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TOPOLOGICAL PROPERTIES OF q-CONVEX SETS

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ABSTRACT. We discuss the topological properties of a certain class of compact sets in a q-complete complex manifold M. These sets—which we call q-convex in M—include, for $q = 0$, the $\mathscr{O}(M)$-convex compact sets in a Stein manifold. Then we show applications of the topological results to the subjects of removable singularities for $\overline{\partial}_b$.

INTRODUCTION

It is known that, if $K \subset \mathbb{C}^n$ is a polynomially convex compact set, the Čech cohomology spaces $H^r(K, \mathbb{C}) = 0$ for $r \geq n$ (see [12, Theorem 2.7.12]). This follows, by an inductive limit consideration, from the parallel result for Runge domains, due to Serre [16] (since K has a neighborhood basis of Runge domains), and was also proved by Browder [6].

From the above property of polynomially convex sets Alexander [2] derived, by merely topological considerations, the following result about the polynomial hull $\widehat{X}$ of a compact set X in the boundary of the open unit ball $\mathbb{B}$ of $\mathbb{C}^n$, $n \geq 2$: the Čech homology spaces $H_i(b\mathbb{B}\backslash X, \mathbb{C}) \cong H_i(\mathbb{B}\backslash \widehat{X}, \mathbb{C})$ for $i \leq n-2$, in particular $b\mathbb{B}\backslash X$ and $\mathbb{B}\backslash\widehat{X}$ have the same number (not necessarily finite) of connected components.[1]

Moreover recently Alexander and Stout [3] proved, by arguments of different nature, that the latter property extends to a more general context. The result of [3] can be stated as follows: Let D be a relatively compact C^2-bounded strongly pseudoconvex domain in a Stein manifold M $(\dim M \geq 2)$, with $\overline{D}$ being $\mathscr{O}(M)$-convex, and X a compact set in bD; then, if $\widehat{X}_M$ denotes the $\mathscr{O}(M)$-hull of X, $bD\backslash X$ and $D\backslash\widehat{X}_M$ have the same number of connected components.

In this paper we pursue the investigation about topological properties of this sort for a suitable class of compact sets in a q-complete complex manifold M, namely the compact sets which are q-convex in M, according to the definition given below. These sets are a quite natural generalization to the setting of a q-complete manifold of the $\mathscr{O}(M)$-convex sets in a Stein manifold.

Our results generalize and improve the above mentioned topological results and include a characterization of Behnke-Stein type of the $(n-1)$-convex compact sets in an arbitrary noncompact complex manifold.

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[1]Some consequences of this fact are shown in [5].

Moreover we show applications relating to the subject of removable singularities for the tangential Cauchy-Riemann operator.

1. q-CONVEX SETS

Let M be a noncompact complex manifold of dimension $n \geq 2$ (connected, with a countable topology).

As is well known, M is said to be q-complete $(0 \leq q \leq n-1)$ in case there exists a C^∞ strongly q-plurisubharmonic proper function $\rho: M \to \mathbb{R}$. In particular M is 0-complete if and only if it is Stein, and every noncompact complex manifold is $(n-1)$-complete (see [9]). We also recall that q-completeness entails the vanishing of the Dolbeault cohomology spaces $H^{r,s}_{\overline{\partial}}(M)$, $r \geq 0$, $s > q$ and of the de Rham cohomology spaces $H^j_{DR}(M, \mathbb{C})$, $j > n+q$ (see [4, 20]).

If M is q-complete and $K \subset M$ is a compact set, let us say that K is q-*convex* in M if the following holds: Given arbitrarily an open neighborhood ω of K, it is possible to find a C^∞ strongly q-plurisubharmonic proper function $u: M \to \mathbb{R}$ such that $K \subset \{x \in M: u(x) < 0\} \subset\subset \omega$.

We have already considered this notion of q-convexity previously [13]. It appears as a generalization to a q-complete manifold of the standard notion of $\mathscr{O}(M)$-convexity relative to a compact set in a Stein manifold. In fact, if M is Stein, a compact set $K \subset M$ is 0-convex according to the above definition if and only if $K = \widehat{K}_M$ $(= \mathscr{O}(M)$-hull of $K)$ (see [12, Theorems 5.1.6 and 5.2.10]).

We point out that, though the $(n-1)$-completeness condition does not impose any restriction to a noncompact complex manifold M, not every compact set $K \subset M$ is $(n-1)$-convex in M (see Theorem 2 below).

Our first result is the following generalization of the result of Serre and Browder mentioned at the beginning.

Theorem 1. *Let M be a q-complete manifold of dimension $n \geq 2$ $(0 \leq q \leq n-1)$ and $K \subset M$ a compact set which is q-convex in M. Then the restriction map*

$$H^{n+q}(M, \mathbb{C}) \to H^{n+q}(K, \mathbb{C})$$

is surjective, moreover $H^r(K, \mathbb{C}) = 0$ for $r > n+q$.

Proof. The basic result on which Theorem 1 depends is the Andreotti-Grauert approximation theorem [4, Theorem 12] (see also [11, Theorem 12.11]). This implies the following fact:

(∗) Let $u: M \to \mathbb{R}$ be a C^∞ strongly q-plurisubharmonic proper function and set $M_0(u) = \{x \in M: u(x) < 0\}$; then, if W is any holomorphic vector bundle on M of finite rank, the restriction maps

$$Z^{r,s}_{\overline{\partial}}(M, W) \to Z^{r,s}_{\overline{\partial}}(M_0(u), W), \qquad r \geq 0, s \geq q,$$

have dense images, with respect to the natural Fréchet-Schwartz topologies of the spaces of C^∞ W-valued differential forms.

Moreover $M_0(u)$ is q-complete, since the function $(-u)^{-1}: M_0(u) \to \mathbb{R}$ is strongly q-plurisubharmonic and proper.

Then, in view of (∗) (for $W = M \times \mathbb{C}$) and of the vanishing of $H^{r,s}_{\overline{\partial}}(M)$ and $H^{r,s}_{\overline{\partial}}(M_0(u))$ for $r \geq 0$ and $s > q$, a standard argument (see [19, §6]) shows that the following is true too:

(**) The restriction map $H_{DR}^{n+q}(M, \mathbb{C}) \to H_{DR}^{n+q}(M_0(u), \mathbb{C})$ has dense image and $H_{DR}^r(M_0(u), \mathbb{C}) = 0$ for $r > n+q$.

Now, as K is q-convex in M, we can find a neighborhood basis of K of open sets of the form $M_0(u)$, such that, in addition, 0 is a regular value of u. For every such $M_0(u)$, $\overline{M_0(u)}$ is a (possibly disconnected) compact C^∞ manifold with boundary, and hence the Čech (or singular) cohomology spaces $H^j(\overline{M_0(u)}, \mathbb{C})$ and $H^j(bM_0(u), \mathbb{C})$, $j \geq 0$, are finite dimensional (see [7, p. 103]). Then, by the cohomology sequence with compact supports

$$\cdots \to H_c^j(M_0(u), \mathbb{C}) \to H^j(\overline{M_0(u)}, \mathbb{C}) \to H^j(bM_0(u), \mathbb{C}) \to \cdots,$$

also the spaces $H_c^j(M_0(u), \mathbb{C})$, $j \geq 0$, are finite dimensional, and by the Poincaré duality and the de Rham isomorphism, so are the spaces $H_{DR}^{2n-j}(M_0(u), \mathbb{C})$, $j \geq 0$, too. Therefore, since every vector subspace of a finite-dimensional Hausdorff vector space is closed (see [10, p. 38]), (**) implies that the restriction map $H_{DR}^{n+q}(M, \mathbb{C}) \to H_{DR}^{n+q}(M_0(u), \mathbb{C})$ is in fact surjective (at least in the case under consideration that 0 is a regular value of u).

Then the conclusion follows at once by the de Rham isomorphism and an inductive limit consideration. Q.E.D.

Our next theorem gives a characterization of $(n-1)$-convex compact sets in an arbitrary noncompact complex manifold, and, in particular, shows that for $q = n-1$ the converse of Theorem 1 is true too.

Theorem 2. *Let M be a noncompact complex manifold of dimension $n \geq 2$ and $K \subset M$ a compact set. The following conditions on K are equivalent*:

(a) *K is $(n-1)$-convex in M*;

(b) *The restriction map $H^{2n-1}(M, \mathbb{C}) \to H^{2n-1}(K, \mathbb{C})$ is surjective*;

(c) *$M \backslash K$ has no relatively compact connected components*;

(d) *Every C^∞ form in $Z_{\bar\partial}^{r, n-1}(K, W)$ can be uniformly approximated on K together with all derivatives of the coefficients by C^∞ forms in $Z_{\bar\partial}^{r, n-1}(M, W)$ ($r \geq 0$, W any holomorphic vector bundle on M of finite rank).*

Moreover, in case $H^{2n-1}(M, \mathbb{C}) = 0$, the above conditions are also equivalent to the following one:

(e) *$M \backslash K$ is connected.*

Proof. We first prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and that (d) $\Rightarrow$ (b). That (a) $\Rightarrow$ (b) follows at once from Theorem 1. Then assume that (b) is valid. Since M is noncompact, $H^{2n}(M, \mathbb{C}) = 0$ (see [7, p. 260]), hence the relative cohomology sequence

$$\cdots \to H^j(M, \mathbb{C}) \to H^j(K, \mathbb{C}) \to H^{j+1}(M, K; \mathbb{C}) \to \cdots$$

implies that $H^{2n}(M, K; \mathbb{C}) = 0$. On the other hand

$$H^{2n}(M, K; \mathbb{C}) \cong H_\Phi^{2n}(M \backslash K, \mathbb{C}),$$

where Φ is the paracompactifying family of supports in $M \backslash K$ made up by the subsets of $M \backslash K$ which are closed in M (see [8, p. 235]). Now, if there existed a relatively compact connected component U of $M \backslash K$, $H_{\Phi|U}^{2n}(U, \mathbb{C}) = H_c^{2n}(U, \mathbb{C}) \cong \mathbb{C}$ would be a direct summand of $H_\Phi^{2n}(M \backslash K, \mathbb{C})$, and the latter would not be zero. Therefore (b) $\Rightarrow$ (c).

The proof that (d) $\Rightarrow$ (b) is essentially the same as the proof of Theorem 1, using condition (d) in place of the Andreotti-Grauert approximation theorem. The vanishing of $H_{\overline{\partial}}^{r,s}(M)$ and $H_{\overline{\partial}}^{r,s}(K)$ for $r \geq 0$ and $s \geq n$, which is also needed, follows from the quoted result of [9] (see also below).

On the other hand, by resorting again to the Andreotti-Grauert approximation theorem, we get immediately, via an inductive limit consideration, that (a) $\Rightarrow$ (d).

Next let us prove that (c) $\Rightarrow$ (a), which will conclude the proof that (a), (b), (c), and (d) are equivalent. Thus assume that (c) is valid. Choose a C^∞ Hermitian metric G on M, and, in accordance with Greene-Wu [9], define a Laplacian Δ_0 on M by setting $\Delta_0 = \sum_{\alpha,\beta}^{1,n} G^{\alpha\beta}\partial^2/\partial z_\alpha \partial \overline{z}_\beta$ for every allowable system $(z_1, \dots, z_n)$ of local complex coordinates. It is proved in [9], using the Lax-Malgrange approximation theorem of Runge type applied to Δ_0, that there exists a nonnegative C^∞ strongly Δ_0-subharmonic proper function $\tau_0 \colon M \to \mathbb{R}$; and since $\Delta_0\tau_0$ is the trace of the Levi form of τ_0, the latter is strongly $(n-1)$-plurisubharmonic (which shows M to be $(n-1)$-complete). Moreover, after adding a constant, we may assume that τ_0 is strictly positive.

We are to prove that, given an open neighborhood ω of K, it is possible to find a C^∞ strongly $(n-1)$-plurisubharmonic proper function $u \colon M \to \mathbb{R}$ such that $K \subset \{u < 0\} \subset\subset \omega$.

Let ω_1 be an open neighborhood of K with $\omega_1 \subset\subset \omega$, and choose a positive real number R large enough so that $\omega_1 \subset M_R(\tau_0) = \{x \in M \colon \tau_0(x) < R\}$.

Since $M\backslash K$ has no relatively compact connected components, if x is any point in $M\backslash K$ we can, by the Lax-Malgrange theorem for Δ_0 again, find a Δ_0-harmonic function $h \colon M \to \mathbb{R}$ with $h(x) > 1$ and $h < 1$ on K. Therefore, as $\overline{M_R(\tau_0)}\backslash\omega_1$ is compact, we can choose finitely many Δ_0-harmonic functions $h_1, \dots, h_N \colon M \to \mathbb{R}$ which are < 1 on K, so that $\max\{h_1(x), \dots, h_N(x)\} > 1$ for every $x \in \overline{M_R(\tau_0)}\backslash\omega_1$. Set moreover $h_0 = \tau_0/R$. Clearly, we can choose a positive integer r large enough so that the function $u = h_0^{2r} + \dots + h_N^{2r} - 1$ is < 0 on K and is > 0 on $M\backslash\omega_1$. Hence $u \colon M \to \mathbb{R}$ is a C^∞ proper function such that $K \subset \{u < 0\} \subset\subset \omega$. There remains only to show that u is strongly $(n-1)$-plurisubharmonic. As a matter of fact we have, in the domain of every system of local coordinates $(z_1, \dots, z_n)$,

$$\Delta_0 u = 2r \sum_{j=0}^{N} \left\{ (2r-1) h_j^{2(r-1)} \sum_{\alpha,\beta}^{1,n} G^{\alpha\beta} \frac{\partial h_j}{\partial z_\alpha} \overline{\frac{\partial h_j}{\partial z_\beta}} + h_j^{2r-1} \Delta_0 h_j \right\},$$

and since h_0, $\Delta_0 h_0$ are strictly positive, $\Delta_0 h_j = 0$, $j = 1, \dots, N$, and $(G^{\alpha\beta}(z))$ is a positive definite Hermitian matrix, it follows that $\Delta_0 u$ is strictly positive. Thus u is strongly Δ_0-subharmonic, and consequently strongly $(n-1)$-plurisubharmonic, which concludes the proof that (c) $\Rightarrow$ (a).

There remains to prove the last statement of the theorem. Since clearly (e) $\Rightarrow$ (c), it suffices to show that, under the additional assumption $H^{2n-1}(M, \mathbb{C}) = 0$, (b) $\Rightarrow$ (e). As a matter of fact we have $H^{2n-1}(K, \mathbb{C}) = 0$, and therefore the exact cohomology sequence with compact supports

$$\cdots \to H^j(K, \mathbb{C}) \to H_c^{j+1}(M\backslash K, \mathbb{C}) \to H^{j+1}(M, \mathbb{C}) \to \cdots$$

implies that $H_c^{2n}(M\backslash K, \mathbb{C}) \cong H_c^{2n}(M, \mathbb{C}) \cong \mathbb{C}$, and this shows $M\backslash K$ to be connected. Q.E.D.

The above theorem improves a result obtained previously by Silva [17], using different techniques, in the direction of extending to $n \geq 2$ the classical Behnke-Stein theorem that characterizes the holomorphically convex compact sets in a noncompact Riemann surface. The result of [17] (generalized to complex spaces in [18]) is essentially equivalent to proving that (c) $\Leftrightarrow$ (d). An earlier approximation result of this type in the case $M = \mathbb{C}^n$ and $K = \overline{D}$ ($D \subset\subset \mathbb{C}^n$ a domain with smooth boundary) is due to Weinstock [21]. A proof that (c) $\Leftrightarrow$ (d) in the case $M = \mathbb{C}^n$ can also be found in [1].

2. Separation properties of q-convex sets

This section is devoted to discuss the topological properties of q-convex sets of the kind of the results of Alexander and of Alexander-Stout mentioned in the Introduction.

In the first place we have

Proposition 1. *Let M be a $(n-2)$-complete manifold of dimension $n \geq 2$, such that $H^{2n-2}(M, \mathbb{C}) = 0$, $K \subset M$ a compact set which is $(n-2)$-convex in M and $D \subset\subset M$ an open domain such that $bD \backslash K$ is a $(2n-1)$-dimensional topological manifold. Then, provided $M \backslash (\overline{D} \cup K)$ is connected,*[2] *$H_0(bD\backslash K, \mathbb{C}) \cong H_0(D\backslash K, \mathbb{C})$.*

Proof. By Theorem 1, $H^j(K, \mathbb{C}) = 0$ for $j \geq 2n-2$, and, by Theorem 2, $H^{2n-1}(\overline{D} \cup K, \mathbb{C}) = 0$. Moreover $0 = H^{2n}(\overline{D}, \mathbb{C}) = H^{2n}(\overline{D} \cup K, \mathbb{C})$ because M is noncompact. Therefore the Mayer-Vietoris cohomology sequence

$$\cdots \to H^j(\overline{D} \cup K, \mathbb{C}) \to H^j(\overline{D}, \mathbb{C}) \oplus H^j(K, \mathbb{C}) \to H^j(\overline{D} \cap K, \mathbb{C}) \to \cdots$$

implies that the restriction map $H^j(\overline{D}, \mathbb{C}) \to H^j(\overline{D} \cap K, \mathbb{C})$ is surjective for $j \geq 2n-2$ and is an isomorphism for $j \geq 2n-1$. Then the exact cohomology sequence with compact supports

$$\cdots \to H_c^j(\overline{D} \backslash K, \mathbb{C}) \to H^j(\overline{D}, \mathbb{C}) \to H^j(\overline{D} \cap K, \mathbb{C}) \to \cdots$$

gives $H_c^j(\overline{D} \backslash K, \mathbb{C}) = 0$ for $j \geq 2n-1$. On the other hand there is also an exact cohomology sequence with compact supports

$$\cdots \to H_c^j(D \backslash K, \mathbb{C}) \to H_c^j(\overline{D} \backslash K, \mathbb{C}) \to H_c^j(bD \backslash K, \mathbb{C}) \to \cdots,$$

and hence it follows that $H_c^{2n-1}(bD\backslash K, \mathbb{C}) \cong H_c^{2n}(D \backslash K, \mathbb{C})$. Then, by the Poincaré duality, we also have $H_0(bD\backslash K, \mathbb{C}) \cong H_0(D \backslash K, \mathbb{C})$, which concludes the proof. Q.E.D.

Proposition 1 implies in particular

Corollary 1. *Let M be a Stein manifold of dimension $n \geq 2$, such that $H^{2n-2}(M, \mathbb{C}) = 0$,*[3] *$X \subset M$ a compact set and $D \subset\subset M$ an open domain. Let $\widehat{X}_M$ denote the $\mathcal{O}(M)$-hull of X, and assume that $bD \backslash \widehat{X}_M$ is a $(2n-1)$-dimensional topological manifold. Then, provided $M \backslash (\overline{D} \cup \widehat{X}_M)$ is connected, $H_0(bD \backslash \widehat{X}_M, \mathbb{C}) \cong H_0(D \backslash \widehat{X}_M, \mathbb{C})$.*

[2] Note that, since $H^{2n-1}(M, \mathbb{C}) = 0$ (due to $(n-2)$-completeness), in view of Theorem 2 applied to $\overline{D} \cup K$, this assumption is not more restrictive than the assumption that $M \backslash (\overline{D} \cup K)$ has no relatively compact connected components.

[3] Of course this condition does not impose any restriction to the Stein manifold M if $n \geq 3$.

Proof. Set $K = \widehat{X}_M$. Then M is 0-complete, hence $(n-2)$-complete, and K is 0-convex in M, hence $(n-2)$-convex in M. Therefore the thesis follows at once from Proposition 1. Q.E.D.

We wish to point out that what Proposition 1 means is that the boundary of every connected component of $D\backslash K$ contains exactly one connected component of $bD\backslash K$; however saying that $bD\backslash K$ and $D\backslash K$ have the same number of connected components could be inappropriate, since this number need not be finite. To show an example, consider the open unit ball $\mathbb{B}$ in $\mathbb{C}^n$, $n \geq 2$, take a point $\zeta \in b\mathbb{B}$ and set

$$X = \{\zeta\} \cup \left[b\mathbb{B} \cap \left(\bigcup_{k=1}^{\infty} b\mathbb{B}\left(\zeta, \frac{1}{k}\right)\right)\right],$$

where $\mathbb{B}(\zeta, 1/k)$ is the open ball with center ζ and radius $1/k$. Then X is a compact subset of $b\mathbb{B}$ such that $b\mathbb{B}\backslash X = b\mathbb{B}\backslash\widehat{X}$ is made up by infinitely many connected components.

For a further investigation on the separation properties of q-convex sets we need the following proposition.

Proposition 2. *Let M be a q-complete manifold of dimension $n \geq 2$ $(0 \leq q \leq n-1)$, $K \subset M$ a compact set which is q-convex in M and $F \subset M$ a closed set endowed with a neighborhood basis of p-complete open sets $(0 \leq p \leq n-1-q)$. Then the restriction map $H^{n+p+q}(F, \mathbb{C}) \to H^{n+p+q}(F \cap K, \mathbb{C})$ is surjective, moreover $H^r(F \cap K, \mathbb{C}) = 0$ for $r > n+p+q$.*

Proof. Let Ω_1 be a p-complete open neighborhood of F and $\omega \subset\subset M$ an open neighborhood of K. We shall prove that there exists a C^∞-bounded $(p+q)$-complete open neighborhood $\Omega_2 \subset \Omega_1 \cap \omega$ of $F \cap K$ such that the restriction map $H^{n+p+q}(\Omega_1, \mathbb{C}) \to H^{n+p+q}(\Omega_2, \mathbb{C})$ is onto. By the $(p+q)$-completeness of Ω_2, we shall also have $H^r(\Omega_2, \mathbb{C}) = 0$ for $r > n+p+q$, and the conclusion will then follow by an inductive limit consideration, in view of the fact that, as Ω_1 and ω range through neighborhood bases of F and K respectively, $\Omega_1 \cap \omega$ ranges through a neighborhood basis of $F \cap K$, and hence so does Ω_2 too.

Since Ω_1 is p-complete, there exists a C^∞ strongly p-plurisubharmonic proper function $\varphi: \Omega_1 \to \mathbb{R}$ and, after adding a constant, we may assume that $F \cap K \subset \{\varphi < 0\}$. Moreover, since K is q-convex, there exists a C^∞ strongly q-plurisubharmonic proper function $u: M \to \mathbb{R}$ such that $K \subset \{u < 0\} \subset\subset \omega$. Consider the functions $\exp(\varphi): \Omega_1 \to \mathbb{R}$ and $\exp(u): M \to \mathbb{R}$. Since $\exp(\varphi) < 1$ on $F \cap K$, $\exp(u) < 1$ on K and $\exp(\varphi) \geq 1$ on $\Omega_1\backslash\{\varphi < 0\}$, $\exp(u) > 1$ on $M\backslash\omega$, we can choose a positive integer N large enough so that

$$\exp(N\varphi) + \exp(Nu) \begin{cases} < 1, & \text{on } F \cap K, \\ > 1, & \text{on } \Omega_1\backslash(\{\varphi < 0\} \cap \omega). \end{cases}$$

Then choose a regular value $\alpha \leq 1$ of $\exp(N\varphi)+\exp(Nu)$ such that $\exp(N\varphi)+\exp(Nu) < \alpha$ on $F \cap K$ and set

$$\rho = \exp(N\varphi) + \exp(Nu) - \alpha, \qquad \Omega_2 = \{x \in \Omega_1 : \rho(x) < 0\}.$$

Clearly $F \cap K \subset \Omega_2 \subset \Omega_1 \cap \omega$ and Ω_2 is C^∞-bounded.

Now, as one can easily check, the function $\rho: \Omega_1 \to \mathbb{R}$ is strongly $(p+q)$-plurisubharmonic and proper, and the same is true of the function $(-\rho)^{-1}: \Omega_2$

$\to \mathbb{R}$. Therefore Ω_1 and Ω_2 are both $(p+q)$-complete and $\Omega_2 = [\Omega_1]_0(\rho) = \{x \in \Omega_1 : \rho(x) < 0\}$. It follows, by the same reasoning as in the proof of Theorem 1, that the restriction map $H^{n+p+q}(\Omega_1, \mathbb{C}) \to H^{n+p+q}(\Omega_2, \mathbb{C})$ is surjective, which gives the desired conclusion. Q.E.D.

Now we can prove the following refinement of Proposition 1.

Proposition 3. *Let M be a q-complete manifold of dimension $n \geq 2$ $(0 \leq q \leq n-2)$, $K \subset M$ a compact set which is q-convex in M and $D \subset\subset M$ an open domain such that $\overline{D}$ has a neighborhood basis of p-complete open sets $(0 \leq p \leq n-2-q)$ and $bD \backslash K$ is a topological $(2n-1)$-dimensional manifold. Then $H_i(M \backslash (\overline{D} \cup K), \mathbb{C}) \cong H_i(M, \mathbb{C})$ and $H_i(bD \backslash K, \mathbb{C}) \cong H_i(D \backslash K, \mathbb{C})$ for $i \leq n-2-(p+q)$.*

Proof. Consider the Mayer-Vietoris cohomology sequence

$$\cdots \to H^j(\overline{D} \cup K, \mathbb{C}) \to H^j(\overline{D}, \mathbb{C}) \oplus H^j(K, \mathbb{C}) \to H^j(\overline{D} \cap K, \mathbb{C}) \to \cdots.$$

By Proposition 2 the homomorphism

$$H^{n+p+q}(\overline{D}, \mathbb{C}) \oplus H^{n+p+q}(K, \mathbb{C}) \to H^{n+p+q}(\overline{D} \cap K, \mathbb{C})$$

is surjective, and $H^j(\overline{D} \cap K, \mathbb{C}) = 0$ for $j > n+p+q$. Since moreover $0 = H^j(\overline{D}, \mathbb{C}) = H^j(K, \mathbb{C})$ for $j > n+p+q$, it follows that $H^j(\overline{D} \cup K, \mathbb{C}) = 0$ for $j > n+p+q$. Then consider the cohomology sequence with compact supports

$$\cdots \to H^j_c(M \backslash (\overline{D} \cup K), \mathbb{C}) \to H^j_c(M, \mathbb{C}) \to H^j(\overline{D} \cup K, \mathbb{C}) \to \cdots.$$

This implies, in view of the above,

$$(*) \qquad H^j_c(M \backslash (\overline{D} \cup K), \mathbb{C}) \cong H^j_c(M, \mathbb{C}) \quad \text{for } j \geq n+p+q+2.$$

Next consider the cohomology sequence with compact supports

$$\cdots \to H^j_c(\overline{D} \backslash K, \mathbb{C}) \to H^j(\overline{D}, \mathbb{C}) \to H^j(\overline{D} \cap K, \mathbb{C}) \to \cdots.$$

Since, by Proposition 2, the homomorphism $H^{n+p+q}(\overline{D}, \mathbb{C}) \to H^{n+p+q}(\overline{D} \cap K, \mathbb{C})$ is surjective and $H^j(\overline{D} \cap K, \mathbb{C}) = 0$ for $j > n+p+q$, it follows that $H^j_c(\overline{D} \backslash K, \mathbb{C}) = 0$ for $j \geq n+p+q+1$. Then the cohomology sequence with compact supports

$$\cdots \to H^j_c(D \backslash K, \mathbb{C}) \to H^j_c(\overline{D} \backslash K, \mathbb{C}) \to H^j_c(bD \backslash K, \mathbb{C}) \to \cdots$$

implies

$$(**) \qquad H^j_c(bD \backslash K, \mathbb{C}) \cong H^{j+1}_c(D \backslash K, \mathbb{C}) \quad \text{for } j \geq n+p+q+1.$$

Finally from $(*)$ and $(**)$ we get, via the Poincaré duality,

$$H_{2n-j}(M \backslash (\overline{D} \cup K), \mathbb{C}) \cong H_{2n-j}(M, \mathbb{C})$$

for $j \geq n+p+q+2$, and $H_{2n-1-j}(bD \backslash K, \mathbb{C}) \cong H_{2n-j-1}(D \backslash K, \mathbb{C})$ for $j \geq n+p+q+1$. This is the desired conclusion. Q.E.D.

Proposition 3 implies in particular the following refinement of Corollary 1:

Corollary 2. *Let M be a Stein manifold of dimension $n \geq 2$, $X \subset M$ a compact set and $D \subset\subset M$ an open domain such that $\overline{D}$ has a neighborhood basis of p-complete open sets $(0 \leq p \leq n-2)$ and $bD \setminus \widehat{X}_M$ is a $(2n-1)$-dimensional topological manifold. Then $H_i(M \setminus (\overline{D} \cup \widehat{X}_M), \mathbb{C}) \cong H_i(M, \mathbb{C})$ and $H_i(bD \setminus \widehat{X}_M, \mathbb{C}) \cong H_i(D \setminus \widehat{X}_M, \mathbb{C})$ for $i \leq n-2-p$.*

Proof. It is a straightforward consequence of Proposition 3 for $q = 0$ and $K = \widehat{X}_M$. Q.E.D.

Corollary 2 improves the results of [2 and 3] mentioned in the Introduction.

3. Removable singularities of $\overline{\partial}_b$

The above Propositions 1 and 3 and Corollaries 1 and 2 are of interest in the subject of removable singularities of the tangential Cauchy-Riemann operator $\overline{\partial}_b$. We proved the following theorem in this area [13, Theorem 2.3].

Let M be a $(n-2)$-complete manifold of dimension $n \geq 2$, $D \subset\subset M$ an open domain and $E \subset M$ a compact set which is $(n-2)$-convex in M. Assume that $bD \setminus E$ is connected and of class C^1. Then every continuous CR-function f on $bD \setminus E$ extends continuously to a function $F \in C^0(\overline{D} \setminus E) \cap \mathscr{O}(D \setminus E)$.

Now, by a direct inspection of the proof of this theorem, it appears that the connectedness of $bD \setminus E$ is not necessary; what is really needed in the proof is that $M \setminus (\overline{D} \cup E)$ is connected and every connected component of $bD \setminus E$ is the whole boundary, in $M \setminus E$, of a connected component of $D \setminus E$ (the latter being also necessary). Therefore, in view of Propositions 1 and 3, the following two other versions of the above-mentioned theorem are valid too.

Theorem 3. *Let M be a $(n-2)$-complete manifold of dimension $n \geq 2$ such that $H^{2n-2}(M, \mathbb{C}) = 0$, $D \subset\subset M$ an open domain and $E \subset M$ a compact set which is $(n-2)$-convex in M. Assume that $bD \setminus E$ is of class C^1 and that $M \setminus (\overline{D} \cup E)$ is connected. Then every continuous CR-function f on $bD \setminus E$ extends continuously to a function $F \in C^0(\overline{D} \setminus E) \cap \mathscr{O}(D \setminus E)$.*

Theorem 4. *Let M be a q-complete manifold of dimension $n \geq 2$ $(0 \leq q \leq n-2)$, $D \subset\subset M$ an open domain and $E \subset M$ a compact set which is q-convex in M. Assume that $bD \setminus E$ is of class C^1 and that $\overline{D}$ has a neighborhood basis of $(n-2-q)$-complete open sets. Then every continuous CR-function f on $bD \setminus E$ extends continuously to a function $F \in C^0(\overline{D} \setminus E) \cap \mathscr{O}(D \setminus E)$.*

Let us point out that the result of [13] recalled above is actually more general than is stated here, as it concerns not only CR-functions, but other kinds of CR-objects on $bD \setminus E$ as well, under suitable smoothness assumptions on $bD \setminus E$. The parallel more general versions of Theorems 3 and 4 are valid too.

It is worth noticing the following particular case of both Theorems 3 and 4 (in view of Corollaries 1 and 2 too).

Corollary 3. *Let $D \subset\subset \mathbb{C}^n$, $n \geq 2$, be an open domain and $X \subset \mathbb{C}^n$ a compact set. Assume that $bD \setminus \widehat{X}$ [4] is of class C^1 and $\mathbb{C}^n \setminus (\overline{D} \cup \widehat{X})$ is connected. Then every continuous CR-function f on $bD \setminus \widehat{X}$ extends continuously to a function*

[4] $\widehat{X}$ = polynomial hull of X.

$F \in C^0(\overline{D}\backslash\widehat{X}) \cap \mathscr{O}(D\backslash\widehat{X})$. Moreover the connectedness of $\mathbb{C}^n\backslash(\overline{D} \cup \widehat{X})$ holds in particular if $\overline{D}$ has a neighborhood basis of $(n-2)$-complete open sets.

This corollary shows that the $\overline{\partial}_b$-removability of polynomially convex sets is valid under less restrictive assumptions than it was proved in [14 and 15].

References

1. L. A. Aĭzenberg and Sh. Dautov, *Differential forms orthogonal to holomorphic functions or forms, and their properties*, "Nauka", Novosibirsk, 1975; English transl., Amer. Math. Soc., Providence, R.I., 1983.
2. H. Alexander, *A note on polynomial hulls*, Proc. Amer. Math. Soc. **33** (1972), 389–391.
3. H. Alexander and E. L. Stout, *A note on hulls*, Bull. London Math. Soc. (3) **22** (1990), 258–260.
4. A. Andreotti and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.
5. R. F. Basener, *Complementary components of polynomial hulls*, Proc. Amer. Math. Soc. **69** (1978), 230–232.
6. A. Browder, *Cohomology of maximal ideal spaces*, Bull. Amer. Math. Soc. **67** (1961), 515–516.
7. A. Dold, *Lectures on algebraic topology*, Springer-Verlag, Berlin, Heidelberg and New York, 1972.
8. R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1973.
9. R. E. Greene and H. Wu, *Whitney's imbedding theorem by solutions of elliptic equations and geometric consequences*, Proc. Sympos. Pure Math., vol. 27, Part 2, Amer. Math. Soc., Providence, R.I., 1975, pp. 287–296.
10. A. Grothendieck, *Topological vector spaces*, Gordon and Breach, New York, London and Paris, 1973.
11. G. M. Henkin and J. Leiterer, *Andreotti-Grauert theory by integral formulas*, Akademie-Verlag, Berlin, 1988.
12. L. Hörmander, *An introduction to complex analysis in several variables*, North-Holland, Amsterdam, 1973.
13. G. Lupacciolu, *Some global results on extension of CR-objects in complex manifolds*, Trans. Amer. Math. Soc. **321** (1990), 761–774.
14. ——, *A theorem on holomorphic extension of CR-functions*, Pacific J. Math. **124** (1986), 177–191.
15. J. P. Rosay and E. L. Stout, *Radò's theorem for CR-functions*, Proc. Amer. Math. Soc. **106** (1989), 1017–1026.
16. J. P. Serre, *Une propriété topologique des domaines de Runge*, Proc. Amer. Math. Soc. **6** (1955), 133–134.
17. A. Silva, *A n-Runge approximation theorem*, Boll. Un. Mat. Ital. **8** (1973), 286–289.
18. ——, *Behnke-Stein theorem for analytic spaces*, Trans. Amer. Math. Soc. **199** (1974), 317–326.
19. G. Sorani, *Homologie des q-paires de Runge*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 319–332.
20. G. Sorani and V. Villani, *q-complete spaces and cohomology*, Trans. Amer. Math. Soc. **125** (1966), 432–448.
21. B. M. Weinstock, *An approximation theorem for $\overline{\partial}$-closed forms of type $(n, n-1)$*, Proc. Amer. Math. Soc. **26** (1970), 625–628.

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