PE^1 = U^{2k}(\mathfrak{g}_2) \cap PE^1 \)). It follows from the representation theory of the Heisenberg group that we may choose a nonzero \( y \) in the \( U(\mathfrak{g}_1)U^1(\mathfrak{g}_2, E)U^{2k-1}(\mathfrak{g}_2) \) span of \( y_{w,x} \) such that, under the action of \( U(\mathfrak{g}_1) \), \( y \) transforms according to \( \rho(\mathfrak{g}_1, \alpha, \eta) \), and under the action of \( U^1(\mathfrak{g}_2, E)U^{2k-1}(\mathfrak{g}_2) \), it transforms according to some extension \( \tilde{\rho} \) say of \( \rho(\alpha/2, \eta^2) \) to \( U(\mathfrak{g}_2, E)U^{2k}(\mathfrak{g}_2) \). Now the remainder of the argument is similar to the final portion of the proof of Proposition 2.6.

**Proposition 2.14.** The representation \( \pi^-(1, 1, 1) \) occurs in \( \mathcal{R}_\chi(G_2) \) and pairs with \( \pi(\mathfrak{g}_1, -\text{sgn} A) \) where \( A \) as before is \((-1)^[f+1/2](-1)^{f+1}(-1)^{f-1}f/8 \) with \( f = \log_p(q) \).

**Proof.** Consider the functions in \( Y \) supported on \( L^{-1} \). Then the result follows from a straightforward computation and Theorem 3.3 and Remark 3.4 of [M1] (it also follows from Theorem 3.1 of [M1] and either [RS] or [Wa]).

We now turn to those representations of \( G_2 \) arising from ramified extensions. To deal with these representations we realize \( \omega_x \) in a self-dual lattice model as follows. Let \( L_1 \) be a self-dual lattice in \( V_1 \) and let \( L'_1 \) be a lattice in \( V_1 \) such that \( (L'_1)^* = P^{-1}L_1 \) and \( PL_1 \subseteq L'_1 \subseteq L_1 \). Now set \( M'_1 = \text{Hom}(L_1, P^{-1}) \), \( M_2 = \text{Hom}(L'_1, P) \), and \( L' = M'_1 \cap M_2 \). Then one can check that \( \mathcal{L} = \{L^k\} \) is a self-dual lattice chain in \( W \) of period two such that \( (L^k)^* = L^{-k} \). One can also check that if we set \( M_i = L_i^{(i-1)/2} \) if \( i \) is even and \( M_i = (L_i^{(i-1)/2})^{-1} \) if \( i \) is odd then \( \mathcal{M} = \{M_i\} \) is a self-dual lattice chain in \( V_1 \) of period two such that \( (M_i)^* = M^{-i} \). Set \( \mathfrak{g}_1 = \mathfrak{g}(\mathcal{M}) \), \( \mathfrak{g}_2 = \mathfrak{g}(D^0) \), and \( \mathfrak{g} = \mathfrak{g}(\mathcal{L}) \). Then one checks that for \( l \) a nonnegative integer \( G_i \cap U^l(\mathfrak{g}_i) = U^l(\mathfrak{g}_i) \). In what follows we realize \( \omega_x \) in the self-dual lattice model attached to \( L_0 \).

**Lemma 2.15.** With notation as above let \( k \) be a positive integer and let \( Y_k \) be the set of functions in \( Y \) (the space of \( \omega_x \) in our lattice model) supported on \( L^{-k} \). Then the following hold.

(i) \( U^{2k}(\mathfrak{g}_i) \) for \( i = 1 \) or 2, fixes \( Y_k \) pointwise.

(ii) For \( w \) in \( L^{-k} \), \( y_w \) transforms according to \( \psi_{b_1} \) and \( \psi_{b_2} \) under the actions of \( U^k(\mathfrak{g}_1) \) and \( U^k(\mathfrak{g}_2) \) respectively where \( b_1 = -w\lambda(w)w/2 \) and \( b_2 = w\lambda(w)/2 \). Moreover, \( b_1 \) is an element of \( (\mathcal{P}_1, -)^{-2k+1} \) and \( b_2 \) is an element of \( (\mathcal{P}_2, -)^{-2k+1} \).

**Proof.** Argue as in the proof of Lemma 4.3 of [M1].

**Lemma 2.16.** With notation as above, under our identification of \( A(D^0)_- \) with \( D^0 \), \( N_D(b_2) = \det(b_1)/4 \). Further \( \nu_D(b_2) = -2k + 1 \) if and only if the images \( w_1 \) and \( w_2 \) of \( w \) in \( L^{-k}/M_1^{-k+1} \) for \( i = 1, 2 \) respectively are both nonzero.

**Proof.** This is similar to the proof of Lemma 2.3.

**Proposition 2.17.** Let \( a \) be a nonsquare element of \( F^* \) such that \( \nu_{\mathfrak{g}}(a) = -2k + 1 \) with \( k \) a positive integer. Then a representation of the form \( \pi(a, \eta, \gamma_1, \gamma_2) \) occurs in \( \mathcal{R}_\chi(G_2) \) if and only if it is pseudospherical.
Proof. Let $w$ be an element of $L^{-k}$ such that $w_1$ and $w_2$ as in the previous lemma are nonzero. Then, by the previous lemma, $\nu_D(b_2) = -2k + 1$. Now suppose that there exists $b$ in $\mathfrak{C}^X$ such that $b^2 N_D(b_2) = -a$. Then letting $g$ be an element of $A_1^X$ such that $\det g = b$ one checks that $\chi_gw$ transforms according to $\chi_\alpha$ under the action of $U^k(\mathfrak{A}_2)$ where $\alpha$ is an element of $P_{-2k+1}$ such that $N_D(\alpha) = -a$. If $-N_D(b_2)/a$ is not a square then changing $w_1$ if $k$ is even and $w_2$ if $k$ is odd and a similar argument allows us to find such $g$, $w$, and $\alpha$. In either case, we may assume without loss of generality that we have an element $w$ of $L^{-k}$ such that $\chi_w$ transforms according to $\chi_\alpha$ where $\alpha$ is an element of $P_{-2k+1}$ such that $N_D(\alpha) = -a$. Then one checks that the stabilizer in $G_2$ of $-w + L$ in $W/L$ is $U^k(\mathfrak{A}_2)$. The result then follows from Frobenius reciprocity.

Theorem 2.18. Let $\pi(\mathfrak{A}_1, \alpha, \eta)$ be a representation of $G_1$ such that $F[\alpha]/F$ is ramified. Then $\pi(\mathfrak{A}_1, \alpha, \eta)$ occurs in $\mathfrak{H}_X(G_1)$ if and only if there exists a $w$ in $L^{\nu(\alpha)-1/2}$ such that $\omega_F \lambda(w)w = -2\alpha$. If $\pi(\mathfrak{A}_1, \alpha, \eta)$ occurs in $\mathfrak{H}_X(G_1)$ then it pairs with $\pi(-\det(\alpha)/4, \eta^2, \eta(-1), \eta(-1))$.

Proof. The proof of this result is similar to the proof of Theorem 2.6 only simpler since we are using a self-dual model.

Remark 2.19. (i) As a consequence of Proposition 2.5, Theorem 2.6, Theorem 2.8, Theorem 2.13, Proposition 2.17, and Theorem 2.18 we have that all pseudospherical representations of $G_2$ occur in $\mathfrak{H}_X(G_2)$ and are thus spherical. We also have explicit pairings for these representations.

(ii) It is a straightforward exercise to determine the effect of changing $\chi$. See for example Remark 2.4(iii) of [M1].

References


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