A CONSTRUCTION OF THE SUPERCUSPIDAL REPRESENTATIONS OF $GL_n(F)$, $F$ $p$-ADIC

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ABSTRACT. Let $F$ be a nondiscrete, locally compact, non-Archimedean field. In this paper, we construct all irreducible supercuspidal representations of $G = GL_n(F)$. For each such representation $\pi$ (which we may as well assume is unitary), we give a subgroup $J$ of $G$ that is compact mod the center $Z$ of $G$ and a (finite-dimensional) representation $\sigma$ of $J$ such that inducing $\sigma$ to $G$ gives $\pi$. The proof that all supercuspidals have been constructed appeals to a theorem (the Matching Theorem) that has been proved by global methods.

Let $F$ be a $p$-adic (= locally compact, nondiscrete, non-Archimedean) field. In this paper we prove:

(1.1) Theorem. The irreducible supercuspidal representations of $GL_n(F)$ are all induced from representations of open compact (mod center) subgroups.

In fact, we construct a set of inducing representations for the supercuspidals of $GL_n(F)$. This should make it possible to do further calculations concerning these representations. For example, in this paper we compute their formal degrees explicitly.

The first major breakthrough in constructing supercuspidals for $G = GL_n(F)$ was made by Howe [10], who gave a construction in the case $p \nmid n$ (the "tamely ramified" case). Moy [19] proved that Howe had indeed constructed all the supercuspidals for these $n$. Meanwhile, Carayol [3] gave a construction for prime $n$ (including $n = p$). In all these cases the general outline was the same: one first uses the similarities between $G$ and $D_n^\times$ (where $D_n$ is a central division algebra over $F$ with $[D_n : F] = n^2$) to construct a set of supercuspidals for $GL_n(F)$, and then uses the Matching Theorem of Deligne-Kazhdan-Vigneras [7] (see also [21]) to show that the set is complete. The proof given here uses the same procedure. In using the Matching Theorem, it is necessary to know the number of irreducible representations of $D_n^\times$ with conductor less than a

Received by the editors April 24, 1990 and, in revised form, September 3, 1990 and November 16, 1990.

1980 Mathematics Subject Classification (1985 Revision). Primary 22E50.

The author was supported by NSF Grants DMS-86-03169 and DMS89-02993.

The results of this paper were described in a series of talks at an NSF-sponsored conference at Bowdoin College in August 1989. A partial account of these results (with sketches of some proofs) will appear in the proceedings of this conference.

† Deceased March 19, 1992.
fixed number $m$. This information is provided in [14]. It is therefore not essential in the second step to know $(D_n^\times)^\wedge$, although it is useful. However, the construction of $D_n^\times$ given in [4] is similar to (but simpler than) the one used here. (We remark further in §11 on the logical connections between this paper and [4].)

The procedure described above for showing that one has found all supercuspidals can succeed only for reductive groups of type $A_n$, because only for such groups is there a compact form of the group. For this reason it is important to have “intrinsic” or “local” proofs of the completeness of the construction. For $GL_n(F)$, $n$ the product of two primes, Kutzko and Manderscheid [16] have shown that all supercuspidals are induced. More recently, a proof of completeness for the case $(n, p) = 1$ has been given by Howe and Moy [11], and for the construction in [10] ($n$ prime) by them [11] and Bushnell [2]. These rely on the theory of minimal $K$-types, developed originally by Howe and Moy. The construction in this paper seems well adapted to the minimal $K$-type picture, and it would not be surprising to see a local proof of completeness in the near future.

The problem of constructing supercuspidals exists, of course, for general reductive $p$-adic groups. For $GL_m(D)$, $D$ a local division algebra, the methods of this paper seem to apply with only minor modifications. (These groups are of course of type $A_n$.) Gerardin [8] gives a construction of some unramified supercuspidals; more recently, Morris [18] has given a construction in a situation like the “very cuspidal” case of Carayol; Moy [20] and Asmuth and Keys [1] have analyzed the situation for $GSp_4(F)$, Moy in the case $p \neq 2$ and Asmuth and Keys in general. A complete analysis of the general reductive case will probably depend on advances in the theory of minimal $K$-types.

A very rough idea of the construction is as follows: supercuspidal representations are connected with anisotropic tori, and we should therefore look at maximal compact (mod center) subgroups that contain such tori. In the tamely ramified case (where $n$ is prime to $p$), one then considers certain characters of the torus that are appropriately nondegenerate. One then determines the elements $x$ of the subgroup such that conjugation by $x$ fixes $x$, extends $x$ to this subgroup, and induces to get the supercuspidal. (In some cases there is an additional step, that tensoring by a finite-dimensional representation like a representation of the Heisenberg group. One may also need to tensor with lifts of cuspidal representations of $GL_m(k_f)$, where $k_f$ is the residue class field of an unramified extension of $F$ and $m|n$.) The general situation is similar, but the characters are no longer characters on the torus; instead, they are defined on certain subgroups of the compact open subgroup in a way that associates them with the torus less directly. The key property is that conjugation by elements of the torus fixes the character. That is, we concentrate more on the group of elements fixing $x$ under conjugation than on $x$ itself. Eventually, we define $x$ on a compact-mod-center subgroup $H$ such that only elements of $H$ fix $x$; we then induce to create the supercuspidal. (Again, there may be the additional step of tensoring with a Heisenberg-like representation or with the lift of a cuspidal representation.)

We now fix some notation in order to give a more detailed description of the construction. Let $n = ef$ and let $V = F^n$; denote by $\mathcal{O}$ the ring of integers of $F$ and by $P$ its (unique) maximal ideal. Define a lattice chain
$\mathcal{L} = \{\ldots, L_{-1}, L_0, L_1, \ldots \}$ in $F^n$ by

$$L_0 = \emptyset \oplus \cdots \oplus \emptyset \quad (n \text{ terms})$$
$$L_1 = \emptyset \oplus \cdots \oplus \emptyset \oplus P \oplus \cdots \oplus P \quad (f \text{ P's})$$
$$\cdots$$
$$L_{e-1} = \emptyset \oplus \cdots \oplus \emptyset \oplus P \oplus \cdots \oplus P \quad (f(e - 1) \text{ P's})$$
$$L_e = P \oplus \cdots \oplus P,$$
$$L_{me+j} = P^mL_j \quad (0 \leq j \leq e - 1).$$

Set

$$A_e^j = \{x \in M_e(F) = xL_i \subseteq L_{i+j} \forall i \in \mathbb{Z}\}, \quad j \in \mathbb{Z} ; \quad A_e = A_e^0,$$
$$K_e = \text{group of invertible elements in } A_e ,$$
$$K_e^j = 1 + A_e^j, \quad j \geq 1 ,$$

(1.2) $\varpi_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \cdots & 0 \\ \varpi_F & 0 & \vdots & \cdots & 0 \end{bmatrix} \in \text{GL}_n(F) \quad (\varpi_F \text{ generates } P),$n

$$Z_e = \text{group generated by } \varpi_e .$$

Then $\varpi_e : L_i \rightarrow L_{i+1} \forall i$ and $Z_eK_e$ is the normalizer of $K_e$ in $\text{GL}_n(F)$; $K_e$ and the $K_e^j$ are all compact open normal subgroups of $Z_eK_e$. Furthermore, $\varpi_e^f = \varpi e I$ is central in $\text{GL}_n(F)$ and the groups $Z_eK_e$ are maximal compact (mod center) subgroups of $\text{GL}_n(F)$. If $E$ is a field extension of $F$ with ramification index $e$ and residue class degree $f$, then $E^\times$ embeds into $Z_eK_e$. Therefore each $Z_eK_e$ contains anisotropic maximal tori, and indeed any maximal anisotropic torus is contained in a conjugate of some $Z_eK_e$. (This description of these groups, using lattices, was introduced into the subject by Bushnell.)

Let $k \subseteq \mathcal{O}$ denote the set of solutions to $X^q - X = 0$, where $q$ is the cardinality of the residue class field $\mathcal{O}/P$. The elements in $k$ give representatives for $\mathcal{O}/P$, and we usually identify the two. (This should not cause confusion.) Then

$$K_e^m/K_e^{m+1} \cong A_e^m/A_e^{m+1} \cong (M_f(k))^e \quad \text{if } m \geq 1, \quad \text{under } 1 + y \mapsto y ;$$
$$K_e/K_e^1 \cong \text{GL}_f(k)^e .$$

Write a typical element of $M_f(k)^e$ as $\alpha = (\alpha_0, \ldots , \alpha_{e-1})$, where each $\alpha_i \in M_f(k)$. Then $\varpi_e$ normalizes $K_e^m$ and $K_e^{m+1}$, and hence induces an action on $M_f(k)^e$. This action is independent of $m$; in fact,

$$\varpi_e \alpha \varpi_e^{-1} = \alpha^\varpi e, \quad \alpha^\varpi e = (\alpha_1, \ldots , \alpha_{e-1}, \alpha_0) .$$

A similar formula holds for $K_e/K_e^1$. We also note a few facts about the relation of $K_e^j$ and $K_e^{j'}$ when $e' \mid e$. Let $e = e'e_0$. Then $K_e^j \subseteq K_e^{j'}$ if $(j' - 1)e_0 \geq j - 1$ and $K_e^j \supseteq K_e^{j'}$ if $j'e_0 \leq j$. If $x \in A_e^j \cap A_e^{j'}$, then $\varpi_e x \in A_e^j \cap A_e^{j'+1}$. When $j'e_0 = j$, there are coset representatives for $K_e^j/K_e^{j'+1}$ that are in $K_e^j$ (mod $K_e^{j'+1}$).
For most of the paper, $e$ (and hence $f$) will be fixed, and we will suppress it in some notation. Let $m_e = M_f(k)^e$ and $m_e^\times = \text{GL}_f(k)^e$, so that any $y \in Z_e K_e$ can be written as

$$y = \alpha_0 \varpi^j_e (1 + \alpha_1 \varpi^1_e + \alpha_2 \varpi^2_e + \cdots),$$

where the $\alpha_j$ are preimages in $M_n(\mathcal{O}_F)$ of elements of $m_e$ and $\alpha_0$ is the preimage of an element of $m_e^\times$. (We say more about the choice of preimages below.) We usually write $\varpi$ for $\varpi_e$ and $\sigma$ for $\sigma_e$.

Let $\psi$ be a fixed additive character of $F$, trivial on $P_F$ but not on $\mathcal{O}_F$, and define $\chi_x$ to be the character of $M_n(F)$ given by

$$\chi_x(y) = \psi \circ \text{Tr}(xy).$$

Then it is standard that $x \mapsto \chi_x$ gives an isomorphism of $M = M_n(F)$ with $M^\sim$ under which $(A_e^\sim)^1 = A_e^1$.

We are going to start with sequences of triples $(s_1, e_1, f_1), \ldots, (s_r, e_r, f_r)$ with the following properties:

(i) $s_1 > s_2 > \cdots > s_r \geq 0$;
(ii) $e_1 | e_2 | \cdots | e_r$, $f_1 | f_2 | \cdots | f_r$, and $1 < e_1 f_1 < \cdots < e_r f_r$;
(iii) $e_r = e$ and $f_r = f$;

(1.3) (iv) for all $i$, $f|s_i$ and $e_i(n/e_{i-1}, s_i) = n$

(where $(\cdot, \cdot)$ is the greatest common divisor and $e_0 = 1$);
(v) $e_r = e_{r-1}$ if $s_r = 0$.

We then construct supercuspidal representations $\pi$ of $G$ associated with these triples.

The construction of $\pi$ will occupy §§6–9 of this paper. The following is a brief description (more remarks appear early in §6): We want to induce $\pi$ from a character $\chi$ on a subgroup $H$ that we construct. (As noted earlier, this statement sometimes needs to be modified slightly.) Define $H' = H \cap K_e^{1}$; write $s_i = ft_i$, and assume that $t_1 > 0$. We will have $\chi$ trivial on $K_e^{t_1+1}$, on $K_e^{t_1}$,

$$\chi(1+y) = \chi_x(y)$$

for some $x \in A_e^{t_1}$ (where $x$ is defined mod $A_e^{1-t_1}$). Write

$$x = \alpha_{-t_1} \varpi^{-t_1} + \cdots, \quad \varpi = \varpi_e;$$

the terms after the first are arbitrary. We require that $\alpha_{-t_1} \varpi^{-t_1}$ generates a field over $F$ of ramification index $e_1$ and residue class degree $f_1$. (Condition (1.3)(iv) says that if $\alpha_{-t_1} \varpi^{-t_1}$ generates a field $E_{(t_1)}$, then $e(E_{(t_1)}/F) = e_1$.) We then show that the elements of $G$ commuting with $\chi|_{E_{(t_1)}}$ are those of the form $g = w_1 x w'_1$, where $w_1, w'_1 \in K_e^{t_1}$ and $x$ commutes with $E_{(t_1)}$. We also begin to define $H$ by declaring that $H' = K_e^{t'_1}$, where $t'_1 = [t_1/2]+1$.

Assume now that $t_2 \geq t'_1$. We show that $\chi$ has an extension $\chi_0$ to $K_e^{t'_2}$ such that any element of $G$ commuting with $E_{(t_1)}$ commutes with $\chi_0$. (We say that $w$ commutes with $\chi$ if $\chi(wyw^{-1}) = \chi(y)$ whenever both are defined.) Using this, we construct extensions of $\chi$ to $K_e^{t_2}$ (for now, we denote a typical one by $\chi_1$) with the property that there exists a field $E_{(t_1)}$ of ramification index $e_1$ and residue class degree $f_1$ such that $g \in G$ commutes with $\chi_1$ iff $w = w_1 x w'_1$,..
where $w_1, w'_1 \in K_e^{t_2-t_2+1}$ and $x$ commutes with $E_1$. The extension of $\chi$ to $K_e^{t_2}$ that we want is, however, not $\chi_1$. Any extension of $\chi$ to $K_e^{t_2}$ agreeing with $\chi_1$ on $K_e^{t_2+1}$ is of the form

$$\chi(1+y) = \chi_1(1+y) \psi \circ \text{Tr}(x'_2y),$$

where $x'_2 \in A_e^{t_2} \pmod{A_e^{t_2-t_2}}$. Restrict attention to the elements $y$ commuting with $E_1$. Then there is a unique element $x_2$ in the elements commuting with $E_1$ such that for these elements,

$$\psi \circ \text{Tr}(x'_2y) = \psi \circ \text{Tr}(x_2y).$$

Again, $x_2$ is defined mod $A_e^{t_2-t_2}$. (The construction of $x_2$ from $x'_2$ is essentially the $S_{\alpha}$-map of [15].) We require $x_2$ to be such that $E_1[x_2]$ is a field of ramification index $e_2$ and residue class degree $f_2$ over $F$. It then turns out that there is a field $E_1[x_2]$, with $e(E_1[x_2]/F) = e_2$ and $f(E_1[x_2]/F) = f_2$, such that the elements of $G$ commuting with $\chi$ are precisely those of the form $w_1w_2xw'_2w'_1$, where $w_1, w'_1 \in K_e^{t_2-t_2+1}$, $w_2, w'_2 \in K_e^1$, $w_2$ and $w'_2$ commute with $E_1$, and $x$ commutes with $F[x_2]$. (This does not define $E_1[x_2]$ uniquely.)

We also set $t_2' = [t_2/2] + 1$ and $H^{t_2}_e = H^{t_2}_e (K_e^{t_2} \cap G_1)$, where $G_1 =$ subgroup of elements in $GL_n(F)$ commuting with $E_1$.

We now iterate this. This is, we show that if $t_3 \geq t_1$, then $\chi$ has an extension $\chi_0$ to $K_e^{t_3}$ such that any $w \in G$ commuting with $E_1[x_2]$ commutes with $\chi_0$. (If $t_3 < t_1$, a modified version holds.) We then consider certain extensions $\chi_1$ to $K_e^{t_3}$ with the property that for a field $E_2$ of ramification index $e_2$ and residue class degree $f_2$ over $F$, every $w \in G$ commuting with $E_2$ commutes with $\chi_1$. We require that $\chi(1+y) = \chi_1(1+y) \psi \circ \text{Tr}(x'_3y)$ on $K_e^{t_3}$, where $x'_3$ gives rise to an $x_3$ commuting with $E_2$ such that $e(E_2[x_3]/F) = e_3$ and $f(E_2[x_3]/F) = f_3$, and so on. We also extend the definition of $H$; of course, we need to show that the new definition is consistent with what we have already done. (The details are in Theorems 6.1 and 8.1; the induction hypothesis is complicated so that we can carry along a large number of needed facts. In particular, we compute exactly which elements of $G$ commute with $\chi_1 |_{H_j}$ for each $j$.) In some cases, we eventually extend $\chi$ to a subgroup $H$ of $Z_eK_e$ such that if $w$ commutes with $\chi$, then $w \in H$. (In other cases, noted above, we tensor with Heisenberg representations or cuspidals.) This shows that $\chi$ induces irreducibly to a supercuspidal $\pi$.

The construction is so arranged that at any time, one is dealing with computations involving only some $K_e^{t_j}/K_e^{t_j+1}$. Another consequence of the construction is that the “$t_1/2$ problem” is eliminated. Set $t'_1 = [t_1/2]+1$, as before. Then any character $\chi$ of $K_e^{t_1}/K_e^{t_1+1} \cong A_e^{t_1}/A_e^{t_1+1}$ can be written as $\chi(1+y) = \psi \circ \text{Tr}(xy)$ for some $x \in A_e^{-t_1}$ (defined modulo $A_e^{t_1-t_1}$). This expression for $\chi$ is of great help in analyzing, for example, the elements $w \in GL_n(F)$ commuting with $\chi$. However, if we need to extend $\chi$ to a subgroup of $K_e$ properly containing $K_e^{t_1}$, then $\chi$ no longer has such a simple form. Since the analysis given here does not depend on the above sort of expression for $\chi$, the above difficulty is obviated.

This construction of $\pi$ parallels the construction of irreducibles in [4]. We use counting arguments based on the Matching Theorem to prove that we have
constructed all supercuspidals. Thus we need to compute the number of supercuspidal representations and of other discrete series representations with given conductoral exponent, and compare that number with the corresponding number for division algebras. We also compute the formal degrees of the representations \( \pi \). These latter computations are similar to those in [6]. They serve the purpose of showing that supercuspidals constructed using \( Z e K e \) and those constructed using \( Z e~K e~ \), with \( e~ \neq e \), are distinct.

As mentioned above, the inductive nature of the construction means that there are large numbers of details to verify at each step. Here is a brief description of the main points needing attention. We need to show that when we extend the definition of \( H \), the new definition is consistent with the old; this involves knowledge of the structure of the \( G_i \) and in particular of their relation to each other. The necessary material is developed in \( \S \S 2 \) and 3. Section 2 also gives some terminology that is used throughout the paper. We also need to show that \( \chi \) extends at each step, and, as noted above, we need to be able to compute the set of elements \( X \) with \( \chi^X = \chi \) (on their common domain) at each step in the construction of \( \chi \). The basic lemmas for this are given in \( \S \S 4 \) and 5. The main part of the construction is done in \( \S \S 6 - 9 \); the remaining sections deal with such matters as computing formal degrees and proving completeness. The reader may wish to read the rest of this section and \( \S 2 \) first, and then go to \( \S 6 \), referring to results in the preceding sections as necessary.

This construction is surprisingly close to that given in [10] for the tamely ramified case. Then the character \( \chi \) is always nontrivial on the field \( E_{(j)} \), and one can thus associate irreducible supercuspidals with certain characters of extension fields of \( F \). Furthermore, the geometry (or algebra) of the tamely ramified situation is simpler, and one can arrange to have \( E_{(n)} \subseteq E_{(n-1)} \subseteq \cdots \); this greatly simplifies many arguments.

We shall use some further notational conventions in this paper. Fix the sequence \( 1 = f_1|f_2|\cdots|f_e \). There is an embedding of \( k_f \), the extension field of degree \( f \) over \( k \), in \( M_f(k) \) such that \( k_{f_i} \) is diagonally embedded as block \( f_i \times f_i \) matrices (with all blocks the same), \( k_{f_2} \) is diagonally embedded as block \( f_2 \times f_2 \) matrices, and so on. Fix such an embedding. Then \( (k_f)^e = k_f \times k_f \times \cdots \times k_f \) (\( e \) factors) is embedded in \( m_e = M_f(k)^e \), and \( k_f \) is embedded as the diagonal. For each \( i \), let \( m_{f_i}^e \) be the algebra of elements in \( m_e \) commuting with \( k_{f_i} \). Then the algebra of elements commuting with \( m_{f_i}^e \) is easily seen to be \( k_{f_i}^e \). We are, of course, using coset representations for \( m_e \) in the case where \( \text{char } F = 0 \); in this case, the representatives for elements in \( k_f \) can be taken to be 0 or roots of unity in (an appropriate embedding of) the ramified field \( F_f \) with \( [F_f : F] = f \), and the representatives of elements in \( m_{f_i}^e \) can be taken to commute with the cyclic group \( k_{f_i}^e \), and hence with \( F_{f_i} \). (In this paper, \( F_{f_i} \) always denotes the unramified extension of \( F \) of degree \( f_i \). The elements of \( m_{f_i}^e \) are \( e \)-tuples of \( f \times f \) matrices, each of which is a matrix of \( f_i \times f_i \) blocks where each block is an element of \( k_{f_i} \).) Notice further that the representatives for each \( (k_{f_i}^e)^e \) form a finite group, and that those for \( (k_{f_i}^e)^e \) form a subgroup of those for \( (k_{f_{i+1}}^e)^e \). We can choose the representatives of \( m_e \) so that

(i) they are closed under left multiplication by \( (k_{f_i}^e)^e \);
(ii) if \( \alpha \) represents an element of \( m_e \), \( \gamma \) represents an element of \( k_{f_i}^e \), and \([\alpha, \gamma] \equiv 0 \mod A_{f_i}^1 \), then \([\alpha, \gamma] = 0 \).
(iii) a representative $\alpha$ is invertible iff its image in $m_e$ is invertible. Here is a proof. It suffices to find a set $S$ of representatives in $M_f(k)$ for $M_f(k)$ satisfying

(i') $S$ is closed under multiplication by $k_f^{\times}$;

(ii') if $\alpha$ represents an element of $M_f(k)$, $\gamma$ represents an element of $k_f$, and $[\alpha, \gamma] = 0 \bmod A_f$, then $[\alpha, \gamma] = 0$;

(iii') a representative $\alpha$ is invertible iff its image in $M_f(k)$ is invertible.

(Given such a set, we simply embed $M_f(F)\epsilon$ diagonally in $M_n(F)$ and replicate the representatives.) We can obviously choose representatives satisfying (ii') and (iii'), with 0 representing 0. The elements of $M_f(k)$ divide into disjoint orbits under multiplication by $k_f$, and all elements of an orbit commute with $\gamma \in k_f$ if one element does. For each orbit, choose a single element $\bar{\alpha} \in M_f(k)$, let $\alpha$ be its representative, and redefine the representative of $\beta \alpha$ (for $\beta \in k_f$) by $\beta \alpha$, where $\beta$ is the root of unity in $F_f$ corresponding to $\beta$. Now (i')-(iii') hold. Since we picked the same representatives for each copy of $M_f(k)$, the representatives of $m_e$ are stable under $\sigma$.

We generally do not distinguish between elements of $m_e$ and their representatives; the justification will be given in Lemma 3.4, where we show how to go from congruences mod $A_f$ to equalities in $GL_n(F)$. We also write $m_e^{(\epsilon)}(e_i)$, where $e_i|e$, for the elements $\alpha = (\alpha_0, \ldots, \alpha_{e-1}) \in m_e^{(\epsilon)}$ with $\alpha(e/e_i)+j = \alpha_j \forall j < e-e/e_i$. For general $\alpha = (\alpha_0, \ldots, \alpha_{e-1}) \in m_e$, we define $Tr_e \alpha = \sum_{j=0}^{e-1} \alpha^{(e/e_i)}$; then $Tr_e \alpha \in m_e(e_i)$. We set $Tr(\alpha) = \sum_{j=0}^{e-1} Tr \alpha_j$; however, for $\alpha \in m_e^{(\epsilon)}(e_i)$, we set $Tr^{(\epsilon)}(\alpha) = \sum_{j=0}^{(e/e_i)-1} Tr \alpha_j$. Hence for $\alpha \in m_e$, $Tr^{(\epsilon)} Tr_e \alpha = Tr \alpha$ (in an obvious sense).

I would like to thank Allen Moy and David Manderscheid for valuable conversations and suggestions concerning parts of this paper. I am also deeply indebted to the referee for a superb job of reviewing the manuscript and ferreting out misprints, obscurities, and the like. If errors remain, they are both my responsibility and smaller in number than beforehand.

2

In this section we introduce some notation and terminology and give some results on subgroups of $GL_n(F)$ of the form $GL_{n_0}(E)$, where $E$ is an extension field of $F$ with ramification index $e_0$ and residue class degree $f_0$ (of course, $e_0|e$ and $f_0$ is one of the $f_i$), and $e_0 f_0 n_0 = n$. We say that $E$ is nicely embedded if the following hold:

(i) $F_{f_0} \subseteq E$. (As in §1, $F_{f_0}$ is embedded “diagonally” in $M_n(F)$.)

(ii) There is a prime element $\xi$ in $E$ such that if we write $\xi$ as an $n/f_0 \times n/f_0$ block matrix (with each block consisting of an entry in $F_{f_0}$), then the only nonzero blocks are those with indices $(i, j)$ such that $i \equiv j \bmod n_0$.

Thus if we rearrange the blocks (numbering the rows and columns from 0 to $n/f_0 - 1$) so that they appear in the order $0, n_0, \ldots, (e_0 - 1)n_0, 1, n_0 + 1, \ldots, (e_0 - 1)n_0 + 1, \ldots, n_0 - 1, \ldots, n/f_0 - 1$, then $\xi$ is a “diagonal” matrix of the form $(\xi_0, \ldots, \xi_{n_0 - 1})$, where each $\xi_i$ is an $e_0 \times e_0$ block matrix (whose blocks are elements of $F_{f_0}$). We require:
(iii) The $\xi_i$ are all equal, and $\xi$ generates $A_{e_0}^1$ over $A_{e_0}^0$. For example, assume that $n = 6$, $f_0 = f = 1$, $e_0 = 2$. Then (i) is trivial, (ii) says that $\xi$ is of the form

$$
\begin{bmatrix}
  a & 0 & 0 & b & 0 & 0 \\
  0 & c & 0 & 0 & d & 0 \\
  0 & 0 & e & 0 & 0 & f \\
  g & 0 & 0 & h & 0 & 0 \\
  0 & i & 0 & 0 & j & 0 \\
  0 & 0 & k & 0 & 0 & l \\
\end{bmatrix} (g, i, k \in P_F; \text{ other entries in } \mathcal{D}_F),
$$

and (iii) says that $a = c = e$, $b = d = f$, $g = i = k$, and $h = j = l$. Notice, incidentally, that every element of $E$ is of this form, from our convention about $F_{f_0}$ and the fact that $F_{f_0}[\xi] = E$.

Occasionally we replace (iii) by

(iii') The $\xi_i$ are conjugate in $GL_{e_0}(F_{f_0})$ and generate $A_{e_0}^1$ (over $A_{e_0}^0$) there; we then say that $E$ is embedded.

Recall that the permutation of rows and columns of a matrix in the same way (so that the $i$th row becomes the $j$th and the $i$th column also becomes the $j$th) is achieved by a conjugation, $x \mapsto P^{-1}xP$, where $P$ is the permutation matrix with 1’s in the entries labeled $(i, j)$ and 0’s elsewhere. (A similar result applies to block matrices.) Let $P$ be the permutation implementing the above permutation of rows and columns. Now let $P^{-1}\eta_sP$ be the $n_0 \times n_0$ block matrix (with $e_0 f_0 \times e_0 f_0$ blocks) such that the $(i, j)$ block is 0 unless $j-i \equiv 1 \mod n_0$, is $I$ if $j-i = 1$, and is $\xi_0$ if $j = 0$ and $i = n_0 - 1$. Then $(P^{-1}\eta_sP)^{e_0} = P^{-1}\xi P$, so that $\eta_s^{e_0} = \xi$; moreover, $\eta_s$ is an $n/f_0 \times n/f_0$ block matrix (with block entries in $F_{f_0}$) such that the only nonzero blocks are those satisfying $j-i \equiv 1 \mod n_0$. Thus $\eta_0 = \eta_s^{f_0/e_0}$ generates $A_{e}^1$ over $A_{e}^0$ and satisfies $\eta_0^{e_0/e_0} = \xi$ and $\eta_0 \alpha \eta_0^{-1} = \alpha^\sigma$ if $\alpha \in m_{f_0}(e_0)$. In the example above, the permutation matrix $P$ has the effect of shifting the order of rows and columns (originally from 0 to 5) to 0, 3, 1, 4, 2, 5, so that

$$
P^{-1}\xi P =
\begin{bmatrix}
  a & b & 0 & 0 & 0 & 0 \\
  g & h & 0 & 0 & 0 & 0 \\
  0 & 0 & a & b & 0 & 0 \\
  0 & 0 & g & h & 0 & 0 \\
  0 & 0 & 0 & 0 & a & b \\
  0 & 0 & 0 & 0 & g & h \\
\end{bmatrix}.
$$

Then

$$
P^{-1}\eta_sP =
\begin{bmatrix}
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 \\
  a & b & 0 & 0 & 0 & 0 \\
  g & h & 0 & 0 & 0 & 0 \\
\end{bmatrix},
$$
and

\[
\eta_* = \eta_0 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
a & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & 1 \\
g & 0 & 0 & h & 0
\end{bmatrix}.
\]

Then matrix algebra \( M_{n_0}(E) \) is generated over \( E \) by \( \eta_* \) and \( m_{\eta_0}^d(e_0) \) or by \( \eta_*^d \) and \( m_{n/d_0}(e_0) \) for any \( d \) dividing \( n/f_0 \)—in particular, by \( \eta_0 \) and \( m_{e_0}^d(e_0) \).

When we say that \( E \) is nicely embedded, we shall also assume that \( \eta_0 \) is of the above form. We also say that \( M_{n_0}(E) \) or \( \text{GL}_{n_0}(E) \) is nicely embedded (or embedded) if \( E \) is.

Call \( \Pi \) and \((e_0, f_0)\)-permutation matrix if it is an \( e_0 \times e_0 \) block matrix where all off-diagonal blocks are 0, the diagonal are equal, and the common diagonal \( n_0 f_0 \times n_0 f_0 \) block is an \( n_0 \times n_0 \) permutation block matrix (so that each \( f_0 \times f_0 \) block is \( I \) or \( 0 \)). Thus if \( f_0 = 1 \), then \( \Pi \) is a permutation matrix in which the first \( n_0 \) rows and columns are permuted, the next \( n_0 \) rows and columns are permuted in the same way, and so on. Notice that if \( f_0 | f_0^o \), then every \((e_0, f_0^o)\)-permutation matrix is also an \((e_0, f_0)\)-permutation matrix; a similar statement holds if \( e_0 | e_0^o \). The \((e_0, f_0)\)-permutation matrices are all in \( \text{GL}_{n_0}(E) \).

Say that an element of \( G = \text{GL}_n(F) \) is a power-permutation matrix if each row and each column has only one nonzero entry and if that entry is of the form \( a \omega^j \), \( j \in \mathbb{Z} \) and \( a \in k^\times \). Such matrices are products \( \Pi u \), where \( \Pi \) is a permutation matrix and \( u = \text{diag}(a_0 \omega^{j_0}, \ldots, a_{n-1} \omega^{j_{n-1}}) \). Here is an expression for these matrices in the form \( \sum_{j=0}^{\infty} \alpha_j \omega^j \), \( \alpha_j \in m_e \). Let \( b_{i,j,h} \) (\( 0 \leq i, j < f - 1 \); \( 0 \leq h < e - 1 \)) be the element \((b_0, b_1, \ldots, b_{e-1})\), where \( b_{h'} = 0 \) unless \( h' = h \) and the only nonzero entry of \( b_h \) is a 1 in the \((i, j)\) position. (As above, we label the rows and columns of the \( b_h \) from 0 to \( e - 1 \), and those of elements of \( G \) from 0 to \( n - 1 \); extend the notation cyclically, so that, e.g., the \( j_{i} \)th row and \( j_{j} \)th row of \( b_h \) are the same if \( j_1 = j_2 \mod e \).) The \( b_{i,j,h} \) form, of course, the “obvious” basis for \( m_\epsilon \). A straightforward computation shows that a power-permutation matrix is of the form \( \sum_{l=1}^{n} a_i b_{i,j,l,h} \omega^{rm_l} \), where the \( a_i \in k^\times \) and the sets \( \{ f(h_l + i_l) \}, \{ f(h_l + m_l) + j_l \} \) (\( 1 \leq l \leq n \)) run through the conjugacy classes mod \( n \).

Now consider the group \( G_0 = \text{GL}_{n_0}(E) \), where \( E \) is a nicely embedded extension field, so that \( M_0 = M_{n_0}(E) \) is generated by \( m_{\eta_0}^\phi(e_0) \) and the element \( \eta_0 \) constructed above. Let \( \xi = \eta_{e/e_0}^e \), so that \( \xi \) is central in \( G_0 \). Recall that \( P^{-1} \xi P = \{ \xi_0, \ldots, \xi_0 \} \), a “diagonal” block \( n_0 \times n_0 \) matrix with all diagonal entries the same. Say that \( g \) is a power-permutation matrix of \( G_0 = \text{GL}_{n_0}(E) \) if \( P^{-1} g P \) is of the form \( Q \{ \alpha_0 \omega_0^{r_0}, \alpha_1 \omega_1^{r_1}, \ldots, \alpha_{n_0-1} \omega_{n_0-1}^{r_{n_0-1}} \} \), where \( Q \) is an \( n_0 \times n_0 \) block permutation matrix, the \( \alpha_j \in k_{f_0} \), and the \( r_i \) are integers. This is consistent with our definition for \( \text{GL}_n(F) \). Another description is as follows: let \( b_{i,j,h} \) (\( 0 \leq i, j < f_0 - 1 \); \( 0 \leq h < e/e_0 - 1 \)) be the element \((b_0, \ldots, b_{e-1}) \in m_{e_0}^\phi(e_0) \) such that (a) the \( b_h \) are periodic with period \( e/e_0 \); (b) for \( 0 \leq h' < e/e_0 \), \( b_{h'} = 0 \) unless \( h' = h \); and (c) \( b_h \) has only one nonzero block, an \( I \).
in the \((i, j)\) block. The \(b'_{i,j,h}\) give the "natural" basis for \(m_{e_0}(e_0)\) as a \(k_{f_0}\)-space. We can describe \(\eta_0\) as an \(e_0 \times e_0\) block matrix each of whose blocks is itself a block matrix with \(f_0 \times f_0\) blocks; as a result, \(\eta_0\) is an \(n/f_0 \times n/f_0\) block matrix where the blocks are elements of \(k_{f_0}\), and the \((i, j)\) block is nonzero only if \(j - 1 \equiv f/f_0 \mod n_0\). Hence the \((i, j)\) block of \(\eta_0^m\) is 0 unless \(j - i \equiv cf/f_0 \mod n_0\). Then the power-permutation matrices of \(G^0\) are the elements of the form \(\sum_{i=1}^{n_0} \alpha_i b'_{i,j,h} \eta_0^m\), where the \(\alpha_i \in k_{f_0}\) and the sets \(\{fh_l/f_0 + i_l\}, \{fh_1 + m_l)/f_0 + j_l\} (1 \leq l \leq n_0)\) run through the conjugacy classes \(\mod n_0\). As usual, \(G^0 = (K^1/n_0 \cap G_0) W_0(K^1/n_0 \cap G_0)\), where \(W_0 = \text{group of permutation-power matrices of } G^0\).

An example may help. Let \(n = 6\), \(f = 1\), and \(e_0 = 2\). One possible choice for \(\eta_0\) is

\[
\eta_0 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & \varpi & 0 & 0 \\
0 & \varpi & 0 & 0 & \varpi & 0 \\
0 & 0 & \varpi & 0 & 0 & \varpi \\
\end{bmatrix} .
\]

(Of course, \(\varpi\) determines \(E\).) Write \(\tau = \varpi^2 + \varpi\), so that

\[
\xi_0 = \eta_0^2 = \begin{bmatrix}
\varpi & 0 & 0 & \varpi & 0 & 0 \\
0 & \varpi & 0 & 0 & \varpi & 0 \\
0 & 0 & \varpi & 0 & 0 & \varpi \\
\varpi^2 & 0 & 0 & \tau & 0 & 0 \\
0 & \varpi^2 & 0 & 0 & \tau & 0 \\
0 & 0 & \varpi^2 & 0 & 0 & \tau \\
\end{bmatrix} .
\]

Therefore

\[
\xi_0 = \begin{bmatrix}
\varpi & \varpi \\
\varpi^2 & \tau \\
\end{bmatrix} .
\]

One permutation matrix (corresponding to

\[
Q = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

and \(\varpi_0^2, \xi_0, \xi_0^0)\) is

\[
\begin{bmatrix}
\varpi & 0 & 0 & a & 0 & 0 \\
0 & 0 & \varpi & 0 & 0 & \varpi \\
0 & 1 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & b & 0 & 0 \\
0 & 0 & \varpi & 0 & 0 & \tau \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}, \quad \text{where } \begin{bmatrix}
\varpi & a \\
a & b \\
\end{bmatrix} = \xi_0^2
\]

(thus \(a = \varpi(\varpi + \tau)\) and \(b = \varpi^3 + \tau^2\)). If we had \(n = 12\), \(f = 2\), and \(e_0 = 2\), the situation would be similar, but each entry would be replaced by a \(2 \times 2\) matrix (corresponding to some element of \(k_f\)).

In this paper, we will often construct \(E\) by constructing \(M_{n_0}(E)\). The following proposition is what we need:
(2.1) **Proposition.** Let $e_0|e$ and $f_0|f$. Suppose that $\eta_0 \in \text{GL}_n(F)$ satisfies:

(i) $\eta_0$ generates $A_e$;

(ii) $\eta_0$ normalizes $m_e^{\eta_0}(e_0)$, and conjugation by $\eta_0$ is $\sigma$ there.

Let $M_0$ be the subalgebra of $M_n(F)$ generated by $\eta_0$ and $m_e^{\eta_0}(e_0)$, and let $E_0$ be its center. Write $n_0 = n/e_0 f_0$. Then

(a) $E_0 = F_0[n_0^{e_0}]$, and $E_0$ is an embedded field;

(b) $e(E_0/F) = e_0$ and $f(E_0/F) = f_0$;

(c) $[M_0 : E_0] = n_0^2$.

**Proof.** The hypothesis implies that $\eta_0$ commutes with $k_f$ and hence with $F_f$. Since $1, \eta_0, \ldots, \eta_0^{e-1}$ are linearly independent over $F_f$ (because elements of $F_f \cap A_e$ are in $A_e$), $F_f[\eta]$ is a commutative subalgebra of dimension $\geq ef = n$ in $M_n(F)$. Therefore it is maximal and $1, \eta_0, \ldots, \eta_0^{e-1}$ form an $F_f$-basis for $F_f[\eta_0] \subseteq \text{GL}_e(F_f)$. This shows that the characteristic polynomial for $\eta_0 \subseteq \text{GL}_e(F_f)$ is irreducible; hence $F_f[\eta_0]$ is a field. Then $E' = F_0[n_0^{e_0}]$ is also a field. Write $[E' : F] = h; h \geq e_0 f_0$, since $1, \xi, \xi^2, \ldots, \xi^{e-1}$ are obviously linearly independent over $F_0$. In fact, $e(E'/F) \geq e_0$ and $f(E'/F) \geq f_0$.

Clearly $k_{f_0}$ (hence $F_{f_0}$) and $\xi = n_0^{e_0}$ commute with $m_e^{\eta_0}(e_0)$; hence $E'$ commutes with $m_e^{\eta_0}(e_0)$. We also see immediately that $E'$ commutes with $\eta_0$.

But the algebra generated by $\eta_0$ and $m_e^{\eta_0}(e_0)$ is an $E'$-algebra of dimension $\geq n_0^2$ over $E'$. Let $M'$ be the commutant of $E'$; $M' \cong M_{n/h}[E']$. Since $M_0 \subseteq M'$ (because $\eta_0$ and $m_e^{\eta_0}(e_0)$ commute with $E'$), these inequalities imply that $h = e_0 f_0$ and that $M_0 = M'$. Thus we have $e(E'/F) = e_0$ and $f(E'/F) = f_0$. Because $\xi$ commutes with $M_0$, it must satisfy (i) and (ii) of the criterion for $E'$ to be embedded; because it satisfies an equation of degree $e_0$ over $F_{f_0}$, the $\xi_i$ in (iii) must be conjugate in $\text{GL}_n(F_{f_0})$. \square

Let $E_0$ be nicely embedded, as above, and assume that $f_0 = f$. Let $\eta_0 \in \text{GL}_{n_0}(E_0)$ generate $A_e^1$, normalize $m_e^{\eta_0}(f_0)$, and act as $\sigma$ there. Say that $y = 1 + y_0 \in G$ is $(e_0, f_0)$-pure if there are integers $c, l$ such that $y_0$ can be written as $\gamma y_0^l$ with $\gamma = (\gamma_0, \ldots, \gamma_{e-1}) \in m_e$ and the $\gamma_a = 0$ unless $a \equiv c \mod e/e_0$. Then if one writes $y_0$ as an $e \times e$ block matrix with $f_0 \times f_0$ blocks, the only nonzero blocks are those with indices $(b, b')$ such that $b \equiv c$ and $b' \equiv c + l \mod e/e_0$. It is easy to verify that if $y$ is $(e_0, f_0)$-pure and $g$ is a power-permutation matrix for $E$, then $gyg^{-1}$ is $(e_0, f_0)$-pure; furthermore, any element $w \in K_e^1$ is a product of $(e_0, f_0)$-pure elements (corresponding to distinct pairs $(c, l)$). We will sometimes need this decomposition of $w$. Often $w$ will be in a subgroup $H$; it will always be easy to check when we use the decomposition that the terms $y$ are in $H$ and that if $g$ is a power-permutation matrix for $\text{GL}_{n_0}(E)$, then $gxg^{-1} \in H$ iff $gyg^{-1} \in H$ for every $y \sim$ in the above factorization.

This section contains what might be called approximation lemmas concerning nicely embedded $\text{GL}_n(E_0)$. Our running assumption for this section is that we have embedded fields $E_1, \ldots, E_i$, with $e(E_h/F) = e_h$ and $f(E_h/F) = f_h$; we assume that $1 = e_0|e_1| \cdots |e_i$ and that $1 = f_0|f_1| \cdots |f_i$, but we do not need to...
know that \( e_h f_h > e_{h-1} f_{h-1} \). (In applications, this inequality will hold if \( h < i \), but not necessarily for \( h = i \).) Set \( n_h = n / e_h f_h \). Assume that \( M_h = M_{n_h}(E_h) \) is generated by \( m_{e_h}(e_h) \) and an element \( \eta_h \) such that \( \eta_h \) generates \( A^1_e \), normalizes \( m_{e_h}(e_h) \), and acts as \( \sigma \) there. We let \( E_0 = F \), \( M_0 = M_n(F) \), and \( \eta_0 = \infty \).

(3.1) Lemma. Let \( E_h \), \( M_h \), and \( \eta_h \) be as above, and set \( G_h = \) group of invertible elements of \( M_h \). Suppose in addition that there are integers \( t_1 > t_2 > \cdots > t_i > 0 \) such that for \( h \geq 2 \),

\[
\eta_h = \delta_h \eta_{h-1} + \zeta_h, h-2 + \cdots + \zeta_h, 1 + \zeta_h, 0,
\]

where \( \zeta_{h,j} \in A^{h-j+1+1}_{e} \cap M_j \) and \( \delta_h \in F_{e_h} \). Then:

(i) For every \( x_h \in A^0_e \cap M_h \) (\( h \geq 1 \)), there are elements \( x_{h-1} \in M_{h-1} \), \( \ldots \), \( x_0 \in M_0 \) such that \( x_h = \sum_{j=0}^{h-1} x_j \) and \( x_j \in A^{e_{h-1}+t_h} \), \( 0 \leq j \leq h - 1 \).

(ii) For every \( x_h \in K_{e_h} \cap G_h \) (\( h \geq 1 \)), there are elements \( y_{h-1} \in G_{h-1} \), \( \ldots \), \( y_0 \in G_0 \) such that \( x_h = \chi^{h-1} y_{h-3} \cdots y_0 \) and \( y_j \in K_{e_{h+j+1}+t_h} \), \( 0 \leq j \leq h - 1 \).

(iii) If \( h \geq 1 \) and \( x_h \in Z_e K_e \cap G_h \) but \( x_h \notin K_{e_h} \), assume that \( x_h \in A^r_{e_h} \) but \( x_h \notin A^{t_h+1}_{e_h} \). Then there are elements \( z_{h-1} \in G_{h-1} \), \( \ldots \), \( z_0 \in G_0 \) such that \( x_h = z_{h-1} z_{h-2} \cdots z_0 \), \( z_h \in A^r_{e_h} \cap G_{h-1} \) but \( z_h \notin A^{t_h+1}_{e_h} \), and \( z_j \in K_{e_{h+j+1}^{-t_h}} \), \( 0 \leq j \leq h - 2 \).

(iv) Suppose that \( r_1, r_2 \) are integers and \( 1 \leq h \leq j \leq i \). Then

\[
(A^0_e \cap M_h)(A^0_e \cap M_j) \subseteq (A^0_{e_{r_1}+r_2} \cap M_{h-1}) + (A^0_{e_{r_1}+r_2} \cap M_{h-2}) + \cdots + A^0_{e_{r_1}+r_2+1} - t_h,
\]

and similarly for \( (A^0_e \cap M_j)(A^0_e \cap M_h) \).

(v) Suppose that \( r_1, r_2 \) are integers and that \( 1 \leq h < j \leq i \). Then

\[
(A^0_e \cap M_h)(A^0_e \cap M_j) \subseteq (A^0_{e_{r_1}+r_2} \cap M_{h+1}) + (A^0_{e_{r_1}+r_2+1} - t_h+1) \cap M_{h-1} + \cdots + A^0_{e_{r_1}+r_2+1} - t_h+1,
\]

and similarly for \( (A^0_e \cap M_j)(A^0_e \cap M_h) \).

(vi) For each \( r > 0 \) and each \( h \leq i \),

\[
1 + A^r_e \cap M_{h-1} + A^r_{e_{r_1}+r_2+1} - t_h \cap M_{h-2} + \cdots + A^r_{e_{r_1}+r_2+1} - t_h = (K_{e_h} \cap G_{h-1})(K_{e_{h+1}+1} - t_h \cap G_{h-2}) \cdots K_{e_{r_1}+r_2} - t_h\]

is a group normalized by \( G_h \cap Z_e K_e \); the intersection of this group with \( G_h \) is \( K_{e_h} \cap G_h \).

Proof. We use induction on \( i \). For \( i = 1 \), the only nonvacuous part is the statement in (v) that \( Z_e K_e \) normalizes \( K_{e_h} \), and this is standard. For \( i = 2 \), it suffices to prove (i) when \( x_2 \in A^r_{e_h} \) for some \( r > 0 \) (multiply by a central element if necessary), and it suffices to do this when \( x_2 \) is a power of \( \eta_2 \) (any \( x_2 \) is a sum of powers of \( \eta_2 \) with coefficients in every \( G_h \)). Now (i) is clear because \( \eta_2 \equiv u_1 \) mod \( A^0_{e_{r_1}+r_2+1} \) with \( u_1 \in G_1 \), by hypothesis, and then \( \eta_2 \equiv u_1 \) mod \( A^0_{e_{r_1}+r_2+1} \). For (ii), write \( x_2 = 1 + y_2 \), with \( y_2 \in A^r_{e_h} \), \( r > 0 \); set \( y_2 = y_1 + y_0 \), as in (i). Then

\[
x_2 = (1 + y_1)(1 + (1 + y_1)^{-1} y_0).
\]

Part (iii) is similar: we have \( x = z_1 + z_0 \), as in (i), and then \( x = z_1(1 + z_1^{-1} z_0) \). Part (iv) is trivial unless \( h = j = 2 \). It then suffices to prove that \( \eta_2 \eta_2^r \subseteq A^0_{e_{r_1}+r_2} \cap M_1 + A^0_{e_{r_1}+r_2+1} - t_h \cap M_0 \), and this is clear from (i). Part (v)
is interesting only if \( j = 2 \) and \( h = 1 \). Since \( A_\ell^2 \cap M_2 \subseteq A_\ell^2 \cap M_2 + A_\ell^{g+h-i-2} \), this case is also easy. As for (vi), we need to show first that

\[ K_e^{r+t-i}(G_1) = 1 + (A_e^r \cap M_1) + A_e^{r+t-i}. \]

If \( u_0 \in A_e^{r+t-i} \) and \( u_1 \in A_e^r \cap M_1 \), then \((1+u_0)(1+u_1) = 1 + u_0 + u_0 u_1 \) and \( 1 + u_0 + u_1 = (1 + u_1)(1 + (1 + u_1)^{-1} u_0) \), with \( u_0 + u_0 u_1 \), \((1 + u_1)^{-1} u_0 \in A_e^{r+t-i} \).

This set is clearly a group. That \( G_2 \) intersects it in \( G_2 \cap K_r \) follows from (i). If \( x_2 \in G_2 \cap Z_eK_e \), write \( x_2 = z_0 z_1 \) as in (ii) or (iii). Then \( z_0 \) normalizes the group because all commutators with elements of the group lie in \( K_e^{r+t-i} \); \( z_1 \) obviously normalizes \( K_e^r \cap G_1 \), and \( K_e^{r+t-i} \) is normal in \( Z_eK_e \). This finishes the proof of (vi).

Now assume the result for \( i - 1 \); we prove it for \( i \). For (i) it suffices, as in the case \( i = 2 \), to prove the result for \( \eta_i^r \) when \( r \geq 1 \). For \( r = 1 \), it holds by hypothesis. Assume the result for \( r - 1 \). Multiplying out the expansions of \( \eta_i^{-1} \) and \( \eta_i \), we see that we need to show that

\[ (A_i^{r-1+h-i-1} \cap M_h)(A_{\ell}^{r+i+t-i} \cap M_j) \subseteq (A_e^{r+h-i} \cap M_h) + (A_e^{r+t-i} \cap M_{h-1}) + \ldots + A_e^{r+t-i}. \]

if \( h \leq j \) (or the similar formula if \( h > j \)). If \( h = j \), this is obvious; for \( h < j \), this follows from (v). For (iii), write \( x = x_{i-1} + \ldots + x_0 \), from (i); then

\[ x = x_{i-1}(1 + x_{i-2}^{-1}x_{i-2} + \ldots + x_{i-1}^{-1}x_0). \]

Now (v) shows that

\[ x_{i-1}^{-1}x_{i-2} + \ldots + x_{i-1}^{-1}x_0 \in (A_i^{r-1-i} \cap M_{i-2}) + \ldots + (A_i^{r-i} \cap M_1) + A_i^{r-i}. \]

From (vi), \( x_{i-1}^{-1} x \in (K_e^{r-i} \cap G_{i-2}) \cdots (K_e^{r-i} \cap G_1)K_e^{r-i} \), and (iii) follows. The proof of (ii) is similar, but we use

\[ x = 1 + x_{i-1} + \ldots + x_0 = (1 + x_{i-1})(1 + (1 + x_{i-1})^{-1}(x_{i-2} + \ldots + x_0)). \]

Part (iv) follows from the inductive hypothesis unless either \( h = i \) or \( j = i \). If, say, \( h = i > j \), then \( A_e^r \cap M_i \subseteq (A_e^r \cap M_{i-1}) + (A_e^{r+i-t-i} \cap M_{i-2}) + \ldots + A_e^{r+t-i} \), and we again get the result from the inductive hypothesis. So the only case to check is where \( h = j = i \). Then \( A_e^r \cap M_i)(A_e^r \cap M_i) = (A_e^{r+i} \cap M_i) \), and the result follows from (i). As for (v), we may again assume by induction that \( j = i \). Write \( A_e^r \cap M_i = (A_e^r \cap M_{i-1}) + (A_e^{r+i-1-t-i} \cap M_i) + \ldots + A_e^{r+i-t-i} \), then use (v) (with \( j < i \)) repeatedly and note that \( (A_e^r \cap M_h)(A_e^{2+h-i} \cap M_h) = A_e^{r+2+i-h-i} \cap M_h \).

We still need (vi). Set \( H_r = (K_e^r \cap G_{i-1})(K_e^{r+t-i+1} \cap G_{i-2}) \cdots K_e^{r+t-i} \). Since \( H_r = \{(Z_eK_e \cap G_{i-1})(K_e^{r+t-i+1} \cap G_{i-2}) \cdots K_e^{r+t-i}\} \cap K_e^r \), this is a group by (vi) for the case \( i - 1 \). To see that \( H_r = 1 + A_e^r \cap M_{i-1} + A_e^{r+i-t-i} \cap M_{i-2} + \ldots + A_e^{r+t-i} \), write \( K_e^r \cap G_{i-1} = 1 + A_e^r \cap M_{i-1} \), etc., and use (v). To show that \( G_i \cap Z_eK_e \) normalizes \( H_r \), it suffices to see that \( \eta_i \) and \( G_i \cap K_e \) normalize it. To verify that \( \eta_i \) normalizes \( H_r \), note first that \( \eta_i \in \eta_j(K_e^{r-j-1} \cap G_{j-1}) \cdots K_e^{r-j-i} \); this follows from repeated use of (v). It is now easy to show that conjugation by \( \eta_i \) maps \( K_e^{r+t-i+1} \cap G_j \) into \( H_r \). Since \( G_i \cap K_e/G_i \cap K_e^r \) has coset representatives that are also in every \( G_j \) with \( j < i \), these elements normalize \( H_r \), and we
therefore need only prove that \( G_i \cap K_i^t \) normalizes \( H_r \). Since \((1+a)y(1+a)^{-1} = y + (ay - ya) - (aya - ya^2) + \cdots\), we see from (v) that this element is in \( H_r \). □

(3.2) Corollary. Let \( H_r = (K_e^t \cap G_{i-1})(K_e^{t+1-t_i} \cap G_{i-2}) \cdots K_e^{t+1-t_i} \), let \( j \geq r \), and let \( h \) be the index such that \( t_{h+1} - t_i + r \leq j < t_h - t_i + r \). (If \( j \geq t_i - t_i - r \), then \( h = 0 \).) Then \( H_r \cap K_i^j / H_r \cap K_i^{j+1} \cong G_h \cap K_i^j / G_h \cap K_i^{j+1} \), and

\[
H_r \cap K_i^j = (K_e^t \cap G_h)(K_e^{t+h-t_i} \cap G_{h+1}) \cdots K_e^{t+h-t_i}. 
\]

The subgroups \( K_i^{t-h_i}, K_i^{t-h_i} \cap G_1, \ldots, K_i^{t-h_i} \cap G_{i-2}, K_e \cap G_{i-1} \) normalize \( H_r \), as does \( \eta_i \).

Proof. Any element of \( H_r \) is of the form \( x = 1 + y_{i-1} + \cdots + y_0 \), with \( y_l \in M_l \cap A_{e_i}^{t_i+t_i+1-t_i} \), from (vi). Suppose that \( x \in K_i^j \) as well and that \( j \geq t_i - t_i + r \). Then \( y_{i-1} \) is the only term in the sum not automatically in \( A_{e_i}^{t_i+1-t_i+r} \); since \( x - 1 \in A_{e_i}^{t_i+1-t_i+r} \), we must have \( y \in A_{e_i}^{t_i+1-t_i+r} \). From (i), \( y \in A_{e_i}^{t_i+1-t_i+r} \cap G_{i-1} \subseteq A_{e_i}^{t_i+1-t_i+r} \cap M_{i-2} + \cdots + A_{e_i}^{t_i+1-t_i+r} \cap M_0 \). Thus we may delete \( y_{i-1} \) from the sum (perhaps changing the other \( y_h \)). Proceeding inductively, we see that \( x - 1 + y_h + \cdots + y_0 \), \( y_l \in M_l \cap A_{e_i}^{t_i+t_i+1-t_i} \), and \( y_h \in A_{e_i} \) as well. So modulo \( K_i^{t_i+1} \), \( x \equiv 1 + y_h \), \( y_h \in M_h \cap A_{e_i} \) (and \( y_h \) determined mod \( M_h \cap A_{e_i+1} \)). The second part now follows by a proof like that of (ii). For the last part, suppose that \( y \in H_r \) and \( w \in K_i^{h-t_i} \cap G_{h-1} \), and write \( w = 1 + w_0 \). (As in Lemma 3.1, there are coset representatives for \( K_e \cap G_{i-1}/K_i^t \cap G_{i-1} \) that obviously normalize \( H_r \), so that we may assume \( w \in K_i^t \).) Then \( u w v w^{-1} = y + (w_0 y - y w_0) - (w_0 y w_0 - y w_0^2) + \cdots \); from (iv) and (v) of the lemma, this expression is in \( H_r \). We saw in the course of proving Lemma 3.1 that \( \eta_i \) normalizes \( H_r \); since it also normalizes every \( K_e^t \), it normalizes \( H_r \). □

We now prove a similar result for power-permutation matrices. The hypothesis that \( f_{i-1} = f_i \) is convenient and does not cause any trouble; it holds in most applications, and in the others one can always work with the compositum of \( E_{i-1} \) and \( F_{f_i} \).

(3.3) Lemma. Let notation and hypothesis be as in Lemma 3.1, except that the \( E_h \) are all nicely embedded, \( f_i = f_{i-1} \), and \( \eta_i \equiv \eta_{i-1} \mod A_2^t \). Then for any power-permutation matrix \( x \) of \( G_i \), there is an element \( u \in (K_e^{t_i-t_i+1} \cap G_{i-2}) \cdots (K_e^{t_i-t_i+1} \cap G_0) \) such that \( u x \) is a power-permutation matrix of \( G_{i-1} \).

Proof. All the matrices we will deal with are block matrices with each \( f_i \times f_i \) block in \( F_{f_i} \). Hence there is no loss of generality in working only with matrices commuting with \( F_{f_i} \); thus we may (and do) assume that \( f_i = 1 \). Because of the way we have written permutation matrices, it is more convenient to show that there is an element \( u \) such that \( x u \) is a power-permutation matrix of \( G_{i-1} \); since the inverse of a power-permutation matrix is a power-permutation matrix, this proves the lemma. (One could also give a direct proof, at the cost of complicating indices.)

Before giving the full proof, we give an example to illustrate the procedure. Suppose that \( n = e = 4 \), that \( e_1 = e_2 = 2 \), and that (writing \( \varpi \) and \( \varpi_F \))

\[
\eta_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varpi & 0 & 0 & 0 \\ 0 & \varpi & 0 & 0 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varpi & 0 & \varpi & 0 \\ 0 & \varpi & 0 & \varpi \end{bmatrix}.
\]
Then $\eta_1$ is composed of blocks $\xi_1$ and $\eta_2$ is composed of blocks $\xi_2$, where

$$\xi_1 = \begin{bmatrix} 0 & 0 \\ \omega & 1 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 0 & 0 \\ \omega & 1 \end{bmatrix}.$$ 

Set $a = 1 + \omega$, $b = 2 + \omega$, and $c = 1 + 3\omega + \omega^2$, and consider the power-permutation matrices for $G_1$, $G_2$ corresponding to $[\xi_1^t]$, $[\xi_2^t]$ respectively. They are

$$x_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \omega^2 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \omega \omega^2 & 0 & b \omega^2 & 0 \\ 0 & \omega & 0 & \omega \\ b \omega^3 & 0 & c \omega^2 & 0 \end{bmatrix}.$$ 

Notice that $\xi_2^4 = \xi_1^t [\omega b c]$ and $\xi_2 = \xi_1 [1 1]$. From this, $x_2 = x_1 k$, where

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 1 \\ \omega b & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

it should be clear how $k$ is constructed from the two $2 \times 2$ matrices above.

We return to the proof. We use the notation of §2 for power-permutation matrices, so that $b^t(r, s, h)$ is the element $(b_0, \ldots, b_{e-1})$ in $m_e^t(e_i)$ (recall: $f_i = 1$) such that the $b_{h'}$ are periodic with period $e/e_i$, $b_{h'} = 0$ unless $h' \equiv h \mod e/e_i$, and $b_h$ has only one nonzero entry, a 1 in the $(r, s)$ place. Let $u_c$ be a matrix whose only nonzero entries are at indices $(r, s)$ with $r \equiv s \equiv c \mod n/e_i = n_i$. Then it is easy to check that $b_{r_1, s_1, h}^t u_c = 0$ unless $s_1 + f(h_1 + m_1) \equiv c \mod n_i$. Write $a_l = s_l + f(h_1 + m_1)$. One consequence of the above calculation is the following: let $u$ be a matrix whose only nonzero entries are at indices $(r, s)$ with $r \equiv s \equiv c \mod n_i$, and let $u_{(c)}$ be the matrix whose $(r, s)$ entry is that of $u$ if $r \equiv s \equiv c \mod n_i$ and is 0 otherwise. Then

$$b_{r_1, s_1, h_1}^t u = b_{r_1, s_1, h_1}^t u_{(c)} .$$

Let $x = \sum_{i=1}^{n_i} a_i b_{r_i, s_i, h_i}^t u_{(c)}$ and $x' = \sum_{i=1}^{n_i} a_i b_{r_i, s_i, h_i}^t u_{(c)}$. Write $u(l) = \eta_l m_i \eta_{l-1}$ and set $u = \sum_{i=1}^{n_i} u(l) a_i$. From the above remarks,

$$x u(l) a_i = a_i b_{r_i, s_i, h_i}^t u_{(c)} = a_i b_{r_i, s_i, h_i}^t u_{(c)};$$

therefore $x' = xu$. Furthermore, the construction gives $u - 1 \in (A_e^{s_i - s_i - 1} \cap M_{e_{i-1}}) + \cdots + (A_e^{s_i - s_i - 1} \cap M_0)$. This suffices to prove the result, by Lemma 3.1.

The last result of this section is of a different sort; it lets us replace congruences by equalities in certain circumstances. In it, we assume that the representatives of elements in $m_e$ satisfy the conditions given in §1.

(3.4) Lemma. Let $f_i$ be one of $f_1, \ldots, f_r$, and let $e_0, s$ satisfy $e_0 = e/(e, s)$. Suppose that $\alpha$ is the lift of an element in $k_e^t$ such that $(\alpha \omega^s)^{e_0} = \gamma \omega^{se_0}$, where $\gamma$ is a root of unity generating $F_{f_i}/F$. Assume that $\beta$ is our lifting of $\beta \in m_e$ and that $\alpha \beta \alpha^s - \beta \alpha^s \equiv 0 \mod A_e^t$. Then $\alpha \omega^s$ and $\beta \omega^j$ commute.

Proof. By hypothesis, $\alpha \omega^s \beta \omega^j - \beta \omega^j \alpha \omega^s \equiv 0 \mod A_e^{t+s+1}$. An easy induction gives $(\alpha \omega^s)^{e_0} \beta \omega^j - \beta \omega^j (\alpha \omega^s)^{e_0} \equiv 0 \mod A_e^{t+s+1}$, or (letting $e_0 s = et$)
\[ \gamma \omega_F^j \beta \omega_F^j - \beta \omega_F^j \gamma \omega_F^j \equiv 0 \mod A_e^{j+e_0 s+1}. \]

Since \( \omega_F^j \) is central and \( \gamma \) commutes with \( \omega_F^j \), this gives \( \gamma \beta - \beta \gamma \equiv 0 \mod A_e^j \). Let \( \bar{\gamma} \) be the image of \( \gamma \) in \( k_e^* \). Then \( \bar{\gamma} \beta - \bar{\beta} \gamma = 0 \), or \( \beta \in m_e^{e_0} \). Since \( \gamma \in F_{e_0} \) and we chose representatives of \( m_e^{e_0} \) to commute the representatives of \( k_e^* \), \( \gamma \beta = \beta \gamma \) and the components of \( \beta \) commute with those of \( \alpha \). Therefore

\[ \alpha \omega^s \beta \omega^j = \alpha \beta^{a^i} \omega^{s+j} = \beta \alpha^{\sigma^j} \omega^{s+j} = \beta \omega^j \alpha \omega^s \]
(both \( \alpha \beta^{a^i} \) and \( \beta \alpha^{\sigma^j} \) are in \( m_e^{e_0} \), since \( m_e^{e_0} \) is closed under left multiplication by \( k_e^* \) and under \( \sigma \); as \( \alpha \beta^{a^i} \equiv \beta \alpha^{\sigma^j} \mod A_e^j \), we must have equality). \( \square \)

**Remark.** We often use this result with nicely embedded fields and matrix algebras, replacing \( \omega \) with \( \eta_0 \).

4

In this section we give results about elements commuting with various characters. The first few are like results in [4] and are used to prove that certain characters can be extended; they involve commutators.

(4.1) **Lemma.** Fix \( e, \) and let \( x \in K_e^* \cap (G, G) \), where \( r \geq 1 \) and \( G = \text{GL}_n(F) \). Assume that the residue class field \( k \) has more than two elements. Then there are elements \( u_j \) and \( v_j \) (\( 1 \leq j \leq s \)) such that each \( v_j \in K_e^* \), each \( u_j \) is either \( \alpha \omega \) or \( \alpha \), where \( \alpha_j \in m_e^* \), and

\[ x \equiv (u_1, v_1) \cdots (u_s, v_s) \mod K^{r+1}. \]

**Proof.** Suppose first that \( n \mid f r \), so that \( e \mid r \). We have

\[ (\alpha, 1 + \beta \omega^r) \equiv 1 + (\alpha \beta (\alpha^r)^{-1} - \beta) \omega^r \mod K^{r+1}. \]

Let \( S = \text{span} \) of the elements \( \alpha \beta (\alpha^r)^{-1} - \beta \). Then \( x = 1 + \gamma \omega^r \) is (mod \( K^{r+1} \)) a product of these commutators if \( \gamma \in S \). We show that \( S = m_e \) when \( \text{card} \, K > 2 \). Suppose that \( \delta \perp S \) under the bilinear map \( (\gamma, \delta) = \text{Tr} \gamma \delta \).

Then for all \( \alpha \in m_e^* \) and all \( \beta \in m_e \),

\[ 0 = \text{Tr} (\delta \alpha (\alpha (\alpha^r)^{-1} - \beta). \]

Replacing \( \beta \) by \( \beta \alpha^{\sigma^j} \), we get

\[ 0 = \text{Tr} (\delta \beta - (\delta \beta)^{\sigma^j}). \]

By the hypothesis on \( k \), \( m_e^* \) spans \( m_e \). (If \( f > 1 \), then the matrix \( e_{i,j} \) with one nonzero element, 1 in the \( (i, j) \) place, is \( (I + e_{i,j}) - I \) if \( i \neq j \); if \( i = j \), it is \( (J + e_{i,i}) - J \), where \( J \) is a cyclical permutation matrix. If \( f = 1 \), we need \( k \) to have \( > 2 \) elements; then the construction is easy.) Therefore this last holds for all \( \alpha \in m_e \), and \( \beta \delta = (\delta \beta)^{\sigma^r} \) \forall \beta \in m_e \). Let \( \beta = (\beta_0, \ldots, \beta_{e-1}) \) have \( \beta_j = I \) and \( \beta_i = 0 \) for \( i \neq j \). Then \( \delta = (\delta_0, \ldots, \delta_{e-1}) \) must have \( \delta_j = 0 \). Hence \( \delta = 0 \), and \( S = m_e \).

Assume \( n \mid f r \); let \( f r = n t \), and write \( x = 1 + \gamma \omega^r + \cdots \). A straightforward computation gives

\[ \text{Det} \, x \equiv 1 + (\text{Tr} \gamma) \omega_F^j \mod P^{r+1}. \]
Hence $\text{Tr}(\gamma) = 0$. We also have

$$(\alpha \varpi, 1 + \beta \varpi') = 1 + (\alpha \beta \sigma^{-1} - \beta) \varpi' \mod K_{r+1}'.$$

Thus it suffices to show that the span $S$ of the elements $\alpha \beta \sigma^{-1} - \beta$ is the space $T$ of elements $\gamma$ with $\text{Tr} \gamma = 0$. Since $S \subseteq T$, we need only show that $S^\perp \subseteq T^\perp$ under the bilinear form $(\gamma, \delta) = \text{Tr} \gamma \delta$.

So suppose that $0 = \text{Tr}(\alpha \beta \sigma^{-1} - \beta) \delta = \text{Tr} \beta[(\alpha \sigma^{-1} - \sigma^{-1}) \sigma^{-1} - \delta]$ for all $\alpha \in m_\sigma^r$ and all $\beta \in m_e$. Then $(\alpha \sigma^{-1} - \sigma^{-1}) \sigma^{-1} = \delta$, $\forall \alpha \in m_\sigma^r$. For $\alpha = I$, we see that $\delta \sigma^{-1} = \delta$, or that $\delta = (\delta_0, \delta_0, \ldots, \delta_0)$; it is now also clear that $\delta_0$ must commute with every element of $\text{GL}_e(k)$, so that $\delta$ is central. But then $\delta \in T^\perp$, and we are done. $\Box$

Remark. An obvious induction lets us write $x$ modulo $K^s_\sigma$ ($s > r$) as a product of commutators. Lemma 4.1 (like the other results of this section) will also be applied to $x \in K^s_\sigma \cap (\{Z_e K_e \cap M_{n_0}(E_0)\}, \{Z_e K_e \cap M_{n_0}(E_0)\})$, where $E_0$ is nicely embedded.

The next result depends on Lemma 2.2 of [4]. We give a restatement here in a form that will be useful. Let $\mathcal{A} = \mathbb{Z}[[a, b]]$ be the ring of formal power series in noncommuting variables $a$, $b$, and regard $\mathcal{A}$ as a subring of $\mathbb{R} = \mathcal{A}[a^{-1}, b^{-1}]$. Let $\mathcal{A}_n$ be ideal in $\mathcal{A}$ generated by all words of length $n$. For $r$, $s$ positive integers, give each word in $\mathcal{A}$ a weight by giving $a$ weight $r$ and $b$ weight $s$, and summing the weights of the letters in a word to get the weight of the word (e.g., $abab^2$ has weight $2r + 3s$). Consider

$$x = (1 + a)(1 + b)(1 + a)^{-1}(1 + b)^{-1} = (1 + a)(1 + b)(1 - a + a^2 + \cdots)(1 - b + \cdots).$$

Given any integer $n > 0$, there exist an integer $N > 0$ and elements $c_1, \ldots, c_N$, $d_1, \ldots, d_N \in \mathcal{A}$ such that:

(i) $x$ is congruent mod $\mathcal{A}_n$ to the product of commutators $(c_1, d_1) \cdots (c_N, d_N)$;

(ii) $d_j - 1$ is a word, $c_j$ is one of $a$, $a^{-1}$, $b$, $b^{-1}$, and $c_j d_j c_j^{-1} \in \mathcal{A}$;

(iii) if $d_j - 1$ has weight $\leq 2s$, then $c_j = a$ or $a^{-1}$.

(One can replace (iii) by (iii'): if $d_j - 1$ has weight $\leq 2r$, then $c_j = b$ or $b^{-1}$. Note that if $d_j - 1$ has weight $2s$, then the proof implies that it cannot be $b^2$.)

(4.2) Lemma. With the above notation, suppose that $E_0$ is nicely embedded in $M_n(F)$ and that $\text{card}(k) > 2$. Set $M_0 = M_{n_0}(E_0)$ and let $\eta_0$ be the element of $M_0$ that generates $A^1_e$, normalizes $m_0^\circ(f_0) = m_0$, and acts on $m_0$ as $\sigma$. Let $\chi$ be a character defined on a subgroup $H$ of $K^1_e$ containing some $K_e^r$, with $\chi$ trivial on $K^r_e$, and assume that $H$ is generated by a set of elements of the form $1 + \beta \eta_i$, $\beta \in m_\sigma^r$, and $\eta_i$, a generator of $A^1_e$ over $A^0_e$. Assume also that $\text{GL}_{n_0}(E_0) \cap Z_e K_e$ normalizes $H$ and that $\{x - 1 : x \in H\}$ is closed under multiplication and under multiplication by $m_0^\circ$ and by $\eta_0$.

(a) If $\chi^w = \chi$ for all $w \in m_\sigma^r$ and for $w = \eta_0$, then $\chi^w = \chi$ for any $w \in \text{GL}_{n_0}(E_0) \cap Z_e K_e$.

(b) If

(i) $H \subseteq K^r_e$;
(ii) for all \( j \geq r \) the group \( H \cap K_e^r / H \cap K_e^{r+1} \) has a set of (coset representatives of) generators \( y = 1 + y_0 \) such that for all such \( y \), \( y_0 \) is invertible and normalizes \( \chi \);

(iii) \( \chi^{y_0} = \chi \); and

(iv) \( \chi^w(y) = \chi(y) \) for all \( y \in H \cap K_e^{r+2t} \) when \( w \in m_0^x \) or \( w = \eta_0 \).

then \( \chi^w = \chi \) (on \( H \)) for all \( w \in GL_{n_0}(E_0) \cap K_e^i \).

Proof. (a) Any \( w \in GL_{n_0}(E_0) \cap Z_eK_e \) can be written as

\[
w = \alpha_0\eta_0^j(1 + \alpha_1\eta_0 + \alpha_2\eta_0^2 + \cdots), \quad \alpha_j \in m_0 \text{ and } \alpha_0 \in m_0^x.
\]

Since \( \alpha_0\eta_0^j \) fixes \( \chi \), we may assume that \( \alpha_0 = 1 \) and \( j_0 = 0 \). Furthermore, an easy induction makes it clear that is suffices to consider monomials \( w = 1 + \alpha\eta_0 \), \( j > 0 \). Similarly, it suffices to consider \( \chi^w(y)\chi(y^{-1}) \) for generating elements \( y = 1 + \beta\eta_0 \). If \( \alpha \) is invertible, the result holds by an application of Lemma 2.2 in [4]. If \( \alpha = \beta + \gamma \), where \( \beta, \gamma \in m_0^x \), then it holds because we have

\[
1 + \alpha\omega_0 = (1 + \beta\omega_0)(1 + \gamma\omega_0)(1 - \beta\omega_0\gamma\omega_0) \cdots,
\]

where for any \( s \) the formula holds mod \( K_e^s \) after finitely many terms. Since the result holds for each term on the right (by Lemma 2.2 of [4]), it holds for \( \alpha\omega_0 \). A similar calculation shows that a similar result holds if \( \alpha \) is any linear combination of invertible elements. But if \( k \) has more than two elements, then every element in \( m_0^x \) is a linear combination of invertible elements.

(b) The proof is essentially the same. The main point to notice is that if \( w = 1 + w_0 \in GL_{n_0}(E_0) \cap K_e^i \) and \( 1 + y_0 \in H \), then words in \( w_0 \) and \( y_0 \) that appear in the commutators and have only one \( w_0 \) can be taken care of (according to the result given before the lemma) by conjugating by \( y_0 \), and \( \chi^{y_0} = \chi \). Those with at least two \( w_0 \)'s are all in \( H \cap K_e^{r+2t} \), and there \( \chi^{w_0} = \chi \) as well.

\( \square \)

(4.3) Note. We often can apply the reasoning of Lemma 4.2 even when the hypotheses do not apply. Suppose that we have a subgroup \( H \) of \( K_e^i \), a character \( \chi \) on \( H \), an element \( 1 + y \in H \), an element \( 1 + \eta_0 \), and an element \( 1 + \alpha\eta_0 = 1 + (\alpha_1 + \cdots + \alpha_h)\eta_0^j \) such that:

(i) \( \alpha_i \in m_0^x \), all \( i \), and \( \beta \in m_0^x \);

(ii) if \( w \) is any sum of words of the form \( u_1u_2\cdots u_m \), where \( m \geq 2 \) and each \( u_j \) is either \( y \) or an \( \alpha_i\eta_0^j \), then \( 1 + w \in H \);

(iii) \( \chi^x = \chi \) (in that they agree on their common domain) if \( x = \alpha_i\eta_0^j \);

(iv) \( k \) has more than two elements.

Then the reasoning of Lemma 4.2 shows that \( \chi((1 + \alpha\eta_0^j, 1 + y)) = 1 \). When we use this reasoning we refer to (4.3). It should be clear when we so refer that the conditions are met. A similar argument shows that \( \beta \) need only be a sum of invertible elements.

(4.4) Remark. The restrictions on \( k \) in (4.1)-(4.3) are annoying. In our uses of these results, it is possible to avoid the restrictions. Consider, for example, Lemma 4.2 when \( q = 2 \). Let \( N \) be a large prime (larger than \( n \)), and let \( F^\sim \) be the unramified extension of \( F \) with \( [F^\sim : F] = N \). Choose a character \( \psi^\sim \) of \( F^\sim \) so that \( \psi^\sim|_F = \psi \). Then \( G \) embeds naturally into \( G^\sim = GL_n(F^\sim) \).
and we have subgroups $Z_e^\sim$, $K_e^\sim \subset G^\sim$ defined like $Z_e$, $K_e \subset G$. In our constructions of $\chi$ and $H$, there will always be a corresponding character $\chi^\sim$ on a subgroup $H^\sim \subset K_e^\sim$ such that $H^\sim = H \cap G$ and $\chi^\sim|_H = \chi$; furthermore, the hypotheses of Lemma 4.2 will also hold for $(m_e^\sim)^f(e_0)^x$, the group in $K_e^\sim$ corresponding to $m_e^f(e_0)^x \subset K_e$. Let $E_0^\sim$ be the compositum of $E_0$ and $F^\sim$. Lemma 4.2 says that if $w \in \text{GL}_{n_0}(E_0^\sim) \cap Z_e^\sim K_e^\sim$, then $(\chi^\sim)^w = \chi^\sim$. For $w \in \text{GL}_{n_0}(E) \cap Z_e K_e = \text{GL}_{n_0}(E_0) \cap Z_e^\sim K_e^\sim \cap G$, we then see that $w$ normalizes $H^\sim \cap G = H$ and $\chi^w = \chi$. This means that in all cases where we will apply Lemma 4.2, it will hold even if $q = 2$. The same will apply to our uses of (4.3).

When we use Lemma 4.1 to show that certain characters extend (because they are trivial on commutators), a similar argument extends the results to the case where $q = 2$. In the rest of this paper, we will apply (4.1)–(4.3) to the case $q = 2$ without comment. A similar remark applies to the following lemma.

(4.5) Lemma. With notation as above, assume that $k$ has more than two elements, and let $\chi$ be the character on $K_e^\sim$, trivial on $K_{e+1}^\sim$ $(r \geq j > 0)$, such that $Z_e^\sim K_e^\sim$ commutes with $\chi$. Then $\chi$ factors through $\text{Det}$ (and hence extends to $G$ as a character).

Proof. It suffices to show that $\chi(y) = 1$ if $y \in K_e^\sim$ and $\text{Det} y = 1$. Then $y \in (G, G)$, and Lemma 4.1 shows that $y$ is (mod $K_{e+1}^\sim$) a product of commutators $uvu^{-1}v^{-1}$ with $\chi(uvu^{-1}v^{-1}) = 1$. □

(4.6) Lemma. Let $\chi$ be a character on $K_e^m$, $m \geq 1$, that is trivial on $K_e^{m+1}$. Let $\alpha$ be such that $\chi(1 + \gamma w^m) = \psi \circ \text{Tr}(\alpha^{\sigma^m} \gamma) = \psi \circ \text{Tr}(\alpha^{\sigma^m} \gamma)$, $\forall \gamma \in m_e$, and assume that $\alpha w^{-m}$ generates a nicely embedded field $E_0$ of ramification index $e_0$ and residue class degree $f_0$, and that $\alpha \in k_{f_0}$. Write $M_0 = \text{ring of elements commuting with } \alpha w^{-m}$, $G_0 = G \cap M_0$. Let $j < m$, and let $e_1$, $f_1$ satisfy $e_0|e_1$, $f_0|f_1$, $e|j$. For $\delta \in k_{f_1}^\times$, set

$$\chi_\delta(y) = \chi(\gamma y w^{-1} y^{-1})$$

where $\gamma = 1 + \gamma w^j$ and $w = 1 + \delta w^{m-j}$. Then:

(i) $\chi_\delta(y) = 1$ for all $\delta$ iff $y \in K_e^\sim \cap G_0$.

(ii) $\chi_\delta$ is trivial iff $w \in G_0$.

(iii) The $\chi_\delta$ exhaust the characters $\chi^\#$ on $K_e^\sim$ trivial on $K_e^{j+1}(K_e^\sim \cap G_0)$ and of the form $\chi^\#(1 + \gamma w^j) = \psi \circ \text{Tr}(e^{\gamma \sigma^j})$ for some $e \in k_{f_1}^\times$.

Proof. Consider the $\chi_\delta$, $\delta$ as above. These $\delta$ obviously form a $k_{f_1}$-vector space of dimension $e$; furthermore, we have

$$\chi(\gamma y w^{-1} y^{-1}) = \chi(1 + (\gamma \delta^{\sigma^j} - \delta^j \gamma^{\sigma^m}) w^m)$$

$$= \psi \circ \text{Tr}(\gamma^{\delta^{\sigma^j} - \delta^j \gamma^{\sigma^m}}) = \psi \circ \text{Tr}(\gamma^{\delta^{\sigma^j} - \delta^j \gamma^{\sigma^m}}) = \psi \circ \text{Tr}(\delta^{\sigma^j} - \delta^j \gamma^{\sigma^m})^{\sigma^j}.$$

This is identically 1 for all $\gamma$ iff $[\delta \sigma^{m-j}, \alpha \sigma^m] \equiv 0 \text{ mod } A_{e-1}$. As we saw in Lemma 3.4, this means that $\delta \sigma^{m-j}$ commutes with $\alpha \sigma^m$, which proves (ii). (Notice that $F[(\alpha \sigma^m)\sigma^e] = F_{f_0}$, so that the hypotheses of Lemma 3.4 are satisfied.) To see that the subspace of such $\delta$ has dimension $e/\epsilon_0$ over $k_{f_1}$, written $\delta = (\delta_0, \ldots, \delta_{e-1})$, $\alpha = (\alpha_0, \ldots, \alpha_{e-1})$. Then $\delta^{\sigma^j} = \alpha \sigma^{m-j}$, so that $\delta_i \alpha_{i-j+m} = \delta_{i-m} \alpha_i$ for all $i$ (all entries commute; we extend
the definition of the \( \delta_i \) and \( \alpha_i \) cyclically). Since the \( \alpha_i \) are nonzero, all \( \delta_i \)
with \( l \equiv i \mod m \) are determined by \( \delta_i \). Therefore the subspace of these \( \delta \)
has dimension \( (e, m) \). But \( e/(e, j) = e_0 \).

The characters \( \chi_\delta \) are all trivial on elements \( y \in K^j_e \cap G_0 \), since
\[
\chi(wyw^{-1}y^{-1}) = \psi \circ \text{Tr}(\alpha \gamma^{\sigma^{-m}} - \gamma \alpha^{\sigma^j})^{\sigma_{m-j}},
\]
and the term in parentheses is 0 iff \( [\gamma \varpi^j, \alpha \varpi^{-m}] = 0 \) (see Lemma 3.4). This
proves (i).

If \( \beta \in (m_e^\text{e}((e_1))^\text{x} \), then \( \beta \) commutes with \( \varpi^m \) and with \( \chi \), so that
\[
\chi_\delta(y) = \chi(\varpi^m y \beta^{-1} \varpi^{-1} \beta y^{-1}) = \chi(\varpi^m yw^{-1}y^{-1} \beta^{-1})
\]
\[
= \chi(\beta)(\varpi^m yw^{-1}y^{-1}) = \chi(y).
\]
Hence every \( \chi_j \) commutes with \( (m_e^\text{e}((e_1))^\text{x} \) and is therefore a \( \chi^* \).

The characters \( \chi^* \) satisfy
\[
\chi^*(y) = \chi^*(1 + \varpi^j) = \psi \circ \text{Tr}(\gamma^{\sigma^j})
\]
for some \( \epsilon \in K^e_f \). Since \( \psi^*(y) = 1 \) for \( \varpi^j \in M_0 \), we have \( \psi \circ \text{Tr}(\gamma^{\sigma^j}) = 1 \)
if the entries of \( \gamma \) are periodic with period \( e_0 \), or \( \psi \circ \text{Tr}(\epsilon^0) \gamma \text{Tr}_0 \epsilon = 1 \)
for all such \( \gamma \). Therefore \( \text{Tr}_0 \epsilon_0 = 0 \). Thus the \( \epsilon \) in question form a \( k^f_e \)-space
of dimension \( e - e/e_0 \). Hence every \( \chi^* \) is a \( \chi_\delta \), and (iii) is verified. \( \square \)

Essentially the same proof used to demonstrate (i) and (ii) above also proves:

(4.7) Lemma. Let \( \chi \) be a character on \( K^m_e \) trivial on \( K^{m+1}_e \) and given on
elements \( y = 1 + \varpi y \varpi^{-m} \) \( (y \in m_e) \) by \( \chi(y) = \psi \circ \text{Tr}(\alpha \gamma^{\sigma^m}) \), where \( E_0 = F[\alpha \varpi^{-m}] \)
is a nicely embedded field with \( e(E_0/F) = e_0 \), \( f(E_0/F) = f_0 \), and \( \alpha \in k^e_f \).
Define \( G_0 = \text{subgroup of elements commuting with } \alpha \varpi^{-m} \). Fix \( j, 1 \leq j < m \).
For \( \delta \in m_e \), define \( \chi_\delta \) on \( K^j_e \) by
\[
\chi_\delta(y) = \chi(\varpi y \varpi^{-1} - y),
\]
where \( y = 1 + \varpi y \varpi^{-1} \) and \( w = 1 + \delta \varpi^{m-j} \). (Hence \( \chi_\delta \) is trivial on \( K^{j+1}_e \).) Then:
(i) \( \chi_\delta(y) = 1 \) for all \( \delta \) iff \( y \in K^j_e \cap G_0 \).
(ii) \( \chi_\delta \) is trivial iff \( w \in G_0 \) (i.e., \( \delta \varpi^{m-j} \) commutes with \( E_0 \)). \( \square \)

(4.8) Lemma. Let \( \chi, E_0, \) and \( G_0 \) be as in Lemma 4.7, but define \( \chi_\delta \) (for
\( \delta \in m^e_e \)) by \( \chi_\delta(y) = \chi(\varpi yw^{-1}y^{-1}), \ w = \delta \varpi^j \). Then:
(i) \( \chi_\delta(y) = 1 \) for all \( \delta \) iff \( y \in K^m_e \cap G_0 \).
(ii) \( \chi_\delta \) is trivial iff \( w \in G_0 \).

Proof. We have
\[
\chi_\delta(1 + \varpi y \varpi^{-m}) = \psi \circ \text{Tr}(\alpha \gamma^{\sigma^m}(\delta \gamma^{\sigma^j}(\delta^{-1})^{\sigma^m} - \gamma)
\]
\[
= \psi \circ \text{Tr}(\gamma(\alpha \gamma^{\sigma^m} \delta^{\sigma^j}(\delta^{-1})^{\sigma^m} - \alpha^{\sigma^m})).
\]
This is 0 for all \( \gamma \) iff \( \alpha R^{\sigma^m} \delta^{\sigma^j}(\delta^{-1})^{\sigma^m} - \alpha^{\sigma^m} = 0 \), or iff \( (\alpha \delta^{\sigma^m} \sigma^m) - \alpha^{\sigma^m} = (\alpha \delta^{\sigma^m} \sigma^m) - \alpha^{\sigma^m} \), or iff \( \alpha \varpi^{-m} \) and \( \delta \varpi^j \) commute (this uses Lemma 3.4). The
other half is similar. \( \square \)
This section is concerned with results used to show that if $\chi^* = \chi$, then the choice of $x$ is restricted in some way. The first two lemmas concern conjugacy. Suppose that $x = \alpha x^m$ and $x^* = \beta x^m$, $\alpha, \beta \in (k^*)^e$ (for some $f$), are known to generate nicely embedded fields over $F$ and to be conjugate in $GL_n(F)$. We will need to know that they are conjugate in $Z_e K_e$. If the fields that they generate have ramification index $e_0$, then $e_0$ is the smallest positive integer such that $e|e_0 m$; $x^{e_0} = \gamma x^{e_0 m}$, where $\gamma$ generates an unramified extension over $F$. A similar statement applies to $(x^*)^{e_0} = \delta x^{e_0 m}$. Furthermore, $\gamma$ and $\delta$ are conjugate in $m^e$. If we write $\alpha = (\alpha_0, \ldots, \alpha_{e-1})$, etc.; then $\gamma_0 = \alpha_0 \alpha_m \cdots \alpha_{m(e_0-1)}$ (where we extend the definition of the $\alpha_i$ by making them periodic mod $e$), etc., that is, $\gamma$ is a sort of norm. Furthermore, $e^{-1} \alpha x^m e = (e^{-1} \alpha e^{m}) x^m$. It is not hard to reduce questions about conjugacy of these elements to the case $m = 1$; then the following lemmas give what we need.

For $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{e-1}) \in m_e$, define

$$N_e(\alpha) = \alpha \alpha^e \cdots \alpha^{e-1} = \left(\alpha_0 \alpha_1 \cdots \alpha_{e-1}, \alpha_1 \alpha_2 \cdots \alpha_{e-2}, \ldots, \alpha_{e-1} \alpha_0 \cdots \alpha_{e-2}\right).$$

Note that the entries in $N_e(\alpha)$ have the same determinant, but need not be equal unless $e = 1$. Similarly, we define $N_e(\alpha) = \alpha \alpha^e \cdots \alpha^{e-1} = \left(\alpha^e \alpha_1 \cdots \alpha_{e-1}, \alpha_1 \alpha_2 \cdots \alpha_{e-2}, \ldots, \alpha_{e-1} \alpha_0 \cdots \alpha_{e-2}\right)$; thus $\gamma = N_e(\alpha)$ above if $m = e/e_0$.

(5.1) Lemma. Let $\alpha, \beta \in (m_e)^x$; set $\gamma = N_e(\alpha)$, $\delta = N_e(\beta)$, and write $\gamma = (\gamma_0, \ldots, \gamma_{e-1})$, etc. Suppose that $\gamma_0, \delta_0$ are conjugate elements in $GL_e(k)$. Then there exists $e \in m^e$ such that $e^{-1} \alpha e^e \equiv \beta$ mod $A_1^e$. If $\alpha, \beta \in (k^e)^f$, then there exists $e \in K_e$ with $e^{-1} \alpha e^e = \beta$.

Proof. Let $\gamma = (\gamma_0, \ldots, \gamma_{e-1})$ and $\delta = (\delta_0, \ldots, \delta_{e-1})$. We show first that the lemma holds if $\gamma_0 = \delta_0$. Set $e = (e_0, \ldots, e_{e-1})$; evidently we need to satisfy

$$e_0^{-1} \alpha_0 e_1 \equiv \beta_1$$
$$e_1^{-1} \alpha_1 e_2 \equiv \beta_2$$
$$\vdots$$
$$e_{e-1}^{-1} \alpha_{e-1} e_0 \equiv \beta_{e-1}.$$ 

Set $e_0 = I$. Then the first $(e - 1)$ equations give

$$e_1 \equiv \alpha_0^{-1} \beta_0,$$
$$e_2 \equiv \alpha_1^{-1} \alpha_0^{-1} \beta_0 \beta_1,$$
$$\vdots$$
$$e_{e-1} \equiv \alpha_{e-2}^{-1} \cdots \alpha_1^{-1} \alpha_0^{-1} \beta_0 \beta_1 \cdots \beta_{e-2}.$$ 

But now the last equation holds as well, since it amounts to the statement that $\gamma_0 = \delta_0$.

In general, we know that $\delta_0 \equiv \zeta_0^{-1} \gamma_0 \zeta_0^e$ for some matrix $\zeta_0 \in GL_f(k)$. Set $\zeta = (\zeta_0, \zeta_0, \ldots, \zeta_0)$. Then $\zeta^{-1} \alpha \zeta$ and $\beta$ are $\sigma$-conjugate by the first part of the proof.
If $\alpha, \beta \in (k^\times_f)^e$, then there is a $\zeta_0 \in \text{GL}_f(\mathcal{O}_F)$ such that conjugating by $\zeta_0$ is an automorphism of $F_f$ taking $\gamma_0$ to $\delta_0$. We use $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_s)$ to reduce to the case where $\gamma_0 = \delta_0$. The rest of the proof now goes as before, but the congruences can be replaced by equalities because all elements $\alpha_i, \beta_i, e_i$ are in $k_f$. □

(5.2) Lemma. Suppose that $\alpha \omega_f^j$ and $\beta \omega_f^j$ (with $\alpha, \beta \in (k^\times_f)^e$) generate fields over $F$ and are conjugate in $G$. Then they are conjugate in $K_f$.

Proof. Let $(j, e) = r, e = re_1$, so that $F[\alpha \omega_f^j]$ has ramification index $e_1$ over $F$; let the residue class degree be $f_1$. Let $(\alpha \omega_f^j)^{e_1} = \gamma \omega_f^{je_1}$ and $(\beta \omega_f^j)^{e_1} = \delta \omega_f^{je_1}$. We have $\gamma = (\gamma_0, \ldots, \gamma_{e_1-1})$ and $\delta = (\delta_0, \ldots, \delta_{e_1-1})$, where the $\gamma_i, \delta_i$ are all conjugate elements of $k_f$. Thus we may (by conjugating) assume that $\gamma = \delta$. Since $\gamma = N_e \alpha, \delta = N_e \beta$, and conjugating, e.g., $\alpha \omega_f^j$ by $e^{-1} \in m_e^{e_1}(e_i)$ gives $e^{-1} \alpha e^{\sigma_i} \omega_f^j$ (where $\sigma_i$ and $e_i$ generate the same group), Lemma 5.1 (applied with $e_i$, to the elements $\alpha_j$ with $j$ in a fixed congruence class mod $e_1$) gives the result. □

We use these lemmas for a further result about conjugacy which we need later. We will have characters $\chi, \chi^\sim$ defined on $K_e^m$ ($m \geq 1$), trivial on $K_e^{m+1}$, and given on $K_e^m$ by

$$\chi(1 + y) = \psi \circ \text{Tr}(xy), \quad \chi^\sim(1 + y) = \psi \circ \text{Tr}(x^\sim y),$$

with $x = \alpha \omega_f^{-m}, x^\sim = \alpha^\sim \omega_f^{-m}$, and $\alpha, \alpha^\sim \in m_e^\times$.

(5.3) Proposition. Use the above notation. Suppose that $x, x^\sim$ generate fields over $F$ and that for some $w \in G$, $\chi(xw^{-1}u) = \chi^\sim(u)$ for all $u \in K_e^m \cap w^{-1}K_e^m w$. Then $x, x^\sim$ are conjugate in $Z_e K_e$.

Proof (adapted from [10]). By assumption, $\chi w x^\sim$ is trivial on $K_e^m \cap w^{-1}K_e^m w$ and is given by $\chi w x^\sim(1 + y) = \psi \circ \text{Tr}((w^{-1}xw - x^\sim)y)$ on $K_e^m$. Therefore $w^{-1}xw - x^\sim \in (A_e^m \cap w^{-1}A_e w)^\perp = A_e^{1-m} + w^{-1}A_e^{1-m} w$, and there exist $v, v^\sim \in A_e^{1-m}$ with

$$w^{-1}xw - x^\sim = v - w^{-1}vw, \quad w^{-1}(x + v)w = x^\sim + v^\sim.$$

Suppose that $e(F[x]/F) = e_0$ and $f(F[x]/F) = f_0$. Since $e_0$ is the smallest positive integer with $e|e_0m$, $e(F[x]/F) = e_0$ also. Take $e_0$th powers in (5.4), noting that $(x + v)^{e_0} \equiv x^{e_0} \text{mod } A_e^{1-e_0-m}$ (and similarly for $x^\sim, v^\sim$). Writing $z = \omega_f^{-e_0m}(x + v)^{e_0}$ and $z^\sim = \omega_f^{-e_0m}(x^\sim + v^\sim)^{e_0}$, we have

$$w^{-1}zw = z^\sim$$

and $z \equiv \beta \text{ mod } A_e^1, z^\sim \equiv \beta^\sim \text{ mod } A_e^1$, where $\beta$ is a root of unity generating the unramified extension $F_{f_0}$ of $F$ and $\beta^\sim \in F_{f_0}$ is also a root of unity. Then for every $N$, $z^{q^fN} \equiv z^{q^{f(N-1)}} \text{ mod } A_e^N$, as an easy induction shows. Therefore $z^{q^fN} \rightarrow \gamma$, where $\gamma^{q^f} = \gamma$ and $\gamma \equiv \beta \text{ mod } A_e^1$. Similarly, $(z^\sim)^{q^fN} \rightarrow \gamma^\sim$, where $(\gamma^\sim)^{q^f} = \gamma^\sim, \gamma^\sim \equiv \beta^\sim \text{ mod } A_e^1$, and $g \gamma g^{-1} = \gamma^\sim$. Let $\varphi, \Phi$ be the minimal polynomials of $\beta, \gamma$ respectively. Then $\varphi(\gamma) \equiv \Phi(\gamma) \equiv 0 \text{ mod } A_e^1$. If $\varphi, \Phi$ were distinct, they would be relatively prime over $k = \mathcal{O}_F/P_F$ and we would have $\gamma \equiv 0$, a contradiction. So $\varphi = \Phi$, and $\beta, \gamma$ are conjugate. Similarly, $\beta^\sim, \gamma^\sim$ are conjugate, so that $\beta, \beta^\sim$ are conjugate. Therefore $x^{e_0}, x^{e_0^\sim}$ are conjugate.
We may thus conjugate (using Lemma 5.2) to arrange that $x^{e_0} = x^{-e_0}$. Now it is clear that $x$, $x^\sim$ satisfy the same minimal equation. Therefore they are conjugate in $G$; Lemma 5.2 implies that they are conjugate by an element of $Z_e K_e$. □

The next result is quite similar to Lemma 2.18 of [15], but the proof given here seems to be shorter and is certainly more in keeping with the methods of this paper.

(5.5) Lemma. Let $\chi$ be defined on $K^j_e/K^{j+1}_e$, with $\chi = \chi_x$ on $K^j_e$, where $x = x_\sim e^{-1}_\sim$ is such that $F[x]$ is a nicely embedded field. Define $G_{(x)} = \{w \in \GL_n(F) : wx = xw\}$. Let $e(F[x]/F)e_0$ and $f(F[x]/F)f_0$. Suppose that $b$ is a power-permutation matrix of some nicely embedded $\GL_{n_0}(E_0)$, with $e(E_0/F) = e_0$ and $f(E_0/F) = f_0$, and that $\chi^b = \chi$ on their common domain. Let $\chi_k$ , $k \in K^{e-j}_e$. If $\chi^{k,b} = \chi$, then $k_1 bk_1 = k_1' bk'$, with $k_1$, $k_1' \in K^{e-j+1}_e (K^{e-j}_e \cap G_{(x)})$.

Proof. There are some technical issues in the proof that complicate the notation. We need to work with groups $K^j_{n_0}$ as well as the groups $K^j_e$, and we also need to deal with intersections of these groups. We fix $j$ as in the statement of the lemma. Let $u = tf/f_0$ and $h = n/f_0$. We prove the lemma for $K^{e-j}_e \cap K^{u-r}_h$, using induction on $r$. We write the intersection as $K^{(u-r)}_h$, and let $\chi^{(u-r+1)} = K^{(u-r+1)}_h K^{(u-r+1)+1}_h$; calculations with $K^{(u-r)}_h$ are usually modulo $\chi^{(u-r+1)}$. Let $\sigma = \sigma_h$ and let $\eta_0$ be a uniformizer for $A^1_h$ in $\GL_{n_0}(E_0)$ (where $e_0 f_0 n_0 = n$) such that $\eta_0^{n_0} e_0 f_0$ is a prime in $E_0$ and conjugation by $\eta_0$ acts as $s$ on $m_0^{(e_0)}$. Observe that we can also write $x = x_\sim e^{-1}_\sim$, where $\alpha \in k^{e_0}_f \subseteq k_0 \subseteq m_h$; this shows that $\chi \equiv 1$ on $K^{e+1}_h$.

As will soon be clear, in our calculations we are concerned only with $k$, $k_1$ modulo $K^{(u-r+1)}_h$. (If, e.g., $k' \in K^{(u-r+1)}_h$, then $k'y^{k'-1}y^{-1} \in \Ker \chi$ for all elements $y$ under consideration.) We therefore write $k = 1 + \delta_0 e^{-u-r}_0$, $k_1 = 1 + \epsilon_0 e^{-u}_0$, where $\delta = (\delta_0, \ldots, \delta_{h-1})$, $\epsilon = (\epsilon_0, \ldots, \epsilon_{h-1})$, and the $\delta_i$, $\epsilon_i \in M_{f_0}(k)$. Notice that some of the $\delta_i$, $\epsilon_i$ may need to be 0, since in general $K^{(u-r)}_h \neq K^{u-r}_h$. If, however, $\delta_i$ need not be 0, then it can be any element of $k_0$, and similarly for $\epsilon_i$. We write $\delta_i^*$ for the element of $\delta_0, \ldots, \delta_{h-1}$ with $\delta_i^* = \delta_i$ and all other $\delta_i^* = 0$; we define $\epsilon_i^*$ similarly (and use a similar convention for other elements of $m_h$). Since we are working modulo $\chi^{(u-r+1)}_h$, we may also write $k = (1 + \delta_0^* e_0^{-u-r}) \cdots (1 + \delta_{h-1}^* e_0^{-u-r})$, and we may permute the order of the factors. Similar remarks apply to $k_1$.

We will be looking at the effect of $\chi$ on elements $y \in K^j_e \cap K^j_h = L^r$, for convenience. Write $(L^{r+1}) = L^{(r+1)} K^{e+1}_h$; it will be clear that we are interested only in $y$ modulo $(L^{r+1})$. Thus we set $y = 1 + y_0^\gamma \eta^\gamma_0$ and $\gamma = (\gamma_0, \ldots, \gamma_{h-1})$, the $\gamma_i$ arbitrary elements of $M_{f_0}(k)$. Mod $K^{e+1}_h K^{e+1}_h$,

$$kyk^{-1} \equiv y(1 + (\delta_0 e^{-u-r} - \epsilon_0 e^{-u}) \eta_0^u).$$

Note that $1 + \delta_i^* e_0^{-u-r}$ and $1 + \gamma_i^* \eta_0^\gamma$ are $(e_0, f_0)$-pure.

For $y \in L^r$, whether $byb^{-1} \in L^r$ or not is obviously important in determining $\chi^{k,b}(y)$. If $y = 1 + \gamma_0^* \eta_0^\gamma$, then the fact that $byb^{-1}$ is $(e_0, f_0)$-pure means that exactly one of the following three statements holds (independently
of \( \gamma_i^* \), provided that \( \gamma_i^* \neq 0 \); we consider only those \( i \) for which \( \gamma_i^* \neq 0 \) for some \( y \in L(r) \):

(i) \( b y b^{-1} \notin L(r) \);
(ii) \( b y b^{-1} \in \mathcal{L}(r+1) \);
(iii) \( b y b^{-1} \equiv 1 + \beta^* y_0^* \mod \mathcal{L}(r+1) \), where \( 1 + \beta^* y_0^* \) is an \((e_0, f_0)\)-pure element (like \( y \)) in \( L(r) \), and \( \beta^* \neq 0 \).

We refer to the index \( i \) as a “down” index for \( r \) if (i) holds, as an “up” index for \( r \) if (ii) holds, and as a “steady” index for \( r \) if (iii) holds. (We add “with respect to \( b^* \)” when necessary for clarity.)

An example of up, down, and steady indices may help; to keep the matrices of manageable size, the example will not involve a character. Let \( n = e_0 = 3 \), \( j_0 = 1 \) (here, \( h = e = n \)), so that we do not have any complications with pairs of congruence subgroups, \( t \) (= 9), and \( j_0 = 7 \). Let

\[
\begin{bmatrix}
0 & w & 0 \\
0 & 0 & 1 \\
w & 0 & 0
\end{bmatrix}
\]

\( w = w_F \).

Since

\[
y = 1 + (\gamma_0, \gamma_1, \gamma_2) w_3^2 = 1 + \begin{bmatrix}
0 & \delta_0 w^2 & 0 \\
0 & 0 & \delta_1 w^2 \\
\delta_2 w^3 & 0 & 0
\end{bmatrix}
\]

satisfies \( b y b^{-1} = 1 + (0, 0, \delta_0) w_3^2 + (0, \delta_2, 0) w_3^2 + (\delta_1, 0, 0) w_3^{10} \), we see that

0 is a down index, 1 an up index, and 2 a steady index for \( r \) with respect to \( b \).

If \( r \) is a multiple of \( n_0 \), then every index is steady for \( r \); in particular, every index is steady for \( u \). Furthermore, since \((\gamma_i^* y_0^*)(\delta_{r+i} y_0^*)\) is of the form \( \zeta_i^* y_0^* \) (where \( \zeta_i^* \) is in general nonzero unless \( \gamma_i^* \) or \( \delta_{r+i} \) must be 0), and since \( i + r \) is steady for \( u \), we see that:

- If \( i \) is down for \( r \), then \( i + r \) is up for \( u - r \);
- If \( i \) is up for \( r \), then \( i + r \) is down for \( u - r \);
- If \( i \) is ready for \( r \), then \( i + r \) is steady for \( u - r \).

It is also clear that if no element of the form \( 1 + \gamma_i^* y_0^* \) (\( \gamma_i^* \neq 0 \)) is in \( L(r) \), then no element of the form \( 1 + \beta_{i+r}^* y_0^* \) (\( \beta_{i+r}^* \neq 0 \)) is in \( K(u-r) \). It is not hard to see that if \( i \) is steady for \( r \) with respect to \( b \), so that \( b(1+\gamma_i^* y_0^*)b^{-1} \equiv 1+\beta_i^* y_0^* \) for some \( i' \), then \( i' \) is steady for \( r \) with respect to \( b^{-1} \), and conversely.

The idea of the proof is this: suppose that \( k \) has only one nonzero index (previous remarks show that we can reduce to this case). If it is an up index for \( u - r \), then \( bk = k_0 b \), where \( k_0 \in K^{u-r+1} \), and we are done. If it is a down index for \( u - r \), then we show directly that \( k \in G(x) \). If it is a steady index for \( u - r \), then \( bk = k_0 b \), where \( k_0 \) corresponds to a steady index; we show that the entry for \( k_1 k_0 \) in that index is in \( G(x) K^{u-r+1} \). That takes care of \( k \) (and of the steady indices for \( k_1 \)). To deal with \( k_1 \), we need to look at elements of \( L(r+1) \) that are conjugated by \( b \) into \( L(r) \); equivalently and more easily, we repeat the analysis for \( (k_1 b k)^{-1} \).

Here are the details. Suppose first that \( i \) is down for \( r \). Then \( i + r \) is up for \( u - r \), and \( b(1+\delta_{i+r}^* y_0^*)b^{-1} \in K^{u-r+1} \). Hence \( b(1+\delta_{i+r}^* y_0^*) = k_{i+r} b \), where \( k_{i+r} \) is of the form required by the lemma. Thus we may assume that \( \delta_{i+r}^* = 0 \) if \( i + r \) is up for \( u = r \).
If \(i\) is up for \(r\), then \(i + r\) is down for \(u - r\). Write \(k = k(i+r)k'k''\), where \(k(i+r) = 1 + \delta_{i+r}^\ast\eta_0^{-r}\) and \(k' = 1 + (\delta - \delta_{i+r}^\ast)\eta_0^{-r}\), so that \(k'' \in \mathcal{K}(u-r+1)\). Let \(y = (k'')^{-1}(1 + \gamma_i^\ast\eta_0^{-1})k'' = (k'')^{-1}y_{i}k''\), say. Then \(k'\) and \(y\) commute, so that

\[
kyk^{-1} = k(i+r)y_{i}k^{-1}(i+r) \equiv y_{i} \mod K_e^e,
\]

and \(bkyk^{-1}b^{-1} \in K(k+1)\) because \(i\) is an up index for \(r\). Since \(k_1 \in K(u-r)\), \(bkyk^{-1}b^{-1} \equiv k_1bkyk^{-1}b^{-1}k_1^{-1} \mod K_{u+1}^u\). Therefore

\[
\chi(bkyk^{-1}b^{-1}) = \chi(k_1bkyk^{-1}b^{-1}k_1^{-1}).
\]

By hypothesis, \(\chi(bkyk^{-1}b^{-1}) = \chi(kyk^{-1})\). So we must have \(\chi(kyk^{-1}) = \chi(y)\), or

\[
\chi(k(i+r)y_{i}k^{-1}(i+r)) = \chi(kyk^{-1}) = \chi(y) = \chi(y_{i}).
\]

The last equality follows from the fact that \(k'' \in \mathcal{K}(u-r+1)\), so that \(y \equiv y_{i}\) modulo \(K^u_e\). Thus

\[
1 = \chi(k(i+r)y_{i}k^{-1}(i+r)) = \chi(1 + (\delta_{i+r}^\ast\eta_0^{-r} - \gamma_i^\ast\delta_{i+r}^\ast)\eta_0^r)\] (for all \(\gamma_i^\ast\)),

and Lemma 4.7 says that \(k(i+r) \in G(x)\) (because \(\chi(1 + (\delta_{i+r}^\ast\eta_0^{-r} - \gamma_i^\ast)\eta_0^r) = 1\) if \(e = (e_1, \ldots, e_{r-1})\) and \(e_{r}^* = 0\).

Now suppose that \(i\) is steady for \(r\) and that \(b(1 + \gamma_i^\ast\eta_0^{-r})b^{-1} \equiv 1 + \beta_i^\ast\eta_0^r\) mod \(\mathcal{K}^{r+1}\). Then \(b(1 + \delta_{i+r}^\ast\eta_0^{-r})b^{-1} \equiv 1 + \zeta_{i+r}^\ast\eta_0^{-r}\) (i.e., the only nonzero entries of the coefficient \(\eta_0^{-r}\) in the expansion of \(b(1 + \delta_{i+r}^\ast\eta_0^{-r})b^{-1}\) are in entries \(\equiv r + i'\) mod \(h/e_0\)). As in the case of indices that are up for \(u - r\), we may replace \(b(1 + \delta_{i+r}^\ast\eta_0^{-r})b^{-1}\) by \((1 + \zeta_{i+r}^\ast\eta_0^{-r})b\) and assume that \(\delta_{i+r}^\ast = 0\). We may thus assume that \(k = k'i'k''\), where \(k'\) now has entries only in indices that are down for \(u - r\) (so that \(k' \in G(x) \cap K^{(u-r)}\)) and \(k'' \in \mathcal{K}(u-r+1)\). Then for \(y = (k'k'')^{-1}(1 + \gamma_i^\ast\eta_0^{-r})k'k'' = (k'k'')^{-1}y_{i}k'k''\), we have \(\chi(y) = \chi(y_{i})\). So

\[
\chi(y) = \chi(k_2bky'k''y'k'k'')^{-1}b^{-1}k_2^{-1} = \chi(k_2byb^{-1}k_2^{-1})
\]

\[
= \chi(byb^{-1})\chi(1 + (\zeta_{i+r}^\ast\gamma_i^\ast\eta_0^{-r} - \gamma_i^\ast\zeta_{i+r}^\ast)\eta_0^r),
\]

by a calculation like the one done above. Since \(\chi(byb^{-1}) = \chi(y_{i})\) by hypothesis, we must have

\[
\chi(1 + (\zeta_{i+r}^\ast\gamma_i^\ast\eta_0^{-r} - \gamma_i^\ast\zeta_{i+r}^\ast)\eta_0^r) = 1.
\]

Because \(\gamma_i^\ast\) can be arbitrary (with entries only in indices \(\equiv i'\) mod \(h/e_0\)), we conclude by Lemma 4.7 (as above) that \(1 + \zeta_{i+r}^\ast\eta_0^r \in G(x)\).

We have shown that if \(\chi k_i b k = \chi\) on their common domain, then we may assume that \(k \in (G(x) \cap K^{(u-r)})'K^{(u-r+1)}\) and that \(k_1 = k'i'k''\), where \(k' \in \mathcal{K}(u-r+1)\), \(k'' \in K^{(r)} \cap G(x)\), and \(k_1 = 1 + \varepsilon_0\eta_0^{-r}\) with the only nonzero entries for \(\varepsilon\) in the indices that are up or down for \(b^{-1}\). Apply the above argument to \(k^{-1}b^{-1}k_1^{-1}\). Evidently \(k^{-1}(k'i'k''i^{-1}) \in (G(x) \cap K^{(u-r)}')K^{(u-r+1)}\), so that we need only concern ourselves with \(b^{-1}(k'i')^{-1}\). Since

\[
(k'i')^{-1} \equiv 1 - \varepsilon_0\eta_0^{-r} \mod (G(x) \cap K^{(u-r)})'K^{(u-r+1)},
\]

the above proof shows that \(b^{-1}(k'i')^{-1} = k'^{-1}b^{-1}k'^{-1}\), where \(k'^{-1}, k'^{-1} \in \mathcal{K}(u-r+1)\). Hence \(k'i'b = k'^{-1}b k_0\), where \(k_0, k'_0 \in \mathcal{K}(u-r+1)\); the lemma follows. \(\Box\)
We now start the construction of the representations \( \rho_i \) that induce to supercuspidals. We begin with the sequence of triples \((s_i, e_i, f_i)\), \(1 \leq i \leq r\), satisfying the properties of (1.3). For the moment, we assume that \( s_1 > 0 \); the case \( s_1 = 0 \) will be dealt with later. Let \( t_i = s_i / f_i \) and \( n_i = n / e_i f_i \); set \( e_0 = f_0 = 1 \). For \( t_i > 0 \), set \( t'_i = [t_i/2] + 1 \). Let \( t_0 \) be the largest index with \( t_0 \leq 2t_{r_0 + 1} \) or, equivalently, \( t'_{r_0} > t_{r_0 + 1} \) (if no such index exists, set \( t_0 = r \)). Write \( t''_i = t_i + 1 - t'_i \), so that \( t''_i = t'_i \) if \( t_i \) is odd and \( t''_i = t'_i - 1 \) if \( t_i \) is even. If \( t_i = 0 \) (so that \( i = r \)), set \( t''_r = t'_r = 0 \). Define numbers \( C_j \) by

\[
C_i = \frac{f_i - 1}{f_i} \sum_{df_i-1|f_i} \mu(f_i/f_i - d) \left(q^{f_i - d} - 1\right), \quad \mu = \text{Möbius function};
\]

\[
C_j = \begin{cases} 
1 & \text{if } t_{i+1} < j < t_i \text{ and } e_i j e_i; \\
q^{f_i} & \text{if } t_{i+1} < j < t_i \text{ and } e_i j e_i; \\
q^f & \text{if } 0 < j < t_i; \\
q^{f-1} & \text{if } j = 0 < t_i.
\end{cases}
\]

(The \( C_i \) depend on \( (s_i, e_i, f_i) \), but we do not indicate the dependence in the notation.)

The major result in the construction, given next, is analogous to Theorem 3.1 of [4]. Before stating it, we give an indication of the strategy to be followed in the construction. We begin with a character \( \chi \) defined on \( K_{e_1}^{t_1} \) and trivial on \( K_{e_1+1}^{t_1} \). Such a \( \chi \) can be defined by

\[
\chi(1 + y) = \psi \circ \text{Tr}(\alpha \omega e^{-t_1} y) \quad \forall y \in A_{e_1}^{t_1},
\]

for a unique \( \alpha \in m_e \). (In fact, \( \chi(1 + y) = \psi \circ \text{Tr}(xy) \) for any \( x \equiv \alpha \omega e^{-t_1} \mod A_{e_1}^{t_1} \). We are making the simplest choice of \( x \), in some sense.) We require that \( \alpha \omega e^{-t_1} \) generate a field \( E(t_1) \) with \( e(E(t_1)/F) = e_1 \) and \( f(E(t_1)/F) = f_1 \). Up to conjugacy, this turns out to give \( C_i \) choices for \( \alpha \). We then compute the elements \( g \in G \) such that \( \chi(g y \omega^{-1}) = \chi(y) \) on their common domain (we say that such an element commutes with \( \chi \)). This set includes \( G(t_1) \), the group of elements commuting with \( E(t_1) \). \( (G(t_1) \cong \text{GL}_{n/e_1 f_1}(E(t_1))) \).

The next step is to show that there is an extension \( \chi_0 \) of \( \chi \) to \( K_{e_1}^{t_1-1} \) such that \( g \) commutes with \( \chi_0 \) if \( g \in G(t_1) \). The point of this step is that any extension of \( \chi \) to \( K_{e_1}^{t_1-1} \) must be of the form \( \chi = \chi_0 \chi_1 \), where \( \chi_1 \) is trivial on \( K_{e_1}^{t_1} \). Therefore we can find a convenient expression for \( \chi_1 \) like the one given above for \( \chi \) on \( K_{e_1}^{t_1} \). It turns out that two extensions \( \chi_0 \chi_1 \) and \( \chi_0 \chi_1' \) are conjugate if they agree on \( K_{e_1}^{t_1-1} \cap G(t_1) \). Therefore we need only look at \( \chi_1 \) on this subgroup. It is given there by

\[
\chi(1 + y) = \psi \circ \text{Tr}(\alpha_1 \eta(t_1^{-1}) y) \quad \forall y \in A_{e_1}^{t_1-1} \cap M(t_1),
\]

where \( M(t_1) \cong M_{n/e_1 f_1}(E(t_1)) \) is the set of elements in \( M_n(F) \) commuting with \( E(t_1) \), \( \alpha_1 \in m_e \cap M(t_1) \), and \( \eta(t_1) \in G(t_1) \) is an element that plays a role in \( G(t_1) \) like that played by \( \omega e \) in \( G \). If \( t_2 < t_1 - 1 \), we require that \( \alpha_1 \eta(t_1) \in E(t_1) \); if \( t_2 = t_1 - 1 \), we require that \( \alpha_1 \eta(t_1) \) generate an extension field of \( E(t_1) \) whose ramification index and residue class degree (over \( F \)) are \( e_2 \) and \( f_2 \) respectively.
It turns out that there are $C_{t_i - 1}$ choices of $\alpha_1$ (up to conjugacy) in each case. Let $\chi = \chi_0 \chi_1$. We now compute the set of elements of $G$ that commute with $\chi$. In the first case ($t_2 < t_1 - 1$), this set contains the group $G_{t_1 - 1}$ of all elements of $G$ commuting with some field $E_{t_1 - 1}$ such that $e(E_{t_1 - 1}/F) = e_1$ and $f(E_{t_1 - 1}/F) = f_1$, but we need not have $E_{t_1 - 1} = E_{t_1}$. Similarly, in the case where $t_2 = t_1 - 1$ there is a field $E_{t_1 - 1} = E_{t_2}$ such that $e(E_{t_2}/F) = e_2$, $f(E_{t_2}/F) = f_2$, and any element $g$ commuting with $E_{t_2}$ commutes with $\chi$, but $E_{t_2}$ need not be the extension field of $E_1$ generated by $\alpha_1 \eta_{(t_1)}^{-t_1}$. It is the fact that these fields can vary in odd ways that makes the wild case ($p \mid n$) more difficult than the tame case ($p \nmid n$); we say more about the tame case later.

Having found $E_{t_1 - 1}$, we now continue to $K_{t_1 - 2}^e$ and repeat the procedure. There are some variations that occur as we continue the induction. For example, $K_{t_1}^e/K_{t_1 + 1}^e$ is an Abelian group, but $K_{t_1 - 1}^e/K_{t_1 + 1}^e$ is not; we cannot extend $\chi$ as a character to all of $K_{t_1 - 1}^e$. This means that as we go along we need to extend the definition of the group $H$ on which $\chi$ is defined. As this sketch undoubtedly suggests, there are large numbers of other details to verify in the course of the proof.

The reader may wonder why this procedure is natural or even reasonable. It was prompted by the need to solve three technical problems. First of all, supercuspidal representations should correspond somehow with characters associated to maximal anisotropic tori, but in the wild case nonconjugate tori (= maximal embedded subfields) can lie very close together, and the characters naturally associated with one of these tori tend to be characters on subgroups of $GL_n$ that may be trivial on the intersection with the associated torus itself. The above procedure does associate $\chi$ with tori, by associating $\chi$ with the fields $E_{(t_2)}$; one can think of the induction as providing a sequence of fields that approximate the field associated with $\chi$. A second problem is that describing the character $\chi$ becomes increasingly difficult as one goes on. On $K_{t_1}^e$, $\chi(1 + y)$ can be given as $\psi \circ \text{Tr}(xy)$ for an appropriate $x$, but for larger groups there seems no easy way to describe $\chi$. The procedure given here obviates the need for a detailed description of $\chi$; one simply needs to describe how $\chi$ extends one layer at a time, and the necessary information is given by $\chi_0$. Thirdly, describing the set of elements commuting with $\chi$ becomes increasingly difficult as time goes on. It becomes manageable in this inductive procedure because if we have defined $\chi$ on $H \cap K_{e - 1}^e$, then any element that commutes with $\chi$ there also commutes with $\chi|_{H \cap K_{e}^e}$. This simplifies computations considerably. In fact, we will see that virtually every calculation reduces to one on $A_e/A_{e - 1}$.

(6.1) **Theorem.** There exist $\prod_{j=t_0}^{t_1} C_j$ choices of fields $E_1, \ldots, E_{r_0}$, elements $\eta(t_1), \ldots, \eta(t_{r_0}) \in Z_e K_e$, and characters $\chi$ on subgroups $H_0$, nonconjugate under $Z_e K_e$ and trivial on $K_{e + 1}^e$, with the following properties (in the rest of this statement we use the notational convention that for an index $j$, $i$ is the largest index with $t_i \geq j$):

1. $\eta(j) \gamma \eta(j)^{-1} = \gamma^\sigma$ for all $\gamma \in m_{t_1}(e_j)$.
2. $\eta(j)$ generates $A_{e_j}^e$ (as an ideal of $A_e$). Indeed, $\eta(j) \equiv e_j \omega_e \mod A_{e_j}^2$, with $e_j \in (k_{t_j}^e)^e$. 

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(3) \( E_j = F_j[\eta_j^{e_j/e_j}] \) is a nicely embedded field extension of \( F \) with ramification index \( e_j \) and residue class degree \( f_j \). (Recall: \([F_j : F] = f_j \), and \( F_j \) is unramified over \( F \).) This means also that \( \eta_j \) and \( \eta_j^{e_j/e_j} \) are related as described at the start of §2.

(4) Let \( M_j = M_{n_j}(E_j) \subseteq G \) and \( M_i = M_{(i+1)}(E_{i+1}) \) (and \( M_{r_0} = M_{(r_0)} \)). Then \( M_j \) is generated over \( E_j \) by \( \eta_j \) and \( m_i^{e_i}(e_i) \).

Write \( G_j = \langle \text{group of invertible elements of } M_j \rangle = \GL_{n_j}(E_j) \), \( G_i = \langle \text{group of invertible elements of } M_i \rangle = \GL_{(i+1)}(E_{i+1}) \) for \( i < r_0 \), \( G_{r_0} = G_{(r_0)} \), etc.

(5) If \( j \neq t_i \), then \( \eta_j \) is of the form \( \eta_{j+1} + y_{i-1} + \cdots + y_0 \), with \( y_g \in M_k \cap A_{t_{i+j}}^{t_i+j-1} \). If \( j = t_i \), then \( \eta_j \) is of the form \( \zeta_i y_{i-1} + y_{i-2} + \cdots + y_0 \), where the \( y_g \) are as before and \( \zeta_i \in (k_{t_i}^e)^e \).

(6) \( H_0 = K_{t_0}^e(K_{t_0}^e \cap G_1) \cdots (K_{t_0}^e \cap G_{r_0-1}) \), and

\[
H_0 \cap K_{t_j}^e = K_{t_j}^e(K_{t_j}^e \cap G_1) \cdots (K_{t_j}^e \cap G_{h-1})(K_{t_j}^e \cap G_h),
\]

where \( h \) is the index with \( t_{i+j} \leq j \leq t_{i+j} \). (We set \( t_0 = t_{r_0} = +\infty \).) In particular, \( H_0 \cap K_{t_j}^e/H_0 \cap K_{j+1}^{t_j+1} \cong K^j \cap G_h/K^j+1 \cap G_h \). We define \( H_j^i = H_0 \cap K_{t_j}^e \).

(7) The set of elements in \( Z_eK_e \) commuting with \( \chi \mid_{H_0} \) is

\[
K_{c_1}^j(K_{c_1}^j \cap G_1) \cdots (K_{c_1}^j \cap G_{i-1})(Z_{c_1}K_e \cap G_j),
\]

where \( c_1 \) is the smaller of \( t_{i+j} \) and \( t_i + 1 - j \); this set is a group and normalizes \( H_0 \). Furthermore, \( \chi(gyg^{-1}) = \chi(y) \) whenever \( g \in G_{n_j}(E_j) = G_j \) and \( y, gyg^{-1} \in H_0 \) (i.e., \( G_j \) commutes with \( \chi \mid_{H_0} \)).

(8) Let \( t_{j+1} \leq t_i < t_{j+1} \), so that \( G_{i-1} \) commutes with \( \chi \) on \( H_0 \cap G_{t_{j+1}} \) and \( H_0 \cap G_{t_{j+1}} \cong G_h \cap K_{t_j}^e / G_h \cap K_{t_{j+1}}^e \). Then one can write \( \chi \mid_{H_0} \) as \( \chi_{0,i} \chi_{1,i} \), where

(i) \( G_{i-1} \) commutes with \( \chi_{0,i} \);
(ii) \( \chi_{1,i} \) is trivial on \( H_0^{t_i+1} \);
(iii) On \( H_0^{t_i} \cap G_{i-1} \) (see the note below), we have

\[
\chi_{1,i}(1 + y_{t_{i-1}}) = \psi \circ \text{Tr}(\xi_{t_{i-1}})(\alpha_{t_{i-1}}^\sigma y), \quad y \in m_{t_{i-1}}^{e_i-1}(e_{t_{i-1}}),
\]

where \( E_{t_{i-1}}[\alpha_{t_{i-1}}, y_{t_{i-1}}] = E_{t_{i-1}} \) is a nicely embedded field satisfying \( e(E_{t_{i-1}}/F) = e_i, f(E_{t_{i-1}}/F) = f_i \);
(iv) the matrix algebra \( M_{t_{i-1}} \) of elements commuting with \( E_{t_{i-1}} \) has an element \( \eta_{t_{i-1}} \) generating \( A^1_{t_{i-1}} \) such that \( \eta_{t_{i-1}} \), \( m_i^{e_i}(e_i) \) generate \( M_{t_{i-1}} \) as in Proposition 2.1, and \( \eta_{t_{i-1}} \equiv \eta_{t_{i-1}} \mod A^1_e \). Note that in (ii) and (iii) we are describing \( \chi_{1,i} \) only on part of its domain, namely on \( H_0^{t_i+1}(H_0^{t_i} \cap G_{i-1}) \subseteq H_0^{t_i} = H_i^{t_i+1}(H_0^{t_i} \cap G_{h-1}) \).

(9) Suppose that \( j_0 < j \); let \( g \geq i \) satisfy \( e_j \mid_j \geq g \). Let \( \chi^g \) be a character on \( G_{i-1} \cap K_0^{j_0} \) trivial on \( G_i \cap K_0^{j_0} \) and on \( G_i \cap K_e^{j_0} \), of the form

\[
\chi^g(1 + y_{t_{i-1}}) = \psi \circ \text{Tr}(\beta^y y_{t_{i-1}}), \quad \beta \in k_{t_{i-1}}^e.
\]
Then there exists \( w = 1 + \delta_i \eta_{i-1}^{j_0} \), with \( \delta_i \in k_{s_j}^e \) and \( \delta_s^{c_{i-1}} = \delta_1 \), such that \( \chi(wyw^{-1}y^{-1}) = \chi^s(y), \ y \in G_{i-1} \cap K_e^{j_0} \).

(10) If \( t_1 < j \leq t_i \), then any extension of \( \chi|_{H_0^i} \) to a character on \( H_0^j \) extends to a character on \( H_0^j \).

(11) The \( \chi \) are nonconjugate under \( Z e K_e \), in that if \( \chi_1, \chi_2 \) are distinct characters on \( H_0^1, H_0^2 \) respectively, then for any \( z \in Z e K_e \), \( \chi_1 \) and \( \chi_2 \) do not agree on their common domain.

Proof. The proof is by backwards induction on \( j \); it is long and complicated. There are five main steps:

1. Proving the theory for \( j = t_1 \).
2. Proving a technical lemma that says roughly that if we have \( \chi \) defined on \( H_0^j \) so that \( G(j) \) commutes with \( \chi \), then \( \chi \) has an extension \( \chi_0 \) to \( H_0^{j-1} \) such that \( G(j) \) commutes with \( \chi_0 \). The value of this lemma, as noted in the remarks before the statement of the theorem, is that any extension of \( \chi \) to \( H_0^{j-1} \) differs from \( \chi_0 \) by a character that is trivial on \( H_0^j \). This gives us a concrete way of describing all such extensions.
3. Proving the inductive step (to \( j - 1 \)) in the case where \( e \{ (j - 1) e_i \) (as before, \( i \) is the largest index with \( j \leq t_i \)). This is the easiest step.
4. Proving the inductive step in the case where \( e \{ (j - 1) e_i \), but \( j - 1 \neq t_i^+1 \).
5. Proving the theorem in the case where \( j - 1 = t_i^+1 \).

In this section we do the first two steps; the rest of the proof is given in \( \S 7 \).

Before beginning the work of the proof, we shall set some notation and attempt to explain the meaning of (1)–(11). Properties (1)–(4) give a useful working description of the groups \( G(j) \) and the fields \( E(j) \), and (5) shows that these groups are related in a way that makes the results of \( \S 3 \) applicable. Property (6) gives a description of the group \( H_0 \) on which the character is defined. (Note that \( H_0 \) is not given at the start of the theorem, but is instead defined inductively in the course of the proof. When we reach level \( t_i \), \( H_0^{t_i} \) is defined; the definition involves \( G_{i-1} = G_{i+1} \) in a critical way. Property (5) and Lemma 3.1 show that, for instance, the definition of \( H_0^{t_i-1} \) does not change when we later define \( H_0^{t_i} \).) Property (7) gives important information about the elements of \( G \) that commute with \( \chi|_{H_0^j} \); a full description of this set is given later, in Theorem 8.1. Properties (8) and (9) give technical information about \( \chi \) used in proving some of the other properties; specifically, (8) is used so that we can apply Lemmas 4.7 and 4.8 when considering elements commuting with \( \chi \) (as we must when proving (7), and (9) is used to construct \( \eta_{j-1} \) from \( \eta(j) \). Property (10) means that we have a certain amount of freedom in extending \( \chi \). The nonconjugacy statement in (11) will be used to show that the supercuspidals constructed are all distinct.

Our list of properties is redundant, in that some of the properties imply others. For instance, (8) \( \Rightarrow \) (9) by Lemma 4.7, and (5) \( \Rightarrow \) (6) by Lemma 3.1 and Corollary 3.2 (the first part of (6) is definition). Next, (5) also implies that the set in (7) is a group, because of Lemma 3.1, since the \( c_i \) in (7) always satisfy \( c_{i-1} - c_i \leq t_{i-1} - t_i \). Finally, (6) and (7) imply (10), since the commutator subgroup \( (H_0^j, H_0^j) \subseteq H_0^j \) and an argument using (4.3) shows that \( \chi \equiv 1 \) on
the commutator. (If \( x = 1 + \gamma \eta_i \) and \( y = 1 + \delta \eta_i \), where \( r, s \geq t_i' \) and \( \gamma, \delta \in (m^c_i(e))^{\times} \), then all words in \( (x - 1) \) and \( (y - 1) \) are in \( H^1_i \), and Lemma 4.2 applies.) Thus we shall not prove (6), (9), or (10) in the induction.

In the inductive part of the proof, we assume the result for \( j ; i \) is the largest index with \( t_{i} \geq j \), and \( h \) is the index with \( t'_{h+1} < j \leq t'_h \). The reason for listing \( h \) is that \( H^{h-1}_0/H^0_0 \cong G_h \cap K^{h-1}_c / G_h \cap K^h_c \cong m^c_h(e_h) \); however, \( h \) does not play a major role in any arguments. We generally use \( g \) and \( l \) as indices.

At times, we shall use (4.4) to deal with the case where \( k \) has two elements. What we need to know is that for some large prime \( N \), if we work with \( \text{GL}_n(F_N) \) (where \( F_N \) is the unramified extension of degree \( N \)), then we can perform the construction (on the composita of the fields \( E_{(j)} \) with \( F_N \)), getting a character \( \chi^\sim \) on a group \( H^\sim_0 \) such that \( H_0 = H^\sim_0 \cap \text{GL}_n(F) \) and \( \chi^\sim |_{H_0} = \chi \). It should not be hard to see that this is always the case.

When \( j - 1 \) is not one of the “jump indices” \( t_{i+1} \), the objects \( E_{(j-1)} \), \( G_{(j-1)} \), etc. satisfy the properties for \( E_{(j)} \), \( G_{(j)} \), etc. It may be used to regard, e.g., \( E_{(t_1)}, E_{(t_1-1)}, \ldots, E_{(t_2+2)} \) as successive approximations to \( E_1 = E_{(t_1+1)} \).

If \( p \nmid n \) (the “tamely ramified” case, treated in [10]), it turns out that we can always take \( E_{(j-1)} = E_{(j)} \) when \( j - 1 \) is not a jump index. (This is the point of the “geometrical” lemmas in the first part of [10], which show, e.g., that \( M_{(t_1)} \oplus M_{(t_1)} = M \), where \( M_{(t_1)} = \{ x \in M : \text{Tr}(xy) = 0 \forall y \in M_{(t_1)} \} \).) Furthermore, at a jump index, we have \( E_{(j-1)} = E_{(j-1)} \) (see property (8), (iii)), so that \( E_1 \subset E_2 \subset \cdots \subset E_n \). This simplifies the description of \( \chi \) and also simplifies many details of the construction. The reader may wish to compare the construction that follows with that of [10] when both apply.

The proof is so arranged that at most points in the argument we need be concerned only with a character \( \chi \) (related to \( \chi \)) defined on some \( H^h_0 \) and trivial on \( H^{h+1}_0 \). (This may indicate the importance of (8).) We often need an argument, using Lemma 4.2, to reduce to this situation, but it may help to keep this organizational principle in mind.

We now give the proof for the case \( j = t_1 \). The characters on \( K^{t_1}_c \) trivial on \( K^{t_1+1}_c \) are of the form

\[
\chi(1 + y) = \psi \circ \text{Tr}(\alpha_1 \omega^e_{-t_1} y), \quad y \in A^{t_1}_c \text{ and } \alpha_1 \in m^c_e.
\]

We require that \( \alpha_1 \in k^{c_e} \), that \( F[\alpha_1 \omega^e_{-t_1}] = E_{(t_1)} \) have ramification index \( e_1 \) and residue class degree \( f_1 \), and that \( F_{f_1} \subseteq E_{(t_1)} \). Up to conjugacy, the number of choices for \( \alpha_1 \) is the number of primitive elements for \( k^{c_e} / k \), with elements equivalent if they have the same minimal equation, namely \( C_{t_1} \). We fix one representative for each conjugacy class. Notice that \( (\alpha_1 \omega^e_{-t_1})^{e_1} = \gamma_1 \omega^e_{-a} \), say, where the entries of \( \gamma_1 \) are all equal. Since the entries of \( \alpha_1 \) all commute and the \( l \)th entry of \( \gamma_1 \) depends only on those entries of \( \alpha_1 \) with indices \( \equiv l \mod e/e_1 \), so that \( \gamma_1 = N_{e_1}(\alpha_1) \), we may arrange to have the first \( e/e_1 \) entries of \( \alpha_1 \) equal, the next \( e/e_1 \) entries equal, and so on. Pick one \( \alpha_1 \) of this form in each conjugacy class; for this choice, \( \alpha_1 \omega^e_{-t_1} \) commutes with every \((e_1, f_1)\)-permutation matrix, and \( E_{(t_1)} \) is nicely embedded. For appropriate \( a, b \in \mathbb{Z}, (\alpha_1 \omega^e_{-t_1})^{a} \omega^b_{k} \) is of the form \( \delta \omega^e_{-e_1} \), since \((t_1, e) = e/e_1 \); \( F[\delta \omega^e_{-e_1}] = E_{(t_1)} \). Furthermore, \( \delta = (\delta_0, \delta_1, \ldots, \delta_{e-1}) \in (k^{c_e})^e \), with \( \delta_0 = \delta_1 = \cdots = \delta_{e/e_1-1} \), \( \delta_{e/e_1} = \cdots = \delta_{2e/e_1-1} = \cdots \), etc. Let \( \gamma_1 = (1, \ldots, 1, \delta_{e/e_1}, 1, \ldots, \delta_{e/e_1}) \),
where there are \( e/e_1 - 1 \) ones between the \( \delta \)'s, and set \( \eta_{(t_1)} = \gamma_1 \varpi_e \). Then
\[ \eta_{(t_1)}^{e/e_1} = \delta \varpi_e^{e/e_1} \], so that \( F_{\eta_{(t_1)}^{e/e_1}} = E_{(t_1)} \). We can easily insure that \( \gamma_1 \in k_{f_1}^e \).

We next check properties (1)–(11). Property (1) is easy, since \( \alpha_1 \) commutes with \( m_f \); (2) is immediate, and it is also clear that (3) holds. As for (4), note that \( \eta_{(t_1)} \) and \( m_f(e_1) \) commute with \( \alpha_1 \varpi_e^{-t_1} \) and are therefore in \( M_{(t_1)} \). They generate a vector space of dimension \( n_2^2 \) over \( E_{(t_1)} \), and hence generate \( M_{(j)} \). (5) is vacuous, and we checked (11) above; for (8), let \( \chi_{0,1} \equiv 1 \). For the second part of (7), set \( \alpha_1 \varpi_e^{-t_1} = u \). If \( g \in G_{(t_1)} \), then \([g, u] = 0\). So if \( y, gyg^{-1} \in A_{e_1}^{\theta} \), then \( \chi(1 + gyg^{-1}) = \psi \circ \text{Tr}(ugyg^{-1}) = \psi \circ \text{Tr}(g^{-1}ugy) = \psi \circ \text{Tr}(uy) = \chi(1 + y) \). The first part of (7) is a bit more work. It is easy to check that \( K_e^1 \) and \( Z_e K_e \cap G_{(t_1)} \) commute with \( \chi|_{K_e^1} \); note that \( (K_e^1, K_e^1) \subseteq K_e^{1 + 1} \).

Conversely, any element of \( K_e Z_e \) can be written uniquely as
\[ w = (1 + \beta_1 \varpi + \cdots) \beta_0 \varpi^{j_0} = w_1 w_0, \quad \beta_i \in m_e \text{ and } \beta_0 \in m_0^e \].

We know that \( w_1 \) commutes with \( \chi \); thus the hypothesis says that \( \chi w_0 = \chi \). By Lemma 4.8, \( w_0 \) commutes with \( u \).

We now give the technical result of the second step. It is analogous to Lemma 3.8 of [4].

(6.2) Lemma. Retain the notation of Theorem 6.1. Let \( \chi \) (as above) be defined on \( H_0^{j_0} \), \( j > t' \), and let \( i, h \) be as defined above in the proof of Theorem 6.1. Assume that \( G_{(j)} \) commutes with \( \chi \). Then \( \chi \) has an extension \( \chi_0 \) to \( H_0^{j-1} \) such that \( G_{(j)} \) commutes with \( \chi_0 \).

Remark. We shall actually prove slightly more about \( \chi_0 \), since we shall show that \( G_{(j)} \) commutes with \( \chi_0 \) once a fairly small subset of \( G_{(j)} \) commutes with \( \chi_0 \). This may be useful elsewhere. In addition, the proof uses little about \( G_{(j)} \) except that \( E_{(j)} \) is nicely embedded, \( G_{(j)} \) commutes with \( \chi \), and elements of \( G_{(j)} \) have the “normalizing” property described at the start of the proof; thus we can often apply the result to groups other than \( G_{(j)} \).

Proof of the lemma. In this proof, we assume that \( q \neq 2 \); when \( q = 2 \), we need to modify the proof as noted earlier.

We begin by indicating something about when elements of \( G_{(j)} \) “normalize” \( H_0^{j-1} \). Specifically, we show that if \( x \in G_{(j)} \), \( y \in H_0^{j-1} \), and \( xyx^{-1} \in K_e^{j-1} \), then \( xyx^{-1} \in H_0^{j-1} \). From (5) and (6) (or the proof of (7)), \( G_{(j)} \cap K_e \) normalizes \( H_0^{j-1} \). Thus we may assume that \( x = b \) is a power-permutation matrix in \( G_{(j)} \). We may also assume that \( y \in G_{(j)} \cap K_e^{j-1} \) for some \( l \). Suppose that \( l = h-1 \). Repeated use of (5) shows that \( \eta_{(j)} \) can be written in the form
\[ \eta_{(j)} = \zeta_{h} \eta_{h} + y_{h} + y_{h-1} + \cdots + y_{0}, \quad \zeta_{h-1} \in (k_{f_1}^e)^e, \]
where \( y_{g} \in M_{g} \cap A_{e_1}^{l_1-\tau_{1}+2} \). [From (5), we get \( \eta_{(j)} = \zeta_{(j+1)} \eta_{(j+1)} + y_{l-1} + \cdots + y_{0}, \ y_{g} \in M_{g} \cap A_{e_1}^{l_1-\tau_{1}+2} \). Apply (5) to the first term; we get \( \eta_{(j)} = \zeta_{(j+2)} \eta_{(j+2)} + y_{l-2} + \cdots + y_{0} \), \( y_{g} \in M_{g} \cap A_{e_1}^{l_1-\tau_{1}+2} \), where the \( y_{g} \) may have changed from one line to the next. Repeating this procedure, we eventually get
\[ \eta_{(j)} = \zeta_{l-1} \eta_{l-1} + y_{l-1} + \cdots + y_{0}, \quad y_{g} \in M_{g} \cap A_{e_1}^{l_1-\tau_{1}+2} \]
again, the $y'_g$ may have changed. Now use (5) and Lemma 3.3 to replace
\( \eta_{i-1} \) with
\[ \zeta_{(t_i+2)} \eta_{(t_i+2)} + y'_{(t_i+2)} + \cdots + y'_0, \]
where
\[ y''_g \in M_g \cap A^{e_{x-1}-t_i+1}, \]
\[ z_{(t_i+2)} \in M_{(t_i+2)} \cap A^2, \]
and
\[ z_g \in M_g \cap A^{e_{x-1}-t_i+2}. \]
Combining terms, we have
\[ \eta_{(j)} = \zeta_{(t_i+2)} \eta_{(t_i+2)} + y'_{(t_i+2)} + \cdots + y'_0, \]
where
\[ \zeta_{(t_i+2)} \in (k_{x+1})^e, \]
\[ y'_{(t_i+2)} \in M_{(t_i+2)} \cap A^2, \]
and
\[ y'_g \in M_g \cap A^{e_{x-1}-t_i+1}. \]
Continue inductively; we get
\[ \eta_{(j)} = \zeta_h \eta_h + y'_{h} + \cdots + y'_0, \]
where
\[ \eta_g \in M_g \cap A^{e_{x-1}-t_i+1}, \]
as desired.

Now use Lemma 3.3: there is a power-permutation matrix $b'$ (for the
compositum of $F_{t_i}$ and $E_{h}$) such that for some
\[ k' \in K^1 \cap G_h(K^{e_{x-1}-t_i+1} \cap G_{h-2}) \cdots K^{1-t_i+1}, \]
\[ b = k'b'. \]
It is obvious that $b'y'_{h-1} \in H^{i-1}$ if it is in $K^{i-1}$, and Corollary 3.2 says that
$k$ normalizes $H^{i-1}$. The argument for the other $G_i \cap K^{i-1}$ is essentially the
same.

It may be worth noting why the lemma is true when $h = 0$ (though what
follows is not a proof). Then $j > t'_i$ and $H^{i-1} = K^{i-1}$, and we can write $\chi$ on $K^{i}$ as
\[ (6.3) \quad \chi(1 + y) = \psi \circ \text{Tr}(xy), \quad x \in A^{e_{x-1}}, \quad x \text{ determined mod } A^{e_{x-1}}. \]
It is possible (though not obvious in our inductive process) to choose $x$ so
that $F[x] = E(j)$. Assuming this, the proof of the lemma is easy: use (6.3)
(with this same $x$) to define $\chi$ on $K^{e_{x-1}}$. In fact, we could also replace $x$ by
\[ x + x_0, \]
where $x_0 \in A^{e_{x-1} - j} \cap E(j)$. This choice will change $\chi$ on $K^{e_{x-1}}$ only if
\[ x_0 \notin A^{e_{x-1} - j} \cap E(j), \]
which suggests that the extension is unique if $A^{e_{x-1} - j} \cap E(j) = A^{e_{x-1} - j} \cap E(j)$. This fact may help explain the division below into cases.

Before entering into the body of the proof, we simplify notation. We will
generally be concerned with elements representing $H^{i-1}_0/H^{i}_0 \equiv m^{e_h}(e_h)$. Since
all the elements we will be concerned with will commute with $k_{f_h}$, we may as
well assume that $f_h = 1$. Similarly, the elements we deal with will all have
the $e/e_h$-periodicity of elements of $m^{e_h}(e_h)$, and it will therefore not affect
the proof if we assume that $e_h = 1$. The real effect of these assumptions is that
we can work with $m_e$ instead of $m^{e_h}(e_h)$. In a sense, we are assuming that
$h = 0$, and the discussion of power-permutation matrices given at the start of
the proof shows that this assumption does not affect whether the elements we
deal with in the proof actually lie in $H^{i-1}_0$. Of course, we must be careful to give
a proof that is valid without these simplifications; for instance, the reasoning
given above would be impermissible.

Because we want to use the decomposition $G(j) = B(j)W(j)B(j)$, where $B(j)$ is
the Iwahori subgroup and $W_{(j)}$ is the group of power-permutation matrices, we
need to deal with another set of congruence subgroups. Set $d = n/f_i$, and write
$H^{i-1}_0 \cap K^{i}_{n/f_i} = H_i$. We define $\chi_0$ to $H_i$ by backwards induction, assuming that
it is already defined on $H_{i-1}$ (to conserve notation, the extension to $H_{i-1}$ will
be called $\chi$). Write $d$ for $n/f$, and let $\eta \in G_{(j)}$ generate $A_d^1$ and induce $\sigma$ on $m_d^1(e_i)$. The coset representatives for $H_i/H_{i+1}$ can be taken as elements $1 + \gamma \eta^i$, $\gamma = (\gamma_0, \ldots, \gamma_{d-1}) \in m_d$; when $d \neq e$, some of the $\gamma_j$ may have to be 0, but for a given index $h$ either $\gamma_h$ must be 0 or there is no restriction on $\gamma_j$ (except, of course, that it commute with $k_{f_{i}}^d$; since we are in $m_d \cong M_d(k)^{n/f}$, this means that $\gamma_j \in k_{f_{i}}^d$). In particular, $(k_{f_{i}}^d)^d$ normalizes $H_i$. However, if $f/f_{i}^d/l$, then there is no requirement that any $\gamma_j$ be 0.

The proof divides into several large steps:

1. We find an extension $\chi_1$ to $H_i$ such that $(k_{f_{i}}^d)^d \cap m_d^1(e_i) \cong K_d \cap G_{(j)} / K_d \cap G_{(j)}$ commutes with $\chi_1$.

2. If $l$ is not divisible by $d/e_i$, then $\chi_1$ is the only extension of $\chi$ with the property in 1. We use this to show that every power-permutation matrix in $G_{(j)}$ commutes with $\chi$. In particular, $\eta$ commutes with $\chi$, and Lemma 4.2 now implies that $K_d^j \cap G_{(j)} = B_{(j)}$ commutes with $\chi$. (Here $B_{(j)} \subseteq K_{e} \cap G_{(j)}$ normalizes $H_i$.) That proves the lemma in this case, with $\chi = \chi_0$.

3a. If $l$ is divisible by $d/e_i$, then $\chi_1$ is not unique, but we can choose coset representatives for $H_i/H_{i+1}$ that remain in $H_i$ under conjugation by any element of $W_{(j)}$. We prove next that there is an extension $\chi_0$ such that $\eta$ commutes with $\chi_0$; $\chi_0$ is also not unique, but it is easy to describe the other possible choices. We also show that any “diagonal” matrix whose diagonal elements are powers of $\eta$ commutes with $\chi_0$.

3b. In view of 3a, we need only prove that permutation matrices commute with $\chi_0$. (This uses the fact that we have cotosef representatives as described in 3a; we therefore need to prove that $\chi_0 \cap W_{(j)}$ only for a set of generators $b$ for $W_{(j)}$.) We prove this last fact to complete the proof.

1. Write $(k_{f_{i}}^d)^d \cap m_d^1(e_i) \cong K_d \cap G_{(j)} / K_d \cap G_{(j)}$ commutes with $\chi_1$.

2. We assume that $d/e_i \not| l$. Then $\chi_1$ is the only extension of $\chi$ commuting with $\chi$. For every extension of $\chi$ to $H_i$ is of the form $\chi^{-}(y) = \chi_1(y)\chi_2(y)$, where $\chi_1$ is as above and $\chi_2$ is trivial on $H_{i+1}$. Thus $\chi_2(1 + \gamma \eta^i) = \psi \circ \text{Tr}(\alpha \eta^i \gamma)$ for some $\alpha \in m_d$, and the same calculation as for Lemma 4.8 shows that $(k_{f_{i}}^d)^d \cap m_d^1(e_i) \cong K_d \cap G_{(j)} / K_d \cap G_{(j)}$ commutes with $\chi_2$ if $\alpha \eta^i \gamma$ commutes with $\alpha \eta^{-1}$. This is impossible if $d/e_i \not| l$. (We need $\gamma \alpha = \alpha \gamma \sigma^{-1}$ for all $\gamma \in (k_{f_{i}}^d)^d \cap m_d^1(e_i)$, hence all $\gamma \in K_d^d \cap m_d^1(e_i)$. Let $\gamma$ have a 1 in the $j$th entry and 0's elsewhere; if $\alpha = (\alpha_0, \ldots, \alpha_{d-1})$, then $\gamma \alpha = (0, \ldots, 0, \alpha_j, 0, \ldots, 0)$, while $\alpha \gamma \sigma^{-1}$ has a 0 as its $j$th entry if $d/e_i \not| l$. So $\alpha_j = 0$ for all $j$. See Lemma 4.1 for a similar argument.) In the rest of the proof for this case, we use only this property of $\chi_1$.

As noted above, it now suffices to show that $W_{(j)}$ commutes with $\chi_1$. For
this, it suffices to prove that $\chi^b_0(y) = \chi_1(y)$ for power-permutation matrices $b$ and $(e_i, f_i)$-pure elements $y$ such that both sides are defined, since such elements $y$ generate $H_I$. Because $b$ normalizes $(k_{f_i}^{x})^{d/e_i}$, it is easy to see that $b^{-1}H_I b \cap H_I$ is normalized by $(k_{f_i}^{x})^{d/e_i}$ and that $\chi^b_0$ is fixed by $(k_{f_i}^{x})^{d/e_i}$ on this group. To see that $\chi^b_0 = \chi_0$ there, it suffices to show that $\chi$ has exactly one $(k_{f_i}^{x})^{d/e_i}$-stable extension from $b^{-1}H_{I+1} b \cap H_{I+1}$ to $b^{-1}H_I b \cap H_I$, since $\chi_1$ is already known to be one such extension. So let $\chi_1$ be any $(k_{f_i}^{x})^{d/e_i}$-stable extension of $\chi^b_0 \chi^{-1} (\equiv 1)$ to $b^{-1}H_I b \cap H_I$; we will prove that it is trivial.

This step, too, is done by induction: we assume that any such extension is trivial on $b^{-1}H_I b \cap H_I \cap K_d'$ ($r \geq 1$) and prove it trivial on $b^{-1}H_I b \cap H_I \cap K_{d-1}'$. Suppose that $y \in b^{-1}H_I b \cap H_I \cap K_d'$, but $y \notin b^{-1}H_{I+1} b \cap H_{I+1}$. Then either $y$ or $y z_{i-1}$ is in $K_d$ but not in $K_{d-1}$; assume the former for definiteness. As noted above, we may assume that $y$ is $(e_i, f_i)$-pure. If $y \in K_d'$, the inductive hypothesis says that $\chi^b_0 \chi^{-1}(y) = 1$. We show next that $(r - 1)$ is not divisible by $d/e_i$. If $(r - 1)$ is divisible by $d/e_i$, then $y$ and $y y_{i-1}$ both lie in $K_{d-1}'$ but not in $K_{d-1}'$, since $\eta^{-1}$ is central in $G_{(j)}$ and if $b \in W_{(j-1)}$ and $0 \neq \alpha \in m_d$, then $\alpha g^{-1} \in A_d$ and $\notin A_d'$. Since $y \in H_I$ and $y \notin H_{I+1}$, $l = r - 1$. But by hypothesis, $l$ is not divisible by $d/e_i$. This means that if $d/e_i | (r - 1)$, then $b^{-1}H_I b \cap H_I \cap K_d' = b^{-1}H_I b \cap H_I \cap K_{d-1}'$, and the induction extends.

So we may assume that $(r - 1)$ is not divisible by $d/e_i$. Let $y = 1 + y_0$, $y_0 = \gamma \eta y_{i-1}$, be a new addition to the domain of $\chi_1$ such that $y$ is $(e_i, f_i)$-pure and the nonzero entries are $\equiv c$ mod $n_i$. $(n_i = n/e_i f_i = d/e_i)$. Assume for convenience that char $k \neq 2$ (the modifications for char $k = 2$ are easy).

Let $\zeta \in m_d^\times((e_i)^\times)$ have $I$'s in every entry except those $\equiv c$ mod $n_i$, and $2I$'s in those. Then $\zeta y \zeta^{-1} = 1 + 2y_0$, and

$$\chi_1(\zeta y \zeta^{-1} y^{-1}) = 1, \quad \zeta y \zeta^{-1} y^{-1} \equiv y \mod K_d'.$$

Since $\chi_1(\zeta y \zeta^{-1} y^{-1}) = \chi_1(y)$ by the inductive hypothesis, $\chi_1(y) = 1$. This extends the induction and proves the result (with $\chi_0 = \chi_1$) when $d/e_i | 1$.

3a. Assume that $d/e_i | l$. Then, as noted above, the elements of $H_I / H_{I+1}$ can be picked to be stable under the power-permutation matrices $W_{(j)}$ for $G_{(j)}$. We change notation and let $\chi_{\alpha}(1 + y) = \psi \circ \text{Tr}(\alpha \eta_{(j)}^{-1 - j} y)$ for $y \in K_d^{-1}$. Since $\eta_{(j)}^{-1 - j} = e_0 \varphi^{1 - j} + \cdots, e_0 \in k_{d/e_i}^{f_i}$ (from 6.1(2)), the argument at the start of step 2 shows that $(k_{f_i}^{x})^{d/e_i}$ commutes with $\chi_\alpha$ iff $\alpha \in k_{d/e_i}^{f_i}$. For $y \in k_{f_i}$ and $y = \gamma \eta^{j-1}$, Lemma 4.8 gives

$$\chi_0^\eta(1 + y)\chi_1^{-1}(1 + y) = \chi_1(1 + \eta y \eta^{-1})\chi_1(1 + y)^{-1} = 1,$$

because $\eta$ and $y$ commute. Since $k_{f_i}^{x}$ fixes $\chi_\eta^\eta$, it fixes $\chi_\eta^\eta \chi_1^{-1}$; therefore

$$\chi_0^\eta = \chi_1 \chi_\beta, \quad \beta \in k_{f_i}^{x} \quad \text{and} \quad \beta \perp k_{f_i} \text{under } (\alpha, \beta) = \text{Tr } \alpha \beta.$$

We also know that there are $q^{d/ef_i}$ characters $\chi_\alpha$ of $H_I / H_{I+1}$. They form a commutative group on which $\eta$ acts by conjugation. The fixed elements are the $\chi_\alpha$ with $\alpha \in k_{f_i}^{x}$ (i.e., all components equal), since $\chi_0^\eta = \chi_\alpha$ with $\alpha^\sigma = \alpha$. Thus there are $q^{d/ef_{(d-1)}}$ characters $\chi_\alpha^\eta \chi_1^{-1}$. All of these annihilate the elements
$1 + \gamma \eta^{j-1}, \gamma \in k_{f_i}$, since

$$\chi_{\alpha}^{J} \chi_{\alpha}^{-1} (1 + \gamma \eta^{j-1}) = \psi \circ \text{Tr} (\gamma^\sigma - \gamma)^{a^{1-j}},$$

and $\gamma^\sigma = \gamma$ if $\gamma \in k_{f_i}$. Hence the $\chi_{\alpha}^{J} \chi_{\alpha}^{-1}$ exhaust the characters on $H_l$ annihilating $H_{l+1}$ and the $1 + \gamma \eta^{j-1}, \gamma \in k_{f_i}$. Since $\chi_{-\beta}$ is such a character, $\chi_{\alpha}^{J} \chi_{\alpha}^{-1} = \chi_{-\beta}$ for some $\alpha$. Set $\chi_0 = \chi_1 \chi_{\alpha}$. Then $\chi_0$ commutes with $\eta$ and with $(k_{f_i}^{x})^{d/e_i}$. (These properties of $\chi_0$ are what we use in further analyses of this case.)

It now suffices to show that $b$ commutes with $\chi_0$ whenever $b \in W(j)$. Elements of $W(j)$ take elements $\gamma \eta^{j-1}, \gamma \in m_\sigma$, to elements $\equiv \gamma' \eta^{j-1} \mod H_{l+d}$, as noted above. It follows easily that if $y \in H_l$ and $byb^{-1} \in K_{l}',$ then $y \in H_{l+1}$. We therefore need only prove that $\chi_0(y') = \chi_0(byb^{-1})$ for our coset representatives $y = 1 + \gamma \eta^l$. It suffices to check this when $y$ has only one nonzero entry, at (say) the $c$th place. It also suffices to assume that $b$ is a permutation matrix in $G(j)$ or a “diagonal” matrix,

$$P\eta P^{-1} = \begin{bmatrix} \gamma_0 & \xi \xi_0 & \cdots & \xi \xi_0 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \\ \end{bmatrix},$$

$P$ the permutation matrix as in §2, $y_h \in k_{f_i}$, where $\xi$ is defined by saying that $Pn^h P^{-1}$ is a “diagonal” matrix of $\xi$’s. (Recall: $n_i = n/e_i f_i$. Each entry is an $e_i \times e_i$ block matrix whose $f_i \times f_i$ blocks are in $k_{f_i}$; the blocks corresponding to $\xi^{a_i}$ have indices $\equiv l \mod n_i$.) For “diagonal” $b$, let $l \equiv c \mod n_i$. Then

$$\chi_0(gyg^{-1}) = \chi_0(\gamma_h \eta^{d,a_i} \cdot y(\gamma_h \eta^{d,a_i})^{-1}) = \chi_0(y),$$

which takes care of this case.

Here is a brief illustration of this last point. For the matrix

$$\eta = \eta_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega & 0 & \omega & 0 \\ 0 & \omega & 0 & \omega \\ \end{bmatrix},$$

used in the example for Lemma 3.3, we have

$$\xi = \begin{bmatrix} 0 & 1 \\ \omega & \omega \\ \end{bmatrix}.$$

The “diagonal” matrix corresponding to $[\xi^t 0 \xi]$ is

$$b_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & a \omega^2 & 0 & b \omega^2 \\ \omega & 0 & \omega & 0 \\ 0 & b \omega^3 & 0 & c \omega^2 \\ \end{bmatrix}, \quad a, b, c \text{ as in } \S 3.$$

It is easy to see that for $1 + \gamma \eta^{2h}$ with $\gamma = (0, \gamma_1, 0, \gamma_3)$, so that the matrix for $\gamma \eta^{2j}$ has entries only in the starred positions of

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \end{bmatrix},$$
we have \( b_1(1 + \gamma \eta^{2h})b_1^{-1} = \eta^4(1 + \gamma \eta^{2h})\eta^{-4} \), since \( b_1 \) and \( \eta^4 \) have the same entries in the rows and columns with the starred entries.

3b. The permutation matrices in \( G_{ij} \) are \((e_i, f_i)\)-permutation matrices and normalize \((k_f)^d\). So if \( b \) is one of these permutation matrices, then \((k_f)^d\) also fixes \( \chi_0^b \) and must therefore be of the form \( \chi_0 \chi_\alpha \), \( \alpha \in (k_f)^d \). The map \( \phi: b \mapsto \alpha \) (which we regard as a map from \( \mathcal{S}_{n_i} \) to \((k_f)^d\); \( \mathcal{S}_{n_i} \) acts on each block of \( n_i \) symbols in the same way) has the following properties:

(i) \( \phi \) is a 1-cocycle: \( \phi(uv) = \phi(u)v + \phi(v) \);
(ii) \( \phi(u) = 0 \) if \( u \) is the cyclic transposition \((0, 1, \ldots, n_i - 1)\) (then \( u \) is the product of \( \eta \) and a "diagonal" element);
(iii) \( \phi(v_1) = (\alpha_0, \ldots, \alpha_{d-1}) \) has \( \alpha_b = 0 \) if \( b \) is fixed by \( v_1 \). (We then have \( v_1v_1^{-1} = y \) if \( y \) is \((e_i, f_i)\)-pure with nonzero entries only at indices \( \equiv b \mod n_i \).)

It is clear that (i) and (iii) imply

(iv) If \( \phi(v_2) = (\beta_0, \ldots, \beta_{d-1}) \) and \( v_1, v_2 \) both take \( b \) to \( c \), then \( \alpha_b = \beta_b \).

(Write \( v_2 = (v_2v_1^{-1})v_1 \).)

The permutation \( v \) of order 2 given by

\[
(0, n_i - 1)(1, n_i - 2)\cdots \left( \frac{n_i - 2}{2}, \frac{n_i}{2} \right) \quad \text{if } n_i \text{ is even}
\]

and

\[
(0, n_i - 1)(1, n_i - 2)\cdots \left( \frac{n_i - 3}{2}, \frac{n_i + 1}{2} \right) \quad \text{if } n_i \text{ is odd}
\]
satisfies \( vuv = u^{-1} \), where \( u \) is as in (ii). Therefore \( \phi(v)^u = \phi(v) \), which implies that \( \phi(v) = \beta \in k_f \).

Suppose first that \( p \neq 2 \). Since \( v \) has order 2 and \( \phi(v)^u = \phi(v) \), \( 2\beta = 0 \) or \( \beta = 0 \). From (iv), \( \phi((0, n_i - 1)) = 0 \). Conjugating by \( u \), we get \( \phi((b_b, b + 1)) = 0 \) for all \( b \). It follows that \( \phi \equiv 0 \).

If \( p = 2 \), we work a bit harder. If \( n \) is odd, then \( \beta = 0 \) because \((n_i - 1)/2\) is fixed by \( v \). We get \( \phi \equiv 0 \) as in the case \( p \neq 2 \). If \( n \) is even, then we have \( \phi((0, n_i - 1)) = (0, 0, 0, \ldots, 0, \beta) \), and conjugation with \( u \) gives \( \phi((b_b, b + 1)) = (0, 0, 0, \ldots, 0, 0, \beta, 0, 0, \ldots) \), with the \( \beta \)'s in the \( b_b \), \( b + 1 \) places. We now have

\[
\phi((0, 1, 2)) = \phi((1, 2))^{(0,1)} + \phi((0, 1)) = \phi((1, 2)),
\]

\[
\phi((0, 1, 2, 3)) = \phi((2, 3)(0, 1, 2)) = \phi((2, 3))^{(0,1,2)} + \phi((0, 1, 2)) = \phi((0, 1)) + \phi((2, 3)) \] (as one sees by calculating the terms),

\[
\phi((0, 1, 2, 3, 4)) = \phi((0, 1)) + \phi((3, 4)),
\]

\[
\vdots
\]

\[
\phi((0, 1, \ldots, n_i - 1)) = \phi((0, 1)) + \phi((2, 3)) + \cdots + \phi((n_i - 2, n_i - 1)) = \beta.
\]

Hence \( \beta = 0 \), and we finish as before. \( \square \)

For the rest of the proof of Theorem 6.1, we make the notational convention that if \( \chi \) is the given representation on \( K_f \), then \( \chi_0 \) is a (fixed) extension of \( \chi \) to \( K_{e,-1} \) with the property of the lemma. (We shall then construct an extension of \( \chi \) to \( K_{e,-1} \), which we also call \( \chi \); often \( \chi \neq \chi_0 \).)
We return to the proof of Theorem 6.1. Assume that we have defined \( \chi \) on \( H^i_0 \), that \( t'_{h-1} \geq j > t'_h \), and that \( i \) is the largest index with \( j \leq t_i \). We need to extend \( \chi \) to \( H^{-1}_0 \); the procedure is slightly different in each of the three remaining steps, which we also think of as different cases. The hard part is generally in finding \( \eta_{(j-1)} \); in constructing it, we usually verify most of (1)-(11).

(a) (Step 3). Assume that \( j - 1 > t_i+1 \) and \( e^{\frac{1}{(j-1)}}e_t \). Then set \( \chi = \chi_0 \) and \( \eta_{(j-1)} = \eta_{(j)} \). It follows that \( E_{(j-1)} = E_{(j)} \), etc.; all of (1)-(11) are vacuous or trivial except for (7), which we now treat. (Occasionally the treatment that follows is slightly more complicated than necessary; this is to ensure that essentially the same proof applies to the other cases). Note that \( C_{j-1} = 1 \).

It is easy enough to check that \( K^{e^2+j}_0 \cap G_{i-1} \) commutes with \( \chi|_{H^i_0} \) when \( h < c < i \); one applies Lemma 4.2. (For instance, if \( x \in K^{e^2+j}_0 \cap G_{i-1} \) and \( x - 1 \) is invertible, then \( \chi^x = \chi \) on \( K^{e^2+j}_0 \), while if \( y \in H^i_0 \) and \( y - 1 \) is invertible mod \( H^i_0 \), then \( \chi^y = \chi \) on \( K^{e^2+j}_0 \).) The proof that \( K^{e^2+j}_0 \cap G_{i-1} \) commutes with \( \chi|_{H^i_0} \) when \( 0 \leq c < h \) \((G_0 = G)\) is the same. We have constructed \( \chi (= \chi_0) \) so that \( G_{(j-1)} (= G_{(j)}) \) commutes with \( \chi \) on \( H^{-1}_0 \). Conversely, suppose that \( w \in Z_eK_e \) commutes with \( \chi \). Then \( w \) commutes with \( \chi|_{H^i_0} \), and (7) applied to \( \chi|_{H^i_0} \) implies that we can write

\[
\begin{align*}
w &= \beta_0 \eta_{(j)}^0 (1 + \beta_1 \eta_{(j)} + \cdots + \beta_{i-j} \eta_{(j)}^{i-j} + \beta_{i-j+1} \eta_{i-j+1}^{i-j+1} + \cdots) \\
&= \beta_0 \eta_{(j-1)}^0 (1 + \beta_1 \eta_{(j-1)} + \cdots + \beta_{i-j} \eta_{(j-1)}^{i-j} + \beta_{i-j+1} \eta_{i-j+1}^{i-j+1} + \cdots),
\end{align*}
\]

with \( \beta_g \in m_e \) for all \( g \), \( \beta_0 \in m_e^* \), and \( \beta_g \in m^f(e_i) \) for \( g \leq t_i - j \), \( \beta_g \in m^f_{i-1}(e_{i-1}) \) for \( t_i - j < g \leq t_{i-1} - j \), etc. We need to prove first that \( \mod A_g^{e^2+j+2} \), \( \beta_{i-j} \eta_{i-j+1}^{i-j+1} \equiv \) some \( u \in M_{i-j+1} \). Dividing by an element known to commute with \( \chi \) and writing \( \beta \) for \( \beta_{i-j+1} \), we may assume that \( w = 1 + \beta \eta_{i-j+1}^{i-j+1} + \cdots = w_1 w_0 \), where \( w_0 = 1 + \beta \eta_{i-j+1}^{i-j+1} \)

We show first that \( \chi^{w_1}(y) = \chi(y) \) if \( y = 1 + \gamma \eta_{i-j}^{i-j} \) with \( \gamma \in m^f_{i-1}(e_{i-1}) \). Note that if \( g \leq i - 1 \), then \( \eta_{i-1} \equiv \zeta \eta_{i-1} \) for some \( \zeta \in k^s_j \), by an induction using (5). Because \( \zeta \) and \( \gamma \) are both in \( m^f(e_g) \), it follows that \( y \) is congruent mod \( K^e_i \) to an element of \( G_g \). In fact, since \( y \) and \( \zeta \) are both in \( H_0 \), this congruence also holds mod \( H^i_0 \).

Since \( w_1 \in (K_e^{i-j+1} \cap G_{i-1})(K_e^{i-j+1} \cap G_i) \cdots (K_e^{i-j+1} \cap G_{h-1}) \), it suffices to prove that \( \chi^{w_1}(y) = \chi(y) \) if \( w_1 \) is in any one of the factors. If \( w_1 \in (K_e^{i-j+1} \cap G_{i-1}) \), then \( \chi^{w_1}(y) = \chi(y) \) by Lemma 4.2 (since \( G_{i-1} \) commutes with \( \chi \) on \( H^i_0 \)). If \( w_1 \in (K_e^{i-j+1} \cap G_i) \), then \( (w_1, y) \in H^i_0 \), and we are concerned with \( \chi \) on that group. Write \( \chi = \chi_0 \cdot 1 \chi_1 \cdot 1 \cdots \chi_{i-1} \cdot 1 \cdot \chi_{i-1} \cdots \chi_1 \cdot 1 \) as in (8). Then \( \chi_0 \cdot 1 \cdots \chi_{i-1} \cdot 1 \) is 1 by Lemma 4.2, since \( G_{i-1} \) commutes with \( \chi_{i-1} \cdot 1 \), while \( \chi_{i-1} \cdot 1 \cdots \chi_1 \cdot 1 \) is 1 by Lemma 4.7, since \( y \) is congruent mod \( H^i_0 \) to an element of \( G_{i-1} \). This proves the claim for \( w_1 \in (K_e^{i-j+1} \cap G_{i-2}) \), and the proof for the other factors is nearly the same.

Therefore \( \chi^w \cdot \chi^{-1}(y) = \chi(w_0 y w_0^{-1} y^{-1}) \) and \( w_0 y w_0^{-1} y^{-1} \in H^i_0 \). Write
\[ \chi|_{H^j_0} = \chi_0, \chi_1, \] as in (8). Since \( G_{i-1} \) commutes with \( \chi_0, \chi_1 \),
\[ \chi_0, (w_0yw_0^{-1}y^{-1}) = 1. \]

By (8) and Lemma 4.7, \( \chi_1, (w_0yw_0^{-1}y^{-1}) = 1 \) for all \( y \) only if \( w_0 \) is congruent mod \( K_{e_{i-1}} \) to an element \( w_0' \) of \( G_{i-1} \), the group of invertible elements in the algebra \( \mathcal{M}_{i-1} \) defined in (8). (Note that \( y \) is congruent modulo \( H^j_0 \) to an element of \( G_{i-1} \).) By (iv) of (8), together with (5), \( w_0 \equiv w' \mod K_{e_{i-1}} \), where \( w' \in G_{(j)} \cap K_{e_{i-1}} \). We already know that \( \chi w' = \chi \).

Dividing by \( w' \), we may write
\[ w = 1 + \beta_{t_i-j+2} \eta_{i-2}^{t_i-j+2} + \cdots + \beta_{t_i-j} \eta_{i-2}^{t_i-j} + \beta_{t_i-j+1} \eta_{i-2}^{t_i-j+1} + \cdots \]
(where the \( \beta_i \) may be different from what they were in the previous expression for \( w \)). The inductive hypothesis lets us write \( w = 1 + \beta_{t_i-j+1} \eta_{i-1}^{t_i-j+1} + \cdots \), \( \beta_{t_i-j+1} \in m_{e_{i-1}}^{h_{i-1}}(f_{i-1}) \), etc.; an argument like the one above shows that \( w \) is congruent modulo \( K_{e_{i-1}} \) to an element in \( G_{i-1} \). We continue inductively to get (7). This concludes the proof in case (a).

(b) (Step 4). Assume that \( j-1 > t_{i+1} \) and \( e(j-1)e_i \). Set
\[ \chi(1+y) = \chi_0(1+y) \psi \circ \text{Tr}(\alpha' \eta_{(j-1)}^{t_{i-1}} y) = \chi_0(1+y) \chi_{\alpha'}(1+y), \]
where \( \alpha' \in m_{e_{i-1}} \) is such that \( \alpha = \text{Tr}_{e_i} \alpha' \in k_{j-1} \). (Note that \( y = \gamma \eta_{(j-1)}^{t_{i-1}} \) for some \( \gamma \in m_{e_{i-1}}^{h_{i-1}}(e_{i-1}) \). There is therefore some ambiguity in \( \alpha' \); what determines \( \chi \) is not \( \alpha' \) but \( \text{Tr}_{e_{i-1}} \alpha' \). We could find an \( \alpha'' \in m_{e_{i-1}}^{h_{i-1}}(e_{i-1}) \) that gives the character, but then we would need to take traces over \( F_{i-1} \). There seems to be no advantage in introducing the additional notation.) We choose one \( \alpha' \) for each \( \alpha \), with the added requirements that each component of \( \alpha' \) is in \( k_{j-1} \); that the first \( e_i \) components of \( \alpha' \) are all the same, the second \( e_i \) components are all the same, etc.; and that 0 corresponds to 0. For instance, we might make all components of \( \alpha' \) beyond the first \( e_i \) equal to 0. (If \( \alpha' = 0 \), then of course \( \chi = \chi_0 \).) This gives us \( q^{j-1} = C_{j-1} \) choices for \( \chi \). (It is in fact the case that if \( \alpha = 0 \), then \( \chi \) and \( \chi_0 \) and conjugate regardless of what \( \alpha' \) is chosen, but we shall not need this fact here. It is proved by applying Lemma 4.7 repeatedly.)

We shall show below that these extensions (for different \( \alpha \)) are nonconjugate under \( Z_{e_{j-1}} K_{e_{j-1}} \). A computation in the proof of Lemma 6.2, case (2), shows that if \( \gamma \in (k_{e_{j-1}}^*)^{h_{j-1}} \), then \( \gamma^T = \chi \); similarly, \( \chi \) is fixed by all \( (e_i, f_i) \)-permutation matrices because \( \alpha' \) is.

We next produce \( \eta_{(j-1)} \) and \( E_{(j-1)} \). This is a complicated affair because we need to insure that \( E_{(j-1)} \) is nicely embedded. An example may help to explain the problem that may arise. Let \( n = e = 4, f = 1, e_1 = 2, s_1 = t_1 = 10, \) and \( s_2 = t_2 = 8 \). Suppose that for \( \omega = \omega_4 \) and \( \gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3) \), \( \gamma_j \in k \), we have
\[ \chi(1 + \gamma \omega^{10}) = \psi \left( \sum_{j=0}^{3} \gamma_j \right). \]

Then we may take \( E_{(10)} = F[\omega^2] \) and \( \eta_{(10)} = \omega \). From case (a), \( E_{(9)} = E_{(10)} \). For level 8, we produce \( \chi_0 \) and set \( \chi = \chi_0 \chi_1 \), where
\[ \chi_1(1 + \gamma \omega^8) = \psi(\gamma_0 + \gamma_1) \text{ if } \gamma \omega^8 \in G_{(9)}. \]
This last condition means that \( \gamma = (\gamma_0, \gamma_1, \gamma_0, \gamma_1) \). Then \( \chi_1 = \psi \circ \text{Tr}^{(2)}(\alpha \gamma) \), where we may take \( \alpha = (1, 1, 1, 1) \in m^1_4(2) \). We need \( \alpha' \). If \( p \not= 2 \), we can take \( \alpha' = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \). It is then clear that \( m^1_4(2) \) and \( \varpi \) commute with \( \chi_1 \), so that \( E_{(\gamma)} = E_{(10)} \). (This is one reason that the tamely ramified case is easier than the general case.) But if \( p = 2 \), this choice is impossible. We may take \( \alpha' = (1, 1, 0, 0) \); recall that the first two entries of \( \alpha' \) must be equal. Now (since \( \text{char } k = 2 \), so that \( + \) and \( - \) are the same)

\[
\chi^{\varpi}_1 \chi^{-1}_1 (1 + \gamma \varpi^8) = \psi \left( \sum_{j=0}^{3} \gamma_j \right), \quad \gamma \in m^4_4,
\]

as a calculation shows. But for \( \delta = (\delta_0, \delta_1, \delta_2, \delta_3) \) and \( w = 1 + \delta \varpi^2 \),

\[
\chi^w_0 \chi^{-1}_0 (1 + \gamma \varpi^8) = \psi \left( \sum_{j=0}^{3} \gamma_j (\delta_j + \delta_{j+2}) \right)
\]

(extend \( \delta_j \) periodically with period 4). So \( \chi^{\varpi w} = \chi \) if \( \delta_j + \delta_{j+2} = 1 \), so that \( \delta = (\delta_0, \delta_1, \delta_0 + 1, \delta_1 + 1) \). If we take \( \delta_0 = \delta_1 = 1 \), then \( F((\varpi w)^2) \) is nicely embedded and gives \( E_{(\gamma)} \). But if not, \( F((\varpi w)^2) \) need not be nicely embedded. The following is one way of forcing the nice embedding of the field.

We begin not with \( \eta_{(j-1)} \), but with something closer to \( \eta^{\epsilon_{(j)}}_{(j)} \). Let \( \theta_{(j)} = \eta^{\epsilon_{(j)}}_{(j)} \). We set

\[
\theta_{(j-1)} = \theta_{(j)} (1 + \delta_{i-1} \eta_{i-1}^{\epsilon_{(j)}}) \cdots (1 + \delta_{h-1} \eta_{h-1}^{\epsilon_{(j)}}),
\]

where \( \delta_g \in k_g^{\epsilon} \), \( \delta_g^{\epsilon_{(j)}} = \delta_g \), and the \( \delta_g \) are chosen so that \( \theta_{(j-1)} \) commutes with \( \chi \). We choose the \( \delta_g \) inductively, as follows: \( \chi^{\theta_{(j)}} \chi^{-1} \) is trivial on \( H_0^1(H_0^{i-1} \cap G_{(j)}) \), and by (9) we can choose \( \delta_{i-1} \) so that if \( w_i = \theta_{(j)} (1 + \delta_{i-1} \eta_{i-1}^{\epsilon_{(j)}}) \), then \( \chi^w \chi^{-1} \) is trivial on \( H_0^1(H_0^{i-1} \cap G_{(i-1)}) \). Now (9) lets us choose \( \delta_{i-2} \) so that if \( w_2 = w_i (1 + \delta_{i-2} \eta_{i-2}^{\epsilon_{(j)}}) \), then \( \chi^w \chi^{-1} \) is trivial on \( H_0^1(H_0^{i-2} \cap G_{(i-2)}) \). Continue.

The matrix \( \theta \) has a special and useful form. Regard elements of \( M \) as \( n/f_i \times n/f_i \) block matrices (with \( f_i \times f_i \) blocks), and let \( P \) be the block permutation matrix taking \( (0, 1, \ldots, n/f_i - 1) \) to \( (0, n_i, \ldots, (e_i - 1)n_i, 1, n_i + 1, \ldots, n_i - 1, n_i - 1, n_i - 2, \ldots, n/f_i - 1) \), as in \( \S 2 \). Then each term \( P^{-1} \theta_{(j)} P \), \( P^{-1} (1 + \delta_{i-1} \eta_{i-1}^{\epsilon_{(j)}}) P \), \ldots, \( P^{-1} (1 + \delta_{h-1} \eta_{h-1}^{\epsilon_{(j)}}) P \) is a "diagonal" block matrix consisting of \( n_i \) blocks, each block an \( e_i \times e_i \) block matrix of \( f_i \times f_i \) blocks that (are) elements of \( F^r_{(j)} \). The same statement therefore holds for \( P^{-1} \theta_{(j-1)} P = \xi \) (say). Let the blocks for \( \xi \) be \( (\xi_0, \ldots, \xi_{n_i-1}) \). Similarly, set \( P^{-1} \theta_{(j)} P = \xi' = (\xi'_0, \ldots, \xi'_0) \) and \( P^{-1} (1 + \delta_g \eta_g^{\epsilon_{(j)}}) P = \tau_g = (\tau_{g,0}, \ldots, \tau_{g,n_i-1}) \). (Since \( \theta_{(j)} \in E_{(j)} \) and \( E_{(j)} \) is nicely embedded, the entries for \( \xi' \) are all the same.) We then have

\[
\xi_l = \xi' \tau_{i-1,l} \cdots \tau_{i-l,1}, \quad 0 \leq l \leq n_i - 1.
\]

If we now write \( \mu_g = (\tau_{g,0}, 1, \ldots, 1) \), then \( P \mu_g \) commutes with \( \chi_{|H_i} \). (For if \( P^{-1} \delta_g P = (\delta_g, \ldots, \delta_g, n_i-1) \), then \( P \mu_g \) is \( (1 + \delta_g \eta_g^{\epsilon_{(j)}}) \), where \( P^{-1} \delta_g P = (\delta_g, 0, \ldots, 0) \); thus \( \delta_g \in m_f^e(e_g) \). Now use property (7).) Since
$P(\xi', 1, \ldots, 1)P^{-1}$ also commutes with $\chi|_{H_j}$, we see that $\kappa = P(\xi_0, 1, \ldots, 1)P^{-1}$ commutes with $\chi|_{H_j}$. We can prove that $\kappa$ commutes with $\chi$ on $H_{j-1}^{-1}$ by showing that $\chi^k(y) = \chi(y)$ for a set of generators $y$ of the group $H_{j-1}^{-1}/H_j$. We choose the $y$ to be $(e_i, f_i)$-pure. Then $\kappa y k^{-1} = \theta_{(j-1)} y \theta_{(j-1)}^{-1}$ if the entries of $y - 1$ (as a block matrix with $f_i \times f_i$ blocks) are in rows and columns with indices divisible by $n/e_i f_i$, and $\kappa y k^{-1} = y$ otherwise. In either case, $\chi^k(y) = \chi(y)$.

We are now almost done with the construction of $\eta_{(j-1)}$. Let $\lambda$ be the $n_i \times n_i$ block matrix

$$
\lambda = \begin{bmatrix}
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & I \\
\xi_0 & 0 & \cdots & 0 & 0
\end{bmatrix},
$$

and let $\zeta_{(j-1)} = PLP^{-1}$, $\eta_{(j-1)} = \zeta_{(j-1)}^{(f_i)}$. Since $\lambda^{n_i}$ is the $n_i \times n_i$ block matrix $(\xi_0, \ldots, \xi_0)$, $F_{(j-1)}[\zeta_{(j-1)}^{n_i}] = E_{(j-1)}$ is nicely embedded. (Note: $n_i = e f/e_i f_i$.)

We also have

$$
\eta_{(j-1)} = \eta_{(j)} P^{-1} \mu_{j-1} \cdots \mu_{h-1} P = \eta_{(j)} (1 + \delta_i^{(f_i)} \eta_{j_i}^{f_i}) \cdots (1 + \delta_i^{(f_i)} \eta_{h-1}^{f_i});
$$

since each $(1 + \delta_i^{(f_i)} \eta_{j_i}^{f_i})$ commutes with $m_i^{f_i}(e_i)$, conjugation by $\eta_{(j-1)}$ acts as $\sigma$ on $m_i^{f_i}(e_i)$. Furthermore, Lemma 3.3 applies to $E_{(j)}$, $E_{(j-1)}$, so that the power-permutation matrices for $G_{(j-1)} = \text{GL}_{n_i}(E_{(j-1)})$ all commute with $\chi|_{H_0^j}$. Thus it suffices to prove that for any power-permutation matrix $b$, $\chi^b(y) = \chi(y)$ when $y$ is a set of $(e_i, f_i)$-pure generators of $H_0^j/H_0^i$, as before. Since the elements $b y b^{-1}$ are then also a set of $(e_i, f_i)$-pure generators of $H_0^j/H_0^i$, it suffices to consider a set of $b$ generating the power-permutation matrices, for instance $\kappa$ and the $(e_i, f_i)$-permutation matrices. But we have already verified that $\chi^b(y) = \chi(y)$ for these elements.

In the course of constructing $\eta_{(j-1)}$ and $E_{(j-1)}$, we have verified properties (1)–(3), (5), and half of (7). Furthermore, (4) is clear, (8)–(10) follow directly from the inductive hypothesis, and the other part of (7) is proved almost exactly as in case (a). (Note that (7.1) holds because of (5) and Lemma 3.1; the $\beta_i$ for $l \geq t_i - j + 1$ may change from one line to the next.) So we need only verify (11), that different choices for $\alpha'$ give nonconjugate extensions of $\chi$. Consider two extensions $\chi = \chi_0 \cdot \chi_0^\alpha$, $\chi_0 \cdot \chi_0^\beta$, where, e.g., $\chi_0^\alpha(1 + y) = \psi \circ \text{Tr}(\alpha y^{(j_i)^{-1}}) y$, $y \in K_{j_i}^{-1}$. Suppose that these are conjugate by $x \in Z_e K_e$ and that $\alpha' \neq \beta'$. Then $x$ commutes with $\chi_0|_{K_{j_i}^{-1}}$, but not with $\chi_0|_{K_{j_i}^{-1}}$, and (7) (for $j, j - 1$) shows that we may take $x = (1 + \delta_i^{(e_i)} \eta_{j_i}) \cdots (1 + \delta_h^{(e_i)} \eta_{h-1}^{(e_i)})$ with $\delta_i \in m_i^{(e_i)}(e_i)$. We show that $\chi^x(y) = \chi(y)$ for all $y = 1 + \gamma \eta_{(j-1)}$ with $\gamma \in m_i^{(e_i)}(e_i)$. The proof is like a part of that used in (7). We may work with one factor of $x$ at a time; thus we assume first that $x = 1 + \delta_i^{(e_i)} \eta_{j_i}^{(e_i)}$. Then $(x, y) \in H_0^i$. Write $\chi$ on $H_0^i$ as $\chi_0, i \chi_1, i$, as in (8). Since $x \in G_{(i)}$, Lemma 4.2(b) implies that $\chi_0, i((x, y)) = 1$; since $y$ is congruent mod $H_0^i$ to an element of $G_{(i)}$ (by the same reasoning as in (a)), (4.7) implies that $\chi_1, i((x, y)) = 1$. Thus
\(\chi^x(y) = \chi(y)\) in this case, and the same argument (with only indices altered) applies to the other factors.

Therefore \(\chi^x = \chi\) on \(G_{(j-1)} \cap K_e\), so that \(\text{Tr}_{e_0}(\beta') = \text{Tr}_{e_0}(\alpha')\). By construction, \(\alpha' = \beta'\), contradicting our assumption that \(\alpha' \neq \beta'\). This concludes the proof in case (b).

(c) (Step 5) Assume that \(j - 1 = t_{i+1}\); then \(e | (j - 1)t_{i+1}\). We let \(\eta_i = \eta_{(j)}, E_i = E_{(j)}, \text{ and } M_i = M_{(j)}\); we set

\[
\chi(1 + y) = \chi_0(1 + y)^{\psi} \circ \text{Tr}(\alpha' \eta_i^{1-j} y) = \chi_0(1 + y)(1 + y),
\]
say, where \(\alpha'\) has the property that if \(\alpha_{i+1} = \text{Tr}_{e_i}\alpha', \text{ then } E_i[\alpha_{i+1} \eta_i^{1-j}] = E_{(j-1)} = E_{(t_{i+1})}\) has ramification index \(e_{i+1}/e_i\) and residue class degree \(f_{i+1}/f_i\) over \(E_i\). (The condition on the ramification index follows from (1.3).) We also assume that all entries of \(\alpha'\) are in \(k_t\). An argument like that for the case of \(t_1\) shows that we may assume that the first \(e/e_{i+1}\) entries of \(\alpha'\) are all the same, the next \(e/e_{i+1}\) are all the same, and so on. This makes \(E_{(j-1)}\) nicely embedded. Then \(\chi'\) commutes with all \((e_{i+1}, f_{i+1})\)-permutation matrices, and hence \(\chi\) does as well. This proves part of the second part of (7). We define \(\chi_{0, i+1} = \chi_0, \chi_{1, i+1} = \chi'\) in (8). Choose one \(\alpha \eta_i^{1-j} \) from each \(Z_k E_k\)-conjugacy class and one \(\alpha'\) for each \(\alpha\). Write \(f_{i+1} = f_{i+1}^r f_i\).

Suppose that \(w = \delta_0 \eta_i^h(1 + \delta_1 \eta_i + \cdots) = w_1w_2 \in Z_k E_k\) commutes with \(\chi\). We show that \(w_1 = \delta_0 \eta_i^h \) commutes with \(E_{(t_{i+1})}\). We may assume that \(w\) commutes with \(\chi\) on \(K_{t_{i+1}}\), so that \(\delta_0 \eta_i^h \) commutes with \(E_i\), from (7); hence we need only prove that \(\delta_0 \eta_i^h \) commutes with \(\alpha \eta_i^{1-j}\). Write \(y = 1 + \gamma \eta_i^{1-j} = 1 + y_1\), with \(\gamma \in m_{e_i}(\xi_i)\). Then

\[
\chi'(wyw^{-1}y^{-1}) = \chi_0(wyw^{-1}y^{-1})\chi'(wyw^{-1}y^{-1}).
\]
But \(\chi_0(wyw^{-1}y^{-1}) = 1\) because \(w \in G_i\), and \(\chi^w = \chi^{w_1}\) because commutators \((y, w_2)\) are in \(K_{t_{i+1}}\). Now Lemma 4.8 shows that \(w_1\) commutes with \(\alpha \eta_i^{1-j}\). Note that for \(h = 0\), this means that \(\delta_0 \in (m_{e_i}(\xi_i))^{x}\); conversely, the calculation shows that \((m_{e_i}(\xi_i))^{x}\) commutes with \(\chi\).

We next construct \(\eta_{(j-1)}\). The procedure is like that used in (b), and we omit details when the calculations are essentially the same as in (b). We begin with \(\theta_i = \eta_i^{e/e_i}\),

\[
\theta_{(j-1)} = \delta_{i} \eta_i(1 + \delta_{i-1} \eta_i^{t_{i-1}-t_{i+1}+1})(1 + \delta_{i-2} \eta_i^{t_{i-2}-t_{i+1}+1}) \cdots (1 + \delta_{i-h} \eta_i^{t_{i-h}-t_{i+1}+1}),
\]
where the \(\delta_i \in k_{t_{i+1}}\) are to be determined. Let \(\tau_i\) be a prime element in \(F_i[\alpha \eta_i^{-t_{i+1}}]\) of the form \((\alpha \eta_i^{-t_{i+1}})^g e_i^{b/e_i}\) (where \((e/e_i)b - t_i a = e/e_{i+1}\); this is possible by (1.3), since \(n/f = e\) and \(s_i/f = t_i\), chosen so that \(F_i[\tau_i] = F_i[\alpha \eta_i^{-t_{i+1}}]\); this element is then of the form \(\delta \eta_i\). The calculation given just before the start of this construction shows that \(\chi^0 \chi^{-1}(1 + \gamma \eta_i^{t_{i+1}}) = 1 \forall \gamma \in m_{e_i}(\xi_i)\). We now choose the \(\delta_g\) inductively, using (8) and Lemma 4.7 as in case (b), to make \(\delta_g^{t_{i+1-j}} = \chi\). Because the \(t_g\), \(g \leq i+1\), are all divisible by \(f_{i+1}\), it is straightforward to check that the terms \(\delta_g \eta_i^{t_{i+1}-t_{i+1}}\) all commute with \(m_{e_i}^{t_{i+1}}(\xi_i)\).

Furthermore, if we regard elements of \(M\) as \(n/f_{i+1} \times n/f_{i+1}\) block matrices and let \(P\) be the block permutation matrix taking \((0, 1, \ldots, n/f_{i+1} - 1)\) to
(0, n_{i+1}, \ldots, (e_{i+1} - 1)n_{i+1}, 1, n_{i+1} + 1, 2n_{i+1} + 1, \ldots, n_{i+1} - 1, \ldots, n/f_{i+1} - 1), as in §2 (with n_{i+1}e_{i+1}f_{i+1} = n), then P^{-1}\theta_{(j-1)}P = \chi is a "diagonal" block matrix consisting of $n_{i+1}$ blocks, each block an $e_{i+1} \times e_{i+1}$ block of elements of $F_{f_{i+1}}$ (embedded as $f_{i+1} \times f_{i+1}$ matrices). Let the blocks for $\xi$ be $(\xi_0, \ldots, \xi_{n_{i+1}-1})$; notice that $P^{-1}\tau_i P$ is of the form $(\xi_0', \xi_0', \ldots)$. Just as in case (b), we show that for $\kappa = P(\xi_0, 1, \ldots, 1)P^{-1}$, $\chi^\kappa = \chi$ on their common domain, and that if $\lambda$ is the $n_{i+1} \times n_{i+1}$ block matrix

$$
\begin{bmatrix}
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & I \\
\xi_0 & 0 & \cdots & 0 & 0
\end{bmatrix}
$$

$\xi_{(j-1)} = P\lambda P^{-1}$, $\eta_{(j-1)} = \xi_{(j-1)}^{f_{i+1}}$, and $E_{(j-1)} = F_{f_{i+1}}[r_{n_{i+1}}^{(j-1)}]$, then $E_{(j-1)}$ is nicely embedded and $\eta_{(j-1)}$ acts as $\sigma$ on $m_{e_{i+1}}^{f_{i+1}}(e_{i+1})$. (Note that $n_{i+1} = e/e_{i+1} \cdot f/f_{i+1}$. Since Lemma 3.3 applies to $E_{(j-1)}$ and $E_{(j-1)}$, the power-permutation matrices for $G_{(j-1)} = \text{GL}^{n_{i+1}}_{(j-1)}$ all commute with $\chi |_{H_0^j}$. Therefore it suffices to prove that for any power-permutation matrix $b$ of $G_{(j-1)}$, $\chi^b(y) = \chi(y)$ when $y$ runs through a set of $(e_{i+1}, f_{i+1})$-pure elements generating $H_0^j/H_0^j$. Because $byb^{-1} \in H_0^j$ for all such $b$, $y$, it suffices to consider a set of $x$ generating the power-permutation matrices, for instance $\kappa$ and the $(e_i, f_i)$-permutation matrices. We have verified that $\chi^b(y) = \chi(y)$ for these matrices, so that $G_{(j-1)}$ commutes with $\chi$; this is part of (7). We also set $\eta_{(i)} = (P\lambda' P^{-1})f/f_{i+1}$, where

$$\lambda = \begin{bmatrix}
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & I \\
\xi_0 & 0 & \cdots & 0 & 0
\end{bmatrix};$$

then $(\eta_{(i)})^{e/e_{i+1}} = \tau_i$.

The construction shows, as in case (b), that (1)-(6) and the second part of (7) all hold; (8) is also clear. For the first part of (7), one inclusion goes just as in (b). To prove the other, suppose that $w$ commutes with $\chi$. Since $w$ commutes with $\chi |_{K_i'}$, we have

$$w = \delta_0'\eta_i^{h_0'}(1 + \delta_1'\eta_1 + \cdots), \quad \delta_0 \in m_{e_i}(e_i).$$

But $\delta_0'\eta_i^{h_0'}$ must commute with $\alpha \eta_i^{x_j}$, as we saw above; from (6) and (8), we can write

$$w = \varepsilon_0 \eta_i^{h_0}(1 + \varepsilon_1 \eta_i + \cdots), \quad \varepsilon_0 \in (m_{e_i}(e_{i+1}))^x.$$

Since $\varepsilon_0 \eta_i^{h_0}$ commutes with $\chi$, we may delete it; the proof now goes as in (b).

That leaves (11). Let $N_0^j$, $N_0^{j-1}$ be the groups of elements of $Z_e K_e$ commuting with $\chi$ on $H_0^j$, $H_0^{j-1}$ respectively. If $\chi$ and $\chi^x = \chi_0 \chi^\sim$ (where $\chi^\sim$ is
given by \( \beta' \) just as \( \chi' \) is given by \( \alpha' \) are conjugate by an element \( w \in \mathbf{Z}_e K_e \), then \( w \in N_0^j \) and \( w \) is determined mod \( N_0^{i-1} \). So we may write
\[
w = \delta_1 \eta_1^0 \cdots \delta_{i-1} \eta_{i-1}^{l_{i-1}+1} \cdots \delta_{h-1} \eta_{h-1}^{l_{h-1}+1}, \quad \delta_g \in (k_{J_1})^e.
\]
Write \( \beta_{i+1} = \text{Tr}_{e} \beta' \). Conjugation by \( \delta_i \eta_i^0 \) takes \( \alpha_{i+1} \eta_i^{1-j} \) to \( \beta_{i+1} \eta_i^{1-j} \) (since a calculation, using Lemma 4.7 and like some done in the course of verifying (7), shows that \( \delta_i \eta_i^{1-j} w \) fixes \( \chi \) on \( K_{e}^{j-1} \cap G_i \)). Conversely, if \( \alpha_{i+1} \eta_i^{1-j} \) and \( \beta_{i+1} \eta_i^{1-j} \) are conjugate, then they are conjugate under some \( \delta_i \eta_i^0 \), by Lemma 5.2. Since we picked only one element from each such conjugacy class, \( \alpha_{i+1} = \beta_{i+1} \). But then \( \alpha_{i+1} = \beta_{i+1} \) (as in case (b)), and \( \chi = \chi^\#. \) The number of choices for \( \alpha \) is equal to the number of nonconjugate nonzero \( \gamma \) generating \( k_{J_1}/k_{J_0} \).

This completes the proof of Theorem 6.1. □

8

The next task is to compute which elements of \( G \) commute with \( \chi \) on \( H_0 \).

As before, we write \( W(j) \) for the group of power-permutation matrices in \( G(j) \), and \( W_0 \) for those in \( G_0 \). Recall that \( G(j) = (K_{n/f} \cap G(j))W(j)(K_{n/f} \cap G(j)) \), where \( i \) is the largest index with \( t_i \geq j \). We have seen that \( G(j) \) commutes with \( x \) on \( H_{0}^j \) for \( j > r_0 \).

For each \( j \), let \( N^j = \{ x \in \mathbf{Z}_e K_e : x = \chi^x \text{ on } H_{0}^j \} \). By Theorem 6.1,
\[
N^j = K_{e}^{j}(G_0 \cap K_{e}^{j}) \cdots (G_{i-1} \cap K_{e}^{j}) (G(j) \cap \mathbf{Z}_e K_e),
\]
where \( i \) is the largest index with \( t_i \geq j \) and \( c_i = t_i' \) if \( j \leq t_i' \), \( c_i = t_i - j + 1 \) otherwise.

(8.1) Theorem. For all \( j \geq r_0 \), \( \chi \mid_{H_{0}^j} \) intertwines with \( \chi \mid_{x^1H_{0}^j} \) if and only if \( x \in N^j G(j) N^j \).

Proof. We essentially proved "if" in Theorem 6.1, since \( N^j \) normalizes \( H_0 \cap K_{e}^{j} \). The converse uses backwards induction on \( j \). If \( j = t_1 \), the claim follows from Theorem 2.4 of [15]. For the inductive step, we assume the result for \( j \) and prove it for \( j - 1 \). Assume first that \( t_{i+1} < j - 1 \). From the inductive hypothesis, \( x \in N^j G(j) N^j \). Write \( G(j) = (G(j) \cap K_{e}^{j}) W(j)(G(j) \cap K_{e}^{j}) \) and use Lemmas 3.1-3.3 to see that \( x \in N^j W_{j-1} N^j \). Write \( x = k_1 g k_2^{-1} \), with \( g \in W(j_{-1}) \) and \( k_1, k_2 \in N^j \); set \( \chi_k^1 = \chi x_1, \chi_k^2 = \chi x_2 \), where \( x_1, x_2 \) are trivial on \( H_0^j \) and on \( G(j) \cap K_{e}^{j-1} \).

(We saw in proving (11) in steps (4) and (5) of Theorem 6.1 that \( \chi_k^j(y) = \chi(y) \) if \( k_{h} \in N^j \) and \( y \in (G(j) \cap K_{e}^{j-1}) \). For \( y \in H_0^{j-1} \), choose \( v \in H_0^{j-1} \) so that \( k_{2}^{-1} v k_2 = y \); assume \( g y g^{-1} \in H_0^{j-1} \). Then
\[
\chi_k^1(g y g^{-1}) = (k_1 g k_2^{-1} v k_2^{-1} g^{-1} k^{-1}) = \chi(v) = \chi_k^1(y),
\]
or \( \chi_k^1(y) = \chi(y) \). Let \( H_{j-1} = (k_1 g)^{-1} (H_0^{j-1}) k_1 g \cap H_0^{j-1} \). After dividing \( k_1, k_l \) by terms known to commute with \( \chi \) and to normalize \( H_{j-1} \), we may write
\[
k_l = (1 + \delta_{l, i-1} \eta_{i-1}^{l_{i-1}+1}) k_l^*, \quad l = 1, 2, \quad k_l^* \in N^j \cap K_{e}^{j-l+2}.
\]
Assume that we show that \((1 + \delta_{l,i-1} \eta_{l-i+j}^{-1})g(1 + \delta_{2,i-1} \eta_{l-i+j-1}^{-1})^{-1} = k'\eta g k_2^{-1},\)
where \(k'_l, k_2 \in N^j_0.\) Again dividing by terms known to commute with \(x\)
and to normalize \(H_0^{-1},\) we get
\[
k_l = (1 + \delta_{l,i-2} \eta_{l-i-1}^{-1})k_i^*, \quad l = 1, 2, \quad k_i^* \in N^j \cap K_{e_i}^{-j-1}.
\]
and we need a similar argument. (The procedure stops at \(1 + \delta_{l,h-1} \eta_{h^{-1}}^{-1},\)
where, as before, \(h\) is the smallest index with \(t_{h+1} \leq j - 1.)\) So assume
inductively that, modulo elements in \(N^j,\)
\[
k_l = (1 + \delta_{l,a-1} \eta_{a-1}^{-j} + 1)k_i^*, \quad l = 1, 2, \quad k_i^* \in N^j \cap K_{e_i}^{-j+2}.
\]
Since \(g\) commutes with \(x,\) we need to show that if \(x_1(gyg^{-1}) = x_2(y)\) for all
\(y \in H_0\), then \(k_1gk_2^{-1} = k'_1gk_2^{-1}, \quad k'_1 = (1 + \delta_{l,a-1} \eta_{a-1}^{-j+1})k_i^* \) \(l = 1, 2, \) with
\(k_i^* \in N^j \cap K_{e_i}^{-j+2} \) and \(1 + \delta_{l,a-1} \eta_{a-1}^{-j+1}, \quad 1 + \delta_{2,a-1} \eta_{a-1}^{-j+1} \in G_0 \cap K_{e_a}^{j}K_{e_a}^{-j+1} ;\)
this will extend the induction.

We want to apply Lemma 5.4. However, that lemma does not apply to the
present situation, because \(x\) is not of the fairly simple form assumed in the
lemma. The idea of the proof is to create a new character, \(x^*\), on \(G_{i}^* = G_{i}^*\)
(the group of invertible elements of \(M_{(a)}^*;\) see Theorem 6.1, (8)(iv)), and to
show that we can find a power-permutation matrix \(g_0 \in W_{a}^*\) (the corresponding
group of power-permutation matrices) such that
\[
(1 + \delta_{l,a-1} \eta_{a-1}^{-j+1})g_0(1 + \delta_{2,a-1} \eta_{a-1}^{-j+1})
\]
commutes with \(x^*\). Lemma 5.4 will apply to this situation, and we will have
\[
g_0 = k_0g, \quad k_0 \in N^j_0 \quad \text{(by Lemma 3.3)},
\]
\[
k_1g_0k_2 = k'_1g_0k'_2, \quad k'_1, k'_2 \in G_0 \cap K_{e_i}^{-j+1}K_{e_i}^{-j+2}.
\]
Then some simple algebra will complete the inductive step.

Here are the details. Let \(x_0\) be the extension of \(x_{H_0^{i+t_0}}\) to \(H_0^{i-1},\) guaranteed
by repeated use of Lemma 6.2, such that \(G_{a-1} = x_0.\) Restrict
attention to \(H_0^{i-1}(G_{a-1} \cap K_{e_i}^{j}) = H^* \). On these elements, \(x^* \) \(= \) \(x^*\)
\(= 1 + \delta_{l,a-1} \eta_{a-1}^{-j+1}, \) and a similar claim holds for \(k_2;\) since \(k^* = k^* \in N^j \cap K_{e_i}^{j+2}
) already, we may assume that \(k_j = 1 + \delta_{l,a-1} \eta_{a-1}^{-j+1}, \) \(l = 1, 2.\) On the extension
to \(H^* \cap K_{e_i}^{j^*}, \) \(x = x_0 x^*\), where \(x_0(1 + \gamma \eta_{a-1}^{j}) = \psi \circ \text{Tr}(e) \) \(= \alpha \sigma^{-i}a\) and \(x^*\) is
trivial on \(H^* \cap K_{e_i}^{j^*}.\) Extend \(x^*\) to \(x^*\) on \(H^*\) so as to commute with \(G_a\)
by using Lemma 6.2 repeatedly; let \(x^* = x_0 x_{0^*}\).

By using (5) and applying Lemma 3.3 to the subalgebra \(M_{a-1}^* = M_{(a)}^*\) of
\(M_{a-1}^*\) commuting with \(a \eta_{a-1}^{-j} \) and with \(F_{i} = \) we need the compositum with \(F_{i}\)
to apply the lemma), we can find an element \(g_0 \in W_{a-1}^*\) such that \(m = g^{-1}g_0 \in K_{e_i}^j.\)
In fact, Lemmas 3.1 and 3.3 and Theorem 6.1(6) show that for \(y \in H^*,
myg_{m-1} \in H_0.\) Thus for any \(y \in G_{a-1} \cap K_{e_i}^{j-1}, \) \(g_0^{-1}yg_0 \in H^* \) \(\iff y \in H_0\).
Furthermore, \(g_0\) commutes with \(x^*,\) and a calculation shows that \((x^*)^k_1 = \) \(x^*x_1,\) \((x^*)^k_2 = x^*x_2.\) Since \(x_2\) is defined on \(H_0^j\) and is trivial on \(H_0^{j-1},\) and since
\(m \in K_{e_i}^j,\) we have \(x_2^m = x_2;\) therefore \(x_0g_0 = x_1 \) \(= x_2\) on \(H^*,\) and so
CONSTRUCTION OF SUPERCUSPIDAL REPRESENTATIONS

((\chi^*)^k_1gok_2^{-1} = \chi^* where both are defined. Since \(G_{a-1}\) commutes with \(\chi_0\), we have \((\chi^*)^{k_1gok_2^{-1}} = \chi^*\) where both are defined.

Now Lemma 5.4 applies (with \(G_{a-1}\) playing the role of \(GL_n\)); we have \(\{k_1gok_2^{-1} = k'_1gok_2'^{-1}\), where \(k'_1, k'_2 \in (K_{e,j}a^{-j} \cap G_{a})(G_{a-1} \cap K_{e, a-j+1}^{-1}) \subseteq (K_{e^{-j}}a^{-j} \cap G_{a})(G_{a-1} \cap K_{e^{-j-j+1}})\) (this last by Theorem 6.1(8) and Lemma 3.1).

Use \(k\) to denote an element of \(K_{e,j}a^{-j} = (K_{e,j}a^{-j} \cap G_{a})(G_{a-1} \cap K_{e,a-j+1})\) (which may change from equation to equation). Then, since \((K_{e,a^{-j}}a^{-j}, m) \subseteq K_{e,a^{-j+1}}(a^{-1})\) (and similarly for \(m^{-1}\)), we get

\[
k_1gok_2^{-1} = k_1gok_2^{-1}(k_2m_1^{-1}k_2^{-1}m_1^{-1} = k_1gok_2^{-1}k_1m_1^{-1} = k_1gok_2'^{-1}m_1^{-1}k_1
\]

and the induction continues. This completes the proof when \(j - 1 \neq t_i + 1\).

If \(j - 1 = t_i + 1\), then restrict attention to \(H_0/K_{i}^{a} \cap G_{i}\); write \(x = k_1gok_2^{-1}\), with \(g \in W_j = W_i\) and \(k_1, k_2 \in N_j\). We have \(\chi = \chi_0\chi_1\), \(\chi_0\) as in Lemma 6.2 and \(\chi_1 = 1 + a\cdot H_0 = \psi \circ \text{Tr}(\alpha g_{a^{-j}})\), where \(E_i[\alpha g_{a^{-j}}] = \text{field of ramification}

index \(e_{i-j}\) and residue class degree \(f_{i+1}\) over \(F\). Elements of \(N_j\) commute with \(\chi_1\), and \(G_j\) commutes with \(\chi_0\); also, \(\chi_0 = \chi_0\) on \(G_i \cap K_{e,j}a^{-j-1}\) (as we saw in the proof of (11) in Theorem 6.1, part (4)). So on the above elements, \(\chi = \chi_0\chi_1\), or \(\chi_1 = \chi_1\). Theorem 2.4 of [15] (or Lemma 6 of [10]) says that \(g = k_1gok_2\), where \(k_1, k_2 \in G_i \cap K_{e,j}a^{-j-1}\) and \(g_1\) is a power-permutation matrix for the group of elements in \(G_i\) commuting with \(E_i\). By Lemma 3.3, we may assume that \(g_1 \in \hat{W}_{j-1}\) at the cost of changing the \(k_i\) to elements of \((G_i \cap K_{e,j}a^{-j-1}) \cap N_j \subseteq N_j\). The rest of the proof for this case is the same as in the previous one. □

We now describe the basic building blocks for the supercuspidals. The following theorem does not give complete information on the set of elements commuting with the representations we construct, since (v) and (vi) apply only to the restriction to \(H \cap K_{e,j}a^{-j}\).

(8.2) Proposition. Given the \((s_1, e_1, f_j)\) of (1.3), define the \(t_i, t'_i, t''_i\), and \(C_j\) as for Theorem 6.1. There are \(e \cdot \prod_{j=0}^{j} C_j\) pairs \((H, \rho)\), where \(H\) is a subgroup of \(Z_{c}K_{e,j}\) and \(\rho\) is an irreducible representation of \(H\), plus fields \(E_j\), matrix subalgebras \(M_j\), and groups \(G_j = \text{group of invertible elements of } \rho_j\), \(0 \leq j \leq t_i\), such that \((E_i = E_i(t_i + 1), M_i = M_i(t_i + 1), G_i = G_i(t_i + 1)\) for \(1 \leq i < r; E_r = E_r;\) etc.\):

(i) \(e(E_i/F) = e_i\), and \(f(E_i/F) = f_i\);

(ii) \(M_j\) is algebra of elements commuting with \(E_{i}\) (hence \(M_r = E_r\));

(iii) \(H = K_{e,j} \cap G_1 \cdots \cap G_{r-1}\) \(G_r\);

(iv) \(\rho\) is a character on \(H\), and a character on \(H\) tensored with a cuspidal representation of \(K_{e,j} \cap G_1 \cdots \cap G_{r-1} \subseteq GL_{f_{r-1}}(k_{f_{r-1}})\) (extended as a character to \(Z_{e}\)) if \(t_r = 0\), and \(K_{e,j}^{t_i+1}(\pi F) \subseteq \text{Ker} \rho\); 

(v) for \(j \geq 1\), set

\[
H_j = H \cap K_{j}, \quad N_j = K_{e,j}^b(K_{e,j} \cap G_1) \cdots (K_{e,j}^b \cap G_{r-1})(Z_{c}K_{e,j} \cap G_j),
\]

where \(b_l = \min(t'_l, t_l + 1 - j)\) and \(i = \text{largest index with } t_i \geq j\). Then \(x\) commutes with \(\rho_{H_j}^x\) if \(x \in N_jW_jN_j\) (as before, \(W_j = \text{group of permutation-power matrices for } G_j\)).
(vi) distinct \( (H, \chi) \) are nonconjugate in \( \mathbb{Z}_e K_e \).

**Proof.** We construct the \( (H, \chi) \) by a double induction on \( n \) and \( s_1 \). If \( n = 1 \), then there are no triples \( (s_i, e_i, f_i) \), and \( \chi \) is trivial on \( F^x \). If \( s_1 = 0 \), then \( r = 1, e_1 = 1 \), and \( f_1 = n \), and we take \( \rho \) to be a cuspidal representation of \( K_1/K_1^1 \cong \text{GL}_n(k) \), extended to be \( 1 \) on \( \varpi \). It is known (see, e.g., [22]) that there are \( c_0 \) nonconjugate representations of this sort, all of dimension \( \prod_{j=1}^{n-1} (q_j - 1) \). We set \( E_1 = F_n \). The other parts are now immediate. (In fact, Theorem 4.1 of [3] shows that \( \rho^x = \rho \Leftrightarrow x \in \mathbb{Z}_1 K_1 \).)

Assume now that \( s_1 \geq 1 \), and let \( s_0 \) be the smallest index with \( 2s_{r_0 + 1} \leq s_r \). We use Theorem 6.1 to obtain pairs \( (H_0, \chi) \) and fields \( E_j \), \( 1 \leq i \leq r_0 \), together with the corresponding \( M_i \), \( G_i \); \( H_0 = K_e^{i_1} (K_e^{i_1} \cap G_1) \cdots (K_e^{i_0} \cap G_{r_0 - 1}) \), and the set of elements commuting with \( \chi \) is \( J_0 G_{r_0} J_0 \), where

\[
J_0 = K_e^{i_r} (K_e^{i_r} \cap G_1) \cdots (K_e^{i_r_0} \cap G_{r_0 - 1}).
\]

If \( r_0 = r \), then \( G_{r_0} = E_x^x \) and \( E_x^x \) normalizes \( J_0 \). Let \( E_j = E_r \) for \( j < r \), and extend \( \chi \) to a character \( \rho \) on \( H = H_0 E_x^x \) trivial on \( \varpi \); this is possible in \( [H : H_0(\varpi)] = e \cdot \prod_{j=0}^{r_0 - 1} C_i \) ways. Because \( E_x^x \) commutes with \( \chi \) and normalizes \( J_0 \), one checks easily that \( E_x^x J_0 \) commutes with \( \rho \); indeed, \( \rho^x = \rho \Leftrightarrow x \in E_x^x N^1 \). This completes the proof in this case.

If \( r_0 < r \), we extend \( \chi \) to a character on \( H_0 (G_{r_0} \cap \mathbb{Z}_e K_e) \) that is trivial on \( (G_{r_0} \cap \mathbb{Z}_e K_e) \cap H_0 \) and is hence fixed by \( G_{r_0} \); note that \( (G_{r_0} \cap \mathbb{Z}_e K_e) \) normalizes \( H_0 \), by Theorem 6.1. Lemma 4.1 says that the extension is possible, because any element of \( (G_{r_0} \cap \mathbb{Z}_e K_e) \cap H_0 \) is a product of commutators in \( (G_{r_0} \cap \mathbb{Z}_e K_e) \cap (G_{r_0} \cap H_0) \), and we know that \( \chi \) is 1 on these elements. There are \( \prod_{j=r_0+1}^{r_0} C_j \) different restrictions of these extensions to \( H_0 (G_{r_0} \cap K_e^{r_0+1}) \), as one sees by calculating the index of the commutator subgroup. (If \( \chi_1 \) is one such character defined on \( H_0 (G_{r_0} \cap K_e^{r_0}) \), the number of such extensions of \( \chi_1 \) to \( H_0 (G_{r_0} \cap K_e^{r_0}) \) is \( \frac{[G_{r_0} \cap K_e^{r_0} : (G_{r_0} \cap K_e^{r_0})(\text{SL}_{r_0} \cap K_e^{r_0})]}{[G_{r_0} \cap K_e^{r_0} : (G_{r_0} \cap K_e^{r_0})(\text{SL}_{r_0} \cap K_e^{r_0})]} \), where \( \text{SL}_{r_0} \) is the special linear group corresponding to \( G_{r_0} \); this index is \( q_0^{r_0} \) if \( (j - 1)e_0 \) is a multiple of \( e \) and 1 otherwise.) Choose one extension (to be called \( \chi \)) for each restriction, and form the tensor product \( \rho = \chi \otimes \rho_0 \), where \( \rho_0 \) is a representation of a subgroup \( H_1 \) of \( G_{r_0} \cap \mathbb{Z}_e K_e \) satisfying (i)–(vi) for the triples \( (s_i, e_i, f_i) \), \( 1 \leq i \leq r \), constructed via the inductive hypothesis (and extended to be trivial on \( H_0 (G_{r_0} \cap K_e^{r_0+1}) \); \( H = H_0 H_1 \), and the \( E_j \), \( M_j \), \( G_j \) are as defined inductively for \( \rho_0 \). One checks that there are \( e \cdot \prod_{j=r_0+1}^{r_0} C_j \), such representations (note that these representations need be trivial only on \( \varpi \), not on \( \eta_{r_0}^{n/e_0 f_0} \)).

The points that still need checking are (v) and (vi). Let \( H^1 = H \cap K_e^{1} \) and \( \rho^1 = \rho|_{H_1} \). Then \( J_0 = K_e^{i_r} (K_e^{i_r} \cap G_1) \cdots (K_e^{i_r} \cap G_{r_0 - 1}) \) commutes with \( \rho \). For if \( y \in H_0 (G_{r_0} \cap K_e^{i_r}) \) and \( w \in K_e^{i_r} \cap G_{i-1} \), then \( \chi (y wy^{-1} w^{-1}) = 1 \) by Lemma 4.2(b) because \( 2t_i + 1 \geq t_i + 1 \) and \( w \) commutes with \( \chi \) on \( H_0^{i+1} \). So to prove (v), we need (in view of Theorem 6.1(7)) only determine which elements of \( G_{r_0} \) commute with \( \rho^1 \), and now (v) follows from the induction hypothesis. The argument for (vi) is essentially the same. If \( \rho^1 \), \( \rho^{-1} \) are conjugate, then they agree on \( H_0 \), by Theorem 6.1, and, since \( N^{i_0} = N^{i_0+1} \),
they agree on $H \cap K_e^{r_0+1}$. On this group, both are multiples of the same character $\chi$. We have fixed an extension of $\chi$ to $H^1$, which we also call $\chi$, and $\rho \otimes \chi^{-1}, \rho^\sim \otimes \chi^{-1}$ are conjugate. Now the inductive hypothesis proves that $\rho^1 \cong \rho^\sim$.

Remark. We have not shown that representations corresponding to different sequences of triples are nonconjugate. For fixed $e$ and $f$, different sequences of $s_i$ lead to pairs $(H, \rho)$ whose restrictions to their respective $H^1$ are nonconjugate over $Z_eK_e$ because one can read off the $s_i$ from the indices $[H \cap K_e^i : H \cap K_e^{i+1}]$; since the $s_i$ determine the $e_i$, we can have conjugacy only if the sequences of $s_i$ and $e_i$ agree. For the $f_i$, one can use the arguments proving (11) in the proof of Theorem 6.1 or use the fact that the indices $[H \cap K_e^j : H \cap K_e^{j+1}]$, $[N^j : N^{j+1}]$ determine the $f_i$. We say more about this in the next section, and we deal with different $e$, $f$ in §10.

The next step is to extend $\rho$ to a representation $\rho_1$ on $J$ that is a multiple of $\rho$ on $H \cap K_e^1$. The procedure is fairly standard and goes back to [10]; here is a sketch. Suppose for definiteness that $t_1$ is even (if it is odd, then $t_1'' = t_1'$ and there is nothing to do at this step). Pick a cross-section for $HK_e^{t_1''}/H$, and extend $\rho$ to $HK_e^{t_1''}$ by making it 1 on the cross-section. Then $\rho$ is a projective representation on $HK_e^{t_1''}$. The cocycle defining the multiplier projects to a cocycle on $HK_e^{t_1''}/H \cap K_e^1$. This group can be regarded as a subgroup of the semidirect product of a symplectic group with $K_e^{t_1''}/H \cap K_e^{t_1''}$. When $p$ is odd, the cocycle is the inverse of that for the Weil (or oscillator) representation; a unified treatment (including $p = 2$) is given in [8], and the cocycle is computed explicitly in [5] for the case of division algebras. Tensor $\rho$ with the Weil representation $\lambda_1$ to produce an ordinary irreducible representation on $HK_e^{t_1''}$. Repeat this construction for all even $t_i > 0$ to get $\rho_1$. Further details are given in the Appendix to this paper. As noted there, distinct $\rho$ produce distinct $\rho_1$.

(9.1) Theorem. (a) Let $\rho_1$ be as above. Then $\pi = \text{Ind}^G_{J} \rho_1$ is irreducible.

(b) Let $(\rho_1, J)$ and $(\rho_1^\sim, J^\sim)$ be distinct representations, constructed as above on subgroups of $Z_eK_e$ from completely satisfactory representations. Then the induced representations $\pi$ and $\pi^\sim$ on $G$ are distinct.

Proof. (a) From [17], we need to show that if $x \in G$ is such that $\rho_1^x$ and $\rho_1$ intertwine on their common domain, then $x \in J$; Proposition 8.2 says that if $s_r > 0$, this is already true for $\rho_1|_{H^1}$. When $s_r = 0$, Proposition 8.2 says that $x \in J^1G_{r-1}J^1$. Since $J^1 \subseteq J$, we need only consider $x \in G_{r-1}$. The argument of Lemma 14 of [10] shows that $x$ also intertwines $\rho_0$ with itself, where $\rho_0$ is the lift of the cuspidal representation of $GL_{n-1}(k_{f_{r-1}})$ used in constructing $\rho$. From Theorem 4.1 of [3], $x \in J$.

For (b), let the triples associated with $\rho_1^\sim$ be $(s^\sim_i, e^\sim_i, f^\sim_i), 1 \leq i \leq r^\sim$; we assume that $e^\sim_i = e$. Suppose that $\rho_1^x$ and $\rho_1^\sim$ intertwine on their common domain. By restricting to $H \cap K_e^1$ and $H^\sim \cap K_e^1$, we may assume that $\rho^x$ and $\rho^\sim$ intertwine on their common domain. Without loss of generality, we may assume that $s_1 \geq s_1^\sim$. Assume that $\rho(1 + y) = \psi \circ \text{Tr}(\alpha \omega^{-i}y)$ and $\rho^\sim(1 + y) = \psi \circ \text{Tr}(\alpha \omega^{-i}y)$.

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\( \text{Tr}(\alpha^\sim \varpi^{-i} y) \) for \( y \in \mathcal{A}^i_e \). From Proposition 5.3, \((s_i, e_i, f_i) = (s_i^\sim, e_i^\sim, f_i^\sim)\) and \( \alpha \varpi^{-i}, \alpha^\sim \varpi^{-i} \) are conjugate by an element of \( Z_e K_e \). Since we chose only one element from each conjugacy class, \( \rho = \rho^\sim \) on \( K_e^i = K_e^i \cap H = K_e^i \cap H^\sim \).

Assume inductively that \((s_i, e_i, f_i) = (s_i^\sim, e_i^\sim, f_i^\sim)\) for \( i \leq i \) (where \( i \) is the largest index with \( s_i \geq j \)), that \( E_{(h)} = E_{(h)}^\sim \) for \( h \leq j \) (so that, in particular, \( K_e^j \cap J = K_e^j \cap J^\sim \)), and that \( \rho = \rho^\sim \) on this group. We want to extend this result to \( j - 1 \). From Proposition 8.2, \( x \in N^j W_j N^j \). Let \( N^j_1 = N^j \cap K_s^{1, f_s} \). Then \( W_j \) contains a set of coset representatives for \( N_j^j / N_j^1 \), so that \( x \in N_j^j W_j N_j^1 \). Write \( x = k b k' \) (\( k, k' \in N_j^1, b \in W_j \)), let \( \rho|_{K_e^j \cap H} \) be a multiple of \( \chi \), and write \( \chi = \chi_0 \chi_{a'} = \chi_0 \chi_b' \) as in the proof of Theorem 6.1; write \( \chi^\sim = \chi_0 \chi_{b'} \) correspondingly.

We saw in the proof of Theorem 6.1 (especially in the construction of \( \eta_{(j-1)} \)) that for \( y \in K_e^j \cap G_j \) and \( k_0 \in N_j^1 \), \( \chi(k_0 y k_0^{-1}) = \chi(y) \); similarly, \( \chi^\sim(k_0 y k_0^{-1}) = \chi^\sim(y) \). For \( w \in k^{-1}(K_e^{j-1} \cap G_j) k \), let \( y = k w k^{-1} \); then

\[ \chi^\sim^j(y) = \chi^\sim^{j-1}(w) = \chi^{j-1}(k^{-1} y k k') = \chi^\sim(y), \]

by the above result with \( k_0 = (kk')^{-1} \). Since \( (\chi^k)^b = (\chi^\sim)^{k-1} \) on their common domain and \( \chi^k(w) = \chi(y) \), we get

\[ \chi^\sim(y) = \chi^{j-1}(w) = \chi^k(w) = \chi^b(y), \quad \text{all } y \in K_e^{j-1} \cap G_j \cap b^{-1}(K_e^{j-1} \cap G_j) b. \]

Since \( G_j \) commutes with \( \chi_0 \), we have \( \chi_0^b = \chi_0 \). Therefore \( \chi^b_{a'} = \chi_{b'} \) on these elements. But for \( y \in K_e^{j-1} \cap G_j \),

\[ \chi_{a'}(y) = \psi \circ \text{Tr}^{(e)}(\gamma a^\sigma^{-i}), \quad \chi_{b'}(y) = \psi \circ \text{Tr}^{(e)}(\gamma b^\sigma^{-i}), \]

where \( \alpha = \text{Tr}_{a'} \alpha' \) and \( \beta = \text{Tr}_{a'} \beta' \). Since \( \alpha \eta_{(j)}^{1-j} \) and \( \beta \eta_{(j)}^{1-j} \) generate fields over \( E_j \), Proposition 5.3 implies that \( \alpha \eta_{(j)}^{1-j} \) and \( \beta \eta_{(j)}^{1-j} \) are conjugate. Because we picked only one element from each conjugacy class, \( \alpha = \beta \); because we picked only one \( a' \) for each \( \alpha \), \( \chi_{a'} = \chi_{b'} \). This also shows that \( t_{i+1} = j - 1 \) iff \( t_{i+1} = j - 1 \) and that \( e_{i+1} = e_{i+1}^\sim, f_{i+1} = f_{i+1}^\sim \) in that case. This extends the induction.

When \( s_r = 0 \), this forces \( s_r^\sim = 0 \). As in (a) we show that \( \rho_{i+1}^r \) intertwines \( \rho_i^\sim \) only if \( x \) intertwines the corresponding cuspidal representations of \( \rho_0, \rho_0^\sim \) by invoking Lemma 14 of [10]; then Theorem 4.1 of [3] shows that we must have \( \rho_0 \cong \rho_0^\sim \) and hence \( \rho \cong \rho^\sim \). This completes the proof. \( \square \)

This section is devoted to computing the formal degrees of the supercuspidal representations computed in the previous section. This is done in three parts. We begin with the data \( (s_1, e_1, f_1), \ldots, (s_r, e_r, f_r) \), and compute \( [Z_e K_e : J] \), \( J \) as defined in (4.5). Then we compute the degree of the irreducible representation \( \tau = \text{Ind}_{J-Z_e K_e} \rho_1, \rho_1 \) as defined just before Theorem 8.2. Finally, we compute the formal degree of the supercuspidal \( \pi \) induced from \( \tau \). The calculations are similar to those in §2 of [6].
Lemma. Use notation as in §§6-8. Then

\[ [Z_eK_e : J] = \begin{cases} 
q^{a_0} c(f)^e / c(f/f_{r-1}) & \text{if } s_r = 0, \\
q^{a_0} c(f)^e / (q^f - 1) & \text{if } s_r > 0, 
\end{cases} \]

where \( c(f/f_i) = (q^f - 1)(q^f_{1/(f/f_i-1)}) \cdots (q^f_{1/(f/f_i-2)}) \cdots (q^f_{1/(f/f_i-1)}) \), \( c(f) = c(f/1), \)

\[ a_0 = \begin{cases} 
\sum_{i=1}^{r} ft''_i (n_{i-1} - n_i) + n(f - 1)/2 - n f + f & \text{if } s_r > 0, \\
\sum_{i=1}^{r} ft''_i (n_{i-1} - n_i) + n(f - 1)/2 - f(f/f_{r-1} - 1)/2 - f n + f n_{r-1} & \text{if } s_r = 0;
\end{cases} \]

\( n_i = n/e_i f_i \), and \( e_0 = f_0 = 1 \).

Proof. We have

\[ [Z_eK_e : J] = [Z_eK_e : JK_e][JK_e : J], \]

\[ [JK_e : J] = [K_e : J \cap K_e] = [K_e : (J \cap K_e)K^1_e]((J \cap K_e)K^1_e : J \cap K_e], \]

\[ [(J \cap K_e)K^1_e : J \cap K_e] = [K^1_e : J \cap K^1_e], \]

and so on; hence

\[ [Z_eK_e : J] = [Z_eK_e : JK_e][K_e : (J \cap K_e)K^1_e] \prod_{j=1}^{\infty} [K^1_e : (J \cap K^1_e)K^{j+1}_e]. \]

(The terms in the product are 1 for \( j \geq t''_r \).) Since \( Z_eK_e \cap G_r \) contains a generator of \( A^1_e \) (regarded as an ideal of \( A_e \)), we have

\[ [Z_eK_e : JK_e] = 1. \]

Next,

\[ [K_e : (J \cap K_e)K^1_e] = [m^\infty_e : m^\infty_e \cap J]; \]

\( m^\infty_e \) has \( q^{e(f-1)/2} c(f)^e \) elements. If \( s_r > 0 \), \( m^\infty_e \cap J = (k^f)^\infty \) has \( q^f - 1 \) elements; if \( s_r = 0 \), \( m^\infty_e \cap J = \text{GL}(f/f_{r-1}) \) has \( c(f/f_{r-1})q^{(f/f_{r-1})/2} \) elements. Finally, if \( t''_{i+1} \leq j < t''_i \) (we take \( t''_{i+1} = 0 \), then

\[ [K^1_e : (J_0 \cap K^1_e)K^{j+1}_e] = [m_e : m_e \cap M_i] = q^{e(f-1)/e_i f_i}. \]

Hence if \( s_r > 0 \), then

\[ [Z_eK_e : J] = q^{a_0} c(f)^e / (q^f - 1), \]

where

\[ a_0 = \frac{e f(f - 1)/2 + e f^2 (t''_1 - 1) - \sum_{i=1}^{r} e f^2 (t''_i - t''_{i+1})/e_i f_i + e f^2 / e_r f_r;}{ \text{if } s_r = 0, \text{ then} } [Z_eK_e : J] = q^{a_0} c(f)^e / c(f/f_{r-1}), \]

where (since \( e_r = e_{r-1} \) in this case)

\[ a_0 = \frac{e f(f - 1)/2 + e f^2 (t''_1 - 1) - f(f/f_{r-1} - 1)/2}{ - \sum_{i=1}^{r-1} e f^2 (t''_i - t''_{i+1})/e_i f_i + e f^2 / e_r f_{r-1}.} \]
In this last expression, we can sum from 1 to $r$ because $t''_r = t''_{r+1} = 0$. Now use partial summation and note that $e_r f_r = ef = n$. □

We now consider $\dim \rho_1$, $\rho_1$ as defined before Theorem 8.2. First of all, $\rho$ (as in Lemma 8.1) has dimension 1 if $s_r > 0$ and dimension $q(f/f_{r-1})/(q^f - 1)$ if $s_r = 0$ (see, e.g., [22]). Second, we tensor by a representation of dimension $[m^f_{r-1}(e_{i-1}) : m^f_i(e_{i})]^{1/2} = q^{(n_i-n_i)/2}$ if $t_i$ is even and $> 0$ and by a representation of dimension 1 if $t_i$ is odd or 0. Combining this with Lemma 10.1, we get

\begin{equation}
\dim \tau = \frac{c(f)e}{q^f - 1} q^{a_1},
\end{equation}

where, for $s_r > 0$,

\begin{equation}
a_1 = \sum_{i=1}^r f t''_i (n_{i-1} - n_i) + \frac{n(f-1)}{2} - n f + f + \sum_{t_i \text{ even}} f(n_{i-1} - n_i)/2
\end{equation}

\begin{equation}
= \sum_{i=1}^r f \left( \frac{t_i + 1}{2} \right) (n_{i-1} - n_i) - \frac{n(f+1)}{2} + f
\end{equation}

(since $t''_i = (t_i + 1)/2$ if $t_i$ is odd and $t''_i = t_i/2$ if $t_i \neq 0$ is even)

\begin{equation}
= \sum_{i=1}^r f \frac{t_i}{2} (n_{i-1} - n_i) - \frac{n-f}{2};
\end{equation}

for $s_r = 0$, we have (similarly)

\begin{equation}
a_1 = \sum_{i=1}^r f t''_i (n_{i-1} - n_i) - \frac{n(f-1)}{2} - \frac{f(f/f_{r-1}-1)}{2}

- fn + fn_{r-1} + \sum_{t_i \text{ even} \neq 0} \frac{f(n_{i-1} - n_i)}{2}

= \sum_{i=1}^r f \left( \frac{t_i + 1}{2} \right) (n_{i-1} - n_i) - \frac{fn}{2} - \frac{(n-f)}{2} + f
\end{equation}

(since $n_{r-1} = f/f_{r-1}$)

\begin{equation}
= \sum_{i=1}^r f \frac{t_i}{2} (n_{i-1} - n_i) - \frac{n-f}{2},
\end{equation}

just as in (10.4). Thus we have proved:

\begin{equation}
\textbf{Lemma.} \text{ Use notation as in §§6–9 and as above. For the representation } \tau = \text{Ind}_{J_1 \leftarrow Z, \kappa} \rho_1, \text{ where } e_i f_i n_i = n.
\end{equation}
We can now compute the formal degree of the supercuspidal \( \pi \) induced from \( \tau \). The appropriate normalization of Haar measure for \( G \) (or, more properly, \( G/Z_1 \)) is that which gives the Steinberg representation formal degree 1. It is standard (see, e.g., [3] or [21]) that this measure gives \( K_1Z_1/Z_1 \) volume \( n^{-1}(q^n - 1)^{-1}c(n) = n^{-1} \prod_{j=1}^{n-1}(q^j - 1) \). Lemma 2.2.7 of [6] then shows that

\[
\text{vol}(Z_cK_c/Z_1) = ec(f)^e c(n)^{-1} \text{vol}(Z_1K_1/Z_1) = f^{-1}(q^n - 1)^{-1}c(f)^e.
\]

Combining this with (10.4), we get

(10.6) **Theorem.** Any supercuspidal representation \( \pi \) associated with data \((s_i, e_i, f_i)\), \(1 \leq i \leq r\), has formal degree

\[
f q^n - 1 \quad q^{-a_i},
\]

\(a_i\) as in (10.4), where \( t_i = s_i/f_i \). \( \square \)

**Remark.** This formula is the same as that of Theorem 2.28 of [6]; in that paper, \( j_i = (t_i + 1)/2 \). It is easiest to use (10.3) to compare the results.

We need one important corollary of Theorem 10.6.

(10.7) **Theorem.** Let \( \pi \) be a supercuspidal representation of \( \text{GL}_n(F) \) associated with the data \((s_i, e_i, f_i)\), \(1 \leq i \leq r\); let \( \pi^\sim \) be another supercuspidal, associated with the data \((s_i^\sim, e_i^\sim, f_i^\sim)\), \(1 \leq i \leq r^\sim\). Let \( f = f_r \) and \( f^\sim = f_r^\sim \). Suppose that \( f \neq f^\sim \). Then \( \pi \) and \( \pi^\sim \) are inequivalent.

**Proof.** Let \( f = f_0f^\sim_* \), where \((f_0, p) = 1 \) and \( f^\sim_* \) is a power of \( p \); write \( f^\sim = f_0^\sim f^\sim^\sim \) similarly. We may assume that \( f_r/f_r^\sim \). Then the largest factors in the formal degree of \( \pi \), \( \pi^\sim \) prime to \( p \) are respectively

\[
f q^n - 1 \quad q^{-f_r^\sim},
\]

\[
f_0^\sim q^n - 1 \quad q^{-f^\sim^\sim} - 1.
\]

By Lemma 4.1 of [6], these numbers are unequal if \( f_0 \neq f_0^\sim \). \( \square \)

In this section we complete the proof of Theorem 1.1 by proving:

(11.1) **Theorem.** Every supercuspidal representation of \( \text{GL}_n(F) \) is obtained by tensoring a character with one of those constructed in Theorem 8.2.

**Proof.** Since every supercuspidal is the tensor product of a unitary supercuspidal trivial on \( \varpi F \) with a quasicharacter, we restrict attention in what follows to this class of supercuspildals. Let \( \mathcal{E}^0(G) \) and \( \mathcal{E}^{sp}(G) \) denote respectively the sets of (equivalence classes of) supercuspidal representations and of (equivalence classes of) discrete series, but not supercuspidal (= generalized special) representations of \( G \) trivial on \( \varpi F \). (In the rest of this proof, we do not distinguish between a representation and its equivalence class.) For an integer \( m > 0 \), write \( \mathcal{E}_m^0(G) \) and \( \mathcal{E}_m^{sp}(G) \) for the subsets of \( \mathcal{E}^0(G) \) and \( \mathcal{E}^{sp}(G) \) respectively with conductoral exponent \( \leq mn \). These, as we shall see, have finite cardinality.

We have already constructed certain elements of \( \mathcal{E}_m^0(G) \). For a sequence \( S = \{(s_1, e_1, f_1), \ldots, (s_r, e_r, f_r)\} \) of triples, define \( \alpha(S) = s_1 \) and \( n(S) = e_rf_r = n \). We have associated to \( S \) a sequence \( C_S \) of supercuspidal representations; its
The cardinality, $|C_S|$, is given in Theorem 8.2. It is proved in [2] that if $\pi \in C_S$, then the conductoral exponent $c(\pi)$ of $\pi$ is $s_1 = \alpha(s)$. We construct other supercuspidals by tensoring these $\pi$ with characters. To avoid repetitions, we proceed as follows: $(G \cap K_{n})/(G \cap K_{n+1})(\text{SL}_{n}(F) \cap K_{n})$ has one element if $n \not| j$ and $q$ elements if $n | j$. In the latter case, we write $j = mn$ and choose $q$ characters $\chi_{m, 1}, \ldots, \chi_{m, q}$ of $G$, trivial on $G \cap K_{mn}$ and distinct on $G \cap K_{m}$. For convenience, we write $C_{S}(m) = C_{S}$ if $mn = \alpha(S)$. Then if $T = \{(s_{1}, e_{1}, f_{1}), \ldots, (s_{r}, e_{r}, f_{r})\}$, we have $C_{S}(m) \cap C_{T}(m) \neq \emptyset$ only if $S = T$, and the elements of $C_{S}(m)$ constructed above are all distinct. Proof: if $s_{r} = e_{r}$, the formal degrees differ, so we may suppose that $s_{r} = e_{r} = e$. Suppose that $\tau_{1} = \pi_{1} \otimes \chi_{a_{1}, h_{1}} \otimes \cdots \otimes \chi_{m_{1}, j_{m_{1}}} \in C_{S}(m)$ and that $\tau_{2} = \pi_{2} \otimes \chi_{b_{2}, h_{2}} \otimes \cdots \otimes \chi_{m_{2}, j_{m_{2}}} \in C_{T}(m)$. Then $\tau_{1}$ is induced from $\rho_{1} = \pi_{1} \otimes \chi_{a_{1}, h_{1}} \otimes \cdots \otimes \chi_{m_{1}, j_{m_{1}}}$, and if $m_{1} > s_{1}$, then there are no characters $\chi_{m_{1}, j_{m}}$ in the tensor product and we already know the result. If $s_{1} = mn > s_{1}$, then $\rho_{2} = \rho_{1}$ is a multiple of $\chi_{m_{1}, j_{m_{1}}} \in C_{T}(m)$, while $\rho_{2}$ is a multiple of $\rho_{2}$ there; Lemma 5.3 implies that $\rho_{1}, \rho_{2}$ are not conjugate, so that $\tau_{1}$ and $\tau_{2}$ are not conjugate. If $s_{1} = mn > s_{1}$, then $\tau_{1}$ is a multiple of $\chi_{m_{1}, j_{m_{1}}} \in C_{T}(m)$, while $\rho_{2}$ is a multiple of $\rho_{2}$ there; Lemma 5.3 implies that $\chi_{m_{1}, j_{m}} = \chi_{m_{2}, j_{m_{2}}}$, and continue inductively.

If $s_{r} = \pi$, the conductor $c(\pi)$ of $\pi$ is shown in [2] to be $m_{0}n$, where $m_{0}$ is the largest index with $h_{m_{0}} \neq 1$. (If $s_{r} = \pi$, we have already computed $c(\pi)$.). Thus all members of $C_{S}(m)$ lie in $E_{m}^{0}(G)$. Let $E_{m}^{1}(G) = \{\tau \in G^{\sim} : \tau \in C_{S}(m)\}$ for some $S$ with $\alpha(S) \leq m$. Since $|C_{S}(m)| = |C_{S}|q^{m - [\alpha(S)/n]}$, we see that

$$|E_{m}^{1}(G)| = \sum_{\alpha(S) \leq m} |C_{S}|q^{m - [\alpha(S)/n]}.$$

We want to prove that $|E_{m}^{1}(G)| = |E_{m}^{0}(G)|$. We argue by induction on $n$, the case $n = 1$ being trivial.

Let $D = D_{n}$ be a central division algebra of dimension $n^{2}$ over its center $F$. The unitary dual of the multiplicative group $D^{\times}$ was constructed in [4]. The representations in $(D^{\times})^{\sim}$ are classified by sequences $S = \{(s_{1}, e_{1}, f_{1}), \ldots, (s_{r}, e_{r}, f_{r})\}$ satisfying (1.3), except that we require only the $e_{r}f_{r}/n$. Let $n(S) = e_{r}f_{r}$ (this is consistent with the previous definition of $n(S)$). Corresponding to $S$ there is a collection of representations $C_{S}^{'} \subseteq (D^{\times})^{\sim}$; the elements $\pi^{'} \in C_{S}^{'}$ all have conductors $c(\pi^{'}) = s_{1} = \alpha(S)$. There are other elements of $(D^{\times})^{\sim}$, but we consider only representations trivial on $w_{F}$, formed by tensoring with characters $\chi_{a_{1}, h_{a_{1}}}, \ldots$, just as for $G$. If we define $C_{S}^{'}(m)$ in a manner analogous to $C_{S}(m)$ (i.e., one tensor with $\chi_{a_{1}, h_{a_{1}}}, \ldots, \chi_{m_{1}, h_{m_{1}}}$), then the formula for $c(\pi^{'})$, $\pi^{'} \in C_{S}^{'}(m)$, is the same as that for elements of $C_{S}(m)$. In particular, elements of $C_{S}^{'}(m)$ have conductors $\leq mn$, and elements of $C_{S}^{'}(m_{1})$, $m_{1} > m$, have conductors $\leq mn$ iff they are in $C_{S}^{'}(m)$. Furthermore, $|C_{S}^{'}(m)| = |C_{S}(m)|$ when $n(S) = n$; this follows from Theorem 5.5 of [4]. Thus if we let $\mathcal{F}_{n}^{m}(D_{n}) = \{\tau \in (D^{\times}_{n})^{\sim} : \tau \in C_{S}^{'}(m)\}$ for some $S$ with $n(S) = n$, then $|\mathcal{F}_{n}^{m}(D_{n})| = |E_{m}^{0}(G)|$. 

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Let \( \mathcal{F}_{m_0}^n(D^n) = \{ \tau \in (D^n_\chi) : \tau \in C_\chi(m) \) for some \( S \) with \( n(S) = n_0 \}, \) where \( n_0 | n \). (For \( n_0 = 1, S = \emptyset, \) and \( C_\emptyset \) is the set of characters trivial on the first congruence subgroup \( K^1 \) that factor through the reduced norm map. There are \( q-1 \) such characters; thus \( |C_\emptyset| = q-1 \) and \( |C_\emptyset(m)| = (q-1)q^{m-1} \). One can see directly that \( |\mathcal{F}_{m_0}^n(D^n)| = |\mathcal{F}_{m_0}^n(D_{n_0})| \); if \( n_0 | n \), \( |\mathcal{F}_{m_0}^n(D_{n_0})| = |\mathcal{F}_{m_0}^n(\text{GL}_{m_0}(F))|, \) by the inductive hypothesis. Let \( \mathcal{F}_{\text{sp}}^n(G) \) be the set of irreducible square-integrable, nonsupercuspidal representations of \( G \) with conductor \( \leq mn \) (i.e., the set of elements of \( \mathcal{F}_{\text{sp}}^n(G) \) with conductor \( \leq mn \)), then

\[
|\mathcal{F}_{\text{sp}}^n(G)| = \sum_{m_0 | n, m_0 \neq n} |\mathcal{F}_{m_0}^n(\text{GL}_{m_0}(F))|,
\]

by [23] (plus Theorem 3.4 of [9], for calculation of conductors). Since

\[
|\mathcal{F}_{\text{sp}}^m(G)| + |\mathcal{F}_{m_0}^n(G)| = \sum_{m_0 | n} |\mathcal{F}_{m_0}^n(D_{m_0})| = |\mathcal{F}_{m_0}^n(G)| + |\mathcal{F}_{m_0}^m(G)|
\]

by what we have just proved, \( |\mathcal{F}_{m_0}^n(G)| = |\mathcal{F}_{m_0}^m(G)| \) and the induction extends. This proves the theorem when \( \text{char } F = 0 \).

For the case \( \text{char } F = p \), we use [13]. Let \( m > 0 \). Then there is a field \( F_0 \), \( \text{char } F_0 = 0 \), such that the Hecke algebras \( \mathcal{H}(\text{GL}_n(F_0)//K_{m_0}^{mn+1}) \), \( \mathcal{H}(\text{GL}_n(F)//K_{mn+1}^{mn+1}) \) are isomorphic. (If \( F = \mathbb{F}_q((x)), q = p^b \), then let \( F_0 = \mathbb{Q}_p(a, w_0), \) where \( \mathbb{Q}_p(a) \) is unramified of degree \( b \) and \( w_0^N = p \) for some large \( N \).) From [13], there is a 1-1 correspondence between the discrete series representations of \( \text{GL}_n(F_0) \) with a \( K_{m_0+1}^{mn+1} \)-fixed vector and those of \( \text{GL}_n(F) \) with a \( K_{mn+1}^{mn+1} \)-fixed vector. (The Hecke algebra isomorphism constructed in [13] is obviously an isometry for appropriate choices of Haar measure. As noted in [12], this gives the correspondence for discrete series representations.)

A discrete series representation \( \pi \) with conductor \( \leq mn \) has a \( K_{mn+1}^{mn+1} \)-fixed vector, since it is induced from a representation on an open subgroup with such a vector. Conversely, if any of the representations we have constructed have a \( K_{mn+1}^{mn+1} \)-fixed vector, they have conductor \( \leq mn \). For, in the terminology of [11], if the representation \( \pi \) has conductor \( j \), then \( \pi \) has a minimal \( K \)-type of level \( j/n \). The uniqueness property of minimal \( K \)-types says that \( \pi \) has no \( K_{j+1}^j \)-fixed vectors. (See also [2].) Since the discrete representations we have constructed with conductors \( \leq mn \) for the two groups correspond 1-1, the theorem for \( \text{GL}_n(F) \) follows. □

Remarks. 1. In [14], Koch computed \( \sum_{m_0 | n} |\mathcal{F}_{m_0}^n(D^n)| \) by calculating the number of conjugacy classes in \( D^n_\chi(\varpi)^{K_{m_0}^{m+1}} \), where \( K_{m_0}^{m+1} \) is the \((m+1)\)th congruence subgroup. One can group conjugacy classes by sequences \( S \) with properties like (1.3), except that \( e_i f_i | n \) and the \( s_i \) increase. Roughly speaking, for the conjugacy class represented by \( x \) there are elements \( x_i \) such that \( F[x_i]/F \) has ramification index \( e_i \) and residue class degree \( f_i \) and such that \( x_i^{-1}x \in P^{s_i} \) (where \( P \) is the prime ideal for the integers of \( D_i \)), but no such congruence (with \( e_i, f_i \)) is possible mod \( P_{s_i+1} \). It is not hard to use results in [14] to show that \( |C_\chi(m)| \) is the number of conjugacy classes corresponding to sequences \((mn-s_1, e_1, f_1), \ldots, (mn-s_r, e_r, f_r) \). This would give a different way of completing the counting argument given above.

2. It is likely that the results of [12] on Hecke algebra isomorphisms apply also to the wildly ramified case. If they did, one could also prove Theorem
11.1 by comparing formal degrees in \((D^\times_n)^J\) and in \(\mathcal{E}_{s}^{0}(\text{GL}_n) \cup \mathcal{E}_{s}^{sp}(\text{GL}_n)\). In fact, the formal degrees of the supercuspidals in Theorem 8.2 are the same as the formal degrees of the irreducible representations of \(D^\times_n\) with the same data \((s_1, e_1, f_i), \ldots, (s_r, e_r, f_r)\), constructed in [4]. This means that once the Hecke algebra isomorphism theorem is extended, one has the following extension of the results in [6]: for any \(n\) and \(s_0\), there is an \(h_0\) such that for \(q = p^h\) with \(h \geq h_0\), the correspondence of the Matching Theorem associates representations of \(D^\times_n\) and \(\text{GL}_n(F)\) with the same data \((s_1, e_1, f_i), \ldots, (s_r, e_r, f_r)\) whenever \(s_1 \leq s_0\).

**Appendix**

Here is the cocycle calculation mentioned at the start of §9.

We assume inductively that we have extended \(\rho\) to a representation (which we also call \(\rho\)) on \(H(J \cap K^J_{e'})\) that is a multiple of a character \(\chi\) on \(H \cap K^J_e\). We want to extend this situation to \(H(J \cap K^J_{e''}) = H(J \cap K^J_{e''}e')\). Write \(H_i = H(J \cap K^J_{e''})\), \(J_i = H(J \cap K^J_{e''}e')\), \(H_I = H_i \cap K^J_e\), and \(J_I = J_i \cap K^J_e\). If \(t_i\) is odd, then \(H_i = J_i\); we therefore assume \(t_i\) even and write \(t_i\) for \(t_i''\). Lemma 6.2 lets us write \(\chi[H_i] = \chi_0 \chi_1\), where \(\chi_0\) extends the character \(\chi_0, i\) of Theorem 6.1(8); \(\chi_0\) extends to a character of \(J'_I\), by Lemma 4.1, since \(G_i\) commutes with \(\chi_0\) and \(J_i' \subset (K_e \cap G_i)(H_i \cap K^J_e)\). We shall assume that \(s_r = 0\); the calculations with \(s_r > 0\) are similar, but easier.

Write \(m = m_{e'-1}^{e''}(e_i-1)\), \(m^x = \) set of elements of \(m\) whose image in \(A_e/A_e^1\) is invertible; we use coset representatives for \(m\), \(m^x\) as described in §1, but we will refine this later. Write \(\eta = \eta_i = \eta_{r-1}\). Coset representatives for \(J'_I/H'_I\) are given by elements \(1 + \alpha \eta^h\), where \(\alpha\) runs through a \(m_{e'}^{e''}(e_i)\)-subspace \(V\) of \(m_{e'}^{e''}(e_i-1)\), and coset representatives for \(J_i/H_i\) by elements \(\delta \eta^h(1 + \alpha \eta^i)\), where \(\delta \in m^x\), \(h \in \mathbb{Z}\), and \(\alpha \in V\). \(H_i^1\) is normal in \(J_i\).

We need two other facts before we start on the calculation:

1. \(\chi_0^\delta = \chi_0\) for \(\delta \in m^x\) (because \(m^x \subset G_i\)), while \(\chi_0^{-1} \chi_0^\eta^h = \psi(h)\) is a character on \(H'_i/H_i^1\). The \(\psi(h)\) are fixed by \(m^x\) because \(\eta_i\) normalizes \(m^x\) (mod elements of \(H_i^1\), which fix \(\psi(h)\)).

2. The alternating bilinear form \(B\) on \(V\) defined by \(B(\alpha_1, \alpha_2) = \chi_1(1 + \alpha_1 \eta^i), (1 + \alpha_2 \eta^i) = \chi_1(1 + (\alpha_1^q \alpha_2 - \alpha_2^q \alpha_1) \eta^{2i})\) is nondegenerate. (This is a calculation using Lemma 4.7.)

For \(x \in J'_I\), \(\rho^x\) and \(\rho\) are equivalent. In fact, a stronger statement holds: if \(x = 1 + \alpha \eta^i\) is one of the cosets representatives in \(J'_I\), then Lemma 4.2 implies that \(\rho^x(y) = \rho(y)\) for \(y \in H_i^1\).

For the coset representatives \(\delta \eta^h\) \(\rho\) is already defined. We extend \(\rho\) to a projective representation \(\rho_i\) on \(J_i\) by

\[
\rho_i(x(1 + \alpha \eta^i)) = \chi_0(1 + \alpha \eta^i) \rho(x), \quad x \in H_i, \quad \alpha \in V.
\]

If we knew that \(\rho\big|_{H_i^1}\) were irreducible, this would be a projective representation by general nonsense. In our case, this needs to be checked, but it falls out of the following computation. (It can also be easily verified directly.) For
We compute:

\[ \rho_i(y_1 \delta_1 \eta^g(1 + \alpha_1 \eta^f)) \rho_i(y_2 \delta_2 \eta^h(1 + \alpha_2 \eta^f)) \]

(A.1)

\[ = \chi_0(1 + \alpha_1 \eta^f) \rho_i(y_1 \delta_1 \eta^g y_2 \delta_2 \eta^h) \]

\[ \cdot [\rho_i(y_1 \delta_1 \eta^g(1 + \alpha_1 \eta^f))y_2 \delta_2 \eta^h(1 + \alpha_2 \eta^f)]^{-1} \]

Write \( y_2^{-1}(1 + \alpha_1 \eta^f)y_2 = z_0(1 + \alpha_1 \eta^f) \), with \( z_0 \in H_i^{t+1} \); \( \chi(z_0) = 1 \) from Lemma 4.2. Then \( \chi([\delta_2 \eta^h]^{-1} z_0 \delta_2 \eta^h) = 1 \), since \( \delta_2 \eta^h \) commutes with \( \chi \) on \( H_i^{t+1} \), and we can omit this term from the calculations. Hence

\[ \rho_i([y_2 \delta_2 \eta^h]^{-1}(1 + \alpha_1 \eta^f)y_2 \delta_2 \eta^h(1 + \alpha_2 \eta^f)) = \rho_i([\delta_2 \eta^h]^{-1}(1 + \alpha_1 \eta^f)\delta_2 \eta^h(1 + \alpha_2 \eta^f)). \]

Write \( [\delta_2 \eta^h]^{-1}(1 + \alpha_1 \eta^f)\delta_2 \eta^h = u(\delta_2, h; \alpha_1)(1 + \alpha_3 \eta^f) \) with \( \alpha_3 \in V \) and \( u(\delta_2, h; \alpha_1) \in H_i^{t+1} \). Then the last expression is

\[ \rho_i(u(\delta_2, h; \alpha_1))\rho_i(1 + \alpha_3 \eta^f)(1 + \alpha_2 \eta^f), \]

which is scalar. Thus (A.1) is scalar (and \( \rho_i \) is projective). Write

\[ (1 + \alpha_3 \eta^f)(1 + \alpha_2 \eta^f) = (1 + (\alpha_2 + \alpha_3) \eta^f) \gamma(\alpha_2, \alpha_3), \]

\[ \gamma(\alpha_2, \alpha_3) \in H_i^{t}. \]

Then

\[ \rho_i((1 + \alpha_3 \eta^f)(1 + \alpha_2 \eta^f)) = \chi_0((1 + (\alpha_2 + \alpha_3) \eta^f) \gamma(\alpha_2, \alpha_3) \chi_1[\gamma(\alpha_2, \alpha_3)])I \]

\[ = \rho_i(1 + \alpha_3 \eta^f) \rho_i(1 + \alpha_2 \eta^f) \chi_1[\gamma(\alpha_2, \alpha_3)], \]

and

\[ \rho_i(1 + \alpha_3 \eta^f) = \chi_0(1 + \alpha_3 \eta^f) = \chi_0(u(\delta_2, h; \alpha_1))^{-1} \psi^{(h)}(1 + \alpha_1 \eta^f). \]

So the scalar in (A.1) is the scalar in

\[ [\rho_i(u(\delta_2, h, \alpha_1))\chi_0(u(\delta_2, h, \alpha_1))^{-1} \psi^{(h)}(1 + \alpha_1 \eta^f) \chi_1[\gamma(\alpha_2, \alpha_3)]]^{-1}; \]

that is,

\[ [\chi_1(\gamma(\alpha_2, \alpha_3)) \psi^{(h)}(1 + \alpha_1 \eta^f) \chi_1(u(\delta_2, h; \alpha_1))]^{-1}. \]

Write \( \zeta(\delta_1, g, \alpha_1; \delta_2, h, \alpha_2) = \psi^{(h)}(1 + \alpha_1 \eta^f) \chi_1(u(\delta_2, h; \alpha_1)) = \chi^*(\delta_2 \eta^h, \alpha_1). \)

This is a 2-cocycle (here, \( (\delta, g, a) \) represents \( \delta \eta^g(1 + \alpha \eta^f) \), since, as is easily checked, the other factor is also a 2-cocycle. We now analyze \( \zeta \).

First of all, we get rid of \( \delta_1 \) and \( \delta_2 \). We can write \( \delta \eta^h = \eta^h e \) for some \( e \in m^\times \). We now refine the choice of coset representatives. Let \( \overline{m} \cong M_{f_i f_i^{-1}}(k_{f_i^{-1}}) \) be the image of \( m \) in \( A_e/A_e \), and similarly for \( \overline{m}^\times \); let \( \overline{\delta} \) in \( \overline{m} \) correspond to \( \delta \) in \( m \); similarly, let \( \overline{V} \cong \) a complement to \( \overline{m}^{l+1}(e_{l+1}) \) in \( \overline{m}^e(e_i) \) (where the bars signify that we are working in the quotient spaces, not with coset representatives). We can choose \( \overline{V} \) to be stable under the action of \( \overline{m}^\times \) given by \( \overline{\alpha} \mapsto \overline{\alpha}(\overline{\delta^*})^{-1} \), \( \overline{\delta} \in \overline{m}^\times \). The reason is that we can regard \( \overline{m}^e_i(\overline{f_i}) \) as \( e/e_i \) copies of \( M_{f_i f_i}(k_{f_i}) \) and \( \overline{m}^{l+1}_{e_{l+1}}(e_{l+1}) \) as \( e/e_{l+1} \) copies of \( M_{f_i f_i}(k_{f_i}) \), embedded in each \( (e_i/e_{l+1}) \)th copy of \( M_{f_i f_i}(k_{f_i}) \); we take the complement in a summand to be the complement under \( \text{Tr} \) when a copy is embedded and to be all of the summand when no copy is embedded, and sum. We now divide \( \overline{V} \) into orbits under the above action and choose coset representatives in each orbit so that for some \( \alpha \) in the orbit, the coset representatives for the other representatives are all of the
form $\delta \alpha (\delta \sigma)^{-1}$ for some $\delta \in m^x$. It is not then true that for arbitrary $1+\alpha \eta'$, we have $\delta (1+\alpha \eta') \delta^{-1} = 1 + \zeta \eta'$ with $\zeta = \text{coset representative for } \delta \alpha (\delta \sigma)^{-1}$, but it is true that if $\zeta$ is that coset representative, then $\delta (1+\alpha \eta')(\delta \sigma)^{-1} = \lambda (1+\zeta \eta')(\lambda \sigma)^{-1}$ with $\lambda \in G_{r-1} \cap K_1^\lambda$. (If $\beta = \delta_0 \alpha \delta_0^{-1}$ with $\delta_0 \in m^x$ and $\zeta = \delta_1 \alpha \delta_1^{-1}$ with $\delta_1 \in m^x$, take $\lambda = \delta_1 \delta_0^{-1}$.) Now let $\eta^{-h}(1+\alpha \eta') \eta^h = (1+\alpha \eta) y$, where $\alpha_0 \in V$ and $y \in G_{r-1} \cap K''_h$, and suppose that $\varepsilon \alpha_0 (\varepsilon \eta)^{-1} = \lambda \alpha_1 (\lambda \eta)^{-1}$, where $\delta \eta^h = \eta^h \varepsilon$, $\alpha_1 \in V$, and $\lambda \in G_{r-1} \cap K_1^\lambda$. Then

$$(\delta \eta^h)^{-1}(1+\alpha_0 \eta') \eta^h \delta = (1+\alpha_1 \eta') \lambda^{-1} \cdot e^{-1} \nu e.$$ 

Since $\psi(h)$ is stable under $m^x$ and $\chi_1$ is stable under $G_{r-1} \cap K_1^h$ (use Lemma 4.8 to show that $\chi$ and $\chi_0$ are stable under $G_{r-1} \cap K_1^h$), it follows that

$$\xi(\delta_1, g, \alpha_1; \delta_2, h, \alpha_2) = \xi(g, \alpha_1; h, \alpha_2).$$

We shall write this last simply as $C(g, \alpha_1; h, \alpha_2)$. It is independent of $g$ and $\alpha_2$; sometimes we write $C(g, \alpha_1; h, \alpha_2) = \chi^*(h, \alpha_1)$.

The cocycle condition gives

$$\chi^*(h, \alpha_1 + \alpha_2) = C(g, \alpha_1 + \alpha_2; h, \alpha_3)$$

$$= C(g, \alpha_1; 0, \alpha_2) C((g, \alpha_1) \cdot (0, \alpha_2); h, \alpha_3)$$

$$= C(g, \alpha_1; (0, \alpha_2) \cdot (h, \alpha_3)) C(0, \alpha_2; h, \alpha_3)$$

$$= \chi^*(h, \alpha_1) \chi^*(h, \alpha_2),$$

and similarly

$$\chi^*(g, \alpha) \chi^*(h, \alpha^g) = \chi^*(g + h, \alpha),$$

where $\alpha^g$ is the element in $V$ such that $1 + \eta^{-g} \alpha \eta^g$ is represented by $1 + \alpha^g \eta^g$. So if we write

$$\varphi(\eta^g, \alpha) = \varphi(g)(\alpha), \quad \varphi : M/M_1 \to \overline{V},$$

then $\varphi$ is a 1-cocycle. If $\varphi$ is a 1-coboundary, then so is $C$, since if $\varphi(g) = \mu^g / \mu$, then $C(g, \alpha_1; h, \alpha_2) = \nu((g, \alpha_1) \cdot (h, \alpha_2)) / \nu(g, \alpha_1) \nu(h, \alpha_2)$ if $\nu(g, \alpha) = \mu(\alpha)$. Thus we examine $\varphi$.

Since $\eta^{e_r-1}$ is a central element in $\text{GL}_n(F)$ times an element of $H_1^f$, it is easy to check that $\varphi$ has order dividing $e_r-1 = e_r$. Because $\overline{V}$ is a $p$-group, standard theory shows that $\varphi$ has order dividing $p$. This, of course, makes $\varphi$ (hence $C$) trivial if $(p, e_r) = 1$. If $p | e_r$, we use the inverse of the other part of our cocycle,

$$C_0(\delta_1 \eta^g(1+\alpha_1 \eta'), \delta_2 \eta^h(1+\alpha_2 \eta')) = \chi_1(z(\alpha_2, \alpha_3)).$$

This is the cocycle corresponding to the Weil representation, and

$$C_0(1+\alpha_1 \eta', 1+\alpha_2 \eta') = B(\alpha_1, \alpha_2)$$

is a nondegenerate bilinear form on $\overline{V} \times \overline{V}$. Hence there exists $\alpha_0$ with

$$C_0(1+\alpha \eta', 1+\alpha \eta') = \varphi(1)(\alpha)^{-1} \quad \forall \alpha \in \overline{V}.$$ 

If we replace $\eta$ by $\eta_*$, then

$$C_0(\delta_1 \eta^g(1+\alpha_1 \eta'), \delta_2 \eta^h(1+\alpha_2 \eta')) = C_0(\delta_1 \eta^g(1+\alpha_1 \eta'), \delta_2 \eta^h(1+\alpha_2 \eta')).$$

That is, the cocycle $(C_0 C)^{-1}$ is the inverse of one for the Weil representation, but we need to use $\eta_*$ instead of $\eta$ to generate the cyclic element. Another
way of saying this is that we are applying an outer automorphism to $J_i/J_i^{1}$. (If $2|\epsilon$, we can use $\eta_\ast$ as a substitute for $\eta_i$ and redefine $E_i, G_i$ accordingly. When $2|\epsilon$, this may not be possible because conjugation by $\eta_\ast$ need not normalize $F_i$.) The outer automorphism depends only on $\chi$, and not on all of $\sigma$. We now tensor with the Weil representation $W$ to get an ordinary irreducible representation on $J_i$. Note that $W$ is trivial on $J_i^{1}$, so that the induction hypotheses apply to $\sigma_i \otimes W$. Since $W$ is irreducible on $K_i^{1}/H_i^{1}$, any operator commuting with $\sigma_i \otimes W$ must be of the form $A \otimes I$; as $\sigma_i$ is irreducible on $H_i$, $A$ must be a multiple of $I$. Therefore $\sigma_i \otimes W$ is irreducible. This concludes the construction.

REFERENCES


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